Two-Dimensional Frequency Domain

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<u>Stability:</u>

A LSI system is stable if and only if its impulse response $h(n_1, n_2)$ satisfies

$$\sum_{n_1}\sum_{n_2}\left|h(n_1,n_2)\right| = S < \infty$$

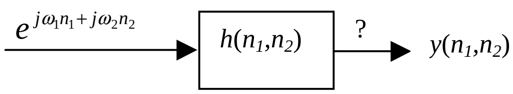
<u>Support:</u>

A sequence $x(n_1,n_2)$ has support on R if

$$x(n_1,n_2) = 0 \text{ for } (n_1,n_2) \notin R$$

A system has support on R if its impulse response has support on R

<u>Frequency response of 2-D systems</u>



$$y(n_1,n_2) = \sum_{k_1}\sum_{k_2} x(n_1-k_1, n_2-k_2)h(k_1,k_2)$$

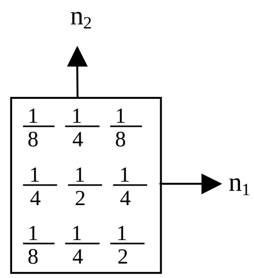
$$= \sum_{k_1}\sum_{k_2} e^{j\omega_1(n_1-k_1)} e^{j\omega_2(n_2-k_2)} h(k_1,k_2)$$

$$= e^{j\omega_1 n_1 + j\omega_2 n_2} \sum_{k_1}\sum_{k_2} h(k_1,k_2) e^{-j(k_1\omega_1 + k_2\omega_2)}$$

$$\mathbf{H}(\omega_1,\omega_2) = \sum_{\mathbf{k_1}}\sum_{\mathbf{k_2}} \mathbf{h}(\mathbf{k_1},\mathbf{k_2})\mathbf{e}^{-\mathbf{j}(\mathbf{k_1}\omega_1 + \mathbf{k_2}\omega_2)}$$

Periodic in both ω_1 and ω_2 with period 2π

<u>Example:</u>

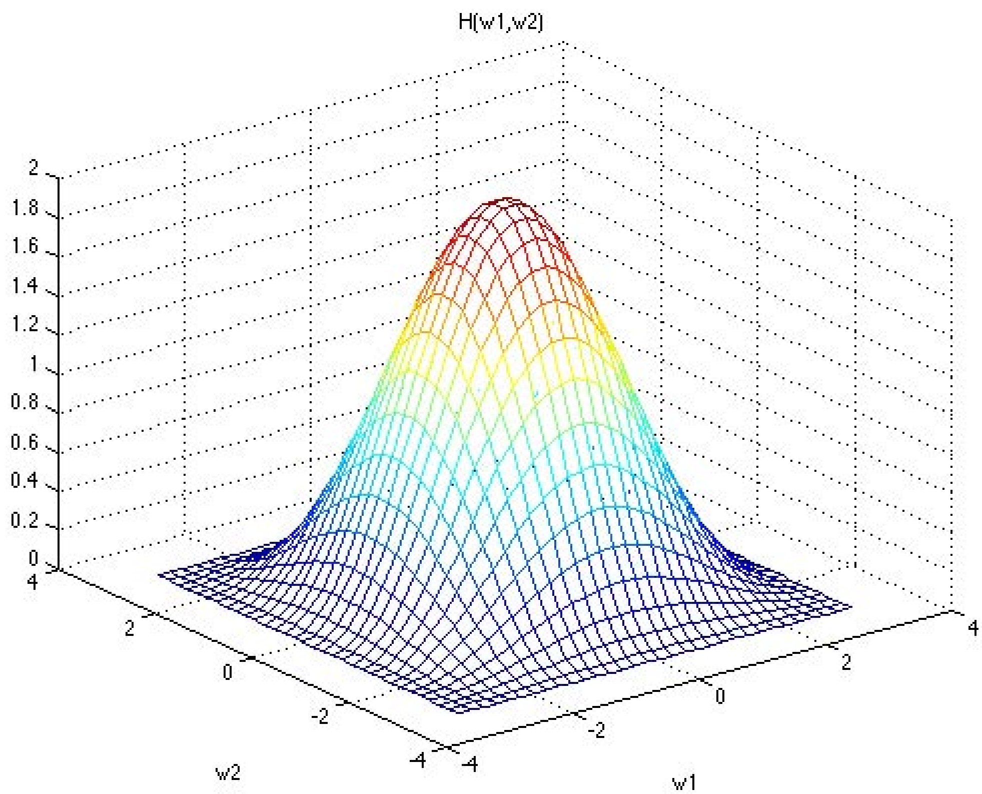


Symmetric (four-fold)
Separable – you can write the impulse response as $h(n_1,n_2) = f(n_1)\, g(n_2)$ where

$$f(-1) = f(1) = \tfrac{1}{4},\quad f(0) = \tfrac{1}{2}$$

$$g(-1) = g(1) = \tfrac{1}{2},\quad g(0) = 1$$

$$\begin{aligned}
H(\omega_1,\omega_2) &= \sum_{n_1}\sum_{n_2} e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} [.5\delta(n_1,n_2) + .25[\delta(n_1-1,n_2) + \delta(n_1+1,n_2) + \delta(n_1,n_2-1) + \delta(n_1,n_2+1)] \\
&\qquad + .125[\delta(n_1-1,n_2-1) + \delta(n_1+1,n_2-1) + \delta(n_1-1,n_2+1) + \delta(n_1+1,n_2+1)]] \\
&= .5 + .25(e^{-j\omega_1} + e^{j\omega_1} + e^{-j\omega_2} + e^{j\omega_2}) + .125(e^{-j\omega_1}e^{-j\omega_2} + e^{j\omega_1}e^{-j\omega_2} + e^{-j\omega_1}e^{j\omega_2} + e^{j\omega_1}e^{j\omega_2}) \\
&= .5(1 + \cos\omega_1 + \cos\omega_2 + \cos\omega_1 \cos\omega_2)
\end{aligned}$$



Inverting the 2-D Frequency Response

The frequency response of a system can be calculated from the system's impulse response.

-The opposite is also true

$$h(n_1,n_2)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}H(\omega_1,\omega_2)e^{jn_1\omega_1}e^{jn_2\omega_2}\,d\omega_1 d\omega_2$$

-Follows from 2-D Fourier series expansion

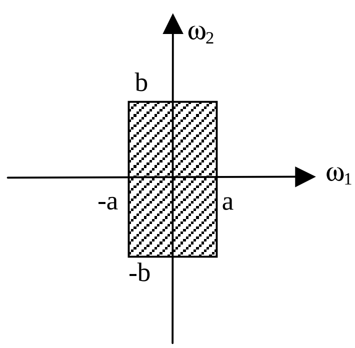
2-D Fourier Transform

$$\mathbf{X}(\omega_1,\omega_2)=\sum_{\mathbf{n_1}}\sum_{\mathbf{n_2}}\mathbf{x}(n_1,n_2)\mathbf{e}^{-j(n_1\omega_1+n_2\omega_2)}$$

$$\mathbf{x}(\mathbf{n_1},\mathbf{n_2})=\frac{\mathbf{1}}{\mathbf{4}\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathbf{X}(\omega_1,\omega_2)\mathbf{e}^{j(n_1\omega_1+n_2\omega_2)}\mathbf{d}\omega_1\mathbf{d}\omega_2$$

Example:

$$H(\omega_1,\omega_2)=\begin{cases}1 & |\omega_1|\le a<\pi,\omega_2\le b<\pi\} \\ 0 & otherwise\end{cases}$$

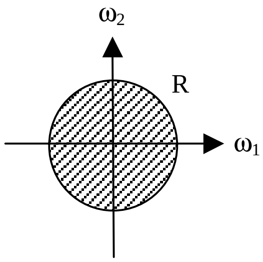


$$h(n_1,n_2)=\frac{1}{4\pi^2}\int_{-a}^{a}\int_{-b}^{b}e^{jn_1\omega_1}e^{jn_2\omega_2}\,d\omega_1 d\omega_2$$

$$=\frac{1}{2\pi}\int_{-a}^{a}e^{jn_1\omega_1}[\frac{1}{2\pi}\int_{-b}^{b}e^{jn_2\omega_2}\,d\omega_2]d\omega_1$$

$$\mathbf{h}(\mathbf{n_1},\mathbf{n_2})=\frac{\mathbf{\sin an_1}}{\pi\mathbf{n_1}}\bullet\frac{\mathbf{\sin bn_2}}{\pi\mathbf{n_2}}$$

Example:



$$H(\omega_1,\omega_2)=\{1, \quad \omega_1^2+\omega_2^2 \le R^2 < \pi^2\}$$
$$\{0, \quad \text{otherwise} \quad \}$$

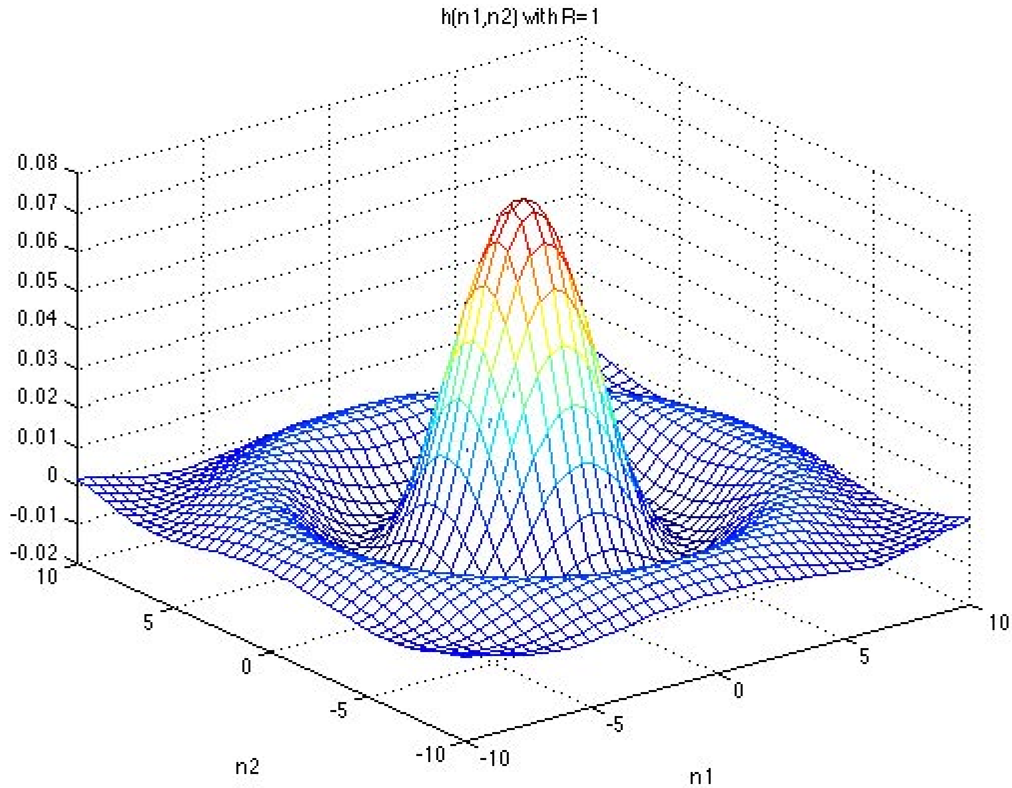
$$\mathrm{h}(\mathrm{n}_1,\mathrm{n}_2)=\frac{1}{4\pi^2}\iint_{circle} e^{jn_1\omega_1}e^{jn_2\omega_2}\,d\omega_1 d\omega_2$$

$$\text{Let } \omega=\sqrt{\omega_1^2+\omega_2^2}, \phi=\tan^{-1}(\tfrac{\omega 2}{\omega 1}), \theta=\tan^{-1}(\tfrac{n_2}{n_1})$$

$$h(n_1,n_2)=\tfrac{1}{4\pi^2}\int_0^{2\pi}\int_0^{R}\omega e^{j(\omega\sqrt{n_1^2+n_2^2}\cos(\theta-\phi))}d\omega d\phi$$

$$=\frac{1}{2\pi}\int_0^R \omega J_0(\omega\sqrt{n_1^2+n_2^2})d\omega$$

$$=\frac{R}{2\pi}\frac{J_1(R\sqrt{n_1^2+n_2^2})}{\sqrt{n_1^2+n_2^2}}$$



Properties of the Fourier Transform

- Linearity
- Shift:

$$x(n_1 - m_1, n_2 - m_2) \Leftrightarrow e^{-j\omega_1 m_1 - j\omega_2 m_2} X(\omega_1, \omega_2)$$

- Modulation: Dual of Above
- Convolution:

$$x(n_1, n_2) ** h(n_1, n_2) \Leftrightarrow X(\omega_1, \omega_2) H(\omega_1, \omega_2)$$

- Multiplication:

$$x(n_1, n_2) v(n_1, n_2) \Leftrightarrow \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1 - \theta_1, \omega_2 - \theta_2) V(\theta_1, \theta_2) d\theta_1 d\theta_2$$

- Differentiation:

$$-jn_1 x(n_1, n_2) \Leftrightarrow \frac{\partial}{\partial \omega_1} X(\omega_1, \omega_2)$$

$$-jn_2 x(n_1, n_2) \Leftrightarrow \frac{\partial}{\partial \omega_2} X(\omega_1, \omega_2)$$

- Transposition:

$$x(n_2, n_1) \Leftrightarrow X(\omega_2, \omega_1)$$

- Reflection:

$$x(-n_1,n_2) \Leftrightarrow X(-\omega_1,\omega_2)$$
$$x(n_1,-n_2) \Leftrightarrow X(\omega_1,-\omega_2)$$
$$x(-n_1,-n_2) \Leftrightarrow X(-\omega_1,-\omega_2)$$

- Conjugation:

$$x^*(n_1,n_2) \Leftrightarrow X^*(-n_1,-n_2)$$

- Parseval's:

$$\sum_{n_1}\sum_{n_2} x(n_1,n_2)w^*(n_1,n_2) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X(\omega_1,\omega_2)W^*(\omega_1,\omega_2)d\omega_1 d\omega_2$$

Substitute $w(n_1,n_2) = x(n_1,n_2)$ for a perhaps more familiar form of the relation.

- Separability:

$$f(n_1)g(n_2) \Leftrightarrow F(\omega_1)G(\omega_2)$$

$$\text{Proof : } \sum_{n_1}\sum_{n_2} f(n_1)g(n_2)e^{-jn_1\omega_1}e^{-jn_2\omega_2}$$
$$= \sum_{n_1} f(n_1)e^{-jn_1\omega_1}\sum_{n_2} g(n_2)e^{-jn_2\omega_2}$$
$$= \mathrm{F}(\omega_1)\mathrm{G}(\omega_2)$$

The result is a little surprising since the product of two functions in the discrete-time domain is a convolution in the frequency domain.

<u>Why not a convolution?</u>

$$\tilde{F}(\omega_1,\omega_2) = F(\omega_1)$$
$$\tilde{G}(\omega_1,\omega_2) = G(\omega_2)$$
$$X(\omega_1,\omega_2) = \tilde{F}(\omega_1,\omega_2)\tilde{G}(\omega_1,\omega_2)$$
$$\Rightarrow x(n_1,n_2) = \tilde{f}(n_1,n_2) ** \tilde{g}(n_1,n_2)$$
$$\text{But } \tilde{f}(n_1,n_2) = f(n_1)\delta(n_2)$$
$$\tilde{g}(n_1,n_2) = \delta(n_1)g(n_2)$$
$$\text{so } [f(n_1)\delta(n_2)] ** [\delta(n_1)g(n_2)] = f(n_1)g(n_2)$$