Multidimensional Sampling

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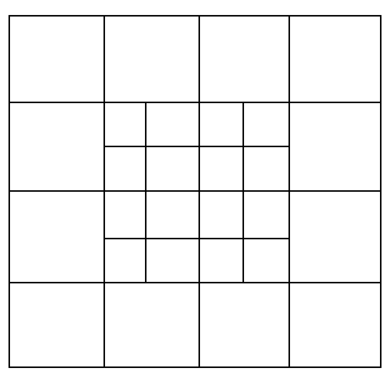
Motivation for the General Case for Sampling

We need the more general case to treat three important applications.

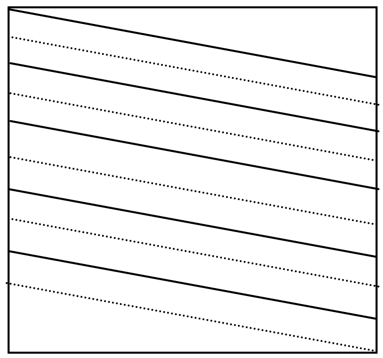
1. Human Vision System: the human vision system is a nonlinear, spatially-varying, non-uniformly sampled system. Rods and cones on the retina, which spatially sample are not arranged in rows and columns.

a. Hexagonal Sampling: when modeled as a linear shift-invariant system, the human visual system is circularly bandlimited (lowpass in radial frequency). The optimal uniform sampling grid is hexagonal. Optimal means that we need the fewest discrete-time samples to sample the continuous-space analog signal without aliasing.

b. Foveated grid: This is based on the fovea in the retina. When you focus on an object, you sample the object at a high resolution, and the resolution falls off away from the point-of-focus. Shown below is a simple example of a foveated grid. The grid is a 4 x 4 uniform sampling with each of the middle four grids subdivided into 4 x 4 grids themselves. The point of focus is at the middle of the grid. We can convert this grid to a uniform grid in several ways. For example, we could start with a rectangular grid and keep the resolution at the point-of-focus. Then, away from the point-of-focus, we can average the pixel values in increasingly larger blocks of samples. This approach allows the use a foveated grid while maintaining compatibility with systems that require rectangular sampling (e.g. image and video compression standards).



2. Television



650 samples /row
362.5 rows /interlace
2 interlaces /frame
30 frames /sec

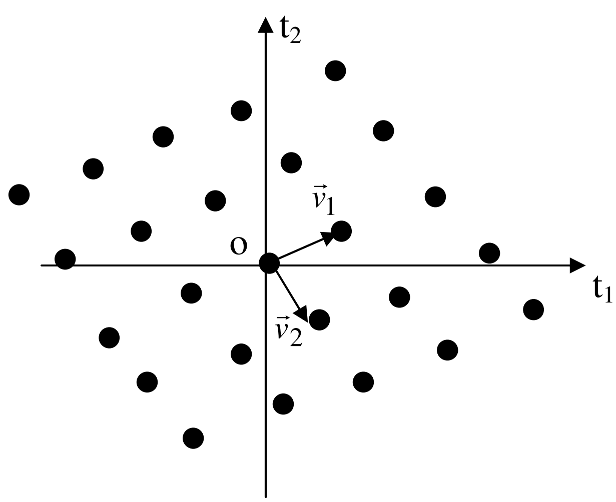
No two samples taken at the same instant of time

Can signals be sampled this way without losing information?

How can we handle

a. standard conversion
b. interlace removal
c. motion compensation

<u>Periodic Sampling Lattices</u>



An M-dimensional periodic sampling lattice (grid) can be formed by taking integer combinations of a set of M linearly independent vectors.

$$t_1 = v_{11}n_1 + v_{12}n_2$$
$$t_2 = v_{21}n_1 + v_{22}n_2 \quad \Leftrightarrow \quad \vec{t} = \mathbf{V}\vec{n} \quad (\ in\ 1-D\ t = Tn\ )$$

where **V** is the sampling matrix (2 x 2 in this case)

$$x(\vec{n}) = x_a(\mathbf{V}\vec{n})$$

In the rectangular case,

$$\mathbf{V} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$$

Continuous Fourier Transform

$$X_a(\vec{\Omega}) = \int_{-\infty}^{+\infty} x_a(\vec{t}) e^{-j\vec{\Omega}^T \vec{t}} d\vec{t}$$

$$x_a(\vec{t}) = \frac{1}{(2\pi)^M} \int_{-\infty}^{+\infty} X_a(\vec{\Omega}) e^{j\vec{\Omega}^T \vec{t}} d\vec{\Omega}$$

Relevant Properties

$$x_a(\vec{t}) * h_a(\vec{t}) \leftrightarrow X_a(\vec{\Omega}) H_a(\vec{\Omega})$$

$$x_a(\vec{t}) p_a(\vec{t}) \leftrightarrow \frac{1}{(2\pi)^M} X_a(\vec{\Omega}) * P_a(\vec{\Omega})$$

Derivation

$p_a(\vec{t})$ is a field of impulses (bed of nails) as before:

$$p_a(\vec{t}) = \sum_{\vec{n}} \delta(\vec{t} - \mathbf{V}\vec{n})$$

$$P_a(\vec{\Omega}) = \frac{(2\pi)^M}{|\det \mathbf{V}|} \sum_{\vec{k}} \delta(\vec{\Omega} - \mathbf{U}\vec{k})$$

where **U** is defined by $\mathbf{V}^T\mathbf{U} = 2\pi\mathbf{I}$ such that **I** is the identity matrix and **U** is the periodicity matrix.

Example: For the rectangular case,

$$\mathbf{U} = \begin{bmatrix} \frac{2\pi}{T_1} & 0 \\ 0 & \frac{2\pi}{T_2} \end{bmatrix}$$

Sample the analog continuous-space signal

$$\tilde{x}_a(\vec{t}) = x_a(t)\, p_a(t)$$

Under integration, the sampled representation simplifies to

$$\tilde{x}_a(\vec{t}) = \sum_{\vec{n}} x(\vec{n})\delta(\vec{t} - \mathbf{V}\vec{n})$$

Taking its Fourier transform

$$\tilde{X}_a(\vec{\Omega}) = F\left\{x_a(\vec{t})p_a(\vec{t})\right\} = \frac{1}{|\det \mathbf{V}|}\sum_{\vec{k}} X_a(\vec{\Omega} - \mathbf{U}\vec{k})$$

The aliased analog spectrum, where $x(\vec{n}) = x_a(\mathbf{V}\vec{n})$, is

$$X(\vec{\omega}) = \sum x(\vec{n})e^{-j\omega^T \vec{n}} = \frac{1}{|\det \mathbf{V}|}\sum_{\vec{k}} X_a\left(\mathbf{V}^{-T}(\vec{\omega} - 2\pi\vec{k})\right)$$

The Discrete-Time Fourier Transform of the sampled signal is

$$X(\vec{\Omega}) = \sum_{\vec{n}} x(\vec{n})e^{-j\vec{\Omega}^T \mathbf{V}\vec{n}} = \frac{1}{|\det \mathbf{V}|}\sum_{\vec{k}} X_a(\vec{\Omega} - \mathbf{U}\vec{k})$$

so $\vec{\omega} = \mathbf{V}^T\vec{\Omega}$ (in 1-D case: $\boldsymbol{\omega} = T\boldsymbol{\Omega}$). The term $|\det \mathbf{V}|$ is the spatial domain area associated with each sample:

$$\frac{1}{|\det \mathbf{V}|} = sampling\ density = \frac{|\det \mathbf{U}|}{(2\pi)^M}\ samples / area$$

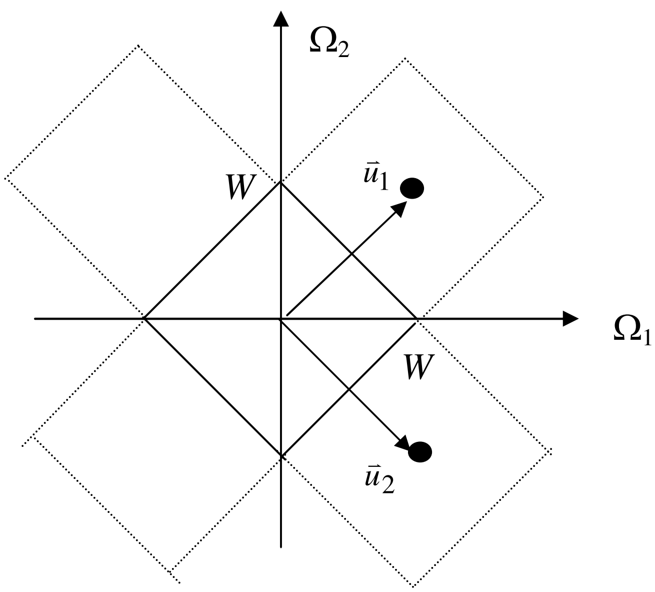
The factor of $(2\pi)^M$ is due to the $\det \alpha\mathbf{A} = \alpha^M \det \mathbf{A}$ for scalar α. If we have control over the sampling lattice, we find **V** by choosing **U** such that

1. there is no aliasing (depends on the bandwidth of $x_a(t)$)
2. $|\det \mathbf{U}|$ is as small as possible

Then $\mathbf{V} = 2\pi\mathbf{U}^{-T}$. **U** is called the aliasing or polar matrix.

A lattice is the set of points generated by a sampling matrix according to $\{\mathbf{V}\vec{n} : \vec{n} \in \vec{\mathrm{I}}\}$. **V** has real-valued elements. In the context of the discrete-space domain, **V** has integer-valued elements. **V** must be non-singular.

Example: In the 2-D frequency domain, consider



How tightly can we "tile the plane" without aliasing?

$$\vec{u}_1 = \begin{bmatrix} W \\ W \end{bmatrix}; \quad \vec{u}_2 = \begin{bmatrix} W \\ -W \end{bmatrix}; \quad \mathbf{U} = \begin{bmatrix} W & W \\ W & -W \end{bmatrix}; \quad \mathbf{V} = \begin{bmatrix} \pi/W & \pi/W \\ \pi/W & -\pi/W \end{bmatrix}$$

$$\frac{|\det \mathbf{U}|}{4\pi^2} = \frac{W^2}{2\pi^2} = \frac{1}{|\det V|} \; (in\ samples/area)$$

<u>Special Case: 1-D</u>

$$x(n) = x_a(Tn)$$

$$\mathbf{V} = T \; (scalar)$$

$$\mathbf{U} = \tfrac{2\pi}{T} \; (scalar)$$

$$X(\omega) = \tfrac{1}{|\det \mathbf{V}|}\sum_{\vec{k}} X_a(\mathbf{V}^{-T}(\vec{\omega} - 2\pi\vec{k})) = \tfrac{1}{T}\sum_{k} X_a(\tfrac{\omega}{T} - \tfrac{2\pi k}{T})$$

$$\text{sampling density} = \frac{1}{T} \text{ (samples/sec)}$$

$$\text{preventing aliasing}: 2\mathrm{W} < \frac{2\pi}{\mathrm{T}} \quad (W < \frac{\pi}{T})$$