Bail - 2044189

Math 202a

Adnan Az
Wednesday

Wm Bade
701 Evans MWF 3-4

Weekly Problem Sets (no late HW) 60%
Final 40%

Consultant: A. Rida MWF 12-1
Office 775 Evans

Books: Munkers "General Topology" (More than required)
Royden

General Topology 3/5
Functional Analysis 2/5 (in Royden)

Concern: subsets $A, B, \ldots$ of a set X
elements or points x, y
$A \cup B$, $A \cap B$, ϕ = empty set
$A^c = \{x \mid x \in X, x \notin A\} = X \setminus A$

DeMorgans Rules Given $\{A_\alpha \mid \alpha \in I\}$

$$\left(\bigcup_{\alpha \in I} A_\alpha\right)^c = \bigcap_{\alpha \in I} A_\alpha^c \quad ; \quad \left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \bigcup A_\alpha^c$$

Defn: A family J of subsets of a set X is a topology on X if
(a) $\phi, X \in J$
(b) J is closed under arbitrary unions
i.e. $V_\alpha \in J$ for all $\alpha \in I$ then $\bigcup_{\alpha \in I} V_\alpha \in J$

(c) $U, V \in J \Rightarrow U \cap V \in J$
the pair (X, J) is a topological space

If T_1 and T_2 are topolgies on X and $T_1 \subseteq T_2$ then we say T_1 is coarser than T_2 and T_2 is finer than T_1.

The sets of T are the open sets of (X, J)

Examples ① $T = \{\emptyset, X\}$

② $T =$ all subsets of X

③ Euclidean Topology on $\mathbb{R}$. Let $J =$ all subsets U of $\mathbb{R}$ such that if $x \in U$ then $\exists\, \epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq U$.

Sets (a,b) are open for Euclidean Topology

Defn Let (X, J) be a top space. A subset B of J is a base (or basis) for J if every set in J is an union of sets in B

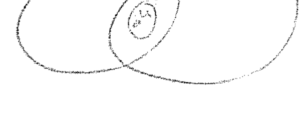
Example: $\mathbb{R}$ w/ Euclidean topology $B = \{(a,b) \mid a < b\}$

A family $C \subseteq J$ is a sub base for J if the family of all finite intersections of sets in C is a base for J.

Example: The set $\{(-\infty, a), (b, \infty) \mid a, b \in \mathbb{R}\}$ is a sub base for the Euclidean topology of $\mathbb{R}$

Theorem Let B be a family of subsets of X and $T =$ all unions of sets in B. Then T is a topology for X if

(i) if $u, V \in B$ and $u \in U \cap V$ then $\exists\, W \in B$ with $u \in W \subseteq U \cap V$



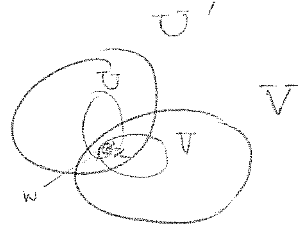
(ii) $X = \cup\{U \mid U \in B\}$

Proof: Let T = all unions of sets in $\mathcal{B}$

(i) $\emptyset \in T$ (logical convention: union of empty collection of subsets in $\mathcal{B}$)

(ii) T is closed under arb. unions of its elements (from defn)
(Show T is closed under finite intersections)

Let $U', V' \in T$, $x \in U' \cap V'$



$\exists\, U, V \in \mathcal{B}$ with $x \in U \subseteq U'$, $x \in V \subseteq V'$

Invoking [Defn (ii)]

$\exists\, W \in \mathcal{B} \subseteq T$ with $x \in W \subseteq U \cap V \subseteq U' \cap V'$

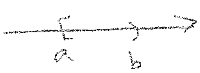
$\therefore U' \cap V'$ is ~~open set~~ in T.

□ QED

Examples ① $\mathbb{R}^2$ with $\mathcal{B} = \{(a,b)\times(c,d) \mid a,b,c,d \in \mathbb{R}\}$ "open boxes"

T = Euclidean Topology for $\mathbb{R}^2$

② $X = \mathbb{R}$ and $\mathcal{B} = \{[a,b) \mid a<b\}$

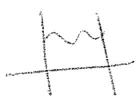


"Sorgenfrey Line"

③ $\mathbb{R}^2$ with $\mathcal{B} = \{[a,b)\times[c,d) \mid a,b,c,d \in \mathbb{R},\ a<b,\ c<d\}$

"Sorgenfrey Plane"

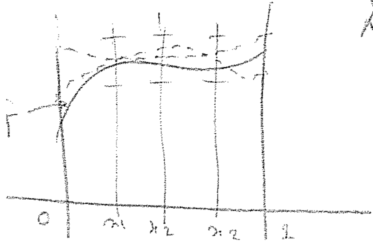
④ Let $X = C([0,1])$ all cont fns on unit interval

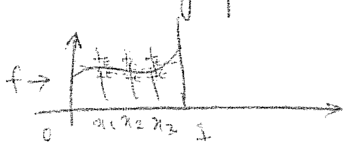


Let $\mathcal{B}$ = all sets of the form

~~$N(x_1, x_2, \ldots, x_n, \epsilon$~~

~~Let $\mathcal{B}$ = collection of all $N($~~ $N(f; x_1, x_2, \ldots, x_n, \epsilon) =$
$\{g \mid |g(x_i) - f(x_i)| < \epsilon \text{ for } 1 \le i \le n\}$





Let $\mathcal{B}$ = collection of all $N(f, x_1, \ldots x_n, \epsilon)$
for $x_1, \ldots, x_n \in [0,1]$ $\epsilon > 0$, $f \in C[0,1]$ (all n)

Then $\mathcal{B}$ is a base for a topology on $C[0,1]$

"Topology of pt wise convergence"

W. Bade

M202a Topology

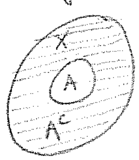
Adnan Tlv
24 Jan '92
Friday

Book available at Humolt / Black Oak (N Gate) / Textbook Exchange (Bancroft)

Definition: Let (X,T) be a top space. A nghbdh of a point x is an open set containing x. $N(x)$ - some open set containing x.

The nghbd systems of a point x = all nbhds of x.

Definition: A set A is a closed set if A^c is open



Theorem: Let (X,T) be given. The family of closed sets for T satisfies

(a) Its closed under finite unions

(b) Its closed under arbitrary intersections

(c) X and $\emptyset$ are closed

Let $\mathcal{F}$ be a family of sets satisfying (a) (b) (c). Then $T = \{F^c \mid F \in \mathcal{F}\}$ is a topology and $\mathcal{F}$ is the family of closed sets for T. (X,T)

Proof: DeMorgans identities

Definition: Let X be a set and $\rho: X \times X \to \mathbb{R}$ be a function satisfying

$\forall x \forall y \quad \rho(x,y) \geq 0$

$\forall x \forall y \quad \rho(x,y) = \rho(y,x)$

$\forall x \forall y \quad \rho(x,y) = 0 \Leftrightarrow x = y$

$\forall x \forall y \forall z \quad \rho(x,y) \leq \rho(x,z) + \rho(z,y)$ $\triangle$ inequality

Then ρ is a "metric" on X

(X, ρ) is a "metric space"

$S(x, \epsilon) = \{y \mid \rho(x,y) < \epsilon\}$ "open ball about x of radius ϵ"

Lemma: The family of all open balls in a metric space forms a basis for a topology

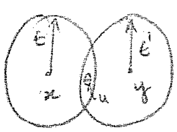
Proof: Recall need to satisfy

(i) $x \in \overset{\in B}{V} \cap \overset{\in B}{U} \Rightarrow \exists\, W \in B \ni x \in W,\ W \subseteq V \cap U$

(ii) $\bigcup \{U \mid U \in B\} = X$

Checking conditions for $B = \{ S(x,\epsilon) \mid x \in X, \epsilon > 0 \}$

clearly (ii) is satisfied (every pt in an open ball)



Let $u \in S(x,\epsilon) \cap S(y,\epsilon')$

Show $\exists\, \delta > 0$ such that $S(u,\delta) \subseteq S(x,\epsilon) \cap S(y,\epsilon')$

Pick δ such that

$\rho(x,u) + \delta < \epsilon$

$\rho(y,u) + \delta < \epsilon'$

Let $v \in S(u,\delta)$

We show $v \in S(x,\epsilon) \cap S(y,\epsilon')$

$\rho(x,v) \leq \rho(x,u) + \rho(u,v)$

$\therefore \rho(x,v) < \epsilon - \delta + \delta = \epsilon$

similarly $\rho(y,v) \leq \rho(y,u) + \rho(u,v)$

$< \epsilon' - \delta + \delta = \epsilon'$

$\square$ QED

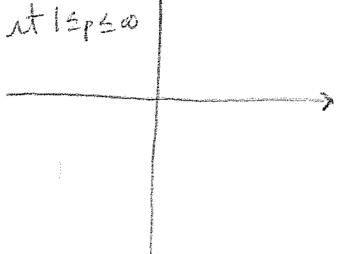
We call T the "metric topology" corresponding to ρ

Remark: Many metrics can yield the same topology

Example (a)

$r = [\xi, \eta]$ $\mathbb{R}^2$

at $1 \leq p \leq \infty$



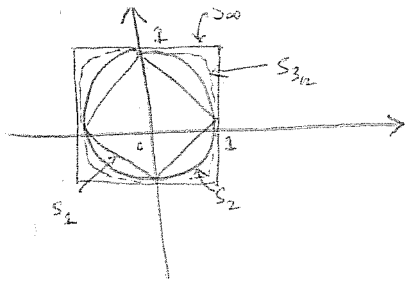
$1 \leq p < \infty$

$\rho_p(x_1, x_2) = [\,|\xi_1 - \xi_2|^p + |\eta_1 - \eta_2|^p\,]^{1/p}$

$p = 2$ gives Euclidean distance

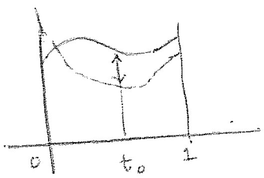
$\rho_\infty(x_1, x_2) = \max[\,|\xi_1 - \xi_2|, |\eta_1 - \eta_2|\,]$

$S_p(0,1)$ look interesting



Minkowskian Geometries - all yield same metr

Example (b) $X = C[0,1]$ with $\rho(f,g) = \max_{t \in [0,1]} |f(t) - g(t)|$



Easily seen to be metric (~~yielding~~ topology of uniform convergence)

Example (c) $X = C(0,1)$ all (possibly unbdd) continuous functions

example $\log(t)$

$$\rho(f,g) = \int_0^1 \frac{|f(t)-g(t)|\, dt}{1+|f(t)-g(t)|} \qquad \int \frac{x}{1+x}\, dx$$

$$\rho(f,g) \le \rho(f,h) + \rho(h,g)$$

Example (d) Non metrizable spaces

Product of c copies of $[0,1]$: no metric whatsoever is definable

c = cardinality of continuum

Bade

M202a Topology

Adnan Aziz
Mon Jan 27, '92

Problems 1-5
Due Monday Feb 3, 1992

x11 paper; one side only

· in fact there are Top Spaces w/no Metric which induces that Topology
· there are trivial metrics on any space (discrete $\rho(x,y) = 1/0$ $x \neq y, x = y$)

5, 4, 3, 2, 1, 0 grades on scores per question

Top Space (X, T)

Definition: The Closure $\bar{A}$ of a set is the intersection of all closed subsets containing A. (Well defined as $X \supseteq A$ and closed sets can be arbitrarily intersected)

Theorem: The closure operation satisfies

(a) $A \subseteq \bar{A}$

(b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

(c) $\bar{\bar{A}} = \bar{A}$

(d) $\bar{X} = X$, $\bar{\phi} = \phi$

(e) $p \in \bar{A} \Leftrightarrow$ every nghbd $N(p)$ intersects A.

Proof: (b) $\bar{A} \subseteq \overline{A \cup B}$; $\bar{B} \subseteq \overline{A \cup B} \Rightarrow \bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$

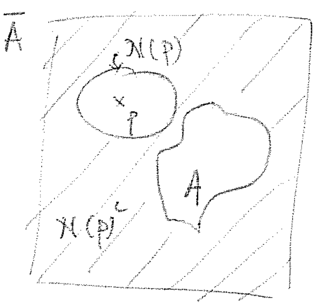
$A \cup B \subseteq \bar{A} \cup \bar{B}$ which is closed!

$\therefore \overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$ $\qquad \therefore \overline{A \cup B} = \bar{A} \cup \bar{B}$ QED

(e) (1) $p \in \bar{A} \Rightarrow$ every nghd of p intersects A

(2) every nghbd of p intersects $A \Rightarrow p \in \bar{A}$

(1) $\Leftrightarrow$ (1') if some nghbd $N(p)$ does not intersect $A \Rightarrow p \notin \bar{A}$

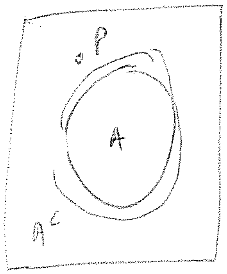


suppose $N(p) \cap A = \phi$

Then $N(p)^c$ is closed and contains A.

$\therefore \bar{A} \subseteq N(p)^c$ and $p \notin N(p)^c \Rightarrow p \notin \bar{A}$

(2)⇔(2') If $p \notin \bar{A} \Rightarrow \exists$ nghbd of p which doesnt intersect A



Then $\bar{A}^c$ is open, contains p and doesnt intersect A.

Theorem: Let X be a set and $A \to \bar{A}$ be a map of subsets of A which satisfies (a) – (d) of last theorem

Let $F = \{ A \mid A = \bar{A} \}$ and T = all complements of sets in F. Then T is a topology for X and $A \to \bar{A}$ is its closure operation.

Definition: Let $A \subseteq X$. A point p is an accumulation point of A if every nghbd $N(p)$ contains points of A different from p.

Example: $[0,1]$ $A = \{ 1/n \mid n \in \mathbb{N} \} \subseteq \mathbb{R}$ metric topology 0 is acc pt of A.

Theorem: The closure $\bar{A}$ of A is the union of A and its set of accumulation points

Proof: Let $A \subseteq X$, B = acc pts of A

clearly $B \subseteq \bar{A}$ $\therefore A \cup B \subseteq \bar{A}$

Let $p \in \bar{A}$. If $p \in A$ we're done

If $p \in \bar{A} \setminus A$ then every nghbd $N(p)$ has $(N(p) \setminus \{p\}) \cap A = \emptyset$

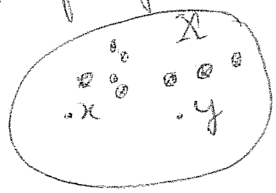
then $p \in B$. $\therefore \bar{A} \subseteq A \cup B$

Seperation Axioms:

(X,T) is a T_1 space if for every $x, y \in X$ ($x \neq y$) $\exists$ open U with $x \in U$, $y \notin U$ ($\therefore$ for every y $\exists$ V with $y \in V$, $x \notin V$) ($x \neq y$)

(X,T) is a T_2 space (Hausdorff space) if for each x, y ($x \neq y$) $\exists$ U, V open with $x \in U$, $y \in V$; $U \cap V = \emptyset$

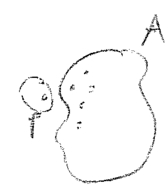
Example: Space which is T_1 but not T_2
Let X be an infinite set. A set U is open $\Leftrightarrow$ it is the complement of a finite set



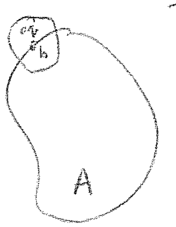
As soon as you put an open set around x $\Rightarrow$ cant have diff non intersecting open set about y.

Factoid: Non T_1 spaces dont deserve to exist, because you can collapse down points which are insperable (ie equiv rel) and get a T_1 space

Theorem: Let (X,T) be T_1 and $A \subseteq X$,
Then (b) the set B of accumulation points of A is closed
(a) If p is an acc point of A, then every nbd of p contains infinitely many pts of A.



Proof: check (a)
For (b), let $q \in \bar{B}$ and $N(q)$ be any nbd of q.
Then $N(q)$ contains a point b of B



$\therefore$ $N(q)$ is a nbd of B
$\therefore$ it contains infinitely many pts of A
$\therefore$ $q \in B$
$\therefore$ $\bar{B} \subseteq B$ so B is closed

K. Bade

M202a Topology

Jan 28 '72
Wednesday

T_1 and T_2 spaces

thereafter all Top spaces are T_1

Spaces w/ a countable base

$\mathbb{R}$: $\mathcal{B} = \{(r,s) \mid r,s \in \mathbb{Q}\}$

$\mathbb{R}\times\mathbb{R}$: $\mathcal{B} = \{(r,s)\times(t,u) \mid r,s,t,u \in \mathbb{Q}\}$

Facts about countable sets

X, Y countable $\Rightarrow X \times Y$ countable

$(\mathbb{R}, \mathcal{D})$ discrete top doesn't have a countable base

l^∞ (HW) has no countable base

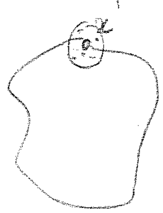
Theorem: Let (X, T) have a countable base and let $A \subseteq X$ be uncountable. Then A contains an accumulation point for A.

Proof: Let B be ~~the~~ a countable base and suppose A has no accumulation point. Let $x \in A$. $\Rightarrow N'(x)$ w/ only $x \in A \Rightarrow$ $\therefore$ con(tains) $N'(x)$

$\therefore \exists$ nghbd $N(x)$ w/ finitely many points of A

$\therefore (T1) \Rightarrow \exists B_x$ in $\mathcal{B}$ s.t. $B_x \cap A = \{x\}$

The map $x \to B_x$ is 1-1 A into B $(\to\leftarrow)$



A

Definition: A set A is dense in a set B if $\bar{A} \supseteq B$.

A set A is dense if it is dense in X.

Example: $\mathbb{Q}$ dense in $\mathbb{R}$

Theorem:

Definition: A topological space is separable if it contains a countable dense set

Example: $\mathbb{R}$ is separable

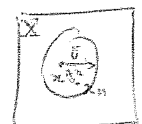
Theorem: A topological space with a countable base is seperable

Proof: Let B be a countable base and enumerate
$B = \{B_n \mid n \in \mathbb{N}\}$
In each B_k pick x_k
Let $A = \{x_n \mid x \in \mathbb{N}\}$ Its dense for if $y \in \bar{A}^c$
then $\exists B_j \in B$ with $B_j \cap \bar{A} = \emptyset$ $(\rightarrow\leftarrow)$
$\because x_j \notin A$.

Remark: There are spaces which are seperable which do not have countable bases. (Examples later)

Theorem: A seperable metric space has a countable base

Proof: Let $A = \{x_n \mid n \in \mathbb{N}\}$ be a countable dense set
For each x_n consider the open balls $S(x_n, r)$ where $r \in \mathbb{Q}^+$ (positive rationals) Let $B = \{S(x_n, r) \mid n \in \mathbb{N}, r \in \mathbb{Q}^+\}$



Clearly B is countable

Let $x \in X$, $\delta > 0$ consider $S(x, \delta)$
[To show $\exists$ some x_n and rational $r > 0$ s.t. ~~x_n~~ $x \in S(x_n, r) \subsetneq S(x, \delta)$

Find rational $r > 0$, so $r < \delta/2$, pick
$x_n \in S(x, r)$. then $x \in S(x_n, r)$
Let $y \in S(x_n, r)$ then $\rho(y, x_n) \leq \rho(y, x_n) + \rho(x_n, x) \leq r + r < \delta$

$\therefore x \in S_n(x_n, r) \subseteq S(x, \delta)$

$\square$ QED

Definition: An open covering (cover) of a set $A \subseteq X$ is a family $\mathcal{A}$ of open sets such that $A \subseteq \bigcup \{ U \mid U \in \mathcal{A} \}$.

An open cover $\mathcal{B}$ of A is a subcover of $\mathcal{A}$ if $\mathcal{B} \subseteq \mathcal{A}$.

Theorem: (Lindelof) If (X, T) has a countable base, then every open covering of a set $A \subseteq X$ has a countable subcovering.

W. Bade

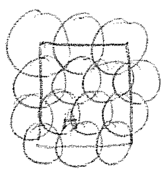
M202a Topology

Jan 31 '9[illegible]
Friday

Problems due Monday

Theorem: (Lindelöf) Let (X, T) have a countable base. Then every open covering $\mathcal{a}$ of a set $A \subseteq X$ has a countable subcovering

$A \subseteq \bigcup \{ U \in \mathcal{a} \}$



Proof: Let $\mathcal{a}$ cover A. Let $\mathcal{C}$ = the family of all sets B in the base such that $B \subseteq U$; $U \in \mathcal{a}$.

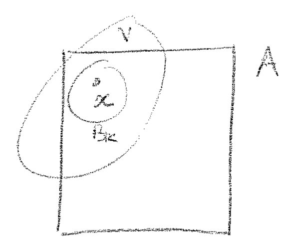
Enumerate $\mathcal{C} = \{ B_k \mid k \in \mathbb{N} \}$. Now for each k let $U_k \in \mathcal{a}$ be such that $B_k \subseteq U_k$

Then we assert the family $\{ U_k \mid k \in \mathbb{N} \}$ is an open covering of A.

Its certainly a subset of $\mathcal{a}$, so if it covers, it is a countable subcovering.

For let $x \in A$. Then $\exists V \in \mathcal{a} \ni x \in V$.

$\exists B_k \in \mathcal{C}$ so that $x \in B_k \subseteq V$



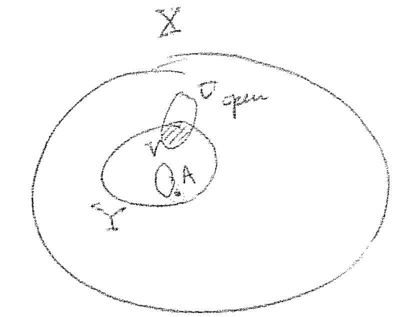
But then $x \in B_k \subseteq U_k$

$\therefore A \subseteq \bigcup_{k=1}^{\infty} U_k$

□ QED

Relative Topology:

Definition: Let (X,T) be a topological space and $Y \subseteq X$.
Let $T_Y = \{ V \subseteq Y \mid V = Y \cap U$ where U is open in $X \}$



Clear that T_Y is a topology on Y, called relative topology of Y.

Theorem: (a) $A \subseteq Y$ is T_Y closed $\iff A = Y \cap K$ where K closed in X.
(b) A point $p \in Y$ is a T_Y accumulation point of $A \subseteq Y$
$\iff$ p is a T accumulation point of T.
(c) The T_Y closure $\bar{A}^Y$ of $A \subseteq Y$ is $\bar{A} \cap Y$.

Remarks: (a) The subset of a metric space is a metric space in its relative topology (use same metric & all's well)
(b) A seperable space X can have a subset Y which is not seperable for the T_Y topology (examples later)
(c) The property of having a countable base drops from X to Y.

Connectedness:

Definition: Sets A and B in a topological space X are seperated in X if $A \cap \bar{B} = \bar{A} \cap B = \emptyset$

Examples: In $\mathbb{R}$ $(-\infty, a)$ and (a, ∞) are seperated
$(-\infty, a)$ and $[a, \infty)$ are not

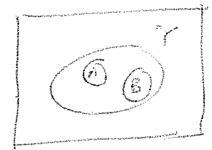
Definition: later. more feeling $\rightarrow$ Two disjoint closed sets are always seperated. Two sets are seperated if and only if neither contains any acc. pt of the other

Lemma: Let $Y \subseteq X$ and $A, B \subseteq Y$. Then A and B are seperated in Y (for T_Y) $\Leftrightarrow$ A and B are seperated in X.

($\Leftarrow$)
Proof: Let A, B be seperable in X. Then $A \cap \bar{B} = \bar{A} \cap B = \emptyset$

$A \cap \bar{B}^Y = (A \cap \bar{B}) \cap Y = \emptyset$ Similarly $\bar{A}^Y \cap B = \emptyset$

($\Rightarrow$) Let A, B be seperable in Y



Then $A \cap \bar{B}^Y = \phi = \bar{A}^Y \cap B$

$$A \cap \bar{B} = A \cap \bar{B} \cap (Y \cup Y^c)$$
$$= \underbrace{(A \cap \bar{B} \cap Y)}_{= \emptyset} \cup \underbrace{(A \cap \bar{B} \cap Y^c)}_{= \emptyset} \quad (\because A \subseteq Y \Rightarrow A \cap Y^c = \emptyset)$$

Definition: A set $Y \subseteq X$ (X, T) is connected if it is not the union $Y = A \cup B$ where A and B are seperated in X (Equiv seperated in any set Z with $Y \subseteq Z$)

Remarks: If $Y = A \cup B$, $A \cap B = \emptyset$ and A and B are seperated then A and B are both open and closed in T_Y.

$A \cup B = Y$, $A \cap B = \emptyset$
$A \cap \bar{B} = \emptyset$ $\therefore$ ~~$A \cap B$~~ $\bar{B} = B$
Similarly $\bar{A} \cap B = \emptyset$ $\therefore A = \bar{A}$

in Y

Theorem: Let $\mathbb{R}$ have its Euclidean topology. The connected sets in $\mathbb{R}$ are precisely the intervals and single points.
[Any kind of intervals $\frac{1}{2}$ open, $\frac{1}{2}$ inf etc]
In $\mathbb{R}^2$: There exists a connected subset of $\mathbb{R}^2$ which is the countable disjoint union of closed intervals

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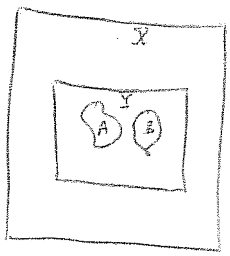
M202a

Feb 3 '92
Monday

$A, B \subseteq X$

A, B separated if $A \cap \bar{B} = \bar{A} \cap B = \phi$



$Y \subseteq X$ is connected if it is not the union
if it is not the union of two separated sets

$\mathbb{R}$: $\mathbb{R}$ is an ordered field in which every set bdd above has a least upper bound. (and any set with a lower bound has a greatest lower bound)

Birkhoff & MacLane "Modern Algebra" Any such set is isomorphic to $\mathbb{R}$

$x < y \Leftrightarrow y - x > 0$; $\rho(x, y) = |x - y|$

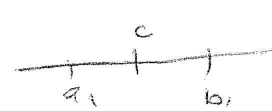
$[a, b)$ $(-\infty, c)$ (a, b) etc are called intervals

Theorem: The connected sets in $\mathbb{R}$ are precisely its intervals and single points.

Proof: show $\mathbb{R}$ is connected (same proof works for any interval)

Suppose $\mathbb{R} = A \cup B$ where $A, B \neq \emptyset$ and $A \cap \bar{B}$, $\bar{A} \cap B = \emptyset$ (separated)

Pick $a_1 \in A$, $b_1 \in B$ and $a_1 < b_1$.



Let c = midpoint of $[a_1, b_1]$

If $c \in A$ let $a_2 = c$ and $b_2 = b_1$

o.w. if $c \in B$ let ~~$a_2 = a_1$~~ $a_2 = a_1$ and $b_2 = c$

Etc.

$\therefore \{a_n\} \subseteq A$ $\{b_n\} \subseteq B$ $\quad a_1 \leq a_2 \leq \cdots \leq b_2 \leq b_1$

$|b_n - a_n| \to 0$ $\left(< \frac{1}{2^n} |(a - b)| \right)$

Let $d = \text{lub}\{a_1, a_2, \ldots\} = \text{glb}\{b_1, b_2, \ldots\}$

A and B are both open and closed in $\mathbb{R}$ $(\because B = \bar{B}$ & $A = \bar{A})$

$\therefore d \in A \cap B$ $(\Rightarrow\Leftarrow)$ separation.

$\Rightarrow \mathbb{R}$ is connected

Let $E \subseteq \mathbb{R}$ be connected, and not a singleton

Let $a, b \in E$, $a < b$ Let $a < c < b$. TST $c \in E$

We assert $c \in E$. For if not,

$(-\infty, c)$ and $(c, +\infty)$ are a pair of sets seperated in $\mathbb{R}$

whose union contains E.

$\therefore E \cap (-\infty, c)$ $\quad E \cap (c, +\infty)$ gives a seperation of E. $(\Rightarrow\Leftarrow)$

$\therefore$ E is an interval (contains all pts in between any of its pts)

Also singletons are clearly connected.

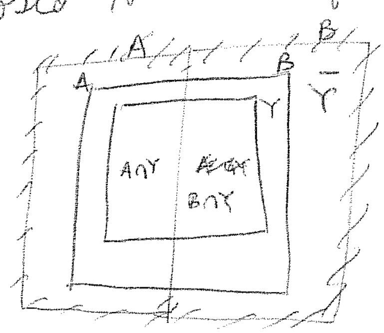
$\mathbb{R}^n$ is connected: But above proof wont work

Theorem: The closure of a connected set is connected.

Let $Y \subseteq X$ be connected

suppose $\bar{Y} = A \cup B$ $\quad A \cap \bar{B} = \bar{A} \cap B = \emptyset$ $\quad (A \neq \emptyset \quad B \neq \emptyset)$

$\therefore A, B$ are disjoint and closed subsets of $\bar{Y}$



$A \cap Y$, $B \cap Y$ are open v closed in Y

Then one is $\emptyset$ since Y is connected

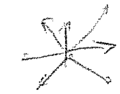
: o.w. Y not connected. Say $B \cap Y = \emptyset$ $\quad \therefore Y \subseteq A$ But A is closed in $\bar{Y}$ $\therefore$ also closed in X

$\because$ its the intersection of $\bar{Y}$ w/a set closed in X.

But then $\bar{Y} \subseteq A$ $\quad \therefore$ B is empty $(\Rightarrow\Leftarrow)$

Theorem: let $\mathcal{A}$ be a family of connected sets in a topological space no two of which are seperated. then $\bigcup\{A \mid A \in \mathcal{A}\}$ is connected

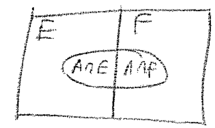
Corollary: $\mathbb{R}^n$ is connected look at all rays from origin



(not sep, not even disjoint) "Path wise connected"

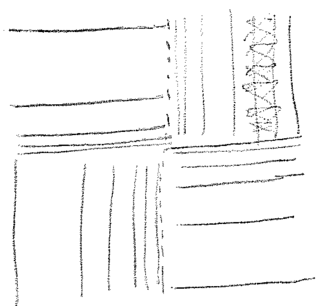
Proof: let $C = \bigcup\{A \mid A \in \mathcal{A}\}$ suppose C not connected. Then $C = E \cup F$ where $E \cap \bar{F} = \bar{E} \cap F = \emptyset$. If $A \in \mathcal{A}$ then $A \subseteq E$ or $A \subseteq F$.

o.w. $A \cap E$, $A \cap F$ gives a seperation of A.



Given $A, B \in \mathcal{A}$ then both are in E or in F

(else seperated) $\therefore$ all are in E or all are in F.



X is connected

If $X = A \cup B$ sep

If A contains one point of a segment it contains the segment $\Rightarrow$ contains all 4 quads

due to Willis Gustin @UCLA

W. Bade M202a Topology February 5 Wednesday

Regular Space

(1) If A is closed & $y \notin A \Rightarrow \exists\, U$ open, V open with $U \cap V = \emptyset$ & $A \subseteq U$, $y \in V$

(2) If A is closed & $y \notin A \Rightarrow \exists\, V$ open with $y \in V$, $\overline{V} \cap A = \emptyset$

(1) $\Leftrightarrow$ (2)

Theorem: A metric space is regular

Proof: Let A be closed $y \notin A$. Then A^c is open

$\Rightarrow \exists\, \delta > 0$ such that $S(y, \delta) \cap A = \emptyset$ $\therefore$ $S(y, \delta/2) \cap A = \emptyset$

Definition: Let (X, T) be a topological space. A component is a maximal connected subset

Theorem: A component is closed. Every connected subset is contained in a component. Distinct componets are seperated.

A connected set ~~xxxx~~ C is maximal if it is not a proper subset of another connected set.

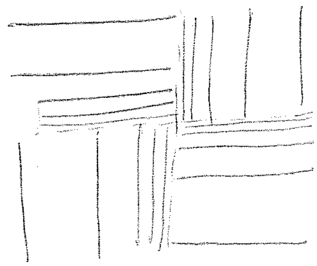
Proof: Closure of a connected set is connected

$\Rightarrow$ comp's are closed

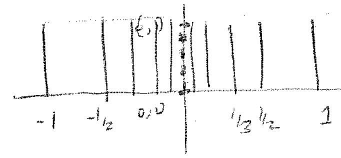
Let A be connected and let $\mathcal{a}$ be the family of all connected sets containing A. Then $\bigcup\{B \mid B \in \mathcal{a}\}$ is connected $\because$ no pair is seperated. Then C is a component containing A.

Let A, B be components. If A, B not seperated, then $A \cup B$ is connected! $\therefore$ Neither A nor B are maximal.

Recall last example



Example 2: It is not true that if E and F are components of X, then $\exists$ a seperation of X say $X = A \cup B$ $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ with $E \subseteq A$ and $F \subseteq B$.



Let $X = \{\pm \frac{1}{n} \times \overset{[0,1]}{I}\} \cup \{0,0\} \cup \{0,1\}$

(0,0) and (0,1) are distinct components and both lie on the same side of any seperation of X.

One part of a seperation of X contains ∞ many line segments

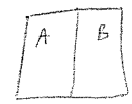
Say A contains an ∞ of lines.

then $(0,0) \in A$, $(0,1) \in A$ ($\because$ A closed)

$X = A \cup B$

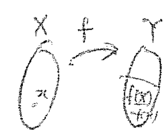
$A \cap \bar{B} = \emptyset$

$\bar{A} \cap B = \emptyset$



$B = \bar{B}$

Continuous Maps:



Definition: Let (X, T) and (Y, T_1) be top spaces and $f: X \to Y$ be a mapping from X to Y.

(a) f is continuous at $x \in X$ if for each open nbd ~~of B~~ V of $f(x)$ in Y, $\exists$ open nbd U of x in X s.t. $f(U) \subseteq V$.

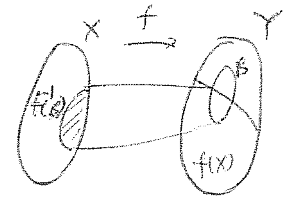
(b) f is continuous if it is continuous at all $x \in X$.

(c) If f is 1-1 onto Y and f^{-1} is continuous, then f is a "homeomorphism", we say X and Y are homeomorphic.

Very weak form of Geometry (Klein) Teacup $\equiv$ Donut.

To show $\mathbb{R}^m$ not homeomorphic $\mathbb{R}^n$ is a hard problem.

Let $f: X \to Y$ be a map. Let $B \subseteq Y$, Then $f^{-1}(B) = \{x \mid f(x) \in B\}$



Rules for f^{-1}:

(1) For all $A \subseteq X$ and $B \subseteq Y$

$$f(f^{-1}(B)) \subseteq B$$
$$f^{-1}(f(A)) \supseteq A$$

(2) $f^{-1}(\bigcup_\alpha B_\alpha) = \bigcup_\alpha f^{-1}(B_\alpha)$

(3) $f^{-1}(\bigcap_\alpha B_\alpha) = \bigcap_\alpha f^{-1}(B_\alpha)$

(4) $f^{-1}(B^c) = [f^{-1}(B)]^c$

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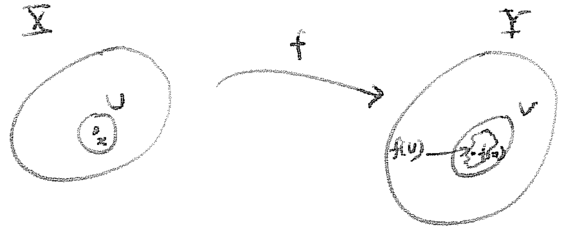
M 202a Topology

Feb 7 '9?
Friday

$f: X \to Y$ is continuous at x if given any open ngbd V of $f(x)$ $\exists\, U$ open ngbd of x s.t. $f(U) \subseteq V$



Theorem: Let (X,T) & (Y,T) be top spaces and $f: X \to Y$

TFAE (a) f is continuous

(b) $f^{-1}(V)$ is open in X for every open V in Y

(c) $f^{-1}(K)$ is closed in X for every closed K in Y

(d) For all $A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$

(e) For all $B \subseteq Y$ $f^{-1}(\bar{B}) \supseteq \overline{f^{-1}(B)}$

Proof: (a) $\Rightarrow$ (b)

Let V be open in Y and $x \in f^{-1}(V)$

By (a) $\exists U_x$ open nbd of x such that $f(U_x) \subseteq V$ $\therefore U_x \subseteq f^{-1}(V)$

$\therefore f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ = union of open sets so $f^{-1}(V)$ is open

(b) $\Rightarrow$ (a)

Let $x \in X$ and V be an open ngbd of $f(x)$. Then by (b) $f^{-1}(V)$ is an open nbd of x, and $f(f^{-1}(V)) \subseteq V$

(b) $\Leftrightarrow$ (c)

Recall if $B \subseteq Y$, then $f^{-1}(B^c) = f^{-1}(B)^c$

(b$\Rightarrow$c) If K is closed in Y $\Rightarrow$ K^c is open in Y

$f^{-1}(K^c) \overset{\text{Rec}}{=} f^{-1}(K)^c$ is open in X

$\Rightarrow f^{-1}(K)$ is closed in X

(c$\Rightarrow$b) similar; closed sets replaced by open ones

(c) $\Rightarrow$ (e) Let $B \subseteq Y$. Then $f^{-1}(B) \subseteq f^{-1}(\bar{B})$ which is closed

$\therefore \; f^{-1}(B) \subseteq f^{-1}(\bar{B})$

(e) $\Rightarrow$ (c) Let F be closed in Y

$\therefore \; f^{-1}(F) = f^{-1}(\bar{F}) \supseteq \overline{f^{-1}(F)}$

$\therefore \; f^{-1}(F)$ is closed

(e) $\Rightarrow$ (d) Have $f^{-1}(\bar{B}) \supseteq \overline{f^{-1}(B)}$ for all B. Let $A \subseteq X$

and take $B = f(A)$

$\therefore \; f^{-1}(\overline{f(A)}) \supseteq \overline{f^{-1}(f(A))} \supseteq \bar{A}$

$\therefore \; f(\bar{A}) \subseteq f[f^{-1}(\overline{f(A)})] \subseteq \overline{f(A)}$

(d) $\Rightarrow$ (e) Suppose $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$

Let $B \subseteq Y$ and $A = f^{-1}(B)$

$\therefore \; f(\overline{f^{-1}(B)}) \subseteq \overline{f(f^{-1}(B))} \subseteq \bar{B}$ using $f(f^{-1}(B)) \subseteq B$

$\therefore \; f^{-1}(\bar{B}) \supseteq f^{-1}(\overline{f(f^{-1}(B))})$

$\supseteq f^{-1}[f(\overline{f^{-1}(B)})] \supseteq \overline{f^{-1}(B)}$

Spaces

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

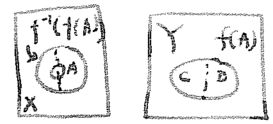
If f and g are continuous, then $g \circ f : X \to Z$ is continuous

Look at open sets & their inverse images.

Theorem: The continuous image of a connected set is connected.

Proof: Let $f: X \to Y$ be continuous and $A \subseteq X$ be connected.

Suppose $f(A) = C \cup D$ where C, D are both open and closed in $f(A)$. Show either C or D is $\emptyset$.



$\therefore$ by continuity of f

$f^{-1}(C)$ and $f^{-1}(D)$ are both open and closed in $f^{-1}(f(A))$

$\therefore$ $A \cap f^{-1}(C)$ and $A \cap f^{-1}(D)$ are disjoint open and closed sets in A.

But A is connected!

$\therefore$ one is empty.

say $A \cap f^{-1}(C) = \emptyset$

$\therefore$ $A \subseteq f^{-1}(D)$

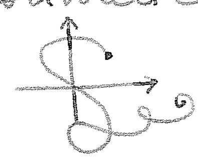
$\therefore$ $f(A) \subseteq D$

$\therefore$ $C = \emptyset$

$\square$ QED

$f([0,1])$ is an arc

continuous $\Rightarrow$ arc is connected



'Arcwise connected'

An arc is a cont. image of $[0,1]$

(X, T) is arcwise connected, if for each $p, q \in X$ $p \neq q$ $\exists$ an arc on X st $f(0) = p$ $f(1) = q$

arcwise connected $\Rightarrow$ connected

(But connected $\not\Rightarrow$ arcwise conn)

Also look at 

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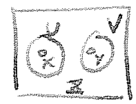
M202a

Feb 10 1992
Monday

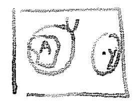
Definition:

Let (X,T) be a topological space

(a) (X,T) is Hausdorff (or T-2) if for each pair $x, y \in X$ $(x \neq y)$ $\exists U, V$ open disjoint so $x \in U$, $y \in V$



(b) (X,T) is regular if for each A closed, $y \notin A$ $\exists U, V$ open so $A \subseteq U$, $y \in V$ and $U \cap V = \emptyset$



(c) (X,T) is normal if given A, B closed $A \cap B = \emptyset \Rightarrow \exists U, V$ open $A \subseteq U$, $B \subseteq V$ And $U \cap V = \emptyset$

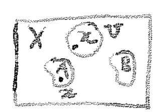
Regular space which is not normal see Munkers

Ex: countable connected Hausdorff
Connected set s.t. remove a pt $\Rightarrow$ totally disconnected

Theorem:

Let (X,T) be regular and have a countable base. Then it is normal.

Proof:



Let A, B be disjoint closed sets. Let $x \notin B$.
Then $\exists$ an open U so that $x \in U$ and $\overline{U} \cap B = \emptyset$
(Equiv defn of regularity)

Let $\mathcal{a} = \{U \text{ open} \mid \overline{U} \cap B = \emptyset\}$

Let $\mathcal{B} = \{V \text{ open} \mid \overline{V} \cap A = \emptyset\}$

Then $\mathcal{a}$ is an open covering of B^c
and $\mathcal{B}$ is an open covering of A^c

Countable base $\Rightarrow$ Lindeloff might be useful

By Lind: We can find a sequence $\{U_n\} \subseteq \mathcal{a}$ s.t. $A \subseteq \bigcup_{n=1}^{\infty} U_n$; $A \subseteq B^c$

$\{V_n\} \subseteq \mathcal{B}$ s.t. $B \subseteq \bigcup_{n=1}^{\infty} V_n$; $B \subseteq A^c$

But V_i and V_j arent necc disjoint. Else we would be done.

Define $U_1^* = U_1 \setminus \overline{V_1}$ $\quad V_1^* = V_1 \setminus \overline{U_1}$

$U_2^* = U_2 \setminus \overline{(V_1 \cup V_2)}$ $\quad V_2^* = V_2 \setminus \overline{(U_1 \cup U_2)}$

. . . . . . $\quad$ . . . .

$U_n^* = U_n \setminus \overline{\left(\bigcup_{i=1}^{n} V_i\right)}$ $\quad V_n^* = V_n \setminus \overline{\left(\bigcup_{i=1}^{n} U_i\right)}$

. . . . . $\quad$ . . . .

Assert $A \subseteq \bigcup_{n=1}^{\infty} U_n^*$ $\quad B \subseteq \bigcup_{n=1}^{\infty} V_n^*$

$\because$ Always throwing away points in A^c similarly B

Show $\left(\bigcup_{n=1}^{\infty} U_n^*\right) \cap \left(\bigcup_{n=1}^{\infty} V_n^*\right) = \emptyset$

Suppose not. Then $\exists\, n, p$ s.t. $U_n^* \cap V_p^* \neq \emptyset$

Can suppose $p \leq n$

But $U_n^* \cap \underbrace{\bigcup_{i=1}^{p} V_i}_{\subseteq \bigcup_{i=1}^{n} V_i^*} = \emptyset$

So $U_n^* \cap V_p = \emptyset$

$\therefore U_n^* \cap V_p^* = \emptyset$ $\quad (\because V_p^* \supseteq V_p)$

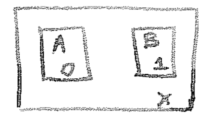
Contradiction

Theorem (Urysohn's Lemma):

Let A, B be disjoint closed sets in a normal. Then $\exists$ a continuous real valued function f on X such that

(1) $0 \leq f(x) \leq 1 \quad (x \in X)$

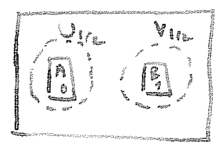
(2) $f(A) = 0 \ , \ f(B) = 1$



Proof:

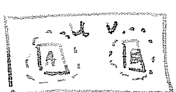
Let $U_{1/2}, V_{1/2}$ be disjoint open sets containing A and B resp. such that

$$A \subseteq U_{1/2} \subseteq \overline{U}_{1/2} \subseteq V_{1/2}^c \ ; \ B \cap V_{1/2}^c = \emptyset$$



Aside:

Normal



$U \cap V = \emptyset$ we are using more: ie $\overline{U} \cap \overline{V} = \emptyset$

$A \cap U^c = \emptyset$

$\exists U_1$ open s.t. $A \subseteq U_1$, $U_1 \cap U^c = \emptyset$

$\therefore U_1^c \cap B = \emptyset$

similarly $\exists V_1$ ~~$\overline{V}_1 \cap B \neq$~~ $\overline{V}_1 \cap \overline{U}_1 = \emptyset$

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M202a Topology

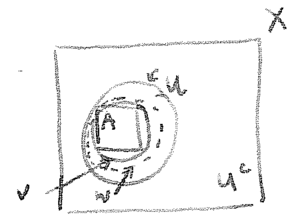
Feb 12. 1992
Wednesday

Zero dimensional space

$U \cap V = \emptyset$ U and V are both open and closed

Cantor set : Remove a dense open set
Remains zero dim dense set

Lemma: Let (X,T) be normal, A closed $A \subseteq U$ open
Then $\exists V$ open so $A \subseteq V \subseteq \bar{V} \subseteq U$



Proof: $A \cap U^c = \emptyset$

$\therefore \exists V, W$ open $W \cap V = \emptyset$
with $A \subseteq V$, $U^c \subseteq W$

$V \subseteq W^c$ which is closed

$\therefore \bar{V} \subseteq W^c$

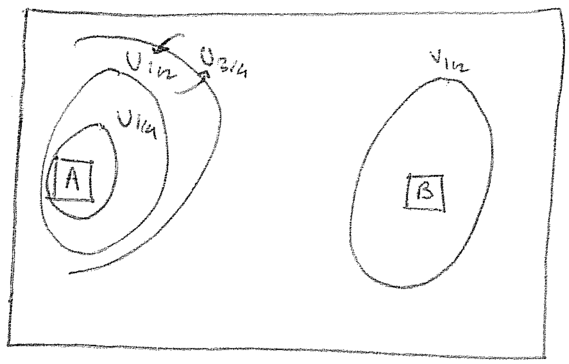
$\therefore A \subseteq V \subseteq \bar{V} \subseteq W^c \subseteq U$

Corollary: Let (X,T) be normal, A, B closed disjoint Then $\exists U, V$ open with

$A \subseteq U$, $B \subseteq V$ and $\bar{U} \cap \bar{V} = \emptyset$

Theorem: (Urysohn's Lemma) Let (X,T) be normal and A, B be closed disjoint in X Then $\exists f : X \to \mathbb{R}$ continuous so

$f(A) = 0$ $f(B) = 1$ $0 \le f(x) \le 1$ $(x \in X)$



Proof:

Let $U_{1/2}, V_{1/2}$ be disjoint open sets $A \subseteq U_{1/2}$, $B \subseteq V_{1/2}$, so

$A \subseteq U_{1/2} \subseteq \bar{U}_{1/2} \subseteq V_{1/2}^c$, $B \cap V_{1/2}^c = \emptyset$

Now $(A, U_{1/2}^c)$, $\{V_{1/2}^c, B\}$ are pairs of disjoint closed sets ∴ can construct open sets $U_{1/4}, U_{3/4}$ so

$A \subseteq U_{1/4} \subseteq \bar{U}_{1/4} \subseteq U_{1/2} \subseteq \bar{U}_{1/2} \subseteq U_{3/4} \subseteq \bar{U}_{3/4}$; $B \cap \bar{U}_{3/4} = \emptyset$

Continuing inductively we construct a family $\{U_r\}$ of open sets indexed on the dyadic rationals in $(0,1)$ [nos $r = \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2^2} + \cdots \frac{\epsilon_n}{2^n}$; $\epsilon_i = 0/1$] such that

(i) if $r < s$, then $U_r \subseteq \bar{U}_r \subseteq U_s$

(ii) for each r, $A \subseteq U_r$; $B \cap \bar{U}_r = \emptyset$

Define:

$f(x) = 0$ if $x \in \cap U_r$

$\quad = \sup\{r \mid x \notin U_r\}$

Clear $f(A) = 0$ (∵ every pt in A is in all U_r)

$f(B) = 1$ (∵ dyadic rationals)

also $0 \leq f(x) \leq 1$

Now we show f is continuous.

Note:

If $x \in U_s$, then $f(x) \le s$

If $x \notin U_r$, then $f(x) \ge r$

If $f(x) = s$ then $x \in \overline{U}_s$

If $f(x) > r$ then $x \notin \overline{U}_r$

(follows directly from defn)

Show f is continuous.

case (a) $0 < f(x) < 1$ let $\epsilon > 0$

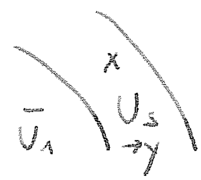
Then $\exists$ diadic rationals $r, s \in (0,1)$ so

$$f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon$$

let $W = U_s \cap \overline{U}_r^c$. Its open, $x \in W$

If $y \in W$, then $r \le f(y) \le s$

$\therefore \; |f(y) - f(x)| \le 2\epsilon$



$\therefore f$ is continuous at x

case (b) $f(x) = 0$

let $\epsilon > 0$, pick $r_\emptyset$ so $0 < r_\emptyset < \epsilon$

$x \in U_{r_\emptyset}$ If $y \in U_{r_\emptyset}$, then $|f(x) - f(y)| \le 2\epsilon$

case (c) $f(x) = 1$

similar to case (b)

Want $s_\emptyset$ so that given ϵ, $1 - s_\emptyset < \epsilon$

Use open set $\overline{U}_{s_\emptyset}^c$ which contains x

If $y \in \overline{U}_{s_\emptyset}^c$ then $|(f(x) - f(y))| < \epsilon$

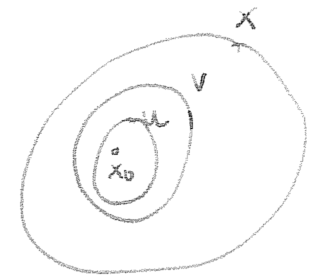
Remarks:

If X is a topological space, $f, g : X \to \mathbb{R}$ are continuous then so is $f \pm g$, fg, and also f/g at any point x when $g(x) \neq 0$.

Proof: (Case of products)

Let $x_0 \in X$, let V be a nbd of x_0 so

$$|f(x) - f(x_0)| < 1 \text{ if } x \in V$$



Δ ineq: (*) $|f(x)g(x) - f(x_0)g(x_0)| \le |f(x)g(x) - f(x)g(x_0)| + |f(x)g(x_0) - f(x_0)g(x_0)|$

$$\le |f(x)|\,|g(x) - g(x_0)| + |g(x_0)|\,|f(x) - f(x_0)|$$

$$\le (1 + |f(x_0)|)\,|g(x) - g(x_0)| + |g(x_0)|\,|f(x) - f(x_0)|$$

above is for $x \in V$

Let $\epsilon > 0$

Pick $U \subseteq V$ so $x_0 \in U$ and $|g(x) - g(x_0)| < \dfrac{\epsilon}{2(1 + |f(x_0)|)}$ and

$$|f(x) - f(x_0)| < \frac{\epsilon}{2(1 + |g(x_0)|)}$$

then if $x \in U$

$$(*) \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Definition:

Let $\{G_n\}$, G be real valued functions on X given $\epsilon > 0$ $\exists N > 0$

then $\lim_{n \to \infty} G_n(x) = G(x)$ uniformly on X if

so that $|G_n(x) - G(x)| < \epsilon \quad (x \in X, n \ge N)$

N. Bade

M202a Topology

14-2-92
Friday

Theorem:

Let $\{G_n\}$ be a sequence of continuous real valued functions on a topological space X. Let $\lim G_n(x) = G(x)$ uniformly on X. Then G is continuous.

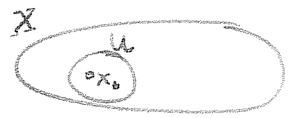
Proof:

Let $\epsilon > 0$. Then $\exists N > 0$ such that $|G_n(x) - G(x)| < \frac{\epsilon}{3}$ $(x \in X, n \geq N)$

Let $x_0 \in X$. We show continuity at this point of G.

G_N is continuous at x_0. Hence $\exists$ open U with $x_0 \in U$

so $|G_N(x) - G_N(x_0)| < \epsilon/3$ $(x \in U)$



(Show U "works" for G at x_0)

Let $x \in U$

$$|G(x) - G(x_0)| \leq \underbrace{|G(x) - G_N(x)|}_{<\frac{\epsilon}{3}} + \underbrace{|G_N(x) - G_N(x_0)|}_{<\frac{\epsilon}{3}} + \underbrace{|G_N(x_0) - G(x_0)|}_{<\frac{\epsilon}{3}}$$

$\therefore |G(x) - G(x_0)| < \epsilon$

$\therefore$ G is cont at x_0

$\therefore$ G is continuous

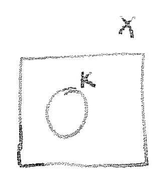
QED

Theorem: (Tietze)

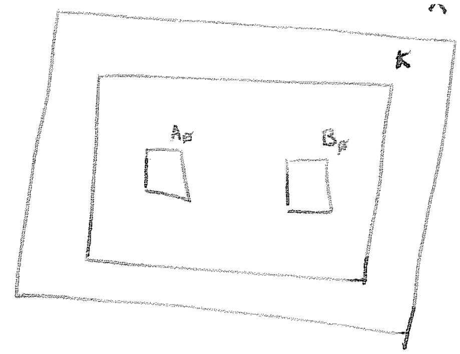
Let (X,T) be a normal space, $K \subseteq X$ be closed and $f: K \to \mathbb{R}$ be a bounded continuous function. Then $\exists$ a cont fn F on X such that

$F(x) = f(x)$ $(x \in K)$

$\sup_{x \in X} |F(x)| = \sup_{x \in K} |f(x)|$



Proof:



Let $M = \sup_{x \in K} |f(x)|$

$\therefore \; -M \le f(x) \le M \quad (x \in K)$

Let $A_\emptyset = \{ x \in K \mid f(x) \le -\frac{M}{3} \}$

$B_\emptyset = \{ x \in K \mid f(x) \ge +\frac{M}{3} \}$

They are closed disjoint sets in X

Recall inverse of open / closed is open / closed $\Rightarrow$ $f^{-1}(a \ge M)$ is closed

By Urysohn's Lemma, $\exists$ continuous function $F_\emptyset$ on X such that $F_0(x) = -\frac{M}{3} \quad (x \in A_0)$

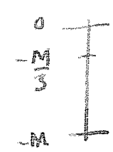
$F_0(x) = \frac{M}{3} \quad (x \in B_0)$

$-\frac{M}{3} \le F_0(x) \le \frac{M}{3} \quad (x \in X)$

Let $f_1(x) = f_0(x) - F_0(x) \quad (x \in K) \quad (f(x) = f_0(x) \text{ for } x \in K)$

Let $M_1 = \sup_{x \in K_1} |f_1(x)|$

Let $x \in A_0$



$|f_0(x) - F_0(x)| = |f_0(x) + \frac{M}{3}|$

$\le \frac{2M}{3}$

For $x \in B_0$

$|f_0(x) - F_0(x)| = |f_0(x) - \frac{M}{3}|$

$\le \frac{2M}{3}$

$x \in K \setminus (A_0 \cup B_0)$

$|f_0(x) - F_0(x)| \le |f_0(x)| + |F_0(x)| \le \frac{2M}{3}$

$M_1 \le \frac{2M}{3}$

Let $A_1 = \{ x \in K \mid f_1(x) \le -\frac{M_1}{3} \} \quad B_1 = \{ x \in K \mid f_1(x) \ge \frac{M}{3} \}$

Get F_1 cont on X with

$F_1(A_1) = -\frac{M_1}{3}, \quad F_1(B_1) = \frac{M_1}{3} \qquad -\frac{M_1}{3} \le F_1(x) \le \frac{M_1}{3} \; (x \in X)$

Let $f_2(x) = f_1(x) - F_1(x) = f_0(x) - [F_0(x) + F_1(x)] \quad (x \in K)$

Let $M_2 = \sup_{x \in K} |f_2(x)| \le \frac{2M_1}{3} \le \left(\frac{2}{3}\right)^2 M$

continuing inductively, we get a sequence $\{F_n\}$ of continuous functions on X with the properties

$$|f(x) - \sum_{i=0}^{n} F_n(x)| \le \left(\frac{2}{3}\right)^{n+1} M \quad (x \in K) \quad (*)$$

$$\sup_{x \in X} |F_n(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^n M$$

The series $\sum_{n=0}^{\infty} F_n(x)$ converges on x.

Convergence is absolute and hence uniform on X to a continuous (last Thm) function on X

$$|F(x)| \le \sum_{n=0}^{\infty} |F_n(x)| \le \frac{M}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = M$$

Also $F(x) = f(x) \quad (x \in K)$ by $(*)$

QED

Remark:

Same approach yields more good theorems

K f_1 f_2

$F_1 = e(f_1)$ $\quad G = e(f_1+f_2)$

$F_2 = e(f_2)$ $\quad$ is $G = F_1 + F_2$? No in general, true in MetricSp

linear extensions, products in extensions

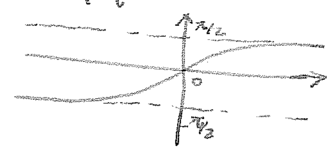
Corollary:

Let (X,T) be normal, $K \subseteq X$ closed and $f: K \to \mathbb{R}$ continuous.
Then $\exists F: X \to \mathbb{R}$ continuous with $F(x) = f(x)$ $x \in K$

Proof:

Can suppose f is unbounded (o.w. we've already proved it)

Consider the function $\tan^{-1}(x)$

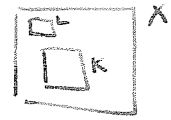


$\therefore$ $-\pi/2 < \tan^{-1}(x) < \pi/2$ on K

Let g be continuous on X with $g(x) = \arctan f(x)$ $(x \in K)$

(By Tietze) with $-\pi/2 \le g(x) \le \pi/2$

Let $L = \{x \in X \mid |g(x)| = \pi/2\}$



We know K and L are disjoint closed sets.

By Urysohn $\exists$ continuous function h such that $h(L) = 0$, $h(K) = 1$, $0 \le h(x) \le 1$, $x \in X$

Let $F(x) = \tan(h(x)\, g(x))$

$\therefore$ on K $F(x) = \tan(1 \cdot \tan^{-1}(f(x))) = f(x)$

$\therefore$ F extends f

QED

"We're getting to the point where we can come up w/ some really interesting problems"

Definition:

A top space (X,T) is compact if every open covering of X has a finite subcovering.

A set $A \subseteq X$ is compact if it is compact in its rel. topology. Because of nature of relative topology, it's equivalent to if every open cover of A in X has a finite subcover.

A set A is conditionally compact if $\bar{A}$ is compact.

A space (X,T) is locally compact if each point x has a ngbd whose closure is compact.

Definition:

A family f of subsets of X has the finite intersection property (f.i.p.) if every finite subfamily has non empty intersection

Remark:

Let $\mathcal{a}$ be a family of open sets

Then $X = \cup\{U_\alpha \mid U \in \mathcal{a}\} \Leftrightarrow X^c = \emptyset = \cap\{U^c \in \mathcal{a}\}$

Theorem:

A topological space is compact $\Leftrightarrow$ every family of closed sets having f.i.p. has non empty intersections.

Remark:

Very convenient test for compactness

Proof:

(a) Let X be compact and $\mathcal{F}$ be a family of closed sets with f.i.p.

(Show $\cap\{F \mid F \in \mathcal{F}\} \neq \emptyset$)

Suppose $\cap\{F \mid F \in \mathcal{F}\} = \emptyset$

Then $\mathcal{A} = \{F^c \mid F \in \mathcal{F}\}$ is an open cover (DeMorgan)
It has no finite subcover because if

$$X \subseteq F_1^c \cup F_2^c \cup \ldots \cup F_n^c$$

$$\bigcap_{i=1}^{n} F_i = \emptyset$$

$\therefore$ F fails f.i.p. contradiction!

(⇐) Suppose each family of closed sets with f.i.p. has non empty intersection, but X isn't compact. Then $\exists$ open cover $\mathcal{A}$ so $X = \cup \{U \mid U \in \mathcal{A}\}$ but $\mathcal{A}$ has no finite subcover. Hence if $\{U_1, U_2, \ldots U_n\} \subseteq \mathcal{A}$ then

$$\left(\bigcup_{i=1}^{n} U_i\right)^c = \bigcap_{i=1}^{n} U_i^c \neq \emptyset$$

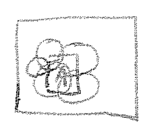
Hence $\mathcal{F} = \{U^c \mid U \in \mathcal{A}\}$ has f.i.p. but has empty intersection (as $\mathcal{A}$ is a covering)

Contradiction!

Recall we assume T-1

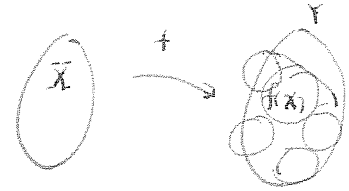
Theorem:
(a) A closed subset of a compact space is compact
(b) The continuous image of a compact set is compact
(c) In a Hausdorff space a compact set is closed

Proof:
(a) Let $A \subseteq X$ be closed, and $\mathcal{A}$ be an open cover of A.
Let $\mathcal{A}' = \mathcal{A} \cup \{A^c\}$ Thus we have a cover of X.



$\therefore$ has a finite subcover
If this finite subcovering of $\mathcal{A}'$ contains A^c throw it out. The result is a finite subcover of A from $\mathcal{A}$.

(b) Let X be compact, $f: X \to Y$ continuous



Let $\mathcal{V}$ be an open cover of $f(X)$

Then $f^{-1}(V)$ is open for all $V \in \mathcal{V}$ and covers X.

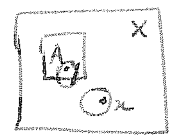
$\therefore$ as X is compact $\exists\, V_1, \dots V_n \in \mathcal{V}$

so $X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_i)$

$$f(X) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(V_i)\right) \subseteq \bigcup_{i=1}^{n} V_i$$

$\therefore \{V_1, \dots, V_n\}$ is a finite subcovering of $f(X)$ by from $\mathcal{V}$.

(c) Let X be Hausdorff, $A \subseteq X$ be compact.



(Show A^c is open)

Let $x \in A^c$ and $y \in A$

Then $\exists$ open nbds $N(y)$ and $N_y(x)$ disjoint

ie $N(y) \cap N_y(x) = \emptyset$

As A is compact $\exists\, y_1, \dots, y_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} N(y_i)$

But then $\bigcap_{i=1}^{n} N_{y_i}(x)$ is open and disjoint from $\bigcup_{i=1}^{n} N(y_i)$

$$\therefore x \in \bigcap_{i=1}^{n} N_{y_i}(x) \subseteq A^c$$

Since x was arbitrary in $A^c \Rightarrow A^c$ is open

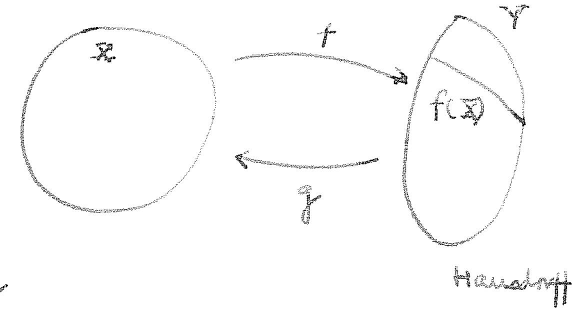
Theorem: (Kelley)

Let X be compact and Y be Hausdorff

Let $f: X \to Y$ be continuous 1-1. Then f is a homeomorphism (onto $f(X)$) [ie f^{-1} is continuous]

Proof:

Note $f(X)$ is compact and closed by (b) and (c)



$g = f^{-1}$ is continuous $\Leftrightarrow$ $g^{-1}(K)$ is closed for every K closed in X.

But $g^{-1}(K) = f(K)$ which is closed $\because$ $f(K)$ is compact in Y

Theorem:

A compact Hausdorff space is normal

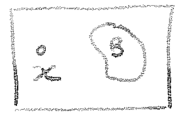
Theorem:

A compact Hausdorff space is normal.

Proof:

Show (X,T) is regular. Let B be closed, $x \notin B$

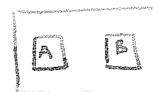
For each $y \in B$ $\exists$ nbd $N(y)$ and nbd $N_y(x)$ disjoint. B is compact $\therefore \exists\ y_1, \dots, y_n$ s.t. $B \subseteq \bigcup_{i=1}^{n} N(y_i)$



Then $x \in \bigcap_{i=1}^{n} N_{y_i}(x)$. These open sets are disjoint.

These open sets are disjoint $\Rightarrow$ space is regular

Let A, B be disjoint closed sets



For each $x \in A$ $\exists$ nbd $N(x)$ and open set U_x with $B \subseteq U_x$ and $N(x) \cap U_x = \emptyset$. As A is compact $\exists x_1 \dots x_n \in A$ so $A \subseteq \bigcup_{j=1}^{m} N(x_j)$

$B \subseteq \bigcap_{j=1}^{m} U_{x_j}$ These are disjoint.

$\boxtimes$ QED

Metric Spaces Again:

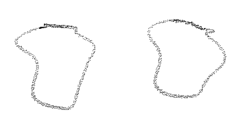
Definition:

Let (X, ρ) be metric and $A \subseteq X$

Define $\rho(X, A) = \inf_{a \in A} \rho(x, a)$

(note: distance $\rho(x, A) = \rho(x, \bar{A})$)

If A, B are sets in X $\rho(A,B) = \inf_{a \in A, b \in B} \rho(a,b)$



Lemma:

The function $x \to \rho(x,A)$ is continuous

Proof:

Idea: $|\rho(x,A) - \rho(y,A)| < \epsilon$ when $\rho(x,y) < \epsilon$

Let $\epsilon > 0$ Pick $a \in A$ so $\rho(x,a) < \rho(x,A) + \epsilon$

suppose $\rho(x,y) < \epsilon$

$$\rho(y,a) \le \rho(y,x) + \rho(x,a)$$

$$\therefore \rho(y,A) \le \rho(y,x) + \rho(x,a) \quad \text{; take inf over all } a \in A$$

$$\Rightarrow \rho(y,A) \le \cancel{\epsilon +} \rho(x,A) + \epsilon$$

$$\Rightarrow \rho(y,A) \le \rho(y,x) + \rho(x,A) + \epsilon$$

$$\le \rho(x,A) + 2\epsilon \quad (*)$$

$$\rho(x,a) \le \rho(x,y) + \rho(y,a)$$

$$\rho(x,A) \le \epsilon + \rho(y,a)$$

$$\le$$

Restart

Pick $b \in A$ so $\cancel{\rho(y,A) < \rho(y,b) + \epsilon}$

$$\rho(y,b) < \rho(y,A) + \epsilon$$

$$\rho(x,b) \le \rho(x,y) + \rho(y,b)$$

$$\rho(x,A) \le \rho(x,y) + \rho(y,b)$$

$$\therefore \rho(x,A) \le \rho(x,y) + \rho(y,A) + \epsilon \le 2\epsilon + \rho(y,A)$$

$$\rho(x,A) \le 2\epsilon + \rho(y,A) \quad (**)$$

$\therefore$ by $(*)$ and $(**)$

$$|\rho(x,A) - \rho(y,A)| \le 2\epsilon$$

Note:

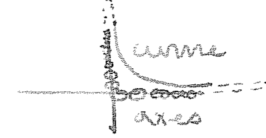
If A and B are disjoint and closed

Note:

If A is closed and $x \notin A$ then $\rho(x, A) > 0$

Theorem:

A metric space is normal (Not entirely obvious)



Proof:

Let A, B be disjoint closed sets

Let $U = \{x \mid \rho(x, A) < \rho(x, B)\}$ open

$V = \{x \mid \rho(x, B) < \rho(x, A)\}$ open

$\rho(x, A)$ cont $\rho(x, B)$ cont

$\Rightarrow f^* - \rho(x, A) + \rho(x, B)$ cont!

$\Rightarrow f^{*-1}((0, \infty))$ is open!

$A \subseteq U$

$B \subseteq V$

$\therefore (X, \rho)$ is normal

(Using continuity of distance function)

Definition:

A sequence in X is a function from $\mathbb{N}$ to X.

We write $\{x_n\}$ for a sequence (Sloppy)

Definition:

A sequence $\{x_n\}$ in X (a top space) converges to x_0 if given any nbd U of x_0, $\exists N_U$ so $x_n \in U$ $\forall n \geq N_U$

A sequence is convergent if it converges to some x_0. We write $\lim x_n = x_0$ or $x_n \to x_0$

Definition:

A point x_0 is a cluster point of some sequence $\{x_n\}$ if given any nbd U of x_0 and $N_U \in \mathbb{N}$ $\exists n \geq N_U$ such that $x_n \in U$.

Note:

$x_n \to x_0 \iff \rho(x_n, x_0) \to 0$

Definition:

Let (X, ρ) be metric. A sequence $\{x_n\}$ in X is a Cauchy sequence if given $\epsilon > 0$ $\exists N_\epsilon > 0$ such that

$\rho(x_m, x_n) < \epsilon$ for all $n, m \geq N_\epsilon$

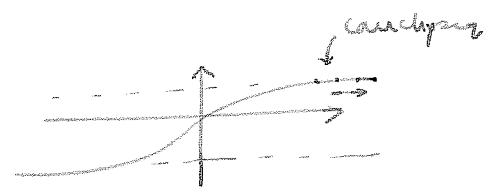
Definition:

X is complete for ρ if every Cauchy sequence in X is convergent

Example:

$\mathbb{R}$ with $\rho(x,y) = |x-y|$ is complete

$\rho'(x,y) = \tan^{-1}(x) - \tan^{-1}(y)$



↳ sequence cauchy for 2^{nd} metric not for first

$\mathbb{R}$ is not complete for ρ' (is for ρ)

With compact spaces there is no problem

W. Bade

M202a - Topology

24-2-92
Monday

Last Time:

Made a sloppy remark

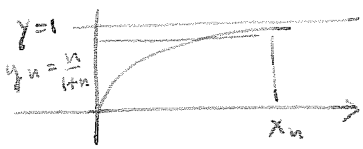
(X, ρ) metric space

$\{x_n\}$ is Cauchy if given $\epsilon > 0$

there is N_ϵ so $\rho(x_n, x_m) < \epsilon$ for all $n, m \geq N_\epsilon$

(X, ρ) is complete if every Cauchy sequence in X converges

$$\rho(x) = \frac{x}{1+x}$$



$[0, \infty) \to [0, 1)$

y_n is Cauchy but has no limit in Y. $(1 \notin [0,1))$

$$\rho'(y_1, y_2) = \frac{|y_1 - y_2|}{1 + |y_1 - y_2|} \quad \text{is a metric}$$

x_0 is a cluster point for a sequence $\{x_n\}$ if given $\epsilon > 0$ and $N \in \mathbb{N}$, $\exists n > N$ such that $\rho(x_n, x_0) < \epsilon$

Remark:

A Cauchy sequence converges to "any" of its cluster points (If a Cauchy sequence has a cluster point it converges to it.)

Proof:

Let x_0 be a cluster point of $\{x_n\}$. Let $\epsilon > 0$.

Then $\exists N_\epsilon$ such that $\rho(x_m, x_n) < \epsilon$ if $m, n \geq N_\epsilon$.

Now $\exists n_0 \geq N_\epsilon$ so $\rho(x_n, x_{n_0}) < \epsilon$ (Cluster Point)

Then $\rho(x_m, x_0) \leq \rho(x_m, x_{n_0}) + \rho(x_{n_0}, x_0)$

$$\leq \epsilon + \epsilon = 2\epsilon \quad \forall n \geq n_0$$

Proposition:
Let (X, ρ) be a metric space.
(a) A convergent sequence is a Cauchy sequence
(b) A Cauchy sequence converges

$\Leftrightarrow$ It has a convergent subsequence

$\Leftrightarrow$ It has a cluster point

(c) A point $x_\phi \in \bar{A} \Leftrightarrow \exists \{x_n\} \subseteq A$ with $x_n \to x$

(d) If x_0 is a cluster point of a sequence then $\exists$ a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to x_ϕ.

Sequence: $f: \mathbb{N} \to X \quad f(n) = x_n$

$\phi: \mathbb{N} \to \mathbb{N}$ strictly monotone increasing

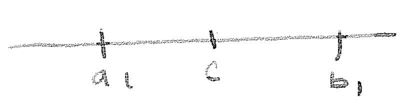
$g: \mathbb{N} \to X$ is a subsequence if $g = f \circ \phi$

Theorem:
$\mathbb{R}$ is complete for the metric $\rho(x,y) = |x-y|$

Proof:
Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$. Then the range of $\{x_n\}$ lies in a bounded set

Let $\epsilon = 1$ then $\exists N_1$ so $\rho(x_n, x_m) < 1$ all $m \geq N_1$ so $x_n \in$ interval of length 2 for $n \geq N_1$. But $\exists$ only finitely many points $x_1, \ldots, x_{N_1 - 1}$ outside

Suppose ~~then~~ $x_n \in [a_1, b_1]$ $(n \in \mathbb{N})$ Let $c =$ midpoint



Then at least one of the $[a_1, c], [c, b_1]$ contains x_n for infinitely many n. Pick the one. Call it $[a_2, b_2]$

Take midpoint and repeat argument. Get sequences

$a_1 \le a_2 \le \dots \quad \le b_2 \le b_1$

with $b_n - a_n \to 0$

Let $d = \text{lub } a_n$

Then d is a cluster point of the sequence $\{x_n\}$ so

$x_n \to d$.

$\therefore \mathbb{R}$ is complete for this metric < Invoke Cauchy @ this step

Theorem:

$\mathbb{R}^n$ is complete for ~~$\rho(x_1, \dots, x_n)$~~ $\rho(x,y) = \left[\sum_{n=1}^{n} |\xi_n - \eta_n|^2\right]^{1/2}$

Proof:

Let $\{x^{(n)}\}$ be Cauchy. Then for each i.

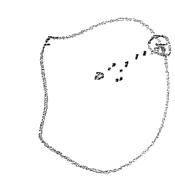
$|\xi_i^{(n)} - \xi_i^{(m)}| \le \rho(x^n, x^m) \to 0$ as $n \to \infty$

$\therefore \xi_i^{(n)} \to \xi_i^0$ for each i. (By completeness of Reals $\mathbb{R}$)

$$\left[\sum_{i=1}^{n} |\xi_i^{(n)} - \xi_i^{(m)}|^2\right]^{1/2} \le \sum_{i=1}^{n} |\xi_i^{(n)} - \xi_0^{(n)}| \to 0$$

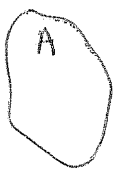
Definition:

(a) A subset A of a topological space (X, T) is sequentially compact if every sequence $\{a_n\}$ in A has a subsequence $\{a_{n_i}\}$ which converges to a point $x_0 \in X$ (in fact in $\bar{A}$)



(k) A subset A of a metric space is totally bounded if for each $\epsilon > 0$ $\exists \{a_1, \ldots, a_n\} \subseteq A$ so $A \subseteq \bigcup_{i=1}^{n} S(a_i, \epsilon)$

$A \subseteq X$ is sequentially compact if every sequence $\{a_n\}$ in A has a subsequence $\{a_{n_i}\}$ converging to some point $x_0 \in X$ ($x_0 \in \bar{A}$)



tot bdd

$\epsilon > 0$

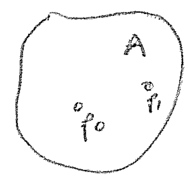
$\exists a_1, \dots, a_n$ so $A \subseteq \bigcup_{n=1}^{n} S(a_i, \epsilon)$

Lemma:

In a metric space, a sequentially compact set is totally bounded and a totally bounded set is seperable in its relative topology

Proof:

Let A be seq cmpct let $p_0 \in A$. Let $d_0 = \sup_{p \in A} \rho(p, p_0)$



Then $d_0 < \infty$ o.w. $\exists \{q_n\}$ in A with $\rho(p_0, q_n) \to \infty$ since A is a seq cmpct

$\Rightarrow \exists$ subsequence $\{q_{n_i}\}$ and point $x_0 \in \bar{A}$ so $q_{n_i} \to x_0$

$\therefore \rho(p_0, q_n) \to \rho(p_0, x_0) = \infty$ (contradiction)

Select $p_1 \in A$ so $\rho(p_0, p_1) \geq \frac{d_0}{2}$

Let $d_1 = \sup_{p \in A} \rho(p, \{p_0, p_1\})$ $\quad d_1 \leq d_0$ (certainly finite)

pick $p_2 \in A$ so $\rho(p_2, \{p_0, p_1\}) \geq \frac{d_1}{2}$

etcetra (inductive construction)

Inductively, get $\{p_i\} \subseteq A$ so if

$$d_i = \sup_{p \in A} \rho(p, \{p_0, \dots, p_i\})$$

then $\rho(p_{i+1}, \{p_0, \dots, p_i\}) \geq \frac{d_i}{2}$

clearly $d_0 \geq d_1 \geq d_2 \geq d_3 \dots$

assert $d_i \to 0$

suppose not

Then $\exists \delta > 0$ so $d_i \geq \delta$ all i.

Then for $\{p_i\}$ sequence

$\rho(p_n, p_m) \geq \delta/2 \quad (n, m \in \mathbb{N})$

$\therefore \{p_n\}$ has no convergent subsequence

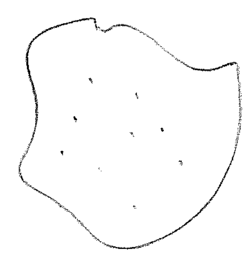
not Cauchy (in the worst way)

Contradicts seq compactness of A

Thus A is totally bounded, for given $\epsilon > 0$

$\exists N$ so that $d_k < \epsilon$ for all $k \geq N$

$\therefore A \subseteq \bigcup_{i=1}^{n} S(p_n, \epsilon)$



if $n \geq N$

QED

Show A tot bdd $\Rightarrow$ A is separable in rel topology

Proof:

Choose $\{\epsilon_n\} \downarrow 0$

For each n, $\exists$ finite set F_n so that

$A \subseteq \bigcup_{x \in F_n} S(x, \epsilon_n)$

Then the union $F = \bigcup_{n=1}^{\infty} F_n$ is a countable dense set in A

Lemma:

If a subset A of a metric space is seq compact so is $\bar{A}$.

Proof:

Let $\{b_n\}$ be a sequence in $\bar{A}$ and for each n, pick $a_n \in A$ so $\rho(a_n, b_n) < 1/n$

Then $\{a_n\}$ has a subsequence $\{a_{n_i}\}$ converging to $x_0 \in \bar{A}$.

$$\rho(b_{n_i}, x_0) \leq \rho(b_{n_i}, a_{n_i}) + \rho(a_{n_i}, x_0)$$
$$\leq \underbrace{1/n_i}_{\to 0} + \underbrace{\rho(a_{n_i}, x_0)}_{\to 0}$$

$\therefore b_{n_i} \to x_0$

QED

Theorem:

For a subset A of a metric space TFAE

(a) A is sequentially compact

(b) $\bar{A}$ is compact

(c) A is totally bounded and $\bar{A}$ is complete

Very useful in analysis

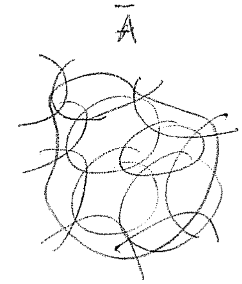
Proof:

(a) $\Rightarrow$ (b)

Let A be sequentially compact. Then $\bar{A}$ is sequentially compact and closed. Then $\bar{A}$ is separable in its rel topology $\therefore$ has a countable base in rel topology (Lindelöf)

Let $\mathcal{G}$ be any open cover of $\bar{A}$ (rel topology)

By Lindelöff $\exists$ a countable subcovering $\{G_n\} \subseteq \mathcal{G}$



$\therefore \bar{A} \subseteq \bigcup_{i=1}^{\infty} G_i$

Show $\exists N$ so that

$\bar{A} \subseteq \bigcup_{i=1}^{N} G_i$

Suppose $\bar{A} \setminus \bigcup_{i=1}^{n} G_i \neq \emptyset$ for all n

Pick $a_n \in \left(\bigcup_{i=1}^{n} G_i\right)^c$ (in rel topology)

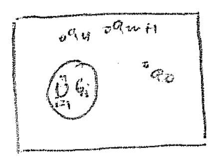
$\{a_n\}$ has a convergent subsequence

converging to $a_0 \in \bar{A}$ But then

$a_0 \in \left(\bigcup_{i=1}^{n} G_i\right)^c$ for each n. (complement is closed)

$\therefore a_0 \in \left(\bigcup_{i=1}^{\infty} G_i\right)^c = \emptyset$ in $\bar{A}$

(because $\bar{A} = \bigcup_{n=1}^{\infty} G_n$)



Contradiction

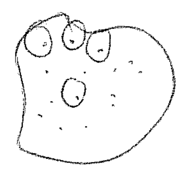
$\boxtimes$ (a)$\Rightarrow$(b)

Proof: (b) $\Rightarrow$ (c)

suppose $\bar{A}$ is compact

Let $\epsilon > 0$ and form the cover $\{S(b,\epsilon) \mid b \in \bar{A}\}$

very wasteful covering ☺



By compactness $\exists b_1, \ldots, b_n$ in B so

$\bar{A} \subseteq \bigcup_{i=1}^{n} S(b_i, \epsilon)$

For each i choose $a_i \in A$ so $\rho(a_i, b_i) < \epsilon$

Then $S(b_i, \epsilon) \subseteq S(a_i, 2\epsilon)$

$\therefore \quad A \subseteq A \subseteq \bigcup_{i=1} S(a_i, 2\epsilon)$

$\therefore$ A is totally bounded ($\because 2\epsilon$ is arb small)

(also $\bar{A}$ " " ")

X is metric space $A \subseteq X$

(a) A is sequentially compact

(b) $\bar{A}$ is compact

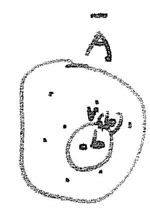
(c) A is ~~totally~~ bounded and $\bar{A}$ is complete

(b) $\Rightarrow$ (c) If $\bar{A}$ is compact, A is totally bounded (done)

Show $\bar{A}$ is complete

Proof:

Suppose not. $\exists$ a Cauchy sequence $\{a_n\}$ with no cluster point in $\bar{A}$



Then for each $b \in \bar{A}$

$\exists$ a nbd of $V(b)$ and $N_b \in \mathbb{N}$ so $a_n \notin V(b)$ for all $n \geq N_b$

$\Rightarrow \exists\, b_1, \ldots, b_n$ so $\bar{A} \subseteq \bigcup_{i=1}^{n} V(b_i)$ Let $N = \max N_{b_i}$

Then $a_n \notin \bar{A}$ for $n \geq N$ (too bad!) ($\rightarrow\leftarrow$)

(c) $\Rightarrow$ (a) A tot bdd, $\bar{A}$ complete $\Rightarrow A$ is seq compact

Let $\{a_n\}$ be a sequence in A

Let $\{a_n\}$ be a sequence in A (Show it has a conv subseq

As A is tot bdd $A \subseteq$ a finite union of open balls of radius 1

Use "shoe box" principal

Then $\{a_n\}$ is in some one of these for ∞ often

$\therefore \exists$ a subsequence $\{a_{1,n}\}$ of $\{a_n\}$ so that

$\{a_{1n}\} \subseteq$ this ball of radius 1 for all n.

$\exists$ a ball of radius $1/2$, $B_{1/2}$ such that $\{a_{1,n}\}$

$\{n \mid a_{1,n} \in B_{1/2}$ for ∞ of $n\}$ Get $\{a_{2,n}\}$ of $\{a_{1,n}\}$ etc

($B_{1/2} \notin B_1$ (not nece)) So for each j, $\{a_{j,n}\}$ is entirely contained in a ball of radius $1/j$.

Consider the sequence $\{\bar{a}_n\}$ where $\bar{a}_n = a_{n,n}$ (Cantor style diagonal argument) Then $\{\bar{a}_n\}$ is a subsequence of $\{a_n\}$ It is Cauchy! since for any integer $N>0$ $\rho(\bar{a}_n, \bar{a}_m) < 2/N$ for $m,n \geq N$

Since $\bar{A}$ is complete, $\{\bar{a}_n\}$ converges to a point of $\bar{A}$ $\therefore$ A is seq compact.

$\boxtimes$ QED (A very good theorem)

Corollaries:

In a Metric Space a set B is compact $\Leftrightarrow$ it is sequentially compact and closed $\Leftrightarrow$ every sequence $\{b_n\}$ in B has a cluster point.

Theorem: (Bolzano-Weierstrass)

A subset of $\mathbb{R}^n$ is sequentially compact $\Leftrightarrow$ it is bounded (inside some big box)

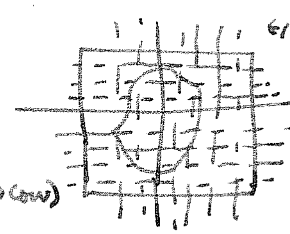
(Based upon metric)

compactness is indep of metric(!!) completeness varies

Interesting

Proof:

In $\mathbb{R}^n$ a bdd set is totally bounded so A is totally bdd and $\bar{A}$ is complete as $\mathbb{R}^n$ is complete ($\because \bar{A}$ complete, $\mathbb{R}^n$ complete $\Rightarrow$ cov)



$\boxtimes$ QED

Clear that a seq compact set is bounded. If not $\exists \{a_n\} \subseteq A$

$\bar{A} \subseteq \mathbb{R}^n$ with $\rho(a_n, 0) \to \infty$ $\therefore$ no conv subsequence

$\uparrow$ complete

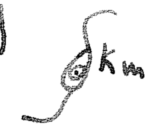
Theorem: (Baire category)

Let (X, ρ) be complete metric space

Suppose $X = \bigcup_{n=1}^{\infty} K_n$ where each K_n is closed.

Rene Baire: Grea
mathematician m
LeBesque, Hilbert

Then some set K_m must contain a non empty open set. (One of them is "fat")



Example: Q in rel top of $\mathbb{R}$

$Q = \bigcup_{r \in Q} \{r\}$ but no such set is open!

Proof:

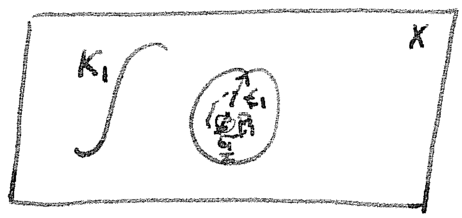
Suppose $X = \bigcup K_n$ K_n is closed. Suppose no K_n contains a non empty open set.

Then $K_1 \neq X$

so K_1^c is open

pick $p_1 \in K_1^c$, $\epsilon_1 < 1/2$

So $S(p_1, \epsilon_1) \cap K_1 = \emptyset$



Then $K_1 \neq X$
so K_1^c is op

K_2 does not contain $S(p_1, \epsilon_1/2)$

$\therefore \exists\, p_2 \in S(p_1, \epsilon_1/2)$ and an $\epsilon_2 < 1/2^2$ so the ball about p_2 $S(p_2, \epsilon_2) \subseteq K_2^c \cap S(p_1, \epsilon_1/2)$

etc.

By induction we get a sequence of balls $S(p_n, \epsilon_n)$ of balls so that $0 < \epsilon_n < 1/2^n$

The ball about $S(p_{n+1}, \epsilon_{n+1}) \subseteq S(p_n, \epsilon_n/2)$

$S(p_n, \epsilon_n) \cap K_n = \emptyset$

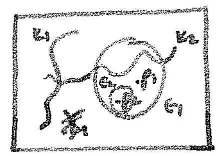
$S(p_n, \epsilon_n) \cap K_j = \emptyset \quad (1 \le j \le n)$

Show $\{p_n\}$ is Cauchy $\therefore$ converges to some p_0, but $p_0 \notin X$ ($\because$ its in each K_n's complement)

Baires Theorem: (Very important)

Let X be a complete metric space if $X = \bigcup_{n=1}^{\infty} K_n$ where each K_n is closed, then some K_n contains a non empty open set.

Proof: (Suppose False)



Get a sequence of open balls $\{S_n(p_n, \epsilon_n)\}$ (by induction) with $0 < \epsilon_n < 1/2^n$

$$S(p_{n+1}, \epsilon_{n+1}) \subseteq S(p_n, \epsilon_n/2)$$

$$S(p_n, \epsilon_n) \cap K_n = \emptyset$$

The sequence $\{p_n\}$ is Cauchy $\because$ if $n < m$

$$\rho(p_n, p_m) \le \rho(p_n, p_{n+1}) + \rho(p_{n+1}, p_{n+2}) + \cdots + \rho(p_{m-1}, p_m)$$
$$< 1/2^{n+1} + 1/2^{n+2} + \cdots 1/2^m$$ add up all those guys
$$< 1/2^n$$

space is complete $\Rightarrow p_n \to p_0$ say

$$\rho(p_n, p_0) \le \rho(p_n, p_m) + \rho(p_m, p_0)$$
$$< [\epsilon_n/2 + \underset{\downarrow 0}{\rho(p_m, p_0)}] \to \epsilon_n/2$$

$\therefore \quad p_0 \in \overline{S(p_n, \epsilon_n/2)}$ ← closure at most
$\subseteq S(p_n, \epsilon_n)$ for any n

$$\therefore p_0 \in \bigcap_{n=1}^{\infty} S(p_n, \epsilon_n)$$

$\therefore p_0 \notin K_n$ for any n But $X = \bigcup_{n=1}^{\infty} K_n$

so $p_0 \notin X$ contradiction.

Corollary: $\nexists\, f: \mathbb{R} \to \mathbb{R}$ whose set of discontinuities is precisely the set I of irrationals

Proof: Recall the set D of discontinuities of any function is a union of closed sets

Suppose $D = \mathbb{I}$ then $\mathbb{I} = \bigcup_{n=1}^{\infty} C_n$ (?) where each C_n is closed

$\therefore \mathbb{R} = \mathbb{I} \cup \mathbb{Q} = \bigcup_{n=1}^{\infty} C_n \cup \bigcup_{r \in \mathbb{Q}} \{r\}$ ← Countable family of closed.

By Baire some one must contain a non empty open set. No $\{r\}$ is open

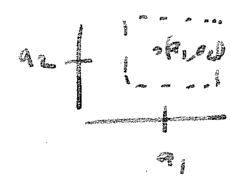
$\therefore$ some C_n is open (has an interior)

But then it contains rational numbers (since it contains an open interval (base) $\therefore$ contains points of $\mathbb{Q}$)

Contradiction

Topological Products:

$A_1 \times A_2 = \{(a_1, a_2) \mid a_i \in A_i,\ i = 1, 2\}$



$A_1 \times A_2$ = the set of all functions $\Phi : \{1,2\} \to A_1 \cup A_2$ such that $\phi(i) \in A_i$

Let $\{A_\alpha \mid \alpha \in I\}$ be a family of sets

Defn: $\prod_{\alpha \in I} A_\alpha = \{ \phi : I \to \bigcup_{\alpha \in I} A_\alpha \mid \text{for each } \alpha\ \ \phi(\alpha) \in A_\alpha \}$

$a \in \prod_{\alpha \in I} A_\alpha$, we write $a = \{a_\alpha\}$ where $a_\alpha = \phi(\alpha) \in A_\alpha$

Is $\prod_{\alpha \in I} A_\alpha$ non empty? Big question

Axiom of Choice (Munkres p 59)

Axiom: (Choice)

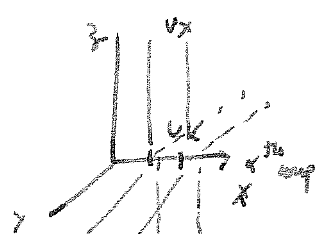
Given a collection $\mathcal{B}$ of sets (not nece disjoint) there exists a function $\phi : \mathcal{B} \to \bigcup_{B\in\mathcal{B}} B$ such that $\phi(B)$ is an elt of B for every $B \in \mathcal{B}$

For each $\alpha \in I$, let (X_α, T_α) be a topological space
How topologize $\prod X_\alpha$?

base for a topology
subbase for a topology T on X:
a family S of sets such that the family $\mathcal{B}$ of all finite intersections of sets in S is a base for T.

Ex. $X=\mathbb{R}$ Then $S=\{(a,\infty),(-\infty,b) \mid a,b\in\mathbb{R}\}$ is a sub base for the Euclidean topology of $\mathbb{R}$.

Defn
For each $\alpha\in I$, define $p_\alpha : \prod_{\beta\in I} X_\beta \to X_\alpha$ by $p_\alpha(\overset{\{a_\beta\}}{\overset{\shortparallel}{a}}) = a_\alpha$
This is the α projection
We define the product topology on $\prod X_\alpha$ as that having as a subbase all sets of the form $p_\alpha^{-1}(U_\alpha)$ where U_α is open in (X_α, T_α)



A base for $\prod X_\alpha$ consists of all sets of the form: Given $\alpha_1,\dots,\alpha_n\in I$

$$U_{\alpha_1}\times U_{\alpha_2}\times\cdots\times U_{\alpha_n}\times \prod_{\alpha\notin\{\alpha_1,\dots,\alpha_n\}} X_\alpha$$

"Weakest topology st every proj map is cont"

Products:

Sets $\{X_\alpha \mid \alpha \in I\}$

$$\prod X_\alpha = \{\phi : I \to \bigcup_{\alpha \in I} X_\alpha \mid \phi(\alpha) \in X_\alpha\}$$

write $\{x_\alpha\}$ for a point $\vec{x}$ in the product

$p_\beta : \prod X_\alpha \to X_\beta \quad p(\vec{x}) = x_\beta \quad x = \{x_\alpha\}$

Product Topology: (X_α, T_α)

Topology generated by the subbase $\mathcal{E}/\mathcal{P}$

$$\{p_\alpha^{-1}(U_\alpha) \mid U_\alpha \in T_\alpha \,;\, \alpha \in I\}$$

base is all sets of the form $\boxed{\bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i})}$

$$p_\alpha^{-1}(U_\alpha) = \{\vec{x} \in \prod X_\beta \mid p_\alpha(\vec{x}) = x_\alpha \in U_\alpha\}$$

Note:

$$\vec{x} = \{x_\alpha\} \in p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots p_{\alpha_n}^{-1}(U_{\alpha_n})$$

$\Leftrightarrow x_{\alpha_i} \in U_{\alpha_i} \; (i = 1 \ldots n)$ and all other coordinates are unrestricted

A base set has the form

$$\boxed{U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \prod_{\alpha \neq \alpha_1, \ldots, \alpha_n} X_\alpha}$$

note: if F, G are finite subsets of the index set I

$$\Big[\bigcap_{\alpha \in F} p_\alpha^{-1}(U_\alpha)\Big] \cap \Big[\bigcap_{\beta \in G} p_\beta^{-1}(U_\beta)\Big] = \bigcap_{\gamma \in F \cup G} p_\gamma^{-1}(U_\gamma)$$

$\therefore$ the intersection of two base sets is a base set

$\therefore$ condition for a base are satisfied

$\therefore$ get a topology on $\prod X_\alpha$ "the product topology"

clearly the projection maps p_β are continuous since if U_β is open in X_β $p_\beta^{-1}(U_\beta)$ is open in $\prod X_\alpha$.

p_β maps an open set in $\prod X_\alpha$ onto an open set in X_β (ie p_β is an open map)

(check p_β maps base sets to open sets in X_β)

Theorem:

~~Let $\{X_\alpha \mid \alpha \in I\}$ and (X, T)~~

Let $\{(Y_\alpha, T_\alpha) \mid \alpha \in I\}$ and (X, T) be topological spaces

A map $f : X \to \prod_{\alpha \in I} Y_\alpha$ is continuous

$\Leftrightarrow$ the maps $p_\beta \circ f : X \to Y_\beta$ are continuous for all $\beta \in I$

Proof:

Its clear if f is continuous so is $p_\beta \circ f$ (composition)

Suppose $p_\beta \circ f$ is continuous for each β

Let x be a point in X and consider

$f(x) \in \prod Y_\beta$

Let $B = V_{\alpha_1} \times \dots \times V_{\alpha_n} \times \prod' Y_\alpha$; $\prod' Y_\alpha$ means product over all $\alpha \neq \alpha_1, \dots, \alpha_n$

be a basis nbd of $f(x) = \{f(x)_\alpha\}$

Here $f(x)_{\alpha_i} \in V_{\alpha_i}$ Find open ~~sets~~ nbd of x $U_{\alpha_i} \in X$ so that $p_{\alpha_i}(U_{\alpha_i}) \subseteq V_{\alpha_i}$ $i = 1 \dots n$ (by cont of $p_\beta \circ f$)

Let $U = \prod_{i=1}^{n} U_{\alpha_i}$: Its open, $x \in U$ and

$f(U) \subseteq B$

☒ QED

Corollary:

Let X be a topological space and for each $\alpha \in I$,

$f_\alpha : X \to Y_\alpha$ be a ~~continuous~~ map

Define

$$f : X \to \prod_{\alpha \in I} Y_\alpha$$

by $f(x) = \{f_\alpha(x)\}$

Then f is continuous $\Leftrightarrow$ each f_α is continuous

Proof:

Clear since $f_\beta = p_\beta \circ f \ \forall \beta$

Theorem: (Tychonoff)

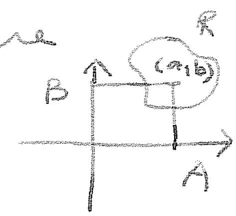
If X_α are compact $(\alpha \in I)$ then $\prod X_\alpha$ is compact

(Equiv to A.C.)

Foundations:

Definition:

Let A, B be sets. A relation (from A to B) is a subset R of $A \times B$. If $(a,b) \in R$ we write aRb.



The Domain of the relation $R = \{a \in A \mid \exists b \in B \text{ s.t. } aRb\}$

The Range of the Relation $R = \{b \in B \mid \exists a \in A \text{ s.t. } aRb\}$

A function is a relation $R \ni (x,y) \in R \ (x,z) \in R \Rightarrow y = z$

We consider relations whose domain & range are in the same set E

$R \subseteq E \times E$ we say R is on E

Definition:

A relation R on E is reflexive

(1) Reflexive if $(x,x) \in R$ $(x \in E)$

(2) Symmetric if $(x,y) \in R \Rightarrow (y,x) \in R$ $(x,y \in E)$

(3) Transitive if $(x,y) \in R$ & $(y,z) \in R \Rightarrow (x,z) \in R$ $(x,y,z \in E)$

A relation which is reflexive, symmetric, and transitive is called an equivalence relation.

An equivalence relation on E partitions E into disjoint sets $E_x = \{y \mid (x,y) \in R\}$

Definition:

A partially ordered set (p.o.s.) $(E, \leq)$ is a non empty set E with a relation $\leq$ on E such that (a,a)

(1) $a \leq a$ $(a \in E)$

(2) $a \leq b$ $b \leq c \to a \leq c$

(3) $a \leq b$ $b \leq a \to a = b$ ← not always assumed (but for us yes)

POS $(E, \leq)$

(1) $a \leq a$ $(a \in E)$

(2) $a \leq b,\ b \leq c \to a \leq c$

(3) $a \leq b,\ b \leq a \to a = b$

Theorem (Zermelo):

Let $(E, \leq)$ be a p.o.s. with the property that every totally ordered subset has a l.u.b.
Let $f : E \to E$ be a map such that

$x \leq f(x)$ $(x \in E)$

Then $\exists\, w \in E$ such that $f(w) = w$

(1) $B \subseteq E$ is ~~totally~~ ordered if for each $x, y \in B$ either $x \leq y$ or $y \leq x$. "Strung out on a long line"

(2) y is an upper bound for $B \subseteq E$ if $b \leq y$ for all $b \in B$.

(3) u is the l.u.b. of $B \subseteq E$ if u is an upper bound for B and $u \leq y$ for every upper bound y of B

(Uniqueness from (3)) Subtle

~~Proof~~:

Let $a \in E$ be fixed
$B \subseteq E$ is admissible if

(1) $a \in B$

(2) $f(B) \subseteq B$

(3) If B_0 is a totally ordered subset of B, the the lub $B_0 \in B$

(4) $a \leq x$ for all $x \in A$

(5) If $x \in P$ and $z \in A$ then either $z \leq x$ or $z \geq f(x)$

(6) P is admissable $Q \hat{=} \{z \in A \mid$ either $z \leq x$ or $z \geq f(x)\}$ (from $x \in P$)

Proof:

Let a be a point of E. Note E is admissable and the intersection of admissable and the intersection of admissable sets is admissable.

Therefore $\exists$ a minimum admissable set A ($\neq \emptyset$ since $a \in A$) (the intersection of all admissable sets)

[We prove that A is totally ordered. If so, we are done for let $w = lub(A)$ Then $w \leq f(w)$ but $f(w) \in A$ so $f(w) \leq w$
$\Rightarrow w = f(w)$]

We next prove (4)

True since if $K = \{x \mid x \in A, a \leq x\}$ then K is admissable

Since $K \subseteq A$, $K = A$ as A is minimal

Notation:

we write $y < x$ if $y \leq x \wedge y \neq x$

$y \geq x$ if $x \leq y$

$y > x$ if $y \geq x \wedge y \neq x$

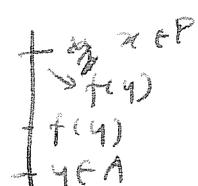
Definition: (Barrier Points)

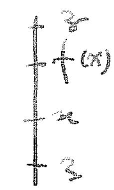
The barrier points P of A

$P = \{x \in A \mid y \in A, y < x \rightarrow f(y) \leq x\}$

Note P is non-empty as $a \in P$ (vacuously)

We prove that barrier points produce "gaps" in A.





Strategy:

Prove P is admissable. Then $P = A$. Then if $x, z \in A$ either $z \leq x$ or $z \geq f(x) \geq x$ so A is tot. ord

Remains to prove (5) and (6)

Proof: (5)

fix $x \in P$

define $Q = \{ z \in A \mid \text{either } z \leq x \text{ or } z \geq f(x) \}$

Prove Q is admissable so $Q = A$. $\therefore$ (5) holds

clearly $a \in Q$ so (1) holds for Q

Also Q satisfies (2) $(f(Q) \subseteq Q)$

To see this let $z \in Q$

so either $z \leq x$ or $\cancel{f(z) \geq x}$ $z \geq f(x)$

Must show $f(z) \leq x$ or $f(z) \geq x$.

cases (1) $z = x$ Then $f(z) = f(x)$ ✓

(2) $z < x$ Then $f(z) \leq \cancel{f}\, x$ (as $x \in P$) ✓

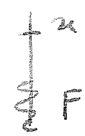
(3) $z \geq f(x)$ Then $f(z) \geq z \geq f(x)$ ✓

$\therefore$ $f(Q) \subseteq Q$ So (2) holds for Q

Also Q satisfies (3)

let $F \subseteq Q$ be a totally ordered subset of Q and $u = \text{lub } F$ ($u \in A$ known)

Must show $u \in Q$ ie $u \leq x$ or $u \geq f(x)$



case (i) $y \leq x$ for all $y \in F$

Then $u \leq x$ ✓

case (ii) suppose $\exists\, y_0 \in F$ such that $y_0 > x$ Hence (as $y_0 \in Q$) $y_0 \geq f(x)$

$\therefore$ $u \geq y_0 \geq f(x)$

$\therefore$ $Q = A$ proving (5)

W. Bade M202a Topology March 9 Monday

Theorem: (Zermelo Fixed Point)

E p.o.s. in which every totally ordered subset has an l.u.b. If $f: E \to E$ satisfies $x \le f(x)$ (any $x \in E$) Then $\exists\, w$ s.o. $w = f(w)$

$a \in E$

A is admissable

(1) $a \in A$

(2) $f(A) \subseteq A$

(3) if $B \subseteq A$ is totally ordered then lub $B \in A$

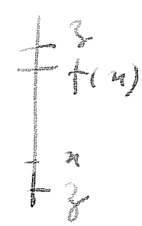
E contains a minimal ad set A

To show A is totally ordered

if so and $x = \text{lub } A$, then $f(x) = x$

$P = \{ x \in A \mid y \in A,\ y < x \to f(y) \le x \}$

(5) If $x \in P$, and $z \in A$ then either $z \le x$ or $z \ge f(x)$



(6) P is admissable!

(1) $a \in P$ vacuously

(2) $f(P) \subseteq P$ (must show if $x \in P$ and if $y \in A$ and $y < f(x)$ then $f(y) \le f(x)$)

From (5) either $y \le x$ or $y \ge f(x)$
not possible

cant have $y \ge f(x)$ so $y \le x$

case 1 $y = x$ then $f(y) = f(x)$ so $f(y) \le f(x)$

case 2 $y < x$ Then as $x \in P$ $f(y) \le x \le f(x)$

$\therefore$ P has property (2)

Now to show P has (3)

let $B \subseteq P$ be totally ordered

and let $v = \text{lub } B$

(Must show ~~fx~~ $v \in P$)

To show $v \in P$, let $y \in A$, $y < v$

Show $f(y) \le v$

From (5) if $b \in B \subseteq P$ then either $y \le b$ or $b \le f(b) \le y$

Last one can't hold for all $b \in B$. For then $b \le f(b) \le y$ is imposs

$v = \text{lub}(B) \le y$ But $y < v$ so

Thus $\exists\, b_0 \in B$ s.t. $y \le b_0$

case (i)

$y < b_0$ Then by defn of P $f(y) \le b_0 \le v$ ✓

case (ii) $y = b_0$ since $y \ne v$, $\exists\, b \in B$ with $y = b_0 < b$

As $b \in P$ $f(y) \le b \le v$

So we've covered all cases

$\therefore$ P is admissable

$\therefore$ $P = A$ (minimality)

$P = A \Rightarrow A$ is totally ordered, since if $x, z \in A = P$

either $z \le x$ or $z \ge f(x) \ge x$

w lub A $f(w) = w$

Vol I Chapter 1 of Linear Operators I
By Dunford & Schwartz

Theorem: (Hausdorff Maximality Theorem)

Every partially ordered set contains a maximal totally ordered subset.

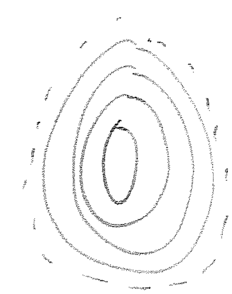
(maximal means not properly included in any other totally ordered subset)

Proof:

Let $\mathcal{E}$ be the family of all totally ordered subsets of E.

Order $\mathcal{E}$ by $\subseteq$. Then $(\mathcal{E}, \subseteq)$ is a p.o.s.

moreover has the property that every totally ordered subset of $\mathcal{E}$ has an l.u.b. (namely the union)



Let $x, y \in$ union of the family of sets

Then $x, y \in$ some set in the family so $x \le y$ or $y \le x$ because that set is totally ordered

$\therefore$ union of the family is totally ordered and $\therefore$ $(\mathcal{E}, \subseteq)$ satisfies the hypothesis of Zermelo's Theorem

(Maximal elt of a p.o.s. E x is a maximal elt of E if $x \le y \to y = x$)

We prove that $(\mathcal{E}, \subseteq)$ contains a maximal elt.

Suppose $(\mathcal{E}, \subseteq)$ has no maximal element

For each $\dot{A} \in \mathcal{E}$, the set $\mathcal{B}_{\dot{A}} = \{\dot{B} \in \mathcal{E} \mid \dot{B} \ge \dot{A}, \dot{B} \ne \dot{A}\}$ is then non empty. By the Axiom of Choice, $\exists$ a function g on $\mathcal{E}$ such that for $\dot{A} \in \mathcal{E}$, $g(\dot{A}) \in \mathcal{B}_{\dot{A}}$

$\therefore$ $g(\dot{A}) \supseteq A$ but $g(\dot{A}) \neq \dot{A}$ for all $\dot{A}$

An enormous contradiction of Zermelos Theorem.

QED

Lemma: (Zorns Lemma)

If every totally ordered subset of a partially ordered set $(E, \leq)$ has an upper bound then $(E, \leq)$ has a maximal element

Proof: By HMP E contains a maximal totally ordered subset E_0. Let x be an upper bound for E_0. If x is not a maximal elt, $\exists y \in E$ with $x \leq y$ & $x \neq y$. Then $y \notin E_0$ so $E_0 \cup \{y\}$ is a totally ordered subset of E properly containing E_0

Contradiction

W Bade — M202a Topology — 3-11-92 Wednesday

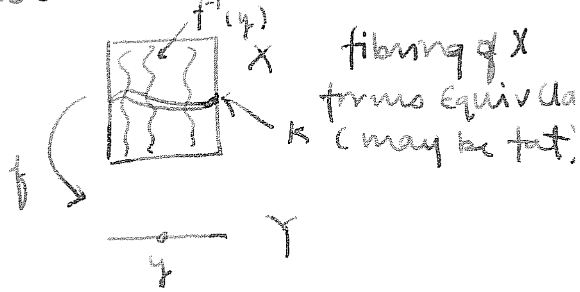
$AC \to HM \to ZL$ ($\to WO \to AC$)

example:

(living on sand?)

Theorem:

Let X, Y be compact Hausdorff spaces and $f: X \to Y$ be a continuous map of X onto Y.
Then $\exists$ a closed set $K \subseteq X$ such that $f(K) = Y$ and there is no closed proper subset L of K with $f(L) = Y$



Run into trouble w/ large spaces.

Would be easy (if countable base)

Proof:

Let $\mathcal{A} = \{E \mid E \subseteq X,\ E$ is closed and $f(E) = Y\}$

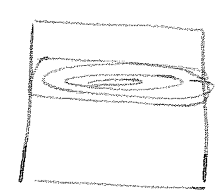
non empty $\because X$ is such a set.

Order $\mathcal{A}$ by $\supseteq$

Then $(\mathcal{A}, \supseteq)$ is a partially ordered set

Let $\mathcal{E}$ be a totally ordered subset of this pos

Let $E_0 = \bigcap \{E \mid E \in \mathcal{E}\}$



Then E_0 is closed

non empty (!) as it has fip.

(courtesy of the total ordering)

We show $f(E_0) = Y$

Zorn: Let $(E, \leq)$ be pos where each totally ordered subset has an upper bound. Then $(E, \leq)$ has a maximal element.

Let $y \in Y$ and ~~consider~~ $f^{-1}(\{y\}) \cap E \neq \emptyset \ \forall E \in \mathcal{E}_0$ (as $f(E) = Y$ for all E) Then the family $\{f^{-1}(\{y\}) \cap E \mid E \in \mathcal{E}\}$ has f.i.p. $\therefore$ has non empty intersection namely $E_\emptyset \cap f^{-1}(\{y\})$

$\therefore f(E_\emptyset) = Y$ (E_0 is a lower bound for E)

By Zorns lemma $(\mathcal{G}, \geq)$ contains a minimal elt.

$\boxtimes$

Let E be a vector space over a field F.

A set $K \subseteq E$ is linearly independent if for each finite set $k_1, \ldots, k_n$ $a_1 k_1 + \cdots + a_n k_n = 0$ with $a_i \in F \Rightarrow$ ~~$k_1 = k_2 = \cdots = k_n$~~ $a_1 = a_2 = \cdots = a_n = 0$

Let $\mathcal{E}$ be a family of all linearly independent subsets of E ordered by inclusion $\subseteq$. Then $(\mathcal{E}, \subseteq)$ is a p.o.s. Let $\mathcal{E}_0$ be a totally ordered subset of $\mathcal{E}$.

Then $\bigcup\{E \mid E \in \mathcal{E}_0\}$ is linearly independent $k_1, \ldots, k_n \in$ this union then they all belong to some set of the collection and are linearly independent there. Hence in the union.

$\therefore (\mathcal{E}_0, \subseteq)$ has an upper bound. By Zorn $(\mathcal{E}, \subseteq)$ contains a maximal element K.

Corollary: Every vector space has a basis. For each $e_0 \in E$ e_0 has a unique representation

$e_0 = \sum_{i=1}^{n} a_i k_i$ where $k_i \in K, a_i \in F$

If e_0 has no such representation then $K \cup \{e_0\}$ is linearly independent

$\Rightarrow K$ non maximal

Uniqueness of the representation follows from linear independence

$$e_0 = \sum_{i=1}^{n} a_i k_i = \sum_{i=1}^{m} a'_j k'_j \quad \text{different rep.}$$

$$\Rightarrow \sum_{i=1}^{n} a_i k_i - \sum_{j=1}^{m} a'_j k'_j = 0$$

$\therefore a_i = 0 = a'_j \quad \forall i,j$ not quite ...

$E = \mathbb{R}$, $\mathbb{F} = \mathbb{Q}$

K is called a Hammel Base for $\mathbb{R}$ (Base in algebraic sense

Hence non measurable. Its descriptive set theory is hard

Definition:

A pos $(E, \leq)$ is well ordered if each non empty subset of E contains a least ~~element~~ member under $\leq$ (contains a lower bound for itself)

Remark: $\exists$ well ordered sets $\mathbb{N}$, all finite sets

A well ordered set is totally ordered (take any pair, one is less than the other)

Theorem: (Zermelo)

Every set E can be well ordered

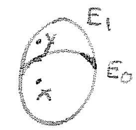
Proof: (Sketch)

Let $\mathcal{E}$ be the family of all pairs $(E_0, \le_0)$ where $E_\emptyset \subseteq E$ and $\le_\emptyset$ is a well ordering for $E_\emptyset$. We define an ordering $\prec$ on $\mathcal{E}$ by declaring $(E_\emptyset, \le_0) \prec (E_1, \le_1) \iff$

(i) $E_\emptyset \subseteq E_1$

(ii) $x, y \in E_\emptyset$ and $x \le_\emptyset y \Rightarrow x \le_1 y$

(iii) $x \in E_\emptyset$, $y \in E_1 \setminus E_0 \Rightarrow x \le_1 y$



[We show $(\mathcal{E}, \prec)$ is a p.o.s. in which every totally ordered subset has an upper bound in $\mathcal{E}$]

[Hence $(\mathcal{E}, \prec)$ has a maximal element $(\tilde{E}, \tilde{\le})$

Then we show ~~$(\tilde{E}, \tilde{\le})$~~ $\tilde{E} = E$]

W. Bade

M202a Topology

March 13
Friday

AC → HM → ZL → WO → AC

Theorem: (Zermelo)

Every set E can be well ordered

Proof:
Let $\mathcal{E}$ be the family of ~~sets~~ all pairs $(E_0, \leq_0)$ where $\leq_0$ is a well ordering of E_0. Define p.o. $<$ on $\mathcal{E}$ by

$$(E_0, \leq_0) < (E_1, \leq_1) \Leftrightarrow$$

(i) $E_0 \subseteq E_1$

(ii) $x, y \in E_0$ and $x \leq_0 y \Rightarrow x \leq_1 y$

(iii) $x \in E_0$, $y \in E_1 \setminus E_0 \Rightarrow x \leq_1 y$

① $(\mathcal{E}, <)$ is pos

② If $\mathcal{E}_0$ is a totally ordered set of $(\mathcal{E}, <)$ then it has an upper bound

(a) ~~Bd~~ Let $B = \bigcup\{A \mid (A, \leq_A) \in \mathcal{E}_0\}$

on B define ordering $\leq'$ by

$x \leq y'$ whenever $x, y \in$ same $(E_0, \leq_0) \in \mathcal{E}_0$

and $x \leq_0 y$ in ordering of $(E_0, \leq_0)$

This is well defined because $\mathcal{E}_0$ is totally ordered

③ Need $x \leq' x$

$x \leq' y \wedge y \leq' x \rightarrow x = y$

$x \leq' y, y \leq' z \rightarrow x \leq' z$

so $(B, \leq')$ is a p.o.s.

④ $(B, \leq')$ is a well ordered set

Let $F \subseteq B$. Show it has a least elt.

Know $F \cap E_0 \neq \emptyset$ for some $(E_0, \leq_0) \in \mathcal{E}_0$

Take least element x_0 of $F \cap E_0$. It works for $(B, \leq')$

ie x_0 is a lower bound for F in $(B, \leq')$

$\therefore (B, \leq')$ is well ordered

⑤ $(B, \leq')$ is an upper bound for $\mathcal{E}_0$

⑥ Hypothesis of Zorns Lemma satisfied

$\therefore (\mathcal{E}, \leq)$ contains a maximal elt $(\hat{E}, \hat{\leq})$

Show if $\hat{E} \neq E$ and $x \in E - \hat{E}$, can extend $(\hat{E}, \hat{\leq})$ to $(\hat{E} \cup \{x\})$ with $y \hat{\leq} x$ for all $y \in \hat{E}$

contradicts maximality of $(\hat{E}, \hat{\leq})$

$\therefore \hat{E} = E$ and $\hat{\leq}$ is a wellording for E.

Theorem: (Anticlimax)

The WO theorem implies the axiom of choice

Proof:

Let $\mathcal{A}$ be a family of non empty sets and well order their union

$B = \bigcup\{A \mid A \in \mathcal{A}\}$ For each $A \in \mathcal{A}$ define

$g(A) =$ least member of A

$g : \mathcal{A} \to \bigcup\mathcal{A} = B$ is a choice function

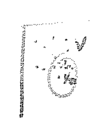
Directed Sets and Nets:

Definition:
A directed set is a set D with a relation $\geq$ satisfying (i) $x \geq x$ (ii) $x \geq y \wedge y \geq z \rightarrow x \geq z$
(iii) if $x, y \in D$, then $\exists z \in D$ with $z \geq x \wedge z \geq y$
(note x & y needn't be comparable)

Example:
(i) $\mathbb{N}$ with $\geq$
(ii) $\mathbb{R}$ with $\geq$

Definition:
A net in a set X is a function $f: D \rightarrow X$ where D is a directed set. A net $f: D \rightarrow X$ is eventually in a set $A \subseteq X$ if $\exists d_A \in D$ such that $f(d) \in A$ for all $d \geq d_A$
Let (X, T) be a topological space. A net $f: D \rightarrow X$ converges to $x_0 \in X$ if f is eventually in any nbd of x_0

Definition:
A point $x_0 \in X$ is a cluster point for a net $f: D \rightarrow A$ if f is not eventually in the complement of any nbd of x_0.



Equivalently if V is any nbd of x_0 and $d_0 \in D$ then $\exists d_1 \geq d_0$ such that $f(d_1) \in V$

Examples:
① $D = \mathbb{N}$ in usual ordering
A net $f: \mathbb{N} \rightarrow X$ is a sequence which converges $\Leftrightarrow$ it converges in the usual sense

② Let (X,T) be a topological space and $x_0 \in X$

Let $D = \{V \mid V \text{ is a nbd of } x_0\}$ ordered by inclusion

D is a directed set and a net $f: D \to X$ has the property that $f(V) \in V \; \forall V$

③ In calculus we say $\lim_{x \to x_0} f(x) = a$

$\Leftrightarrow \forall \epsilon > 0 \; \exists \delta > 0 \ni |f(x) - a| < \epsilon$ whenever $0 \leq |x - x_0| < \delta$

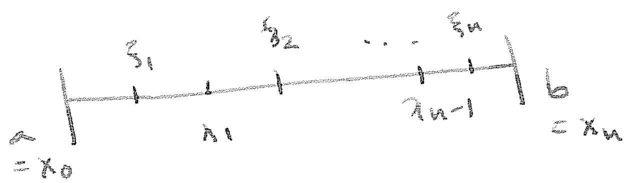
Let D be the directed set $D = \mathbb{R}$ with

$x_1 > x_2 \Leftrightarrow 0 < |x_1 - x_0| \leq |x_2 - x_0|$

and $\lim_{D} f(x) = A \Leftrightarrow \lim_{x \to x_0} f(x) = A$

④ A partition Π of $[a,b]$ is a finite set ~~x_0~~ x_0

$a_1 \leq x_0 \leq \xi_1 \leq x_1 \leq \xi_1 \; \ldots \leq \xi_n \leq x_n \leq b$; $2n$ points



mesh of Π is $\max(x_i - x_{i-1})$. Define $\Pi > \Pi'$ if mesh $\Pi \leq$ mesh Π'

If $\phi: [a,b] \to \mathbb{R}$ is continuous

define the net
$$f(\Pi) = \sum_{i=1}^{n} \phi(\xi_i)(x_i - x_{i-1})$$

Then $\lim_{\Pi} f(\Pi) = \lim_{\Pi} \sum_{1}^{n} \phi(\xi_i)(x_i - x_{i-1}) = \int_a^b \phi(t)\,dt$

Nets

Definition:

Directed set $(D, \geq)$

(1) $d \geq d$

(2) $d \geq e,\ e \geq f \rightarrow d \geq f$

(3) $d_1, d_2 \in D$ $\exists\, d_3 \in D$ with $d_3 \geq d_1,\ d_3 \geq d_2$

Not a partial ordering ($\because$ don't assume $d_1 \geq d_2 \wedge d_2 \geq d_1 \rightarrow d_1 = d_2$

Definition:

A net in X is an $f: D \rightarrow X$

A net is eventually in a set A if $\exists d_0 \in D$ so $f(d) \in A$ for all $d \geq d_0$

In (X, T) a net $f: D \rightarrow X$ converges to x_0 if it is eventually in every nbd of x_0

a cluster point x_0 of $f: D \rightarrow X$ if it is not eventually in the complement of any nbd of x_0

Definition:

For each $i \in I$, let $(D_i, \geq_i)$ be a directed set

$$\prod_{i \in I} D_i = \{ \varphi : I \rightarrow \bigcup D_i \mid \text{for each } i,\ \varphi(i) \in D_i \}$$

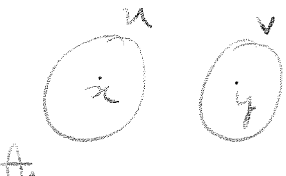
We direct $\prod D_i$ by $\varphi \geq \psi \Leftrightarrow$ for each $i \in I$, $\varphi(i) \geq_i \psi(i)$. This is the product directed set.

Theorem:

A top space (X, T) is a Hausdorff $\Leftrightarrow$ every convergent net has a unique limit.

Proof:

Let X be Hausdorff; $x, y \in X$, $x \neq y$. Let U, V be disjoint nbds of x, y resp. No net can eventually be in both U and V.



Suppose X is not Hausdorff. Then $\exists$ points $x, y \ni$ every nbd of x intersects every nbd of y. Let D = nbds of x, E = nbds of y. Direct them by $\supseteq$. Consider $D \times E$ (superdirect) Then for each ~~point~~ pair $(U, V) \in D \times E$ $\exists f(U,V) \in U \cap V$ (AC!)

Then $f: D \times E$ converges to both x and y.

Notation:

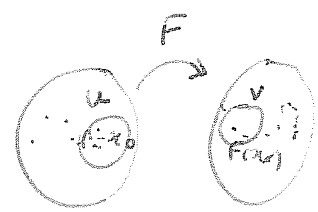
We write $\{x_\alpha\}$ for a net, understanding that $\exists$ directed set D and $f: D \to X$ with $f(\alpha) = x_\alpha$ for each $\alpha \in D$.

Theorem:

Let X, Y be topological spaces and ~~$F: X \to Y$~~. Let $F: X \to Y$. Then F is continuous $\Leftrightarrow$ whenever a net $\{x_\alpha\} \to x_0$, then $\{F(x_\alpha)\} \to F(x_0)$

Proof:

Let F be continuous. Let $\{x_\alpha\} \to x_0$



Let V be a nbd of $F(x_0)$

Then $F^{-1}(V) = U$ is a nbd of x_0

$\therefore \exists \alpha_0$ so $\alpha \geq \alpha_0 \to x_\alpha \in U$

$\therefore F(x_\alpha) \in V$ if $\alpha \geq \alpha_0$

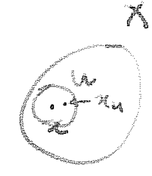
Suppose F is not continuous. Then $\exists x \in X$ and a nbd V of ~~x~~ $F(x)$ such that for each nbd U of x $\exists\, x_U \in U$ with $F(x_U) \notin V$

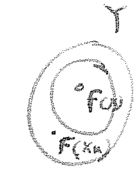
Let $\mathcal{D}$ = nbds of x directed by $\supseteq$.

Then the net $\{x_U\} \to x$

($\because$ eventually in every nbd of x)

But $\{F(x_U)\} \not\to F(x)$ (all members of net are outside V!)





$\boxtimes$ QED

Theorem:

A ~~topological~~ space (X,T) is compact $\Leftrightarrow$ every net in X has a cluster point.

(Generalization of result for compactness in metric space)

Proof:

Let $f: D \to X$ be a net in a compact space

For each $\alpha \in D$, let $A_\alpha = \{f(\beta) \mid \beta \geq \alpha\}$. Then the family $\{\bar{A}_\alpha \mid \alpha \in D\}$ has f.i.p. For given $\alpha_1, \dots, \alpha_n \in D$

$\exists\, \beta \geq \alpha_i \quad (1 \leq i \leq n)$

$\therefore f(\beta) \in \bigcap_{i=1}^{n} \bar{A}_{\alpha_i} \Rightarrow$ Fip

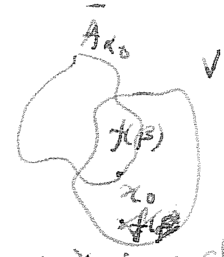
Since X compact, $\bigcap_{\alpha \in D} \bar{A}_\alpha$ is nonempty

Let $x_0 \in \bigcap_{\alpha \in D} \bar{A}_\alpha$

Then x_0 is a cluster point of $f: D \to X$

Let V be any nbd of x_0 and let $\alpha_0 \in D$

Since $x_0 \in \bar{A}_{\alpha_0}$, (every open nbd of x_0 must intersect A_{α_0}) $\exists$ a point $f(\beta)$ of A_{α_0} in V



$\beta \geq \alpha_0$ ... Hence $\exists\, \beta \geq \alpha_0$ so $f(\beta) \in V$ $\therefore x_0$ is a cluster point

We prove the converse.

Suppose every net in X has a cluster point.

(show X is compact)

Let $\mathcal{E}$ be a family of closed sets in X with f.i.p. Enlarge it if nec, to include all finite intersections of its members. Still has f.i.p. Still call it $\mathcal{E}$.

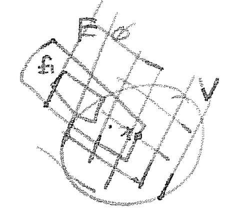
(To show $\bigcap_{E \in \mathcal{E}} E \neq \emptyset$)

Direct $\mathcal{E}$ by $\supseteq$. For each $E \in \mathcal{E}$ select $f(E) \in E$.

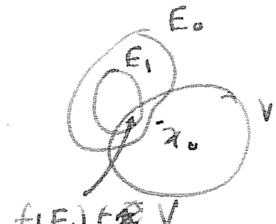
(Here we have a perfectly good net)

The net $f: \mathcal{E} \to X$ has a cluster point x_0.

(show $x_0 \in \bigcap \{E \mid E \in \mathcal{E}\}$) Let $E_0 \in \mathcal{E}$



Let V be a nbd of x_0. By defn of cluster point $\exists E_1 \subseteq E_0$ with $f(E_1) \in V$



Hence every nbd of x_0 contains points of E_0

Note $f(E_1) \in E_1 \subseteq E_0$

But E_0 is closed; so $x_0 \in E_0$

$\boxtimes$ QED

Definition:

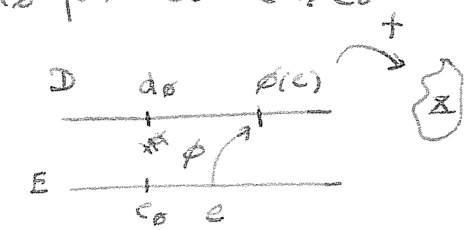
Let $f: D \to X$ be a net in X. A net $g: E \to X$ is a subnet of $f: D \to X$

~~A net~~ $\exists\, \varphi: E \to D$ such that

(1) For each $d_0 \in D$, φ is eventually in $\{d \mid d \geq d_0\}$

(2) $g(e) = f(\varphi(e))$ $(e \in E)$ $g = f \circ \varphi$

(1') Given $d_0 \in D$, $\exists\, e_0 \in E$ s.t. $\varphi(e) \geq d_0$ for all $e \geq e_0$



Note: A net $g: \mathbb{N} \to X$ is a subsequence of a sequence $f: \mathbb{N} \to X$ if $\exists$ a strictly increasing $\varphi: \mathbb{N} \to \mathbb{N}$ such that $g = f \circ \varphi$

Remark: A subnet of a sequence need not be a subsequence

Remark: Sequences are small for dealing w/ topology of metric spaces but for arb spaces nets are required

Theorem: If a net $f: D \to X$ (X, T) converges to x_0 then so does every subnet

Proof:

Let U be any nbd of x_0. Then ~~$\exists d$~~ $\exists\, d_u \in D$ so that $f(d) \in U$ for all $d \geq d_u$. Let $g = f \circ \varphi: E \to X$ be a subnet of f

By (1') $\exists\, e_u$ so $e \geq e_u \to \varphi(e) \geq d_u$

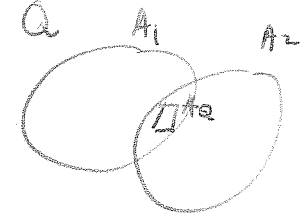
$\therefore$ if $e \geq e_u$ then $g(e) = f(\varphi(e)) \in U$ (as $\varphi(e) \geq d_u$)

Lemma:

Let $f : D \to X$ be a net in X; let Q be a family of non empty subsets of X satisfying

(a) f is not eventually in the complement of any set in Q

(b) The intersection of 2 members of Q contains a member of Q



Then $\exists$ a subnet of f which is eventually in every set in Q

$\Rightarrow Q$ has f.i.p.

doesn't mean has a cluster pt or $\bigcap_{q \in Q} q \neq \emptyset$

eg $\{1/n\}$ in $(0,1)$ space where $Q = (0, \alpha)$ $\alpha \in (0,1)$

Theorem:

If a net in X has a cluster point $x_\emptyset$, then $\exists$ a subnet which converges to $x_\emptyset$

Proof: (Theorem)

Let Q = nbds of $x_\emptyset$

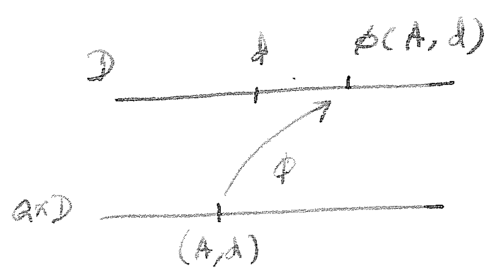
~~(a) says~~ Proved by Lemma

Proof: (Lemma)

Direct Q by $\supseteq$. Form the product directed set (SP) $(Q, \supseteq) \times (D, \geq)$ In this $(A, d) \geq (A_1, d_1)$ iff $A \subseteq A_1$ and $d \geq d_1$. For each pair $(A, d) \in Q \times D$ let $\phi(A, d)$ be a member of D such that

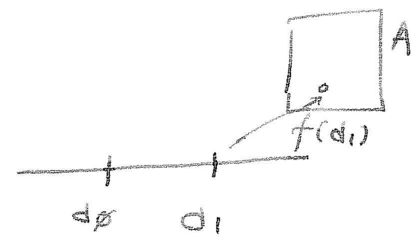
(i) $\phi(A, d) \geq d$

(ii) $f(\phi(A, d)) \in A$



Note:

(1) For each $A \in \mathcal{Q}$ and $d \in D$ $\exists d_1 \geq d$ so $f(d_1) \in A$



Take $\phi(A, d) = d_1$

$g = f \circ \phi$ is a subnet which is eventually in each set of $\mathcal{Q}$.

For given $A_\emptyset \in \mathcal{Q}$, $d_\emptyset \in D$

if $(A, d) \geq (A_\emptyset, d_\emptyset)$ then $A \subseteq A_\emptyset$

so $g(A, d) = f(\phi(A, d)) \in A \subseteq A_\emptyset$

- did I miss something here

Theorem:

A topological space $(X, \mathcal{T})$ is compact $\Leftrightarrow$ every net in X has a convergent subnet

Definition:

A net $f: D \to X$ in a set X is a universal net if for each $A \subseteq X$, f is either eventually in A or in A^c

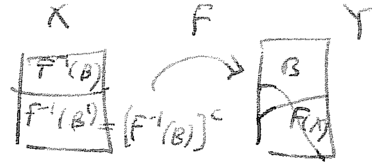
Theorem:

Every net in a set has a universal subnet (consequence of ZL)

Theorem:

(a) If f is universal in X and $F: X \to Y$ then $F \circ f$ is universal in Y

(b) A universal map converges to any of its cluster points ($\therefore$ can have only one if any)



Proof:

Let $B \subseteq Y$ and $A = F^{-1}(B)$

$\therefore A^c = F^{-1}(B^c)$

f is eventually in $F^{-1}(B)$ or $F^{-1}(B^c) = [F^{-1}(B)]^c$

$\therefore F \circ f$ is eventually in B (or in B^c)

N. Bade

M202a Topology

20 March
Friday

Recall

Lemma:

Let $f: D \to X$ be a net and Q be a family satisfying

(a) f is not eventually in the complement of any member of Q

(b) The intersection of two members of Q contains a member of Q

Then $\exists$ a subnet $g: E \to X$ of $f: D \to X$ which is eventually in each member of Q

Recall:

A net $f: D \to X$ is universal in X if fore each $A \subseteq X$, f is either eventually in A or in A^c

Theorem:

Each net in a set has a universal subnet.

Proof:

Let f be the net in X.

Suffices to show $\exists$ a family Q of subsets of X which satisfy

(a) If $A \subseteq X$, either A or $A^c \in Q$

(b) Q is closed under finite intersections

(c) f is not eventually in the complement of any set in Q

If so $\exists$ a subnet eventually in every set in Q

$\therefore$ it is universal!

Consider the family $\mathcal{E}$ of all collections of subsets of X which satisfy (b) and (c). Order $\mathcal{E}$ by $\subseteq$. Then $(\mathcal{E}, \subseteq)$ is a p.o.s. Let $\mathcal{E}_0$ be a totally ordered subset of $\mathcal{E}$. Then the union of all collections in $\mathcal{E}_0$ is a collection which satisfies (b) and (c). (Reasoning: identical to that used in one of the HW problems)

$\therefore$ By Zorns Lemma $\exists$ a family Q of subsets of X which is maximal wrt properties (b) and (c). We show Q satisfies (a)

We assert:

(1) If A_0 is any subset of X such that f is not eventually in the complement of any of the sets $\{A_0 \cap A \mid A \in Q\}$ then $A_0 \in Q$. Otherwise $Q \cup \{A_0\} \cup \{A_0 \cap A \mid A \in Q\}$ is bigger than Q and has (b) and (c)

(2) If f is eventually in a set A_0 then $A_0 \in Q$

(By (1))

(3) If $A \in Q$ and $B \supseteq A$ then $B \in Q$

(By (1))

Let $A \subseteq X$ and suppose $A \notin Q$ (show $A^c \in Q$)

If $A \notin Q$ then by (1) $\exists B \in Q$ such that f is eventually in $(A \cap B)^c$. Hence by (2), $(A \cap B)^c \in Q$

(Refer fig (1))

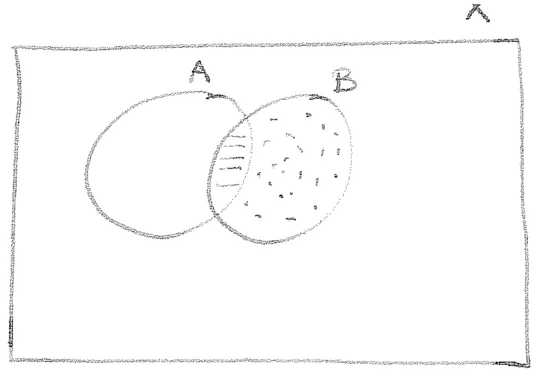


fig 1

But then $B \cap (A \cap B)^c \in Q$

$$\begin{aligned} &\| \\ &B \cap (A^c \cup B^c) = B \cap A^c \end{aligned}$$

Since $A^c \supseteq B \cap A^c$

$A^c \in Q$ by (3)

Bizarre but valid consequence of ZF + AC

Theorem:

A net in a topological product $f: D \to \prod_{\alpha \in I} X_\alpha$ converges to $x_0 \iff pr_\alpha f$ converges to $pr_\alpha x_0$ for all $\alpha \in I$

Proof:

If $f: D \to \prod X_\alpha$ converges to x_0, then for each α, $pr_\alpha f(d) \to$ ~~prf~~ $pr_\alpha(x_0)$ since pr_α is continuous.

$\lim_d pr_\alpha f(d) = pr_\alpha(x_0)$ each α (QED[$\Rightarrow$])

Let V be any nbd of x_0. Then $\exists \alpha_1, \dots, \alpha_n$ and $U_{\alpha_1}, \dots, U_{\alpha_n}$ open in $X_{\alpha_1}, \dots, X_{\alpha_n}$ resp. so

$$x_0 \in \underbrace{\left[\prod_{i=1}^{n} U_{\alpha_i}\right] \times \prod_{\alpha \notin (\alpha_1, \dots, \alpha_n)} X_\alpha}_{\leftarrow \text{ basic open set}} \subseteq V$$

Now $\exists\, d_1, \dots, d_n \in D$ so

$pr_\alpha f(d) \in U_{\alpha_i}$ if $d \geq d_i$

$\therefore$ if $d \geq d_1, d_2, \dots, d_n$ then $f(d) \in V$

$\therefore f$ is eventually in V

$\therefore \lim_D f(d) = x_0$ QED[$\Leftarrow$]

Theorem: (Tychonoff)

Let $X_0 = \prod\{X_\alpha \mid \alpha \in I\}$ be a product of topological spaces with the product topology

Then X_0 is compact $\Leftrightarrow X_\alpha$ is compact.

Proof:

Clear that if X_0 is compact, then for each $\alpha \in I$, $X_\alpha = pr_\alpha(X_0)$ is compact (Being the continuous image of a compact set) QED($\Rightarrow$)

Let all X_α be compact. We saw X_0 is compact if every net in X_0 has a cluster point. Also if every net in X_0 has a convergent subnet.

Let $f: D \to X$ be a net in X_0 and let $g: E \to X_0$ be a universal subnet of f. Then $pr_\alpha g$ is universal in X_α for each α.

Since X_α is compact $pr_\alpha g$ converges in X_α to x_α^0, say. Let x_0 be the point in X_0 such that $pr_\alpha(x_0) = x_\alpha^0$ $\alpha \in I$ ($\because$ Universal net has at most 1 limpt)

By last theorem,

g converges to x_0 in X_0

$\therefore X_0$ is compact QED ($\Leftarrow$)

W. Badi

M202a Topology

March 30 '92
Monday

Theorem: (Tychonoff)

$\prod \{X_\alpha \mid \alpha \in I\}$ is compact $\Leftrightarrow$ if each X_α is compact

Proof:

($\Rightarrow$) If $\prod X_\alpha$ is compact then $X_\alpha = p_\alpha(\prod X_\beta)$ is compact

($\Leftarrow$) Let each X_α be compact. To show $\prod X_\alpha$ is compact it is enough to show every universal net $f: D \to \prod X_\alpha$ converges. Let $f: D \to \prod X_\alpha$ be universal. Then for each β, $p_\beta f$ is universal in X_β and therefore converges to some x_β^0. Let x^0 be $\{x_\beta^0\}$ Then $f \to x^0$ $\therefore \prod X_\alpha$ is compact.

(All the work is concealed in existence of univ subnets.)

There is also a proof using filters & ultrafilters.

Definition:

A topological space (X,T) is metrizable if there exists a metric ρ on X such that the metric topology of (X,ρ) is T.

Theorem:

The countable product of metric spaces is metrizable. If for each $n \in \mathbb{N}$ (X_n, ρ_n) is a metric space, then a metric for $\prod X_n$ which yields the product topology is $\rho(\vec{x},\vec{y})$

$[\vec{x} = \{x_n\}]$ $\quad \rho(\vec{x}^d,\vec{y}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left[\frac{\rho_n(x^d_n, y_n)}{1+\rho_n(x^d_n, y_n)}\right]$ $\quad \vec{x}^d \to \vec{y}$

If each (X_n, ρ_n) is complete so is $\prod X_n$ for ρ.

Proof:

$\phi(t) = \frac{t}{1+t}$ is increasing of $(0,\infty)$

$\therefore$ if $a,b > 0$ then $\frac{a}{1+a} + \frac{b}{1+b} = \frac{a+b+2ab}{1+a+b+ab} \geq \frac{a+b+ab}{1+a+b+ab} \to$

$\to \geq \frac{a+b}{1+a+b}$

Hence if ρ is a metric, so is $\rho' = \rho/_{1+\rho}$, since

$$\frac{\rho(x,y)}{1+\rho(x,y)} < \frac{\rho(x,z)+\rho(z,y)}{1+\rho(x,z)+\rho(z,y)} \leq \frac{\rho(x,z)}{1+\rho(x,z)} + \frac{\rho(z,y)}{1+\rho(z,y)} \quad (\P)$$

If $\rho'(x_\alpha, x_0) \to 0 \iff \rho(x_\alpha, x_0) \to 0$

$\therefore$ ρ and ρ' yield the same topology

$\therefore$ $i : (X,\rho) \to (X,\rho')$ is bicontinuous (homeom.)

identity map: $\therefore$ take net converging to something in X under ρ $\Rightarrow$ converges to something in X under ρ'

Clear $\rho(\vec{x},\vec{y})$ is a metric on $\prod(X_n)$. Let $(\prod X_n, p)$ ① be the product with the product ~~topology~~ and $(\prod X_n, \rho)$ ② be the product with the metric ~~topology~~. Let $\{\vec{x}^d\} \to \vec{x}^0$ be a net converging in ①. Then $\lim_d x_n^d = x_n^0$ $(n \in \mathbb{N})$ (only countably many coordinates!?)

$\therefore \lim_d \rho_n(x_n^d, x_n^0) = 0$ $(n \in \mathbb{N})$

assert $\rho(\vec{x}^d, \vec{x}^0) \to 0$ so $\vec{x}^d \to \vec{x}^0$ in ②

For net $\epsilon > 0$ Pick N so $1/2^N < \epsilon$

$$\rho(\vec{X}^d, \vec{x}^o) \le \frac{\epsilon}{2} + \sum_{n=1}^{N} 1/2^n \frac{\rho_n(X_n^d, x_n^o)}{1 + \rho_n(X_n^d, X_n^o)}$$

Hence $\exists\, d_0 \in D$ so $d \geq d_0 \Rightarrow$ right side $< \epsilon$

$\therefore \vec{x}^d \to \vec{x}^o$ in ② $(\Pi X_n, \rho)$

If $\vec{X}^d \longrightarrow \vec{x}^o$ in ②, then $\rho(\vec{X}^d, \vec{X}^o) \xrightarrow[d]{} 0$

$\therefore \rho_m(\vec{X}_n{}^d, \vec{X}_n^o) \to 0 \quad \therefore \vec{X}_n^d \to \vec{X}_n^o$ so the net converg in ①

(coordinates of net converge $\Rightarrow$ net converges)

argument using nets is much more convenient that one using neighborhoods

W. Badi

M202a Topology

April 1 '92
Wednesday

Theorem:

The countable product of metric spaces is metrizable

Definition:

A top space (X,T) is a Tychonoff space (or completely regular) if given any closed set F and $x_0 \notin F$ $\exists$ continuous function f such that

$f(F)=0$ and $f(x_0)=1$ and $0 \le f(x) \le 1$ on X

Tychonoff is stronger than regular, weaker than normal

A subspace of a Tychonoff space is Tychonoff.
(Not true of a normal space!!)

Locally compact spaces are Tychonoff but need not be normal. (Messy, ordinals ...)

Definition:

A cube is a topological product of unit intervals

$I=[0,1]$

$\prod\{I_\alpha \mid \alpha \in \text{some index set}\}$

compact (by Tychonoff theorem)

Theorem:

A topological space (X,T) is Tychonoff $\Leftrightarrow$ it is homeomorphic to a subspace of a cube

Proof:

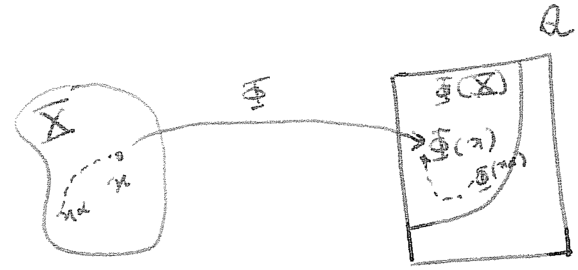
Let (X,T) be Tychonoff and let $\mathcal{F}$ be a family of continuous functions f on X to $[0,1]$ such that for each

F closed and $x_0 \notin F$ $\exists f \in \mathcal{F}$ such that $f(x_0)=1, f(F)$

Define $Q = \prod\{I_f \mid f \in \mathcal{F}\}$, $I_f = I$ $(f \in \mathcal{F})$

Let Q have product topology

Define $\Phi : X \to Q$ by $\Phi(x) = \{f(x) \mid f \in \mathcal{F}\}$



$\Phi(x)(f) = f(x)$ (Better write $x(f)$?

crucial point!!

Let $x_\alpha \to x_0$ in X (net in X)

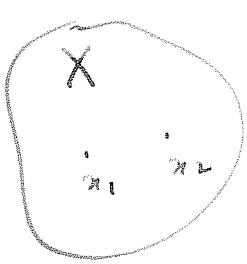
$f(x_\alpha) \to f(x_0)$ for each $f \in \mathcal{F}$ (since f is continuous)

Then $\Phi(x_\alpha)(f) \to \Phi(x_0)(f)$ $\forall f \in \mathcal{F}$

$\therefore \Phi(x_\alpha) \to \Phi(x_0)$ in Q ← why? depends on use of product topology!

$\therefore \Phi : X \to Q$ is continuous

It is one to one because $\mathcal{F}$ contains enough functions f to distinguish points of X.



$f(x_1) = 1$ $f(x_2) = 0$

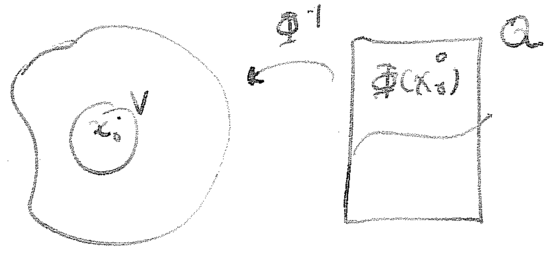
$\Phi(x_1) \neq \Phi(x_2)$

← defined on $\Phi(X) \to X$

(Show Φ^{-1} is continuous) Let $\Phi(x_\alpha) \to \Phi(x_0)$ in Q

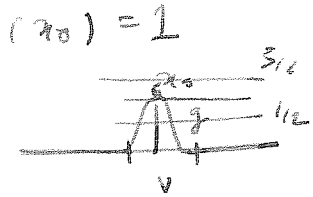
(Show $x_\alpha \to x_0$ in X)

Let V be a nbd of x_0 in X



Let V be a nbd of x_0 in X (Show $\exists\ \alpha_0$ so that $\alpha \geq \alpha_0 \Rightarrow x_\alpha \in V$)

Select $g \in \mathcal{F}$ such that $g(V^c) = 0$; $g(x_0) = 1$



Since $\Phi(x_\alpha) \to \Phi(x_0)$

$g(x_\alpha) \longrightarrow g(x_0)$ (??)

$\therefore\ \exists\ \alpha_0$ so that $|g(x_\alpha) - 1| < 1/2$ if $\alpha \geq \alpha_0$

Then $x_\alpha \in V$ for all $\alpha \geq \alpha_0$

$\therefore\ \alpha \geq \alpha_0 \to x_\alpha \in V$ $\therefore$

$\therefore\ \Phi^{-1}|_X : \Phi(X) \to X$ is continuous

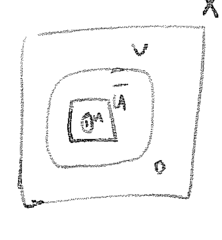
Theorem: (Urysohn Metrization Theorem)

A regular topological space w/a countable base is metrizable

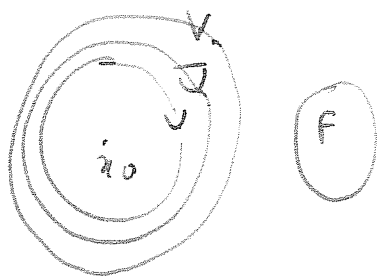
Proof:

By a previous theorem X is normal. Let $\mathcal{B}$ be a countable base for T. Consider a family $\mathcal{A}$ of all pairs (U, V) belonging to $\mathcal{B} \times \mathcal{B}$ such that $U \subseteq \bar{U} \subseteq V$ By normality, For each pair select a function $f: X \to [0, 1]$ so that $f(\bar{U}) = 1$ $f(V^c) = 0$ (f continuous)

Note if ~~x/y~~ F is closed, $x_0 \notin F$ $\exists (U, V) \in \cancel{\mathcal{B}\times\mathcal{B}}\ \mathcal{A}$



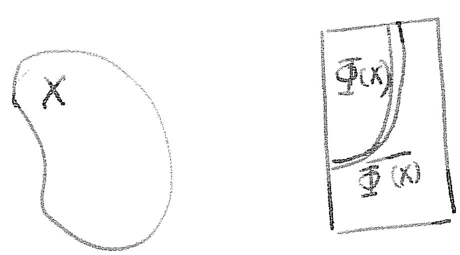
so that $x_0 \in U \subseteq \bar{U} \subseteq V$ $F \subseteq V^c$



Let $\mathcal{F}$ be the collection of such functions (using ZL)
the map $\Phi : X \to \prod\{I_f \mid f \in \mathcal{F}\} = Q$ is a homeomorphism by
the last theorem. Now $\mathcal{F}$ is countable as
$\mathcal{B} \times \mathcal{B}$ is countable and so Q is metrizable
Give X the metric from $\Phi(X)$.

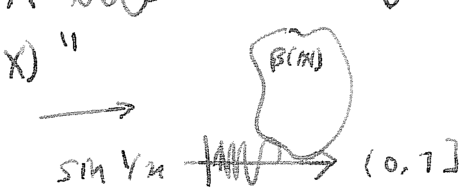
$Q = \prod\{I_n \mid n \in \mathbb{N}\}$ is called the Hilbert cube.

If $\Phi : X \to Q = \prod\{I_f \mid f \in \mathcal{F}\}$
then $\overline{\Phi(X)}$ is a compactification of X



What happens if we take $\mathcal{F}$ = all cont fns from $X \to I$
Stone-Cech compactification ~~theorem~~ $\beta(X)$ of X
has remarkable property!
"Every bdd cont function on X has a unique
continuous extension to $\beta(X)$"

$X = \mathbb{N}$



W Badé

M202a Topology

April 3 '9
Friday

Functional Analysis : Next Wednesday

X : completely regular

$\mathcal{F}$ $\quad f : X \to [0,1]$

enough of these f's so given

F ~~of~~ closed $\quad x_0 \notin F \quad \exists\, f \in \mathcal{F}$

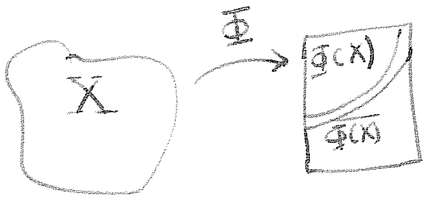
$$f(x_0) = 1 \quad f(F) = 0$$

$$Q_{\mathcal{F}} = \Pi\{ I_f : f \in \mathcal{F}\}$$

$\Phi : X \to Q_{\mathcal{F}}$ by ~~$\Phi(x) = f(x)$~~

$$\Phi(x)(f) = f(x) \quad (f \in \mathcal{F})$$

is a homeomorphism of X into Q



$Q_{\mathcal{F}} \leftarrow$ when $\mathcal{F}$ is <u>all</u> cont $f : X \to [0,1]$

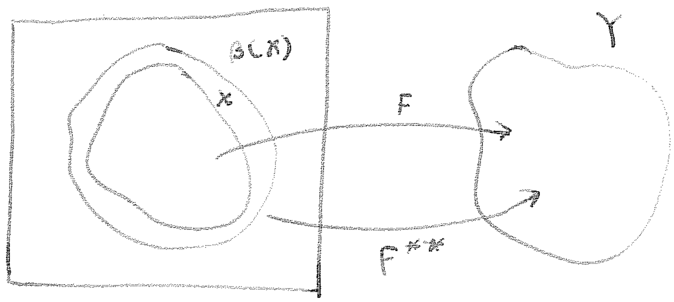
$$\overline{\Phi(X)}^{Q} \triangleq \beta(X)$$

What if $\mathcal{F}$ = all continuous functions $f : X \to [0,1]$

In that case $\overline{\Phi(X)}$ is called the Stone-Čech compactification of X. (By closing the product get a compact set)

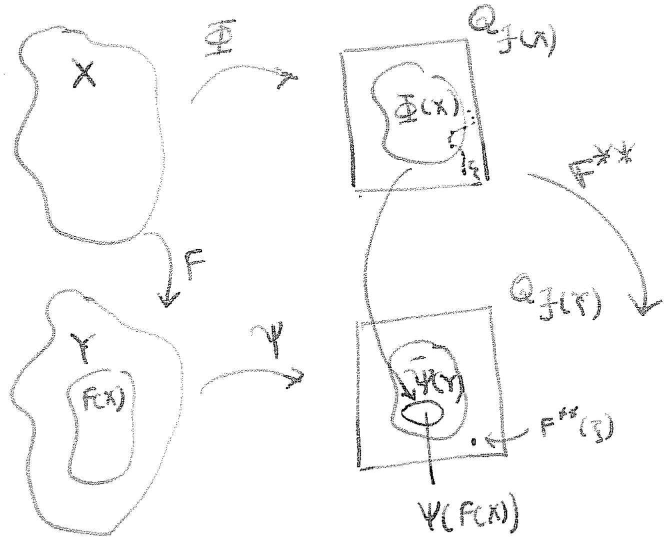
Theorem: (Stone Čech)

If X is a Tychonoff space and $F : X \to Y$ is a continuous map of X into a compact Hausdorff space ~~of~~ Y then there exists a unique continuous extensions of f which carries $\beta(X)$ into Y.



Proof:

Let $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ be all continuous functions from X (resp Y) into $[0,1] = I$ form $Q_{\mathcal{F}(X)}$, $Q_{\mathcal{F}(Y)}$



Let $\Phi : X \to Q_{\mathcal{F}(X)}$ be the embedding homeomorphism of X into $Q_{\mathcal{F}(X)}$

Let $\Psi : Y \to Q_{\mathcal{F}(Y)}$ be the embedding homeomorphism of Y into $Q_{\mathcal{F}(Y)}$

defined by

$$\Phi(x)(f) = f(x) \quad (f \in \mathcal{F}(X))$$

$$\& \ \Psi(y)(g) = g(y) \quad (g \in \mathcal{F}(Y))$$

we define a map $F^{**} : Q_{\mathcal{F}(X)} \to Q_{\mathcal{F}(Y)}$

we have $F : X \to Y$

define $F^* : \mathcal{F}(Y) \to \mathcal{F}(X)$ by $F^*(g)(x) = g(F(x)) = ((g \circ F)(x))$

continuous!

$x \in X$

$g \in \mathcal{F}(Y)$

Now define $F^{**}: Q_{\mathcal{F}(X)} \to Q_{\mathcal{F}(Y)}$ by

$$F^{**}(\xi)(g) = \xi(F^* g)$$

Proof essentially over.

(1.) F^{**} is continuous

Let $\{\xi_\alpha\}$ be a net in $Q_{\mathcal{F}(X)}$ with $\xi_\alpha \to \xi_0$

$$(\therefore \xi_\alpha(f) \to \xi_0(f) \ (f \in \mathcal{F}(X))) \text{ so}$$

$$\therefore \xi_\alpha(F^* g) \to \xi_0(F^* g) \quad (g \in \mathcal{F}(Y))$$

$$\| \qquad\qquad\qquad \|$$

$$(F^{**}\xi_\alpha)(g) \to (F^{**}\xi_0)(g)$$

$$\therefore F^{**}\xi_\alpha \to F^{**}\xi_0$$

$\therefore F^{**}$ is continuous. (Oh, the joy of nets)

(2) $F^{**}\,\Phi(X) = \Psi(F(X))$

Proof $(F^{**}\Phi(x))(g) = \Phi(x)(F^* g) = (F^* g)(x) = g(F(x))$

$$= \Psi(F(x))(g) \quad (g \in \mathcal{F}(Y))$$

We identify X with $\Phi(X)$ in $Q_{\mathcal{F}(X)}$ and identify Y with Y with $\Psi(Y)$ in $Q_{\mathcal{F}(Y)}$

$\therefore F^{**}$ is a continuous extension of F

By continuity of F^{**}

$$F^{**}(\beta(X)) = F^{**}\left(\overline{\Phi(X)}^{Q_{\mathcal{F}(X)}}\right) \subseteq \overline{F^{**}(\Phi(X))}^{Q_{\mathcal{F}(Y)}}$$

) proof of cont maps

$$= \overline{\Psi(F(X))}^{Q_{\mathcal{F}(Y)}} \qquad \cancel{\beta(X)}$$

$$= \cancel{\beta(F(X))}$$

first use of this fact

$$\subseteq \beta(Y) = Y \quad (\text{as } Y \text{ is compact})$$

$$\subseteq \overline{\Psi(Y)}^{Q_{\mathcal{F}(Y)}}$$

$$= \beta(Y) = Y$$ (as Y is compact! Any compactification of a compact set gives back same set)

W. Bade

M202a Topology

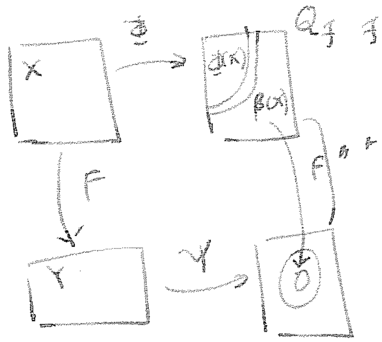
April 6 '9?
Monday

Theorem: (Stone Cech compactification)
X completely regular, then $\exists$ compactification $\beta(X)$ of X

Proof:

$F\ \Phi: X \to Y$ cont

$\exists$ a unique continuous extension $\overset{F^{**}}{\Phi}: \beta(X) \to Y$



$Q_{\mathcal{F}}$ $\mathcal{F}$ = all cont fn $f: X \to [0,1]$

Uniqueness of F^{**} comes from fact that X is dense in $\beta(X)$
(So existence of extension implies it is uniquely determined)

$X \subseteq \beta(X)$

Corollary:
If $F: X \to \mathbb{R}$ is a bounded continuous function i.e. $|F(x)| \le M$ $(x \in X)$, then $\exists$ a unique continuous extension to $\{z \mid |z| \le M\} \leftarrow Y$

Let $X = \mathbb{N}$

Let $\beta(\mathbb{N})$ be its S.C. compactification

singletons are open & closed $\Rightarrow$ remain open & closed
($\because$ imbedding map is a homeomorphism)

Theorem: In $\beta(\mathbb{N})$ the closure $\bar{U}$ of every open set U is open. (Very strange)

Proof: Let U be open in $\beta(\mathbb{N})$ and $A = \mathbb{N} \cap U$
Remember $\mathbb{N}$ is dense in $\beta(\mathbb{N})$. so $\bar{A} = \bar{U}$ (?)

Consider χ_A as a bounded continuous real valued function on $\mathbb{N}$.

Let f be its unique continuous extension to $\beta(\mathbb{N})$

Then f takes only the values 0 & 1 on $\beta(\mathbb{N})$

can't be say $1/2$ $\because$ there would have to be a net of integers going to it. Let $V = \{p \in \beta(\mathbb{N}) \mid f(p) = 0\}$,

$W = \{p \in \beta(\mathbb{N}) \mid f(p) = 1\}$

$\beta(\mathbb{N}) = V \cup W$, $V \cap W = \emptyset$ $\quad$ V, W are both open & closed in $\beta(\mathbb{N})$

$\beta(\mathbb{N}) = V \cup W$

$A \subseteq W$

$A^c \subseteq V$

$\bar{A} = \bar{U} \subseteq W$

$\bar{A^c} \subseteq V$

$\bar{A} \cup \bar{A^c} = \overline{A \cup A^c} = \bar{\mathbb{N}} = W \cup V$

$\parallel$

$\bar{U} \cup \bar{A^c}$ $\quad (\because \bar{A^c} \subseteq V)$

$\therefore$ $\bar{U} = W$ which is open

Definition:

A compact Hausdorff space is extremely disconnected if in X the closure of every open set is open

Facts:

D discrete

$\beta(D)$ is ext disc

Not all ext disc spaces are the SC compactification of discrete spaces

N. Bade

M202a - functional analysis

April 8 '92
Wednesday

X : vector space over either $\mathbb{R}$ or $\mathbb{C}$ +, scalar multiplication

opology on X ; Cont of +, $\circ$ scalar

Banach Space

Let X be a Hausdorff space

Let BC(X) be the vector space of all bdd continuous functions on X (real valued or complex valued)

f is bounded if $\exists M$ so $\sup_{x \in X} |f(x)| \le M$

$$(f+g)(x) = f(x) + g(x) \quad (\forall x \in X)$$

$$(\alpha f)(x) = \alpha \cdot f(x) \quad (x \in X), \alpha \in \text{scalars}$$

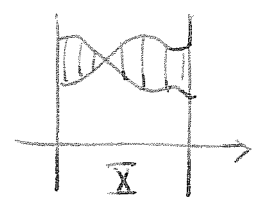
for $f \in BC(X)$, the norm of f $\quad \|f\| = \sup_{x \in X} |f(x)|$

Easy to see

$$\|f+g\| \le \|f\| + \|g\|$$

$$\|\alpha f\| = |\alpha| \, \|f\|$$

$$f = 0 \Leftrightarrow \|f\| = 0$$



then with $\rho(f,g) = \|f-g\| = \sup_{x \in X} |f(x) - g(x)|$

BC(X) is a metric space.

$$\rho(f,h) = \|f-h\| \le \|f-g\| + \|g-h\| = \rho(f,g) + \rho(g,h)$$

Note $\|f\| = \|-f\|$

also $\big| \|f\| - \|g\| \big| \le \|f \pm g\| \le \|f\| + \|g\|$

$f = f - g + g \quad \|f\| \le \|f-g\| + \|g\|$

$\|f\| - \|g\| \le \|f-g\|$

$\|g\| - \|f\| \le \|-(f-g)\| = \|f-g\|$

$\therefore \big| \|f\| - \|g\| \big| \le \|f-g\|$

For + sign replace $\|g\|$ by $\|-g\|$

Lemma: $BC(X)$ is complete for the metric given by the norm

Proof: Let $\{f_n\}$ be Cauchy $\therefore$ given $\epsilon > 0$ $\exists N$

so $\sup_{x \in X} |f_m(x) - f_n(x)| = \|f_m - f_n\| < \epsilon$ if $m, n \ge N$

Given x, $|f_m(x) - f_n(x)| \le \epsilon$ if $m, n \ge N$

$\therefore \lim f_n(x) = f_0(x)$ exists for all $x \in X$

show $f_0 \in BC(X)$ and $\|f_n - f_0\| \to 0$ as $n \to \infty$

Since $|f_m(x) - f_n(x)| < \epsilon$ $(x \in X;\ m, n \ge N)$

fix m, let n run !!

$\therefore |f_m(x) - f_0(x)| \le \epsilon$ $(x \in X,\ m \ge N)$

$\therefore \lim_{m \to \infty} f_m(x) = f_0(x)$ uniformly on X

Recall: uniform limit of continuous functions is continuous

also $\sup_{x \in X} |f_m(x) - f_0(x)| \le \epsilon$ if $m \ge N$ (14-2-92)

$= \|f_m - f_0\|$

$\therefore f_m - f_0 \in BC(X)$

$\therefore f_0 = f_m - (f_m - f_0) \in BC(X)$ and $\|f_m - f_0\| \to 0$ as $m \to \infty$

Let X be compact

Lemma:

If f is continuous on a compact space, then f is bounded & $\sup_{x \in X} |f(x)| = \max_{x \in X} |f(x)|$

($\hookrightarrow \exists x_0$ so $|f(x_0)| = \sup_{x \in X} |f(x)|$)

Proof:

Suppose f is not bounded. Then $\exists \{x_n\} \subseteq X$ so $|f(x_n)| > n$, $(n \in \mathbb{N})$

$x_1 \, x_2 \cdots$

Let x_0 be a cluster point of $\{x_n\}$

Let $\{x_\alpha\}$ be a convergent subnet of the sequence. So $\{x_\alpha\} \to x_0$ then $|f(x_\alpha)| \to \infty = |f(x_0)|$

Contradiction!

Let $M = \sup_{x \in X} |f(x)|$ then $\exists$ sequence $\{x_n\}$ so $|f(x_n)| > M - 1/n$ $(n \in \mathbb{N})$

Let $\{x_\alpha\}$ be a convergent subnet of the sequence $\{x_n\}$

Then $\{x_\alpha\} \to x_0$ say

then $|f(x_\alpha)| \to |f(x_0)| = M$

QED

If X is compact, we denote by $C(X)$ the vector space of all real (or complex) continuous functions on X

Definition:

A normed linear space (over $\mathbb{R}$ or $\mathbb{C}$) is a vector space together with a function $\|\cdot\| : X \to [0, \infty)$ such that

$\|x+y\| \le \|x\| + \|y\| \quad (x, y \in X)$

$\|\alpha x\| = |\alpha| \, \|x\| \quad x \in X, \alpha \in$ scalars

$\|x\| = 0 \iff x = 0$

If X is complete for the metric $\rho(x,y) = \|x-y\|$ then X is a Banach space

Note: $BC(X)$ and $C(X)$ if X is compact are Banach spaces. Stefan Banach

Stone-Weierstrass Theorem

Arzelo-Ascoli Theorem

if $f, g \in C(X)$ pointwise product then $fg \in C(X)$ and

$$\|fg\| = \max_{x \in X} |f(x)g(x)| \le \left(\max_{x \in X} |f(x)|\right) \cdot \left(\max_{x \in X} g(x)\right) = \|f\| \, \|g\|$$

$\therefore$ algebra space

Banach Algebra

W Bade

M202a Functional Analysis

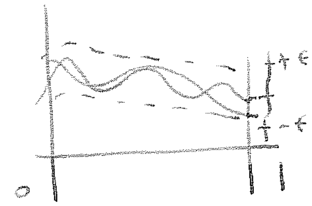
April 10th
Friday

$C(X) = C_{\mathbb{R}}(X)$

X is compact Hausdorff

Let S be a subalgebra of $C(X)$

containing the constant functions



When is S dense in $C(X)$? (for the sup norm!)

Theorem: In $C([0,1])$, the polynomials are dense for the supremum

P = polynomials on $[0,1]$

$\|f\|_\infty = \max_{t \in X} |f(t)|$

note $\{f_n\} \to f_0 \in C(X) \iff \lim_{n\to\infty} f_n(x) = f_0(x)$ uniformly on X

Functional Analysis approach -

functions are points in topological space

Definition:

A subset S of $C(X)$ separates points of X if for each $x_1 \neq x_2$ $\exists f \in S \ni f(x_1) \neq f(x_2)$

Theorem: (Stone-Weirstrass)

Let S be closed subalgebra of $C_{\mathbb{R}}(X)$ which contains the constants. Then $S = C_{\mathbb{R}}(X) \iff S$ separates points of X.

Proof:

(1) Certainly $C(X)$ separates points of X so the condition is necessary (?!) Urysohn's work

(2) We need the fact that $|t|$ can be approximated uniformly by polynomials on $[-1,1]$

Proof Let $\sum_{n=0}^{\infty} c_n t^n$ be the binomial expansion of $(1-t)^{1/2}$

$$(1-t)^{1/2} = 1 - \frac{t}{2} - \frac{t^2}{2^2 2!} - \frac{3t^3}{2^3 3!} - \frac{3.5t^4}{2^4 4!} \cdots$$

Converges absolutely $\therefore$ uniformly on $[-1,1]$

$\therefore$ given $\epsilon > 0$ $\exists N$ so that

$$|(1-t)^{1/2} - q_N(t)| < \epsilon \qquad 0 \le t \le 1$$

where $q_n(t) = \sum_{j=0}^{n} c_j t^j$. Let $P_n(t) = q_n(1-t^2)$

$$\therefore \Big| \underset{\sqrt{t^2}}{\overset{\shortparallel}{|t|}} - q_n(1-t^2) \Big| = \big| |t| - p_n(t) \big| < \epsilon \qquad (-1 \le t \le 1)$$

If $f, g \in C(X)$ define

$(f \vee g)(x) = \max\{f(x), g(x)\}$

$(f \wedge g)(x) = \min\{f(x), g(x)\}$

$|f|(x) = |f(x)|$

all cont functions!

Note: if $\{x_\alpha\} \to x_0$

$$|\max[f(x_\alpha), g(x_\alpha)] - \max[f(x_0), g(x_0)]|$$
$$\le \max[f(x_\alpha) - g(x_\alpha), f(x_0) - g(x_0)]$$

from $\big| \|x\| - \|y\| \big| \le \|x-y\|$ for $\mathbb{R}^2$ w/ norm $x = [s_1, s_2]$
$\|x\| = \max(s_1, s_2)$

$\therefore f \vee g$ is continuous

$\therefore f \wedge g = -[(-f) \vee (-g)]$

is continuous

$|f| = \cancel{f \vee 0}\ (f \vee 0) + (-f \vee 0) \Rightarrow |f|$ continuous

3. If $f \in S$, so ~~does~~ is $|f|$

Proof:
Choose polynomials $\{p_n\}$ in t so that

$\big| |t| - p_n(t) \big| \le 1/n \qquad (-n \le t \le n)$

$\therefore \big| |f|(x) - \underbrace{p_n(f)}_{\in S\ \checkmark}(x) \big| = \big| |f(x)| - p_n(f(x)) \big| \le 1/n$ if $-n \le f(x) \le n$

But f is bounded on X

$\therefore |f| \in S$ as S is closed

4. If $f, g \in S$ then $f \vee g$ and $f \wedge g \in S$

Proof: Note $f \vee g = \frac{f+g}{2} + \frac{|f-g|}{2}$

$f \wedge g = \frac{f+g}{2} - \frac{|f-g|}{2}$

5. Let $F \in C(X)$ (Show $F \in S$)

If $x_1, x_2 \in X$
then $\exists f \in S$ with $f(x_1) = F(x_1)$; $f(x_2) = F(x_2)$

(turn)

Proof: Can suppose $x_1 \neq x_2$ (Else use a multiple of the const)

Pick $g \in S$ with $g(x_1) \neq g(x_2)$

choose $s, t \in \mathbb{R}$ w/

$$s\, g(x_1) + t = F(x_1)$$

$$s\, g(x_2) + t = F(x_2)$$

can. do ~~coordidas~~ as $\begin{vmatrix} g(x_1) & 1 \\ g(x_2) & 1 \end{vmatrix} \neq 0$

W. Bade

X compact Hausdorff

$C_{\mathbb{R}}(X)$ = all real valued cont fns

S = closed subalgebra of $C_{\mathbb{R}}(X)$ containing 1 which separates points

To prove: $S = C_{\mathbb{R}}(X)$

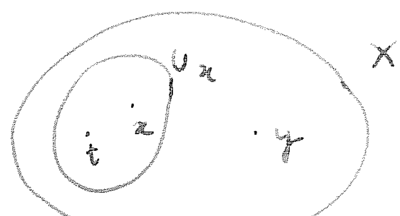
S is closed under $|\cdot|$ and $\wedge$

Given $F \in C_{\mathbb{R}}(X)$, $x_1, x_2 \in X$ $\exists f \in S$ with $f(x_1) = F(x_1)$, $f(x_2) = F(x_2)$

Let $F \in C_{\mathbb{R}}(X)$, $\epsilon > 0$

For $x, y \in X$, let $f_{xy} \in S$,

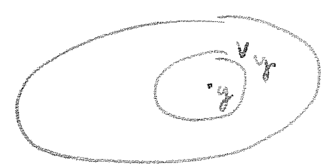
$f_{xy}(x) = F(x)$ $f_{xy}(y) = F(y)$



$\exists$ an open nbd U_x so that

$f_{xy}(t) > F(t) - \epsilon$ $(t \in U_x)$

Suppose $U_{x_1}, \dots, U_{x_n}$ cover X.



Let $f_y = f_{x_1 y} \vee f_{x_2 y} \vee \dots \vee f_{x_n y}$

$f_y \in S$. If $t \in X$, then $t \in$ some U_{x_i}

$\therefore$ (a) $f_y(t) \geq f_{x_i y}(t) > F(t) - \epsilon$ $(t \in X)$

Hence $\exists$ an open nbd V_y so that

(b) $f_y(t) < F(t) + \epsilon$ $(t \in V_y)$ [Since $f_{xy}(y) = F(y)$, $f_y(y) = F(y)$]

Let $V_{y_1}, \dots, V_{y_m}$ cover X and define

$$f = f_{y_1} \wedge f_{y_2} \wedge \dots \wedge f_{y_m} \quad (f \in S)$$

Note

(i) $f(t) > F(t) - \epsilon \quad (t \in X)$

since $f_{y_i}(t) > F(t) - \epsilon$ for all $t \in X$

But $t \in X \Rightarrow t \in$ some V_{y_i}

$$f(t) \leq f_{y_i}(t) < F(t) + \epsilon \quad (\forall t \in X)$$

$\therefore \ |f(t) - F(t)| < \epsilon \quad \forall t \in X$

Since ϵ is arbitrary $F \in \bar{S} = S$

$\therefore S = C_{\mathbb{R}}(X)$ $\boxtimes$ QED

Also true in $C_{\mathbb{C}}(X)$

Theorem: (complex S-W)

Let $C_{\mathbb{C}}(X)$ = all complex valued continuous functions on X (X compact Hausdorff) w/ the sup norm. Let S be a closed subalgebra of $C_{\mathbb{C}}(X)$ which

(i) contains the constants

(ii) separates points

(iii) is closed under conjugates i.e.

$$f \in S \Rightarrow \bar{f} \in S \quad \text{where } f = u + iv \text{ real}, \quad \bar{f} = u - iv$$

Then $S = C_{\mathbb{C}}(X)$

Proof:

let B = all real valued continuous functions in S

If $f \in S$ then $Re(f) = \frac{f+\bar{f}}{2} \in S$

$Im(f) = \frac{f-\bar{f}}{2i} \in S$

If $x, y \in X$, $x \neq y$

$\exists$ $f \in S$ with $f(x) \neq f(y)$

$\therefore$ either $Re(f)$ or $Im(f)$ separates x & y

$\therefore$ B separates points

Clear B is closed

$\therefore$ $B = C_{\mathbb{R}}(X)$

$\therefore$ $S = B + iB = C_{\mathbb{C}}(X)$

$\boxtimes$ QED

Theorem: (Arzela-Ascoli)

We identify the ~~co~~compact sets in $C(X)$

↳ sets whose closures are compact in $C(X)$

For a set A, in a metric space (X, ρ) TFAE

(1) $\bar{A}$ compact

(2) A is seq compact

(3) A is totally bdd & complete

Definition: Let X be a top space and let $\mathcal{K}$ be a family of continuous functions on X. Let $x \in X$ then $\mathcal{K}$ is equicontinuous at x if given $\epsilon > 0$ $\exists$ a nbd $N(x)$ so that

$|f(t) - f(x)| < \epsilon$

for all $t \in N(x)$ and all $f \in K$

We say K is equicontinuous if it is equicontinuous at each $x \in X$

Theorem (Arzela-Ascoli):

Let X be compact Hausdorff and $K \subseteq C(X)$

Then K is sequentially compact $\Leftrightarrow$ K is equicontinuous and bounded in norm

Recall: (Sequential compactness)

$\{f_n\} \subseteq K$

$\exists \{g_i\}$ ~~subsequence~~ which converges uniformly to some $g_0 \in \overline{K}$

Proof:

Let K be bdd in norm & equicontinuous If $\epsilon > 0$ and $x \in X$, then $\exists N(x)$ so that

$$\sup_{f \in K} \sup_{t \in N(x)} |f(t) - f(x)| < \epsilon$$

Let $\{f_n\}$ be a sequence K

Let $\epsilon = 1/8$ By compactness of X a finite family $N(x_1), N(x_2), \ldots, N(x_p)$ cover X

$$\therefore \sup_{n} \sup_{t \in N(x_i)} |f_n(t) - f_n(x_i)| < 1/8 \quad (1 \le i \le p)$$

Consider $\{f_n(x_i)\}_{n=1}^{\infty}$ its bdd in $\mathbb{R}$

By Bolza Weierstrass theorem, this has a convergent subsequence in $\mathbb{R}$ i.e.

$\exists \{g_n\}$ subsequence of $\{f_n\}$ so that

$\lim_{n\to\infty} g_n(x_i)$ exists $(1 \le i \le p)$

N. Bache

M202a Functional Analysis

April 15th
Wednesday

Theorem: (Kadic)

All separable Banach spaces are homeomorphic

$h : X_1 \to X_2$ (Non linear, very complicated map)

Theorem: (Arzelo Ascoli) cont'd

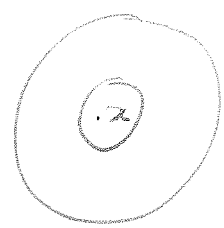
Let X be compact Hausdorff. A subset $K \subseteq C_{\mathbb{R}}(X)$ has compact closure for the norm topology $\Leftrightarrow$ it is bounded in norm and equicontinuous. ($\mathbb{R}$ or $\mathbb{C}$)

Proof:

Bdd + equicont $\to$ seq compactness

Let $\epsilon > 0$ If $x \in X$ $\exists N(x)$

so $\sup_{f \in K} \sup_{t \in N(x)} |f(t) - f(x)| < \epsilon$



Let $\{f_n\}$ be a sequence (show it has a norm convergent subsequence)

Let $\epsilon = 1/8$

$$X = \bigcup_{n=1}^{p} N(x_i)$$

$$\therefore \sup_n \sup_{t \in N(x_i)} |f_n(t) - f_n(x_i)| < 1/8 \quad 1 \le i \le p$$

Hence $\|f\| \le K$ say for $f \in K$

$\therefore |f_n(x_i)| \le K$

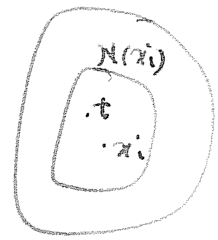
$\exists \{g_n\}$ of $\{f_n\}$

$\lim_{n\to\infty} \{g_n(x_i)\}$ exists $1 \le i \le p$

$\therefore \exists M_1 \in \mathbb{N}$ so

$|g_n(x_i) - g_m(x_i)| < 1/4$ for

$m, n \ge M_1, \; 1 \le i \le p$



Let $t \in X$ $\quad t \in$ some $N(x_i)$

$$|g_n(t) - g_m(t)| \le \underbrace{|g_n(t) - g_n(x_i)|}_{\le 1/8} + \underbrace{|g_n(x_i) - g_m(x_i)|}_{\le 1/4} + \underbrace{|g_m(x_i) - g_m(t)|}_{\le 1/8}$$

$\le \cancel{1/4}$

$\le 1/2.$

Now $\exists$ subsequence $\{h_n\}$ of $\{g_n\}$ and $M_2 > M_1$ so

$|h_m(t) - h_n(t)| < 1/2^2 \quad (m, n \ge M_2, t \in X)$

$\| h_n - h_m \| \le 1/2^2$

etc

$\exists$ subsequence $\{k_n\}$ and $M_3 > M_2$ so that $\|k_m - k_n\| \le 1/2^3$; $m, n \ge M_3$

$\vdots$

The sequence $\cancel{\{f_1, g_2, h_3\}}$ $\{g_1, h_2, k_3, \ldots\}$ is Cauchy in $C(X)$

~~(For $m, n \ge M_q$, all pairs~~

(∵ for some $\mathcal{N}_q$ if y, z are members of the sequence whose indices are $\geq q$, then $\|y - z\| \leq \frac{1}{2}q$)

Hence this subsequence of $\{f_n\}$ converges to some point of K

QED ~~(⇒)~~ (⇐)

Proof: (K compact in $C(X)$ ⇒ K is bdd & equicontinuous)

since $\bar{K}$ is compact, K is totally bounded i.e. given $\epsilon > 0$ $\exists f_1, \dots, f_n \in K$ and $K \subseteq \bigcup_{i=1}^{n} S(f_i, \epsilon)$

Hence K is bdd in norm (weak but true)

Let $x \in X$, pick $\mathcal{N}(x)$ so that $|f_i(t) - f_i(x)| < \epsilon/3$ ($t \in \mathcal{N}(x)$, $i = 1, \dots, n$)

suppose $f \in K$, $t \in \mathcal{N}(x)$

Pick the i ($1 \leq i \leq n$) so that $\|f - f_i\| < \epsilon/3$

Show K is equicontinuous

$$|f(t) - f(x)| \leq |f(t) - f_i(t)| + |f_i(t) - f_i(x)| + |f_i(x) - f(x)|$$
$$\leq \|f - f_i\| + \epsilon/3 + \|f - f_i\|$$
$$\leq \epsilon/3 + \epsilon/3 + \epsilon/3$$
$$\leq \epsilon \quad \text{for all } t \in \mathcal{N}(x), \text{ all } f \in K \Rightarrow \text{Equicont}$$

QED (⇒)

Definition:

A linear space X over $\Phi = \mathbb{R}$ or $\mathbb{C}$ (scalar field) with a Hausdorff topology T is a topological linear space (t.l.s.) if the maps

$[x,y] \to x+y$

$[\alpha, x] \to \alpha x$

are continuous from $X \times X \to X$ & $\Phi \times X \to X$ wrt the product topologies of $X \times X$ & $\Phi \times X$

Note: In a topological linear space if $x_0 \in X$, $\alpha_0 \neq 0$ and $\alpha_0 \in \Phi$, the maps

$x \mapsto x + x_0$, $x \to \alpha_0 x$ are homeomorphisms (with inverses $x \to x - x_0$ and $x \to \alpha_0^{-1} x$)

Hence T is completely determined by nbds of 0

Definition:

A set $K \subseteq X$ is convex if $k_1, k_2 \in K$, $0 \leq t \leq 1$ implies

$tk_1 + (1-t)k_2 \in K$

Definition:

A t.l.s. is locally convex if it has a base for nbds of 0 consisting of convex open sets.

W. Bade

M202a Functional Analysis

April 11 '9?
Friday

Top Linear Space X, T

$[x, y] \to x+y$ $\quad X \times X \to X$

cont fn $\quad F \times X \to X$

$[\alpha, x] \to \alpha x$

$x \to x + y$

topology det by what happens near 0

Theorem:

A Banach space is a locally convex t.l.s. (for the normed topology)

Proof:

$S_r(0) = \{x \mid \|x\| < r\}$ open ball about 0

Its convex

$\|x\| < r$; $\|y\| < r$

$\|tx + (1-t)y\| \leq t\|x\| + (1-t)\|y\| \leq tr + (1-t)r \leq r$

$\in S_r$

Let $\{x_n\} \to x$ $\quad \{y_n\} \to y$

$\|(x_n+y_n) - (x+y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0$

If $\alpha_n \to \alpha_0$ $\quad x_n \to x_0$

$\|\alpha_n x_n - \alpha_0 x_0\| \leq \|\alpha_n x_n - \alpha_n x_0\| + \|\alpha_n x_0 - \alpha_0 x_0\|$

$\leq \sup_n |\alpha_n| \, \|x_n - x_0\| + |\alpha_n - \alpha_0| \, \|x_0\|$

$\to 0$ ($\because$ sup is finite)

PS # 8: any bdd, non empty interior symmetric set in $\mathbb{R}^n$ or $\mathbb{C}^n$ is a unit ball (diff measuring stick in each dirn)

Definition:

Let H be a linear space over $\mathbb{R}$ or $\mathbb{C}$

H is an inner product space if $\exists$ a function of two variables $(\cdot,\cdot) : H \times H \to \mathbb{C}$ satisfying

(1) $(x,x) \overset{\in\mathbb{R}}{\geq} 0$; $(x,x) = 0 \Leftrightarrow x = 0$

(2) $(x+y,z) = (x,z) + (y,z)$

(3) $(\alpha x, y) = \alpha(x,y)$

(4) $(y,x) = \overline{(x,y)}$

Define: $\|x\| = (x,x)^{1/2}$

Note:

$(x, y+z) = (x,y) + (x,z)$

$(x, \alpha y) = \bar{\alpha}(x,y)$

Lemma: (Cauchy-Schwartz $\leq$)

If H is an inner product space then

$|(x,y)| \leq \|x\|\|y\| \quad x,y \in H$

Proof: True if $x = 0$ or $y = 0$ (since $(\alpha x, y) = \alpha(x,y)$)

suppose $y \neq 0$

let $\alpha \in \mathbb{C}$

$0 \leq (x+\alpha y, x+\alpha y)$; Expand using Defn, Note

$= (x, x+\alpha y) + (\alpha y, x+\alpha y)$

$= (x,x) + (x,\alpha y) + (\alpha y, x+\alpha y)$

$= (x,x) + (x,\alpha y) + (\alpha y, x) + (\alpha y, \alpha x)$

$= \|x\|^2 + 2\,\mathrm{Re}(\alpha y, x) + |\alpha|^2\|y\|^2$

$\therefore \|x\|^2 + |\alpha|^2\|y\|^2 \geq -2\,\mathrm{Re}(\alpha y, x) \quad \forall \alpha \in \mathbb{C}$

write $\alpha = re^{i\theta}$

$$\underbrace{\|x\|^2 + \lambda^2 \|y\|^2}_{\text{indep of } \theta} \geq -2\lambda \operatorname{Re}(e^{i\theta} y, x)$$

pick θ to make RHS as big as possible

$$\|x\|^2 + \lambda^2 \|y\|^2 \geq +2\lambda |(x,y)|$$

$\forall \lambda \in \mathbb{R}^+$

take $\lambda = \dfrac{\|x\|}{\|y\|}$

$$\|x\|^2 + \frac{\|x\|^2}{\cancel{\|y\|^2}} \cancel{\|y\|^2} \geq 2 \frac{\|x\|}{\|y\|} |(x,y)|$$

$$\Rightarrow \quad \|x\|^2 \geq \frac{\|x\|}{\|y\|} |(x,y)|$$

$$\Rightarrow \quad \|x\| \cdot \|y\| \geq |(x,y)|$$

☒ QED

Theorem:

An inner product space is a normed space for $\|x\| = (x,x)^{1/2}$

Proof:

$$\|x+y\|^2 = (x+y, x+y) = (x,x) + (y,x) + (x,y) + (y,y)$$

$$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\|$$

$$\leq (\|x\| + \|y\|)^2$$

$$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$$

$$\|\alpha x\| = |\alpha| \cdot \|x\|$$

Definition:

An inner product space which is complete for its norm is a Hilbert space

$\mathbb{R}^n$ or $\mathbb{C}^n$ for the Euclidean norm $\|[\alpha_1, \ldots, \alpha_n]\| = \sqrt{\sum |\alpha_i|^2}$

is a Hilbert Space (Finite Dimensional)

QM: linear operators on inf dim H spaces

Let $1 \le p < \infty$. Then ℓ^p is the space of all sequences $x = \{\xi_n\}$ $\in \mathbb{C}, \mathbb{R}$ such that

$$\|x\|_p = \left[\sum_1^\infty |\xi_n|^p\right]^{1/p} < \infty$$

Clear that $\|\alpha x\|_p = |\alpha| \, \|x\|_p$

Not clear (1) $\|x+y\|_p \le \|x\|_p + \|y\|_p$ (?!)

(2) complete

(ℓ^2 is a Hilbert Space) $(x, y) = \sum_{n=1}^\infty \xi_n \bar{\eta}_n$

N. Bade

M202a Functional analysis

April 20'9
Monday

M202a final
Thursday May 14 12:30-3:30

Examples:

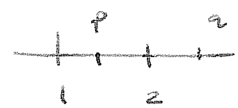
let $1 \le p < \infty$

we denote by ℓ^p the set of all sequences $x = \{\xi_n\}_{n=1}^{\infty}$ s.t.

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |\xi_n|^p\right)^{1/p} < \infty$$

$1 < p < \infty$, let q be defined

$1/p + 1/q = 1$ $\qquad 1 < q < \infty$



Lemma: (Holder $\le$)

let $1 < p < \infty$, $1/p + 1/q = 1$

let $x = \{\xi_n\} \in \ell^p$, $y = \{\eta_n\} \in \ell^q$ Then

$$\sum_{n=1}^{\infty} |\xi_n \eta_n| \le \left(\sum_{n=1}^{\infty} |\xi_n|^p\right)^{1/p} \cdot \left(\sum_{n=1}^{\infty} |\eta_n|^q\right)^{1/q} \left(= \|x\|_p \cdot \|y\|_q\right)$$

Proof:

let $\phi(t) = \dfrac{t^p}{p} + \dfrac{t^{-q}}{q}$ on $(0, \infty)$

$\phi'(t) > 0$ for $t > 1$

< 0 for $0 < t < 1$

$\therefore \min\limits_{t \in (0,\infty)} \phi(t) = \phi(1) = 1$

let $a, b > 0$ set $t = a^{1/q} b^{-1/p}$

$\therefore 1 \le a^{p/q}/bp + b^{q/p}/aq$

$\therefore \quad ab \le [a^{p/q+1}]/p + [b^{q/p+1}]/q \quad = \dfrac{a^{p(1/q+1/p)}}{p} + \dfrac{b^{q(1/q+1/p)}}{q}$

$$= \frac{a^p}{p} + \frac{b^q}{q}$$

we ~~took~~ $a, b > 0$

$\therefore \quad \forall a,b \quad |a \cdot b| \le |\frac{a^p}{p}| + |\frac{b^q}{q}|$

(Holds for complex numbers too!)

let $a = \dfrac{\xi_n}{\|x\|_p}$, $b = \dfrac{\eta_n}{\|y\|_q}$

$$\frac{|\xi_n \eta_n|}{\|x\|_p \cdot \|y\|_q} \le 1/p \frac{|\xi_n|^p}{\|x\|_p^p} + 1/q \frac{|\eta_n|^q}{\|y\|_q^q}$$

$$\therefore \quad |\xi_n \eta_n| \le 1/p\, |\xi_n|^p \cdot \|x\|_p^{1-p} \|y\|_q + 1/q\, |\eta_n|^q \|y\|_q^{1-q} \|x\|_p$$

$$\sum_{n=1}^{\infty} |\xi_n \eta_n| \le 1/p \|x\|_p^{1-p} \|y\|_q \|x\|_p^p + 1/q \|y\|_q^{1-q} \|y\|_q^q \|x\|_q$$

$$= \underbrace{(1/p + 1/q)}_{=1} \|x_p\| \|y\|_q = \|x\|_p \cdot \|y\|_q$$

QED

Lemma: (Minkowski)

Let $1 \le p < \infty$

If $x, y \in l^p$ then $x+y \in l^p$ and ~~$\|x+\|y$~~ $\|x+y\|_p \le \|x\|_p + \|y\|_p$

Proof:

True if $p=1$ as

$$\sum_{n=1}^{\infty} |\xi_n + \eta_n| \le \sum_{n=1}^{\infty} (|\xi_n| + |\eta_n|) = \sum_{n=1}^{\infty} |\xi_n| + \sum_{n=1}^{\infty} |\eta_n|$$

Let $1 < p < \infty$

$$\phi(t) = \frac{(1+t)^p}{1+t^p} \to 1 \text{ as } t \to \infty \text{ or } t \to 0$$

$\therefore$ its bounded on $(0, \infty)$ $\quad \phi(t) \le c \quad (t > 0)$

Let $a, b > 0$, $t = a/b$

$$\therefore (1 + a/b)^p \le c\left(1 + \frac{a^p}{b^p}\right)$$

$$\therefore (a+b)^p \le c(b^p + a^p) \qquad (0 \le a, b < \infty)$$

Let $x, y \in l^p$

$$|\xi_n + \eta_n|^p \le (|\xi| + |\eta|)^p \le c\left[|\xi_n|^p + |\eta_n|^p\right]$$

$$\therefore \sum_{n=1}^{\infty} |\xi_n + \eta_n|^p \le \cancel{\Sigma}\ c\left[\|x\|_p^p + \|y\|_p^p\right]$$

$\therefore x + y \in l^p$

$$\|x+y\|_p^p = \sum_{n=1}^{\infty} |\xi_n + \eta_n|^p \overset{?}{\leq} \underbrace{\sum_{n=1}^{\infty} |\xi_n|\cdot|\xi_n+\eta_n|^{p-1}} + \underbrace{\sum_{n=1}^{\infty} |\eta_n|\cdot|\xi_n+\eta_n|^{p-1}}$$

Apply Holder $\leq$

By Holders inequality

$$\leq \left(\sum_{n=1}^{\infty} |\xi_n|^p\right)^{1/p} \left[\sum_{k=1}^{\infty} |\xi_k+\eta_k|^{q(p-1)}\right]^{1/q}$$

$$+ \left(\sum_{n=1}^{\infty} |\eta_n|^p\right)^{1/p} * \left[\sum_{k=1}^{\infty} |\xi_k+\eta_k|^{q(p-1)}\right]^{1/q}$$

times

$$\therefore \|x+y\|_p^p \leq (\|x\|_p + \|y\|_p)\, \|x+y\|_p^{p/q}$$

$$\|x+y\|_p^{p-p/q} \leq \|x\|_p + \|y\|_p$$

but $p - p/q = 1$

$$\therefore \|x+y\|_p \leq \|x\|_p + \|y\|_p$$

Theorem: If $1 \leq p < \infty$, ℓ^p is a normed linear space

Proof: Note $\|\alpha x\|_p = |\alpha|\,\|x\|_p$ is clear

Theorem:

If $1 \leq p < \infty$, ℓ^p is a Banach space

Proof:

Let $\{x_n\} = \{\xi_i^n\}_{i=1}^{\infty}$ be a Cauchy sequence in ℓ^p

$\therefore \lim_{m,n\to\infty} \|x_m - x_n\|_p = 0$

$\lim_{n,m\to\infty} \left[\sum_{i=1}^{\infty} |\xi_i^m - \xi_i^n|^p\right]^{1/p} = 0$

$\therefore$ for any i_0, $|\xi_{i_0}^m - \xi_{i_0}^n| \leq \|x_m - x_n\|_p \to 0$ as $m,n\to\infty$

$\therefore \lim_{n\to\infty} \xi_i^n = \xi_i^0$ exists (completeness of $\mathbb{R}$ ($\mathbb{C}$))

Following is useful general approach to completeness arguments

Let $\epsilon > 0$. Then $\exists M$ so that $\sum_{i=1}^{\infty} |\xi_i^m - \xi_i^n|^p < \epsilon^p$ $\forall m,n \geq M$

$\sum_{i=1}^{K} |\xi_i^m - \xi_i^n|^p < \epsilon^p$ ($m,n \geq M$, all $K \in \mathbb{N}$)

Let $n \to \infty$

$\sum_{i=1}^{K} |\xi_i^m - \xi_i^0|^p \leq \epsilon^p$ ($m \geq M$, $K \in \mathbb{N}$)

most that could happen!

Let $K \to \infty$

(※) $\sum_{i=1}^{\infty} |\xi_i^m - \xi_i^0|^p \leq \epsilon^p$ ($m \geq M$)

$\therefore \{\xi_i^m - \xi_i^0\} \in \ell^p$

$\therefore x_0 \overset{\text{def}}{=} \{\xi_i^0\} = \underset{\in \ell^p}{\{\xi_i^m\}} - \underset{\in \ell^p}{\{\xi_i^m - \xi_i^0\}}$

$\therefore x_0 \in \ell^p$. Then (※) shows $\lim_{m\to\infty} \|x_m - x_0\|_p = 0$

Note: ℓ^2 is a Hilbert Space for $(x,y) = \sum_{n=1}^{\infty} \xi_n \bar{\eta}_n$

W. Bade

M202a Functional analysis

April 22 '19 Wednesday

ℓ^p is a Banach Space
ℓ^2 is a Hilbert Space

Let X, Y be Banach Spaces
A map $T: X \to Y$ is linear
if $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$ ($\alpha_1, \alpha_2 \in \Phi$, $x_1, x_2 \in X$)

Like Matrices define linear maps for finite dim spaces

We say T is continuous at x_0 if given $\epsilon > 0$ $\exists \delta > 0$ so that
$$\|T(x) - T(x_0)\| < \epsilon \quad \text{if} \quad \|x - x_0\| < \delta$$

Equivalently: $x_n \to x_0$, ($\|x_n - x_0\| \to 0$) then $\|Tx_n - Tx_0\| \to 0$

Note that if T is continuous at x_0, then T continuous at 0.

Let $x_n \to 0$ $\quad (x_0 - x_n \to x_0)$ so $T(x_0 - x_n) \to T(x_0)$
$$T(x_0 - x_n) = T(x_0) - T(x_n)$$

$\therefore T(x_n) \to 0$

Theorem:
Let $T: X \to Y$ be a linear map
TFAE
(1) T is continuous (ie cont @ every x)
(2) T is continuous @ 0
(3) $\exists C > 0$ so $\|Tx\| \le C\|x\|$ ($x \in X$) (T is bounded)

Proof: (1) $\to$ (2) ✓

(2) $\to$ (3)

Let $\epsilon > 0$, $\exists \delta > 0$ so

$\|Tx\| \leq \epsilon$ if $\|x\| \leq \delta$

$\|T(\delta \frac{x}{\|x\|})\| \leq \epsilon \quad (x \in X)$ use linearity

$$\|Tx\| \leq \frac{\|x\|}{\delta} \cdot \epsilon \quad (x \in X)$$

Take $C = \frac{\epsilon}{\delta}$ This works

(3) $\to$ (1)

Let $x_n \to x_0$. Then $\|Tx_n - Tx_0\| = \|T(x_n - x_0)\|$

$\leq C\|x_n - x_0\| \to 0$ as $n \to \infty$

$\therefore T$ is continuous at x_0

$\boxtimes$ QED

Let $\mathcal{B}(X,Y)$ be the set of all continuous linear maps from X to Y (X, Y are normed spaces)

$\mathcal{B}(X,Y)$ is a vector space

$(T_1 + T_2)(x) = T_1(x) + T_2(x) \quad (x \in X)$

$(\alpha T)(x) = \alpha(T(x)) \quad (\alpha \in \mathbb{F}, x \in X)$

Define $\|T\| = \inf\{c \geq 0 \mid \|Tx\| \leq c \cdot \|x\| \quad (x \in X)\}$

~~$\|T_1 + T_2\| \leq \sup \|Tx\|$~~ $\quad \|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$

≤ 1

$$\|T_1 + T_2\| = \sup_{\|x\|=1} \|T_1 x + T_2 x\|$$

$$\leq \sup_{\|x\|=1} \left(\|T_1 x\| + \|T_2 x\|\right)$$

$$\leq \sup_{\|x\|=1} \left(\|T_1 x\|\right) + \sup_{\|x\|=1}\left(\|T_2 x\|\right)$$

$$= \|T_1\| + \|T_2\|$$

$$\|\alpha T\| = \sup_{\|x\|=1} \|\alpha T x\| = |\alpha| \sup_{\|x\|=1} \|Tx\| = |\alpha| \cdot \|T\|$$

$B(X,Y)$ is a normed linear space

Theorem:
Let (X,Y) be normed linear spaces.
Let Y be complete. (ie Y is Banach). Then
$B(X,Y)$ is complete, hence a Banach space.

Proof:
Let $\{T_n\}$ be a Cauchy sequence in $B(X,Y)$
So if $\epsilon > 0$ $\exists N$ so $\|T_m - T_n\| < \epsilon$ $\forall m,n \geq N$
Let $x \in X$, $\boxed{\cancel{\|x\| \leq 1}}$
$\|T_m x - T_n x\| < \epsilon\|x\|$ for $m, n \geq N$
$\therefore \{T_n x\}$ is Cauchy in Y.
Define $T_0 x = \lim_{n \to \infty} T_n x$ $(x \in X)$ pt wise limit
Clear T_0 is a linear map from X to Y
Show T_0 is continuous & $\|T_n - T_0\| \to 0$

Let $\|x\| \le 1$. Then $\|T_m x - T_n x\| < \epsilon \quad (m, n \geq N)$

Let $n \to \infty$

$\therefore \|T_m x - T_0 x\| \le \epsilon \quad (m \geq N)$

$\therefore \|T_m - T_0\| \le \epsilon \quad (m \geq N)$ (like proof for ℓ^p)

$\therefore T_m - T_0 \in \mathcal{B}(X, Y)$ (bdd on unit ball)

so $T_0 = (T_0 - T_m) + T_m \in B(X, Y)$

and $\|T_m - T_0\| \to 0$ as $m \to \infty$

Notation: (X is complete, Banach space)

If $Y = X$ we write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$

$B(X)$ is an algebra (closed under products!)

$T : X \to X$

$U : X \to X$

$(UT)(x) = U(T(x)) \quad x \in X$

"Banach algebra"

$$\|UT\| = \sup_{\|x\|=1} \|UTx\|$$

$$\le \cancel{\|Tx\|} \sup_{\|x\|=1} \|U\| \cdot \|Tx\|$$

$$\le \|U\| \cdot \sup_{\|x\|=1} \|Tx\|$$

$$\le \|U\| \cdot \|T\|$$

$Y = \Phi$ (scalar field eg $\mathbb{R}$ or $\mathbb{C}$)

Write $X^* = \mathcal{B}(X, \Phi)$ "X^* is the dual space of X"

$x^* \in X^*$

$\hookrightarrow$ continuous linear functional

W. Bade

M202a Functional analysis

April 27th Friday

$T : X \to Y$

$\|T\| = \sup_{\|x\| = 1} \|Tx\|$

Examples:

(1) Let $X = l^p$ $\quad 1 \le p < \infty$

$T : X \to X$ by

$T[\xi_1, \xi_2, \ldots] = [\xi_2, \xi_3, \ldots]$ "left shift"

$\|Tx\| = \left[\sum_{n=1}^{\infty} |\xi_n|^p\right]^{1/p}$

$\|Tx\| \le \|x\|$ $\quad \|T\| = \sup_{\|x\| = 1} \|Tx\| = \sup_{x \ne 0} \frac{\|Tx\|}{\|x\|}$

$\therefore \|T\| \le 1$

$\|T\| = 1$ (can find x s.t. $\|x\| = 1$, $Tx = X$)

$x_0 = [0, 1, 0, \ldots]$

$\|x_0\| = 1 = \|Tx_0\|$

(2) $X = l^p$ let ~~ξ~~ $\{\lambda_n\}_{n=1}^{\infty}$ be a bounded sequence (not nec l_p)

$V : [\xi_1, \ldots] = [\lambda_1 \xi_1, \lambda_2 \xi_2, \ldots]$

$\|V\| = \sup_{n \in \mathbb{N}} |\lambda_n|$

(3) consider $X = l^p(\mathbb{Z})$; $x = \{\ldots, \xi_{-1}, \xi_0, \xi_1, \ldots\}$

let $K = \{\ldots, k_{-1}, k_0, k_1, k_2, \ldots\} \in l^1(\mathbb{Z})$

$\|x\| = \left(\sum_{-\infty}^{\infty} |\xi_n|^p\right)^{1/p}$, $\|K\| = \sum_{-\infty}^{\infty} |k_n|$

For $x \in l^p(\mathbb{Z})$ define $K : l^p(\mathbb{Z}) \to l^p(\mathbb{Z})$ by

$Kx = \{\eta_n\}_{n=-\infty}^{\infty}$ where $\eta_n = \sum_{n=-\infty}^{\infty} \xi_{n-m} k_m$

convolution of k and x. $(k * x)$

$|\eta_n| \le \sum_{m=-\infty}^{\infty} |\xi_{n-m}| \cdot |\kappa_m|^{1/p} |\kappa_m|^{1/q}$ $1<p$ taken ($p=1$ easy)

$1/p + 1/q = 1$

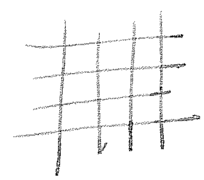
By Hölder, $\le \left[\sum_{m=-\infty}^{\infty} |\xi_{n-m}|^p |\kappa_m|\right]^{1/p} \cdot \left[\sum_{n=-\infty}^{\infty} |\kappa_n|\right]^{1/q}$

/* $\sum |\xi_n \eta_n| \le \left[\sum |\xi_n|^p\right]^{1/p} \left[\sum |\eta_n|^q\right]^{1/q}$ */

$\therefore\ |\eta_n|^p \le \|\kappa\|_1^{p/q} \sum_{m=-\infty}^{\infty} \underbrace{\sum_{n=-\infty}^{\infty} |\xi_{n-m}|^p}_{\|x\|_p^p} \cdot |\kappa_m|$ $\|\kappa\|$

$y_{mn} \ge 0 \quad \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y_{mn} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} y_{mn}$

$\sup_N \sum_{-N}^{N} \quad \sup_P \sum_{-P}^{P} y_{nm}$



$\therefore\ \|\kappa x\|_p^p \le \|\kappa\|_1^{p/q} \|x\|_p^p \|\kappa\|_1$

Note: $1 + p/q = p$

$\|\kappa x\|_p^p \le \|\kappa\|_1^{p/q+1} \|x\|_p^p$

$\|\kappa x\|_p \le \|\kappa\|_1 \|x\|_p$

$\therefore\ K : l^p(\mathbb{Z}) \to l^p(\mathbb{Z})$ is bdd linear map and

$\|K\| \le \|\kappa\|_1$

K is a normal operator on $l^2(\mathbb{Z})$

Theorem:

A finite dimensional normed linear space X is complete

Proof:

Let $[e_1, \dots, e_n]$ be a basis for X

$$x = \sum_{i=1}^{n} \alpha_i e_i \quad \text{(unique)}$$

$$\overset{\longleftarrow \; n \text{ times} \; \longrightarrow}{\text{Let } \Phi^n = \Phi \times \Phi \times \cdots \times \Phi}$$

Define $T : \Phi^n \to X$ by

$$T[\alpha_1, \dots, \alpha_n] \to \sum_{i=1}^{n} \alpha_i e_i$$

Φ^n has Euclidean norm $\left[\sum_{i=1}^{n} |\alpha_i|^2\right]^{1/2}$

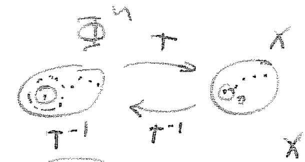
T is continuous, 1-1 and onto X

Assert $T^{-1} : X \to \Phi^n$ is continuous

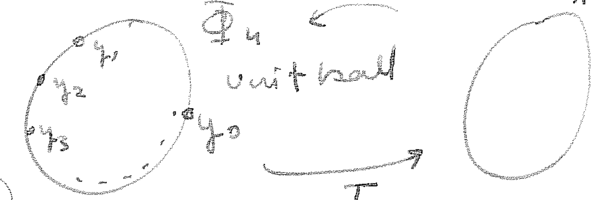
Suppose not!

Then not cont @ $0 \in X$

$\therefore \exists \{x_n\} \in X$ w/ $\|x_n\| \to 0$, and $\|T^{-1}x_n\| \geq \epsilon > 0$



Let $y_n = \dfrac{T^{-1}x_n}{\|T^{-1}x_n\|}$ (Normalizing)



$\|y_n\| = 1$ in Φ^n

But unit ball is compact (?!) wlog
hence by Bolzano Weierstrass, can assume

$y_n \to y_0 = [\alpha_1^0, \dots, \alpha_n^0]$ on ball

$$T y_0 = \sum_{i=1}^{n} \alpha_i^0 e_i = \lim_{m \to \infty} T y_m = \lim_{m \to \infty} \frac{T T^{-1} x_m}{\|T^{-1} x_m\|}$$

$$= \lim_{m \to \infty} \frac{x_m}{\|T^{-1} x_m\|} = 0 \text{ as } x_m \to 0 \;\; \& \;\; \|T^{-1}x_m\| \geq \epsilon > 0$$

bounded away from 0

$\therefore \; T y_0 = 0$ contradiction

$(\because T \text{ is } 1\text{-}1)$

Corollary:

A finite dimensional subspace of a normed linear space is closed.

↳ space itself not nece complete

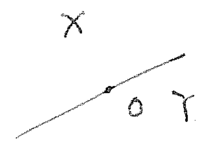
27 April '92
Monday

M202a Functional analysis

X: Normed Linear Space
Φ: scalar field
$B(X, \Phi) = X^*$ (X') (dual) conjugate space of X

Banach space

$B(X,Y) \leftarrow$ complete
$\llcorner$ complete



Question:
Suppose Y is a closed subset of X
~~[illegible]~~
If $x^* \in X^*$, then $y^* = x^*|_Y \in Y^*$

$$\|y^*\| = \sup_{\|y\|=1} |y^*(y)| \leq \sup_{\|x\|=1} |x^*(x)|$$

$$\uparrow$$
$$x^*(y)$$

Suppose $Y \subseteq X$ is a closed subspace and $\forall\, y^* \in Y^*$. Does $\exists\, x^* \in X^*$ such that

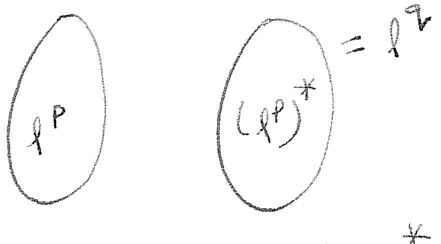
(1) $y^* = x^*|_Y$

(2) $\|y^*\| = \|x^*\|$

(Does $\exists$ a norm preserving extension of y^* to all of X?)

Examples: $x = \{\xi_n\}$ $y = \{\eta_n\}$
(1) ~~[illegible]~~ $X = \ell^p$ $1 < p < \infty$
$X^* = \ell^q$ where $1/p + 1/q = 1$

Pairing $\langle x, y \rangle = \sum_1^\infty \xi_n \eta_n$ $(y = x^*)$



For each $x^* \in (\ell^p)^*$ $\exists$ a unique sequence $\{\eta_n\} \in \ell^q$ so that $x^*(x) = \sum_1^\infty \xi_n \eta_n$

$\delta_n = [0, 0, \ldots, 0, 1, 0, \ldots]$

$\eta_n = x^*(\delta_n)$ $\quad \|x^*\| = \|y\|$ $\quad x^*(\lambda) = \sum_{n=1}^\infty \eta_n \lambda_n$

Theorem:

Let $1 < p < \infty$ For each $x^* \in (\ell^p)^*$ $\exists y \in \ell^q$ $\quad y = \{\eta_n\}$

$1/p + 1/q = 1$ such that

$$x^*(x) = \sum_{n=1}^\infty \xi_n \eta_n$$

Moreover the map $T: x^* = y$ is an injective linear map of $(\ell^p)^*$ onto ℓ^q.

Q 43 $\quad c_0$: $\quad x = \{\xi_n\}$ $\quad \lim_{n \to \infty} \xi_n = 0$

$$\|x\| = \sup_{n \in \mathbb{N}} |\xi_n|$$

Theorem: If $x^* \in C_0^*$, $\exists \underset{\{\eta^n\}}{y} \in \ell^1$ s.t. $x^*(x) = \sum_1^\infty \xi_n \eta_n$ ($x \in c_0$) and onto ℓ^1

$T: c_0^* \to \ell^1$ is a linear injective

$(\ell^1)^* = \ell^\infty$

$(c_0)^* = \ell^1$

$(\ell^\infty)^* \overset{??}{=}$ An absurd example: Banach sp of all additive ... total variation .. finite subsets of $\mathbb{Z}$.

$(c_0)^{**} = l^\infty$

c_0 is not dual of any Banach space

$\because$ c_0: Unit Ball has ~~not~~ extreme pts

dual of Banach space has lots of extrem pts of Unit Ball

How about $C(X)^*$ X compact Hausdorff

$= \mathcal{M}(\mathcal{B}, (X))$ measures of Borel sets of X

$$x^*(x) = \int_X x(t)\,\mu(dt)$$

Theorem: (Hahn Banach)

Let L be a linear space over $\mathbb{R}$ and p be a real valued function ~~of~~ on L satisfying

(1) $p(x+y) = p(x) + p(y)$ $\forall x, y \in L$

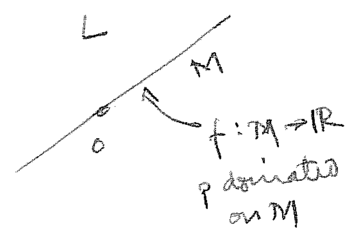
(2) $p(\alpha x) = \alpha p(x)$ $\forall x \in L$; $\alpha \geq 0$

(Kinda like a norm..)

Let f be a real valued linear function on a linear subspace M of L s.t. $f(x) \leq p(x)$ $(x \in M)$. Then $\exists$ a real valued linear function F on all of L s.t.

$F(x) = f(x)$ $(x \in M)$

$F(x) \leq p(x)$ all $x \in L$



~~F extension~~
F: ~~is~~ p dominated extension of f to all of L.

Proof: Let G = set of all real valued linear maps ~~[illegible]~~ satisfying

$\mathcal{D}(g) \supseteq M$ $g(x) = f(x)$ ~~$(x \in dom(G))$~~ $(x \in \mathcal{D}(g))$

$g(x) \leq p(x)$ $x \in \mathcal{D}(g)$

non empty $\because f \in G$ order G by $g_1 \leq g_2$ if g_2 extends g_1

W. Bach

M202a Functional Analysis

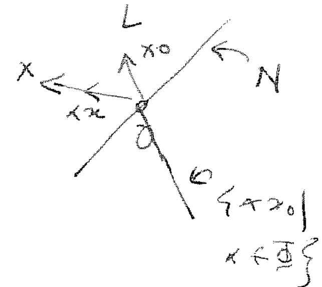
April 29 '9[illegible]
Wednesday

linear space

$f: L \to \Phi$ is linear functional

Let $N = \{x \in L \mid f(x) = 0\}$ subspace of L (closed if L is a T.L.S.) {and f is continuous}

N has algebraic codimension 1; i.e. $\dim(L/N) = 1$



Suppose $x_0 \notin N$ then $f(x_0) \neq 0$ let $x \notin N$

Show $L = \{N + \alpha x_0 \mid \alpha \in \Phi\} = N \oplus A$

Show $x = n + \beta x_0$ some $n, \beta \in \Phi$

Pick α so that $\alpha f(x) = f(x_0)$

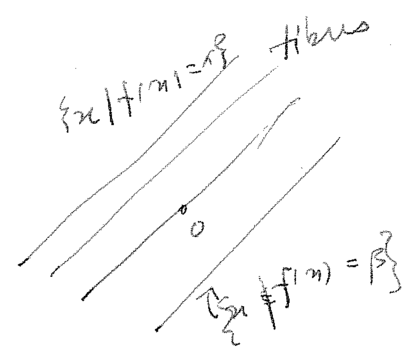
$\therefore \quad f(\alpha x - x_0) = 0$

$\therefore \quad \alpha x - x_0 = n \in N$

$$x = \frac{x_0}{\alpha} + \frac{n}{\alpha} = \frac{x_0}{\alpha} + \underbrace{n_1}_{\in N}$$

If f is a linear functional, $\alpha \in \Phi$

$\{x \mid f(x) = \alpha\}$ is a hyperplane



$\mathbb{R}^3 \to \mathbb{R}$

$$\begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \end{array}$$

A

Null space $(0\ 0\ x)$

Theorem: (Hahn Banach)

Let p be a real valued linear functional on a real linear space L

Suppose

$$p(x+y) \le p(x) + p(y) \qquad (x, y \in L)$$
$$p(\alpha x) = \alpha p(x)$$

Let f be a real valued function defined on a subspace M of L such that

$$f(x) \le p(x) \qquad (x \in M)$$

Then $\exists$ a real valued linear function F on L such that

$$F(x) \le p(x) \qquad (x \in L)$$
$$F(x) = f(x) \qquad (x \in M)$$

Proof:

Let $\mathcal{G}$ be the family of all linear real valued functions ~~xxx~~ such that

(a) $\mathrm{dom}(g)$ is a linear subspace $\supseteq M$

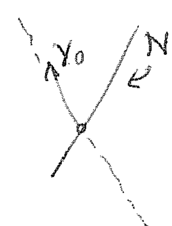
(b) $g(x) = f(x) \qquad (x \in M)$

(c) $g(x) \le p(x) \qquad (x \in \mathrm{dom}\, g)$

"All p dominated extensions of f"

$\mathcal{G} \ne \emptyset$ since $f \in \mathcal{G}$.

There are proper extensions. Let $g \in \mathcal{G}$ with domain N. Suppose $y_0 \notin N$. Let



$$N_0 = N \oplus \{\alpha y_0 \mid \alpha \in \mathbb{R}\}$$

Show $\exists g_0 \in \mathcal{G}$ w/ $\mathrm{dom}(g_0) = N_0$

Need to define $g_0(y+\alpha y_0) = g(y) + c\alpha$ where $c \in$

So g_0 will be linear (choose c so that $g_0 \le p$ on N_0)

If $x, y \in N$

$$\underline{g(y) - g(x)} = g(\overset{\in N}{y-x}) \le p(y-x) = p(y+y_0-y_0-x)$$

$$\le p(y+y_0) + p(-y_0-x)$$

$$\underbrace{-p(-y_0-x) - g(x)}_{\text{indep of } y} \le \underbrace{p(y+y_0) - g(y)}_{\text{indep of } x} \quad (x, y \in N)$$

Take sup on LHS, inf on RHS

$\therefore \exists c \in \mathbb{R}$ so

$$-p(-y_0-x) - g(x) \le c \le p(y+y_0) - g(y)$$

$$c \le p(y+y_0) - g(y) \quad (y \in N)$$

Let $\alpha > 0$.

$$c \le p(y/\alpha + y_0) - g(y/\alpha) \quad \forall y \in N$$

$$\alpha c \le p(y + y_0\alpha) - g(y)$$

$$\underbrace{g(y) + \alpha c}_{= g(y+\alpha y_0)} \le p(y+\alpha y_0) \quad (y \in N)$$

Let $\alpha < 0$

$$-p(-y_0 - y/\alpha) - g(y/\alpha) \le c \quad (y \in N)$$

$$-p(-y_0 + y/|\alpha|) + g(y/|\alpha|) \le c \quad (y \in N)$$

$$-p(-|\alpha| y_0 + y) + g(y) \le c|\alpha| \quad (\cancel{y \in N})$$

$g(y) + \alpha c \le p(\alpha y_0 + y) \qquad (y \in N)$

$\quad g_0(y + \alpha c)$

∴ Proved that its p-dominated

∴ if we define $g_0(y + \alpha y_0) = g(y) + \alpha c$

g_0 is p dominated on $N_0 = N \oplus \{\alpha y_0\}$

∴ g_0 is proper extension of g

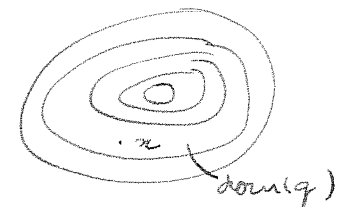
We make $\mathcal{G}$ into a p.o.s. by defining $g \le h$ iff h is a p dominated extension of g

If J is a totally ordered subset of $(\mathcal{G}, \le)$ define $k \in \mathcal{G}$ by $\mathrm{dom}(k) = \bigsqcup_{g \in J} \mathrm{dom}(g)$

∵ of linear ordering, dom(k) is a linear subspace

If $x \in \mathrm{dom}(k)$ then $x \in \mathrm{dom}(g)$ for some $g \in J$

define $k(x) = g(x)$ $\quad$ k is well defined & is a p domin extension of all $g \in J$.

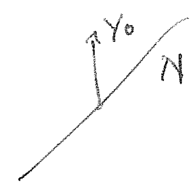


By Zorn, ∃ a maximal extension $F \in \mathcal{G}$.

But then dom(F) = L (∵ o.w. we can get an extension up one more dimension)

↳ N = dom(F)

(extend to $N \oplus \{\alpha y_0\}$)



Theorem: (Hahn Banach)

Let X be a normed linear space and Y be a closed linear subspace of X. Let $y^* \in Y^*$. Then $\exists\, x^* \in X^*$ such that $x^*(y) = y^*(y)$ $(y \in Y)$

$\|x^*\| = \|y^*\|$

Proof:

Suppose first that $\Phi = \mathbb{R}$

Define $p(x) = \|x\| \cdot \|y^*\|$ $(\forall x \in X)$

clear: $p(x_1 + x_2) \le p(x_1) + p(x_2)$

$p(\alpha x) = |\alpha|\, p(x)$

By last theorem, $\exists\, x^* \in X^*$ so that

$x^*(y) = y^*(y)$ $\quad y \in Y$

but also $x^*(x) \le p(x) = \|x\| \cdot \|y^*\|$ $\quad (x \in X)$

Replace $x \to -x$

$$\Rightarrow |x^*(x)| \le \|x\| \cdot \|y^*\| \qquad (x \in X)$$

$$\Rightarrow \|x^*\| \le \|y^*\|$$

but $\|y^*\| \le \|x^*\|$ free

$$\therefore \|x^*\| = \|y^*\|$$

QED

W Bade

M202a Functional Analysis May 1 '92
Friday

Theorem:
Let X be a normed linear space and Y is a closed subspace of X. Let $y^* \in Y^*$ Then $\exists x^* \in X^*$ such that

$$x^*(y) = y^*(y) \quad (y \in Y)$$

"strange fish"

$$\|x^*\| = \|y^*\|$$

Proved for $\Phi = \mathbb{R}$ already

Proof: ~~Proved~~ for $\Phi = \cancel{\mathbb{R}}\ \mathbb{C}$

X is a cx linear space

Its also a real linear space since $\mathbb{R} \subseteq \mathbb{C}$

Given $y^* \in Y^*$. I define 2 real linear functionals f_1, f_2 by

$$y^*(y) = f_1(y) + i f_2(y)$$

$$f_1(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 f(y_1) + \cancel{f}\ \alpha_2 f_1(y_2) \quad (\alpha_1, \alpha_2 \in \mathbb{R})$$

Similarly for f_2

Note: $y^*(iy) = f_1(iy) + i f_2(iy)$

$$\| $$

$$i y^*(y) = i f_1(y) - f_2(y) \qquad (\|f_1\| \leq \|y^*\| \ \because \text{its the real part})$$

$$\therefore\ f_2(y) = -f_1(iy) \quad (y \in \cancel{X}\ Y)$$

By Hahn Banach for real scalars

$\exists$ real linear functional F_1 on X (as real linear sp)

$$F_1(y) = f_1(y) \quad (y \in Y)$$

$$\|F_1\| \leq \|y^*\|$$

Define a complex linear functional x^* in X^* by

$$x^*(x) = F_1(x) - i F_1(ix) \quad (x \in X)$$

$x^*(x+y) = x^*(x) + x^*(y)$

$x^*(\alpha x) = \alpha x^*(x) \quad (\alpha \in \mathbb{R})$

Also $x^*(ix) = F_1(ix) - iF_1(-x)$

$= i x^*(x)$

$\therefore$ x^* is a complex linear functional

Must prove:

$\|x^*\| = \|y^*\|$

(suffices to show: $\|x^*\| \le \|y^*\|$)

Let $x^*(x) = re^{i\theta}$ $\quad r > 0$

$|x^*(x)| = e^{-i\theta} \cdot re^{i\theta}$

$= x^*(e^{-i\theta}x)$ (real)

$= F_1(e^{-i\theta}x) \le \|y^*\| \, \|e^{-i\theta}x\|$

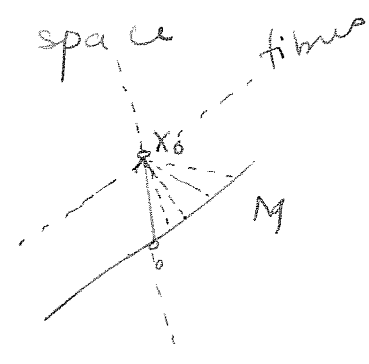
$= \|y^*\| \cdot \|x\|$

Corollary 1:

Let M be a closed subspace of a Banach space and ~~x_0~~ $x_0 \notin M$. Then $\exists x_0^* \in X^*$ such that

$\|x_0^*\| = 1$, $x_0^*(M) = 0$

$x_0^*(x_0) = \text{dist}(x_0, M) = \inf_{y \in M} \|x_0 - y\|$



This is an extremal problem.

Proof: Let $d = \inf_{y \in M} \|x_0 - y\|$. Then $d > 0$ as $x_0 \notin M$. Define y^* on the subspace $Y = M \oplus \{\lambda x_0 \mid \lambda \in \Phi\}$

defined by

$$y^*(y+\lambda x_0) = \lambda d \quad (y \in M)$$

Then y^* is linear $\left(\sup_{z} \frac{|y^* z|}{\|z\|} \quad z \in Y \right)$

$$\|y^*\| = \sup_{y \in M} \frac{|\lambda d|}{\|y + \lambda x_0\|}$$

$$= \sup_{y \in M} \frac{d}{\|y/\lambda + x_0\|}$$ but M is a subspace

$$= \sup_{y \in M} \frac{d}{\|y + x_0\|}$$

$$= \frac{d}{\inf_{y \in M} \|x_0 + y\|}$$

$$= \frac{d}{d}$$

$$= 1$$

$\therefore$ y^* has norm $=1$

Extend it by HB Theorem to x_0^* on X

Then $\|x_0^*\| = 1 \qquad x_0^*(x_0) = d = y^*(x_0)$

Corollary 2

If $x \in X$, then $\|x\| = \max_{\|x^*\|=1} |x^*(x)|$

$$\|x^*\| = \sup_{\|x\|=1} |x^*(x)|$$

Proof:

Let $M = \{0\}$ in last corollary (Perfectly good subspace)

Then $d = \|x\|$ and $\exists\, x^*$ such that

$\|x^*\| = 1$ and $x^*(x) = \|x\|$

clearly $\|x\| \le \sup_{\|x^*\|=1} |x^*(x)|$

$\overset{\text{fixed}}{x^*}(\underset{\text{variable}}{x})$ ← usually ; How about $\overset{\text{variable}}{x^*}(\underset{\text{fixed}}{x})$

For $x \in X$ define $\hat{x} \in X^{**}$ by

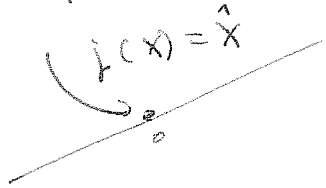
$$\hat{x}(x^*) = x^*(x) \quad \forall\, x^* \in X^*$$

Let $j(x) = \hat{x}$

$j : X \to X^{**}$ embedding of X in X^{**}

$\|j(x)\| = \|\hat{x}\| = \|x\|$

$X \qquad X^* \qquad X^{**}$



"Natural imbedding"

If $j(X) = X^{**}$ we say X is a <u>reflexive</u> Banach space.

ℓ^p $1 < p < \infty$ is reflexive

$(\ell^p)^* = \ell^q$

$(\ell^q)^* = \ell^p$

Final
W Bade

M202a Functional Analysis

May 4 '92

Lemma: Let M be a closed subspace of a Banach space. Let Z be a finite dimensional subspace and $Y = M \oplus Z$. Then Y is closed.

x_0

M

Proof: Can assume $\dim(Z) = 1$. Let $x_0 \notin M$. Let $d = \inf\{\|x_0 - m\| \mid m \in M\}$, $d > 0$. Let $Y = M \oplus \{\lambda x_0 \mid \lambda \in \mathbb{C}\}$. Let $\{y_n\} = \{m_n + \lambda_n x_0\}$ be Cauchy.

Clear can suppose λ_n are distinct.

$$\|y_k - y_p\| = \|m_k - m_p + (\lambda_k - \lambda_p) x_0\| \qquad \geq \inf_{m \in M} \|x_0 - m\| = d$$

$$= |\lambda_k - \lambda_p| \, \Big\| \underbrace{\frac{m_k - m_p}{\lambda_k - \lambda_p}}_{\in M} + x_0 \Big\| \geq |\lambda_k - \lambda_p| \cdot d$$

$\|m_k - m_p\| \geq \|y_k - y_p\| - |\lambda_k - \lambda_p| \|x_0\|$

$\therefore \{\lambda_k\}$ is Cauchy $\therefore \lambda_k \to \lambda_0$

$\{m_k\}$ is Cauchy $m_k \to m_0 \in M$

$\{y_k\} \to z$

$z = m_0 + \lambda_0 x_0 \in Y$

$\therefore Y$ is closed

Final: Thursday MAY 14

(1.) Statements of Theorems
(2.) Proofs of Theorems *Have a choice*/
(3.) Definitions
10 problems

Last Time:

Natural embedding of a space in its dual

Hilbert Spaces:

H: Hilbert Space over $\mathbb{C}$

innerproduct (x,y) on H

$$(\cdot,\cdot): H \times H \to \mathbb{C}$$

$$\|x\| = (x,x)^{1/2} \quad ; \quad x = 0 \Leftrightarrow \|x\| = 0$$

Definition:

If $(x,y) = 0$ we say x and y are orthogonal and write ~~(x,y)~~ $x \perp y$. We write $M \perp N$ if $m \perp n$ for all $m \in M$, $n \in N$. Also

$$A^{\perp} = \{x \mid (x,a) = 0 \text{ all } a \in A\}$$

Clear:

(i) If $A \subseteq H$, then $A^{\perp}$ is a closed subspace (closed $\because$ of cont of inner product)

(ii) $A \subseteq A^{\perp\perp}$

(iii) $A \subseteq B \rightarrow B^{\perp} \subseteq A^{\perp}$

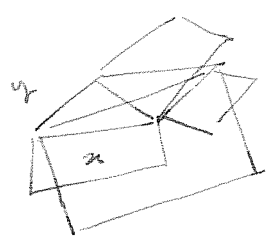
(iv) If A is a linear subspace, $A \cap A^{\perp} = (0)$

Pf: $a \in A \cap A^{\perp}$ $(a,a) = \|a\|^2 = 0 \Rightarrow a = 0$

Lemma: (▱ Law)

If $x, y \in H$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$



Proof: expand both sides (Ex $\|x+y\|^2 = (x+y, x+y)$ etc)

Lemma: (Useful in IE & places like that)

Let K be a closed convex set in a Hilbert space

Then K contains a unique vector k_0 $\bar{c}$ min^{m} nor

Remark:

A Geometric Thm, has to do w/ roundness of unit ball in H spaces.

Proof:

Let $d = \inf_{k \in K} \|k\| = \lim_{n\to\infty} \|k_n\|$

If $0 \in K$ we're done; so

suppose $d > 0$

$\in K$ by convexity of K!

$$\|k_n - k_m\|^2 = 2\|k_n\|^2 + 2\|k_m\|^2 - 4\left\|\frac{k_n + k_m}{2}\right\|^2$$

$$\leq 2\underbrace{\|k_m\|^2}_{\downarrow d^2} + 2\underbrace{\|k_n\|^2}_{\downarrow d^2} - 4d^2$$

$\longrightarrow 0$ as $m, n \to \infty$

Shows $\{k_n\}$ is Cauchy

let $k_0 = \lim_{n\to\infty} k_n$

suppose $k' \in K$ and $\|k'\| = d$

consider $\{k_0, k', k_0, k', \dots\}$ its norm minimizing, &

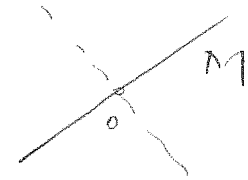
∴ by the last argument it converges.

∴ $k_0 = k'$

QED

Theorem:

Let M be a closed subspace of H. Then



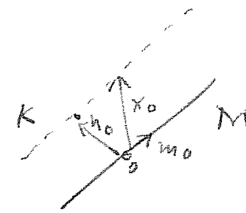
$$H = M \oplus M^\perp$$

i.e. given $x_0 \in H$ $\exists!\, m_0 \in M$, $n_0 \in M^\perp$

$\ni x_0 = m_0 + n_0$

and $\|x_0\|^2 = \|m_0\|^2 + \|n_0\|^2$

Proof:



Let $x_0 \in H$ and $K = \{x_0 - m \mid m \in M\}$

K is convex & closed

Let $n_0 \in K$ have smallest norm.

and $m_0 = x_0 - n_0$

Prove $n_0 \in M^\perp$

define m_0 by $x_0 - m_0 = n_0$ so $m_0 \in M$

Need to show $(n_0, m_0) = 0$

W. Bade

M202a - Functional Analysis

May 6 '9
Wednesda

Final Exam: Rm 9 Evans
1:30 - 3:30 May 14

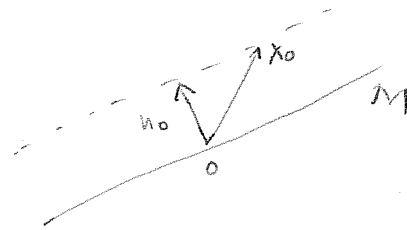
Theorem:

In a Hilbert space if M is a closed linear space then

$H = M \oplus M^{\perp}$

Proof:

Let $x_0 \in H$, $x_0 \notin M$



$K = \{x_0 - m \mid m \in M\}$

contains vector n_0 of smallest norm

Show $n_0 \perp M$. If so $x_0 = m_0 + n_0$ where $m_0 \in M$

Let $m \in M$. Show $(m, n_0) = 0$

Have $\|n_0\| \le \|n_0 - \alpha m\|$ (all $\alpha \in \mathbb{C}$)

$\|n_0 - \alpha m\|^2 - \|n_0\|^2 \geq 0 \quad (\alpha \in \mathbb{C})$

$(n_0 - \alpha m, n_0 - \alpha m) - (n_0, n_0)$; Expand & collect terms

Get $|\alpha|^2 \|m\|^2 - 2 \operatorname{Re} \alpha (m, n_0) \geq 0$

Choose $\alpha = t(n_0, m)$ where $t \geq 0$

Then $t^2 |(n_0, m)| \, \|m\|^2 - 2t |(n_0, m)|^2 \geq 0$ any $t \geq 0$

If $(n_0, m) \neq 0$ then left side < 0 for very small t

Contradiction!

$\therefore m \perp n_0$

$\therefore n_0 \in M^{\perp}$

$\therefore$ any vector $x_0 = \underset{M}{m_0} + \underset{M^{\perp}}{n_0}$

$\|x_0\|^2 = \|m_0\|^2 + \|n_0\|^2$ Follows from dfn "Pythagoras"

Theorem:

Let $\mathcal{H}$ be a Hilbert space and $x^* \in H^*$. Then $\exists!$ vector $y_{x^*} \in H$

$$x^*(x) = (x, y_{x^*}) \quad (x \in \mathcal{H})$$

Consequences of the Baire Category Theorem:

Theorem: (Principle of Uniform Boundedness)

Also known as "Principle of Equicontinuity"

Let X & Y be Banach spaces and for each $\alpha \in A$

let $T_\alpha \in \mathcal{B}(X, Y)$

Suppose $\sup_{\alpha \in A} \|T_\alpha x\| < \infty \quad \forall x \in X$

Then $\sup_{\alpha \in A} \|T_\alpha\| < \infty$ ← (The operators T_α are equicont

For each x $\exists M_x$ so $\|T_\alpha x\| \leq M_x$ for all $\alpha \in A$

$\{T_\alpha\}$; $\{T_\alpha x \mid \alpha \in A\}$

Suppose $\{T_\alpha x \mid \alpha \in A\}$ is bdd for each $x \in X$

Proof:

For each $k \in \mathbb{N}$ let $X_k = \{x \mid \sup_{\alpha \in A} \|T_\alpha x\| \leq k\}$

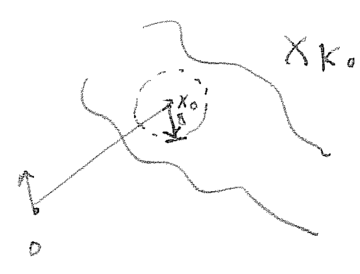
(1) $X = \bigcup_{k=1}^{\infty} X_k$

(2) X_k is closed

∴ By Baire's Theorem some one of these X_k are fat (contain a non empty open set)

i.e. $\exists$ $k_0 \in \mathbb{N}$ and $x_0 \in X_{k_0}$ and $\delta > 0$ so that

$x_0 + x \in X_{k_0}$ if $\|x\| \leq \delta$



$\therefore$ if $\|T_\alpha(x_0 + x)\| \le K_0$ $\quad (\|x\| \le \delta,\ \alpha \in A)$

$$\|T_\alpha x\| \le K_0 + \|T_\alpha x_0\| \quad (\alpha \in A,\ \|x\| \le \delta)$$

suppose $\sup_{\alpha \in A} \|T_\alpha x_0\| = K' > 0$

Then $\|T_\alpha x\| \le K_0 + K_1$ $\quad (\alpha \in A,\ \|x\| \le \delta)$

$\therefore \|T_\alpha x\| \le \dfrac{K + K'}{\delta}$ $\quad (\|x\| = 1)$

$\therefore \sup_\alpha \|T_\alpha\| \le \dfrac{K + K'}{\delta}$

QED

Corollary: ($\mathfrak{X}$ Banach Space)

let $K \subseteq X^*$ suppose

$\sup_{x^* \in K} |x^*(x)| = M_{\|x\|} < \infty$ for each $x \in X$

Then $\sup_{x^* \in K} \|x^*\| < \infty$ (ie K is a bdd set for the norm of X^*)

Proof:

let $Y = \Phi$ +

$x^* \in \mathcal{B}(X, \Phi)$

$K = \{x^*_\alpha \mid \alpha \in A\}$

Corollary: (X Banach space)

let $Q \subseteq X$ and suppose $\sup_{x \in Q} |x^*(x)| < \infty$ for each $x^* \in X^*$ $\rightarrow$ weakly bounded

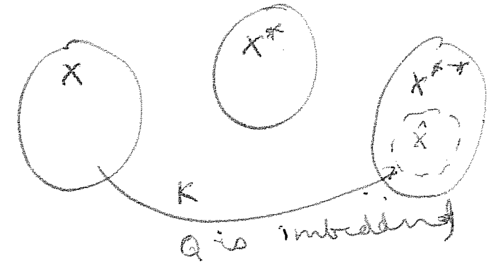
Then $\sup_{x \in Q} \|x\| < \infty$ $\rightarrow$ strongly bounded

Proof:

Let $K : X \to X^{**}$

$K(x) = \hat{x}$

$\hat{x}(x^*) = x^*(x) \quad (x^* \in X^*)$



K is an isometric linear imbedding

We have $\sup_{x \in Q} |x^*(x)| = \sup_{\hat{x} \in K(Q)} |\hat{x}(x^*)| < \infty$ each x^*

(Q is strongly bounded.

By last corollary

$K(Q)$ is bdd in X^{**} so Q is bdd in X

Theorem: (Open Mapping Theorem)

Let X, Y be Banach spaces

Let $T \in B(X, Y)$ s.t. $T(X) = Y$

Then T is an open map (U open in $X \to T(U)$ open in Y)

W. Bade

M202a Functional Analysis

May 8 '92
Friday

Theorem: (Uniform Boundedness)

Let X, Y be Banach spaces and $\{T_\alpha \mid \alpha \in A\} \subseteq B(X,Y)$

Suppose $\sup_{\alpha \in A} \|T_\alpha x\| = M_x < \infty$ for each $x \in X$

Then $\sup_{\alpha \in A} \|T_\alpha\| < \infty$

Theorem: (Banach-Steinhaus)

Let $\{T_n\}$ be a sequence of linear operators in $B(X,Y)$ where X and Y are Banach spaces. Suppose

$\lim_{n\to\infty} T_n x$ ~~exists for~~ $= Tx$ exists for each $x \in X$.

Then $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$ and T is continuous

Proof: By UB thm $\underbrace{\sup_n \|T_n\| < \infty}_{= M}$ (The sequence $T_n x$ is bdd for each x)

If $\epsilon > 0$ then $\|Tx\| \le \|Tx - T_n x\| + \|T_n x\| \le \epsilon + M\|x\|$

for large n. → Follows from fact that $\sup_n \|T_n\| < \infty$ ∴ then convergence is uniform!

∴ $\|T\| \le M$

QED

Theorem: (Banach Steinhaus)

Suppose $\{T_n\} \subseteq B(X,Y)$ and

(a) $\sup_{n \in \mathbb{N}} \|T_n x\| < \infty$ for each $x \in X$

(b) $\lim T_n x$ exists for each x in a dense subspace X.

Then $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$ $Tx = \lim T_n x$ exists for all $x \in X$

and $\|T\| < \infty$

Proof:

Note $M = \sup \|T_n\| < \infty$ by U.B.

Let $x \in X$ and $\epsilon > 0$

Pick $y \in X_0$ with $\|x-y\| < \epsilon$

$\{T_n\}$ is Cauchy because

$$\|T_n x - T_m x\| \leq \|T_n x - T_n y\| + \|T_n y - T_m y\| + \|T_m y - T_m x\|$$

$$\leq \|T_n\| \cdot \|x-y\| + \|T_n y - T_m y\| + \|T_m\| \cdot \|x-y\|$$

$$\leq \underbrace{2M \|x-y\|}_{\leq 2M\epsilon} + \underbrace{\|T_n y - T_m y\|}_{\to 0 \text{ as } m,n \to \infty}$$

$\therefore \lim_{m,n \to \infty} \|T_n x - T_m x\| = 0$

$\therefore \lim_{n \to \infty} T_n x = Tx$ exists for all $x \in X$

and $\|T\| < \infty$ by last theorem

Theorem: (open mapping)

Let X, Y be Banach spaces and $T \in B(X,Y)$

Suppose $T(X) = Y$. Then T maps open sets in X onto open sets in Y.

Proof:

Lemma:

Let $B = B_X(0,1)$ (unit ball in X) Then $\overline{TB}$ contains a nbd of 0 in Y, when T is onto (all hyp as above)

Proof: (Lemma)

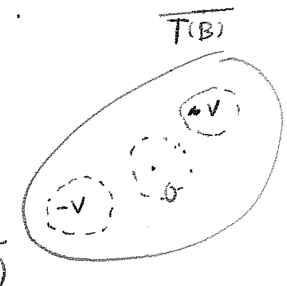
$$X = \bigcup_{n=1}^{\infty} nB$$

$$\therefore\ Y = \bigcup_{n=1}^{\infty} T(nB)$$
$$= \bigcup_{n=1}^{\infty} nT(B) = \bigcup_{n=1}^{\infty} \overline{nT(B)}$$

so for some n_0

$\overline{nT(B)} = n \cdot \overline{T(B)}$ contains an open set (Baire Thm)

Hence $\overline{T(B)}$ contains an open V in Y.



$-V \subseteq \overline{T(B)}$
↳ open

$\therefore\ \frac{1}{2}V + \frac{1}{2}(-V) = \frac{1}{2}(V - V)$ is an
contains in $\overline{T(B)}$
open nbd of 0 in Y
(By convexity of $\overline{T(B)}$) and $v - V$ is open
for each $v \in V \Rightarrow \bigcup_{v \in V}(v - V)$ is open)

QED[Lemma]

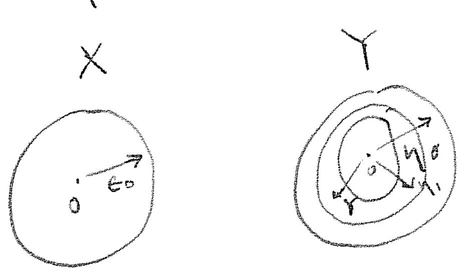
Proof: (Theorem)

Let $\epsilon_0 > 0$, and $\epsilon_i > 0$, $\epsilon_i \downarrow 0$, $\sum_{i=1}^{\infty} \epsilon_i < \epsilon_0$

By lemma, $\exists\ \eta_i > 0$, $\eta_i \downarrow 0$ and

$\overline{T(B_X(0, \epsilon_i))} \supseteq B_Y(0, \eta_i)$ $(i = 0, 1, 2, \ldots)$

We prove $T(B_X(0, 2\epsilon_0)) \supseteq B_Y(0, \eta_0)$



Let $y \in B_Y(0, \eta_0)$
Then $\exists$ $x_0 \in B_X(0, \epsilon_0)$ so that
$\|y - Tx_0\| < \eta_1$

$(\therefore\ y - Tx_0 \in B_Y(0, \eta_1))$

$\therefore \exists x_1 \in B_X(0, \epsilon_1)$ such that

$$\|y - Tx_0 - Tx_1\| = \|(y - Tx_0) - Tx_1\| < \eta_2$$

$$\left(\therefore\ y - T(x_0 + x_1) \in B_Y(0, \eta_2)\right)$$

Continue inductively. We define a sequence $\{x_n\}$

with $\left.\begin{array}{l} x_n \in B_X(0, \epsilon_n) \\ \|y - \sum_{i=1}^{n} Tx_i\| < \eta_{n+1} \end{array}\right\} n \in \mathbb{N}$

The series $\sum_{n=0}^{\infty} x_n$ converges since $K < 1$

$$\left\|\sum_{i=0}^{k} x_i - \sum_{i=0}^{l} x_i\right\| = \left\|\sum_{i=k+1}^{l} x_i\right\| \le \left\|\sum_{i=k+1}^{l} \epsilon_i\right\| \le \sum_{i=k+1}^{\infty} \epsilon_i \to 0 \text{ as } k, l \to \infty$$

Let $x = \sum_{n=0}^{\infty} x_n$

$$\|x\| \le \sum_{n=0}^{\infty} \|x_n\| \le \epsilon_0 + \sum_{i=1}^{\infty} \epsilon_i \ \cancel{\le} < 2\epsilon_0$$

$\therefore x \in B_X(0, 2\epsilon_0)$, and $\|y - \sum_{i=0}^{n} Tx_i\| \le \eta_{i+1} \to 0$

$$\therefore y = T(\Sigma x_i) = Tx$$

QED [Theorem]

Named Theorems:

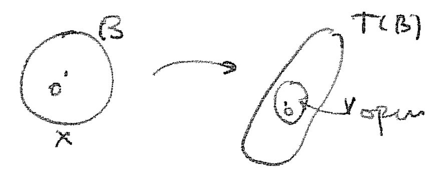
Lindelöff
Urysohn's Lemma
Tietze Ext Thm
Bolzano Wierstrass
Baire Category Thm
Zermelo fixed Pt Thm
Zorns Lemma
A.C.
Zermelo Well Ordering
Tychynoff Thm
Open Mapping Thm
Urysohn Metrization Theorem
Stone Cech Thm
Stone Weierstrass
Arzela Ascoli
Cauchy Schwartz Inequality
Holder ≤
Minkowski's ≤
Hahn Banach
□ Law
Uniform Bdd

Theorem: (Open Mapping Thm)

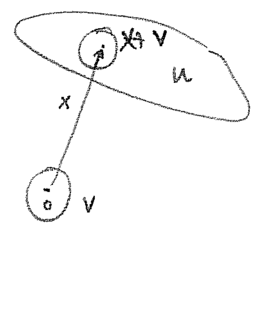
Let X, Y be Banach spaces and $T \in B(X,Y)$ Suppose $T(X) = Y$.
Then T is an open map (U open in $X \Rightarrow T(U)$ open in Y)

Proof:

Proved lemma: If $B = \{x \mid \|x\| \leq 1\}$ then $T(B)$ contains a nbd of Y. $T(B_X(0, 2\epsilon_0)) \supseteq B_Y(0, \eta_0)$



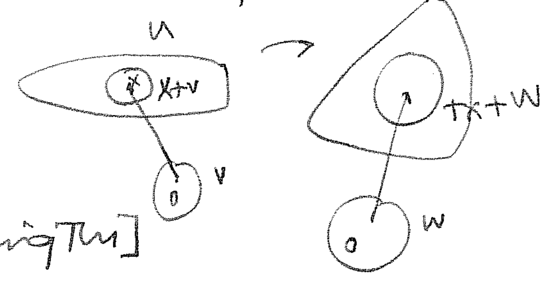
Let U be open in X,
let $x \in U$



Let V be a nbd of 0 in X such that $x + V \subseteq U$

Let W be an open nbd of 0 in Y s.t.
$T(V) \supseteq W$ (By lemma)
$T(U) \supseteq T(x+V) = Tx + T(V) \supseteq Tx + W$ which is open

$\therefore T(U)$ contains an open nbd of each of its points
$\therefore T(U)$ is open



QED [Open Mapping Thm]

Recall the middle of the argument made heavy use of approximation

Corollaries:

Let $T \in B(X,Y)$ and let T be a 1-1 map of X onto Y. Then T^{-1} is continuous

Proof:

T^{-1} will be continuous $\Leftrightarrow (T^{-1})^{-1}(U)$ is open in Y ($U \in T$.

But $(T^{-1})^{-1} = T$ and $T(U)$ is open in U.

Defn:

Let X, Y be Banach spaces and T a linear map whose domain $D(T)$ is a subspace of X (not nec closed) and $T(D(T)) \subseteq Y$. Then T is a closed operator if the graph $\Gamma(T) = \{(x, Tx) \mid x \in D(T)\}$ is closed in $X \times Y$ (Prod Top)

Equivalently T is closed $\Leftrightarrow$ whenever $\{x_n\} \subseteq D(T)$, $x_n \to x_0 \in X$ and $Tx_n \to y_0$ then $x_0 \in D(T)$ and $Tx_0 = y_0$

Theorem: (Closed Graph)

Let T be a closed operator from X to Y for which $D(T) = X$. Then T is continuous.

Proof:

$\Gamma(T)$ is a closed subspace of $X \times Y$

($X \times Y$ is a Banach Space for $[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$ and $\alpha[x_1, y_1] = [\alpha x_1, \alpha y_1]$, $\|X \times Y\| = \|x\| + \|y\|$

Then the map $P: \Gamma(T) \to X$ defined by $P[x, Tx] = x$ ($x \in X$) is continuous, one to one, and onto X.

$\therefore P^{-1}$ is continuous, Hence the composition

$$X \xrightarrow{P^{-1}} \underset{\in \Gamma(T)}{[x, Tx]} \xrightarrow{\text{cont}} Tx$$ is continuous

$\therefore T$ is continuous

Theorem:

Let Y and Z be closed subspaces of a Banach space X. Suppose X is the algebraic direct sum $X = Y \oplus Z$ of Y and Z. Then the map $Px = y$ is continuous. Moreover $Y \oplus Z$ is linearly homeomorphic to $Y \times Z$.

Proof:

P is a closed operator $P : X \to X$

Let $\{x_n\}$ be a sequence in X. $\{x_n\} \to x_0$ $\{Px_n = y_n\} \to y_0$

Let $x_n = y_n + z_n$ Then $z_n = x_n - y_n$ converges to z_0

$y_0 \in Y$, $z_0 \in Z$ as these are closed subspaces

and $x_0 = y_0 + z_0$ $\therefore Px_0 = y_0$

$\therefore P$ is closed $\therefore$ continuous as $\mathcal{D}(P) = X$

$\therefore (I - P)(x) = z$ continuous

$\therefore T : X \to Y \times Z$ is continuous

converse is immediate T^{-1} is continuous so $\mathfrak{X} \longrightarrow \mathfrak{Y} \oplus Z \xrightarrow{P} \mathfrak{Y}$ is continuous

(2) Let $x_n \in \mathfrak{X}$, $x_n \Rightarrow x_0$. Suppose $Px_n = y_n \Rightarrow y_0 \in \mathfrak{Y}$. Then $z_n = x_n - y_n \Rightarrow$ some $z_0 \in Z$ as Z is closed. $\therefore x_0 = y_0 + z_0$, so $Px_0 = y_0$. Hence P is closed and $\therefore$ continuous.

<u>Weak Topologies of Banach spaces</u>

Let $\mathfrak{X}$ be a Banach space. A linear subspace Γ of $\mathfrak{X}^*$ is called <u>total</u> if $x^*(x) = 0$ for all $x^* \in \Gamma$ implies $x = 0$. Clearly a subspace Γ of $\mathfrak{X}^*$ is ~~total~~ iff it separate points of $\mathfrak{X}$. (By H.B. $\mathfrak{X}^*$ is total). We show each Γ determines a locally convex topology on $\mathfrak{X}$.

Let $\mathcal{B}_\Gamma$ be the family of all subsets of $\mathfrak{X}$ of the form $N(x_0; x_1^*, \ldots, x_n^*, \epsilon) = \{x \mid |x_i^*(x) - x_i^*(x_0)| < \epsilon, i = 1, \ldots, n\}$, where $x_0 \in \mathfrak{X}$, $\epsilon > 0$ and $x_1^*, \ldots, x_n^* \in \Gamma$.

<u>Theorem</u>. The family $\mathcal{T}_\Gamma$ of all unions of sets in $\mathcal{B}_\Gamma$ is a topology making $(\mathfrak{X}, \mathcal{T}_\Gamma)$ a locally convex topological linear space.

<u>Proof</u>. If $z_0 \in \underbrace{N(x_0; x_1^*, \ldots, x_n^*, \epsilon)}_{B_1} \cap \underbrace{N(y_0; y_1^*, \ldots, y_m^*, \delta)}_{B_2}$ then $\exists\, \gamma > 0$ such that

$$|x_i^*(z_0) - x_i^*(x_0)| < \epsilon - \gamma \qquad i = 1, \ldots, n$$

$$|y_j^*(z_0) - y_j^*(y_0)| < \delta - \gamma \qquad j = 1, \ldots, m.$$

Form $N(z_0, x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_m^*, \gamma) = B_3$.

Then $z_0 \in B_3 \subseteq B_1 \cap B_2$. We use a result of 202A.

The topology is Hausdorff since if $x_0 \neq y_0$

Choose $x^* \in \Gamma$ with $|x^*(x_0 - y_0)| = \epsilon > 0$. Then

$$\{x \mid |x^*(x) - x^*(x_0)| < \tfrac{\epsilon}{2}\} \cap \{x \mid |x^*(x) - x^*(y_0)| < \tfrac{\epsilon}{2}\} = \phi.$$

Clearly the maps $[x, y] \to x + y$, $[\alpha, x] \to \alpha x$ are continuous since a net $\{x_\alpha\} \to x_0$ iff $\lim_\alpha x^*(x_\alpha) = x^*(x_0)$ for all $x^* \in \Gamma$.

We call $\mathcal{T}_\Gamma$ the $\sigma(\mathfrak{X}, \Gamma)$ topology.

<u>Examples</u> (a) Let $\mathfrak{X}$ be a Banach space, $\Gamma = \mathfrak{X}^*$. Then $\sigma(\mathfrak{X}, \mathfrak{X}^*)$ is called the <u>weak topology</u>.

(b) Let $\mathfrak{X}^*$ be the Banach space in question and $\Gamma = \kappa(\mathfrak{X})$, the natural imbedding. We call the resulting topology the $\sigma(\mathfrak{X}^*, \mathfrak{X})$-topology of $\mathfrak{X}^*$ or the <u>weak star</u> topology of $\mathfrak{X}^*$.

Basic neighborhoods for $\sigma(\mathfrak{X}^*, \mathfrak{X})$ have the form

$$N(x_0^*; x_1, \dots, x_n, \epsilon) = \{x^* \mid |x^*(x_i) - x_0^*(x_i)| < \epsilon \quad i = 1, \dots, n\}$$

Of course there is also the $\sigma(\mathfrak{X}^*, \mathfrak{X}^{**})$ topology, but it is generally stronger.

<u>Remarks</u> ① If $\mathfrak{X}$ is infinite dimensional $\sigma(\mathfrak{X}, \Gamma)$ is always weaker than the norm topology as every $\sigma(\mathfrak{X}, \Gamma)$ nbd. of 0 is an unbounded set.

② For each $x^* \in \Gamma$, let $\Phi_{x^*} = \Phi$ and form the topological product $\mathbb{P} = \prod_{x^* \in \Gamma} \Phi_{x^*}$. Elements of $\mathbb{P}$ are then functions $F : \Gamma \to \Phi$. The map $J : \mathfrak{X} \to \mathbb{P}$ which sends x to the function $F_x : \{x^*(x) \mid x^* \in \Gamma\}$ imbeds $\mathfrak{X}$ one-one into $\mathbb{P}$ and the relative product topology

$F_x = \{x^*(x) \mid x^* \in \Gamma\}$

of $J(\mathfrak{X})$ is exactly the $\sigma(\mathfrak{X}, \Gamma)$ topology. The next result is important

<u>Theorem</u>. (Alaoglu-Bourbaki) The closed unit ball $S^* = \{x^* \mid \|x^*\| \le 1\}$ is compact in the $\sigma(\mathfrak{X}^*, \mathfrak{X})$ topology of $\mathfrak{X}^*$.

<u>Proof</u>. $S^* = \{x^* \mid |x^*(x)| \le \|x\| \text{ all } x \in \mathfrak{X}\}$.

Form $\mathbb{P} = \prod_{x \in \mathfrak{X}} \Phi_x$, $\Phi_x = \Phi$. Let

$$K = \prod_{x \in \mathfrak{X}} D_x \quad , \quad D_x = \{\lambda \in \Phi \mid |\lambda| \le \|x\|\}.$$

Then K is a compact subset of $\mathbb{P}$ by Tychonoff and the map $J : \mathfrak{X}^* \to \mathbb{P}$ defined by

$$J(x^*) = \{x^*(x) \mid x \in \mathfrak{X}\}$$

imbeds $\mathfrak{X}^*$ homeomorphically into $\mathbb{P}$ and S^* into K. It suffices to prove S^* is closed
or $J(S^*)$

in K. Suppose $\{x_\alpha^*\}$ is a net in S^* and

$$F(x) = \lim_\alpha x_\alpha^*(x), \quad x \in \mathfrak{X}$$

(so $F \in \mathbb{P}$). Then F is linear and $|F(x)| \le \|x\|$, $x \in \mathfrak{X}$. Hence $F = J(x_0^*) \in J(S^*)$.

We need a useful general lemma.

<u>Lemma</u> Let $f_1, \ldots, f_n$ be linear functionals on a linear space $\mathfrak{X}$. A functional g is a linear combination of $f_1, \ldots, f_n$ iff it vanishes on the intersection of the null spaces of the f_i.

<u>Proof</u>: Let $N_i = \{x \mid f_i(x) = 0\}$. Suppose $g(\cap N_i) = 0$. Define $T : \mathfrak{X} \to \Phi^n$ by

$$Tx = \{f_1(x), \ldots, f_n(x)\} \qquad x \in \mathfrak{X}.$$

On the range $T(\mathfrak{X})$ of T in Φ^n define the functional Θ by $\Theta(Tx) = g(x)$. Note $Tx = Ty$

$\to T(x-y) = 0 \to f_i(x-y) = 0,\ i = 1, \dots, n \to g(x-y) = 0$

$\to g(x) = g(y)$ $\therefore$ Θ is well defined. Extend Θ to

Ψ on Φ^n. Then $\exists\ \alpha_1, \dots, \alpha_n$ such that

$$\Psi[\xi_1, \dots, \xi_n] = \sum_{i=1}^{n} \alpha_i \cdot \xi_i, \qquad [\xi_1, \dots, \xi_n] \in \Phi^n.$$

$$\therefore\ g(x) = \Psi(Tx) = \sum_{i=1}^{n} \alpha_i f_i(x) \qquad x \in \mathfrak{X}.$$

The converse is clear.

<u>Theorem</u>. Let Γ be a total subspace of $\mathfrak{X}^*$ and $\mathfrak{X}$ have its $\sigma(\mathfrak{X}, \Gamma)$ topology. Then a linear functional f is continuous for the $\sigma(\mathfrak{X}, \Gamma)$ topology iff $f \in \Gamma$.

<u>Proof</u>. If f is continuous it is bounded on some nbd N of 0, i.e. $\exists\ x_1^*, \dots, x_n^* \in \Gamma$,

such that $|f(x)| \le M$ if $|x_i^*(x)| < \epsilon$, $i = 1, \ldots, n$.

i.e. $x_i^*(x) = 0$, $i = 1, \ldots, n \rightarrow f(x) = 0$, since

for all k we have $|x_i^*(kx)| < \epsilon$, $i = 1, \ldots, n$,

so $k|f(x)| = |f(kx)| < M$, i.e. $|f(x)| < \frac{M}{k}$, all k.

Hence by the lemma $f = \sum_{i=1}^{n} \alpha_i \cdot x_i^* \in \Gamma$.

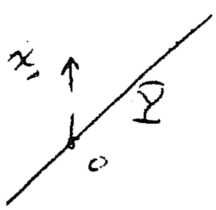
<u>Corollary</u> A linear functional f on $\mathfrak{X}^*$ is continuous for the $\sigma(\mathfrak{X}^*, \mathfrak{X})$ topology iff $f = \hat{x}$ for some $x \in \mathfrak{X}$.

<u>Corollary</u> A linear functional on $\mathfrak{X}$ is continuous for the weak topology (i.e. $\sigma(\mathfrak{X}, \mathfrak{X}^*)$), iff it is continuous for the norm topology.

<u>Corollary</u> For a linear subspace $\mathfrak{Y} \subseteq \mathfrak{X}$, $\mathfrak{Y}$ is norm closed iff it is $\sigma(\mathfrak{X}, \mathfrak{X}^*)$ closed.

Proof: If $\mathfrak{Y}$ is weakly closed its norm closed, since $x_n \Rightarrow x_0$ in norm implies $x_n \to x_0$ weakly.

Conversely, let $\mathfrak{Y}$ be norm closed and $x_0 \notin \mathfrak{Y}$. Then by a Corollary to the Hahn Banach theorem $\exists\, x_0^* \in \mathfrak{X}^*$ with $x^*(\mathfrak{Y}) = 0$, $x_0^*(x_0) = 1$.



Hence $\{x \mid |x_0^*(x) - x_0^*(x_0)| < \frac{1}{2}\}$ is a $\tau(\mathfrak{X}, \mathfrak{X}^*)$ neighborhood of x_0 disjoint from $\mathfrak{Y}$.

Definition: Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Given $y^* \in \mathfrak{Y}^*$ the function $x \to y^*(Tx)$ is continuous. Thus $\exists\; x^* \in \mathfrak{X}$ such that $x^*(x) = y^*(Tx)$, $x \in \mathfrak{X}$. The correspondence $y^* \to x^*$ is linear and $\|x^*\| \le \|y^*\| \|T\|$. We write

$x^* = T^* y^*$. Then $T^*: \mathfrak{Y}^* \to \mathfrak{X}^*$ is in $\mathcal{B}(\mathfrak{Y}^*, \mathfrak{X}^*)$.

We see $\|T^* y^*\| \le \|T\| \, \|y^*\|$, so $\|T^*\| \le \|T\|$. However

$$\|T^*\| = \sup_{\|y^*\|=1} \|T^* y^*\| = \sup_{\substack{\|y^*\|=1 \\ \|x\|=1}} |(T^* y^*)(x)|$$

$$= \sup_{\substack{\|x\|=1 \\ \|y^*\|=1}} |y^*(Tx)| = \sup_{\|x\|=1} \|Tx\| = \|T\|.$$

We will use this notion later. For now we get

<u>Corollary</u>. If $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$, then T is continuous when $\mathfrak{X}$ and $\mathfrak{Y}$ have their weak topologies

<u>Proof</u>: Let $\{x_\alpha\} \to x_0$ for $\sigma(\mathfrak{X}, \mathfrak{X}^*)$. Then $x^*(x_\alpha) \to x^*(x_0)$ all $x^* \in \mathfrak{X}^*$. Hence $(T^* y^*)(x_\alpha) = y^*(Tx_\alpha)$ $\to (T^* y^*)(x_0) = y^*(Tx_0)$ all $y^* \in \mathfrak{Y}^*$

M202a Topology and Functional Analysis FINAL REVIEW May 10

Jan 22 (Wednesday)
- Defn of Topology
- Defn of Base, Subbase
- Thm on B being a base
- Examples

Jan 24 (Friday)
- Defn Nbd, closed set
- Thm on closed sets
- Defn Metric

Jan 27 (Monday)
- Defn Closure, Properties
- Defn Acc pt
- Separation Axioms

Jan 28 (Wednesday)
- Defn Dense, Separable
- Defn open covering
- Thm on Sep Metric Spaces

Jan 31 (Friday)
- Thm Lindelöf (countable Base ⇒ countable subcover exists)
- Defn Rel Topology
- Defn Connectedness, Separated

Feb 3 (Monday)
- Thm connected sets in $\mathbb{R}$ are intervals and singletons
- Thm closure of conn set is conn
- Thm Union of con sets which are not separated is closed [Easy proof by contradiction]
- Cor: $\mathbb{R}^n$ is connected (rays)

Feb 5 (Wednesday)
- Defn component
- Thm closure & comp & conn
- Defn Cont Maps

Feb 7 (Friday)
- Thm TFAE $f: X \to Y$
  (i) f cont (ii) $f^{-1}(V)$ open in X whenever V open in Y
  (iii) $f^{-1}(C)$ closed in X when C is closed in Y
  (iv) $\forall A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$ (v) For all $B \subseteq Y$ $f^{-1}(\bar{B}) \supseteq \overline{f^{-1}(B)}$
- Thm continuous image of a connected set is continuous

Feb 10 (Monday)
- Defn Hausdorff, Regular, Normal
- Thm (X,T) regular + countable base ⇒ (X,T) is normal (use Lindelöf)
  Neat Proof: Do tossing away [see Munkres]

Feb 12 (Wednesday)
- Thm (Urysohn) (X,T) normal A, B closed disjoint ⇒ $\exists f: X \to \mathbb{R}$ cont st $f(A)=0$ $f(B)=1$ $0 \le f(x) \le 1$ [Munkres

Feb 14 (Friday)
- Thm Uniform limit of cont real valued functions is also continuous
- Thm (Tietze) (X,T) normal; $K \subseteq X$ closed $f: K \to \mathbb{R}$ be a bdd continuous function Then $\exists$ cont F on X so that $F(x) = f(x)$ on K, $\sup_{x \in X} |F(x)| = \sup_{x \in K} |f(x)|$ TRICKY!
- Cor Let (X,T) be normal, K closed subset of X and $f: K \to \mathbb{R}$ cont. Then $\exists F: X \to \mathbb{R}$ cont with $F(x) = f(x)$ use Unbdd f; compose $g(x) = \arctan(f(x))$ $(x \in K)$ (neat

Feb 19 (Wednesday)
- Defn compact, cond compact ($\bar{A}$ is compact), locally compact (each pt has nbd whose closure is compact)
- Defn f.i.p.
- Thm relating f.i.p. to compactness
- Thm (a) closed subset of compact space is compact
  (b) continuous image of a compact set is compact
  (c) in a Hausdorff space a compact set is closed
- Thm X compact Y Hausdorff Let $f: X \to Y$ be 1-1 Then f is a homeomorphism onto $f(X)$.

Feb 21 (Friday)
- Thm A compact Hausdorff space is normal
- Defns In a metric space $\rho(x, A) = \inf_{a \in A} \rho(x, a)$; sim $\rho($
- Lemma $f: x \to \rho(x, A)$ is continuous (?!)
- Thm A Metric Space is normal $U = \{x \mid \rho(x, A) < \rho(x,$ $B = \{x \mid \rho(x, B) < \rho(x,$
  uses continuity of dist fn
- Defn Seq, Convergence, cluster pt, Cauchy seq, complete

Feb 24 (Monday)
- Remark Cauchy seq can have at most one limit
- Proposition: Let (X, ρ) be a metric space
  (a) Cauchy Seq ⇐ convergent
  (b) Cauchy Seq converges ⇔ (1) Has a convergent subsequence ⇔ (2) Has a cluster point
  (c) $x_0 \in \bar{A} \Leftrightarrow \exists \{x_n\} \to x_0$ $\{x_n\} \subseteq A$
  (d) x_0 cluster pt of a seq ⇔ ∃ subsequence converging to it
- Thm $\mathbb{R}$ is complete for $\rho(x,y) = |x-y|$ (Bolzano Weierstrass
- Thm $\mathbb{R}^N$ is complete for $\rho(x,y) = \left[\sum_{n=1}^{N} |x_n - y_n|^2\right]^{1/2}$
- Defn Seq compactness $A \subseteq X$ is seq cmpct if every seq in X (in has a subsequence which converges to a pt in X (in
- Defn Totally Bounded $\forall \epsilon \exists \{a_1, \dots a_n\} \in A$ s.t $A \subseteq \bigcup_{i=1}^{n} S(a_i, \epsilon)$

Feb 26 (Wednesday)
- Lemman: In a metric space a seq cmpct set is (Hmm?) separable in its ... totally bdd and a totally bdd set is separable in its ... seq cmpct
- Lemma: If a subset A of a metric space is seq cmpct so is $\bar{A}$. direct construction
- Thm: TFAE (a) A seq cmpct
  (b) $\bar{A}$ is compact
  (c) A is totally bdd and $\bar{A}$ is complete

Feb 28 (Friday)
- TFAE result contd
- Bolzano Weierstrass Theorem A subset of $\mathbb{R}^n$ is seq compact $\Rightarrow$ it is bounded (in a try box)
- Baire category theorem: Let (X, ρ) be a complete metric space $X = \bigcup_{n=1}^{\infty} K_n$ K_n closed. Then $\exists$ i few $\ni$ K_i contains a non-empty open set

March 2 (Monday)
- Cor: $\nexists f: \mathbb{R} \to \mathbb{R}$ whose discont are exactly the set of irrationals
- Defn of Topological Products
- Axiom [Choice]
- Projection Function

March 4 (Wednesday)
- Thm Let $\{(Y_\alpha, T_\alpha) \mid \alpha \in I\}$ and (X, T) be topological spaces; $f: X \to \prod Y_\alpha$ is cont $\Leftrightarrow$ $p_\beta \circ f: X \to Y_\beta$ is cont for each $\beta \in I$
- Thm [Tychonoff] If X_α are compact then so is $\prod X_\alpha$
- Defn Relation, POS

March 6 (Friday)
- Thm [Zermelo F.P.] Let $(E, \leq)$ be a p.o.s. with the property that every totally ordered subset has a l.u.b. Let $f: E \to E$ be a map s.t. $x \leq f(x)$ for any $x \in E$. Then $\exists w \in E$ such that $f(w) = w$

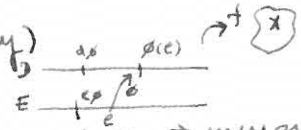
March 8 (Monday)
- ZFP Thm Contd
- Hausdorff Maximality Principle: Proof from AC + ZFP $(\mathcal{E}, \leq)$ totally ordered subsets of E ordered by inclusion. Has l.u.b. (Union) If no maximal elt, take $\dot{A} \in \mathcal{E}$ the set $B_{\dot{A}} = \{B \in \mathcal{E} \mid B \supseteq A, B \neq A\}$ by AC. $\exists f$ on $\mathcal{E}$ so $\dot{A} \in \mathcal{E} \to g(A) \in B_{\dot{A}}$ contradicts ZFP
- Zorns Lemma

March 10 (Wednesday)
- Thm Let X, Y be compact Hausdorff and $f: X \to Y$ be ~~compact~~ a continuous map of X onto Y. Then $\exists$ a closed set $K \subseteq X$ such that $f(K) = Y$ and $\nexists L \subseteq K$, L closed s.t. $f(L) = Y$
- Thm Every vector space has a basis
- Defn Well ordering of a set

March 13 (Friday)
- Thm (Zermelo) Every set can be well ordered
  Proof based on $(E_0, \leq_0) < (E_1, \leq_1)$ if (i) $E_0 \subseteq E_1$ (ii) $x, y \in E_0$ and $x \leq_0 y \Rightarrow x \leq_1 y$ (iii) $x \in E_0, y \in E_1 \setminus E_0 \Rightarrow x \leq_1 y$
  Uses Zorns Lemma
- Thm WO $\to$ AC
- Defn Directed Set, Net, Convergence, Cluster point

March 16 (Monday)
- Thm (X, T) Hausdorff $\Leftrightarrow$ Every conv net has a unique limit
- Thm F cont $\Leftrightarrow$ whenever $\{x_\alpha\} \to x_0 \Rightarrow \{F(x_\alpha)\} \to F(x_0)$
- Thm (X, T) cmpct $\Leftrightarrow$ every net in X has a cluster pt

March 18 (Wednesday)



- Defn of Subnet
- Thm $f: D \to X$ converges to $x_0 \Rightarrow$ every subnet also converges
- Lemma If $\{x_\alpha\}$ is a net in X; Q a family of non empty subsets of X satisfying (a) f not eventually in complement of any set in Q (b) the intersection of 2 members of Q contains a member of Q. Then $\exists$ a subnet of f which is eventually in every set in Q
- Thm If a net has a cluster point x_0, then $\exists$ a subnet which converges to x_0 (From lemma use Q = nbds of x_0
- Thm (Direct proof from above) (X, T) compact $\Leftrightarrow$ Every net has a convergent subnet
- Defn Universal Net $f: D \to X$ for each $A \subset X$ f is either in A or in A^c.
- Thm Every net in a set has a universal subnet
- Thm If f is universal in X and $F: X \to Y$ then $F \circ f$ is universal in Y (From defn)
- Thm A Universal net converges to any of its cluster point

March 20 (Friday)
- Thm Every net has a universal subnet
  (Proof: Let f be net. Suffices to show a family Q of subsets of X which satisfy (a) $A \subset X \Rightarrow A \in Q \vee \bar{A} \in Q$ (b) Q ~~[illegible]~~ is closed under finite intersections (c) f not event in complement of any set in Q)
- Thm A net in a topological product $f: D \to \prod_{\alpha \in I} X_\alpha$ converges to $x_0 \Leftrightarrow pr_\alpha f$ converges to $pr_\alpha x_0$ for all $\alpha \in I$
- Thm (Tychonoff) Let $X_0 = \prod\{X_\alpha \mid \alpha \in I\}$ be a product of top spaces with product top X_0 compact $\Leftrightarrow X_\alpha$ is compact ($\Rightarrow$) trivial

March 30 (Monday)
- Defn Metrizability
- Thm countable product of Metric Spaces is Metri
  Done using nets & homeomorphisms

April 1 (Wednesday)
- Defn Tychonoff space
- Defn cube
- Thm (X, T) Tychonoff $\Leftrightarrow$ Homeom to a subspace of the unit cube
  nets & functions based proof w/ image
- Thm [Urysohn Metrization] Regular space with a countable base is metrizable
  Proof uses $B \times B$ subset $(U, V) \in B \times B$ if $U \subseteq \bar{U} \subseteq V$ for each pair select a fn $f: X \to [0, 1]$ as Urysohn Then map $\Phi: X \to \prod\{I_f \mid f \in \mathcal{F}\}$ is homeomorphism $\mathcal{F}$ countable so Q metrizable; $\overline{\Phi(X)}$ is "Compactification"

April 3 (Friday)

- Thm [Stone Cech] X Tychonoff, and $F: X \to Y$ continuous map of X into a compact Hausdorff space Y then ∃ unique cont ext of f which carries $\beta(X)$ into Y

Defin $F^*: \mathcal{F}(Y) \to \mathcal{F}(X)$ by $F^*(g)(x) = g(F(x)) = (g \circ F)(x)$ $g \in \mathcal{F}(Y)$

$F^{**}: Q_{\mathcal{F}(X)} \to Q_{\mathcal{F}(Y)}$ $F^{**}(\xi)(g) = \xi(F^* g)$

show cont using nets, $F^{**}\Phi(X) = \Psi(F(X))$

∴ F^{**} is a cont ext of F; $F^{**}(\beta(X)) = \beta(Y) = Y$ as Y is compact!! Unique ∵ X dense in $\beta(X)$

April 6 (Monday)

- Thm In $\beta(\mathbb{N})$ the closure $\bar{U}$ of any open set is open
- Cor (SC) $F: X \to \mathbb{R}$ bdd cont fn ⇒ ∃ unique continuous ext to Y (?!)
- Defn "Extremely disc" A cmpct Hausdorff space X is "ED" if in X the closure of any open set is open

April 8 (Wednesday)

- Defn Banach space, BC(X) X Hausdorff
- Thm BC(X) complete normed. X cmpct then C(X) norms

April 10 (Friday)

- Thm (Stone Weierstrass) S closed subalgebra of $C_{\mathbb{R}}(X)$ which separates constants. Then $S = C_{\mathbb{R}}(X) \Leftrightarrow$ S separates constants X cmpct Hausdorff

April 13 (Monday)

- Proof of SW Thm contd Uses cmptness
- Thm Complex SW (closed under conj)

Look at B = all real valued cont fns in S

- Thm (Arzela Ascoli) Let K be a family of cont functions on X. Let $x \in X$ then K is equicont at x if given $\epsilon > 0$ ∃ $N(x)$ s.t. $|f(y) - f(x)| < \epsilon$ ∀ $y \in N(x)$
- Thm (Arzela Ascoli) Let X be compact Hausdorff and $K \subseteq C(X)$ then K is seq compact ⇔ K is equicontinuous and bounded in norm (Diagonal Arg)

April 15 (Wednesday)

- Proof (Arzela Ascoli)
- Defn (t.l.s.)
- Defn t.l.s. "locally convex" Base for nbds of 0 consisting of convex open sets

April 17 (Friday)

- Defn inner prod space
- Lemma Cauchy Schwarz $\|x\|\|y\| \geq |\langle x, y\rangle|$
- Thm inner prod space is normed space for $\|x\| = \sqrt{\langle x, x\rangle}$
- Defn Hilbert space

April 20 (Monday)

- Example $1 \leq p < \infty$ ℓ^p = all p sum seq
- Lemma [Holder ineq] $1 < p < \infty$ $1/p + 1/q = 1$

$x = \{\xi_n\} \in \ell^p$ $y = \{\eta_n\} \in \ell^q$

$\sum_{n=1}^{\infty} |\xi_n \eta_n| \leq \left(\sum_{n=1}^{\infty} |\xi_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |\eta_n|^q\right)^{1/q} = \|x\| \|y\|$