Periodically Sampled Systems

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1 Processing Signals on Arbitrary Lattices

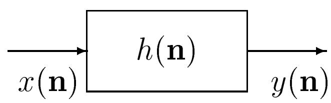

Figure 1: Block diagram of linear shift-invariant filtering input signal x to produce output signal y.

- Almost any signal processing operation that can be performed on rectangularly sampled signals can be performed on arbitrarily sampled ones.
  - Filtering
  - Spectrum Analysis
  - Interpolation
- The response of the LSI system $h(\mathbf{n})$ shown in Fig. 1 is

$$y(\mathbf{n}) = \sum_{\mathbf{k}} x(\mathbf{k})\, h(\mathbf{n}-\mathbf{k}) = \sum_{\mathbf{k}} h(\mathbf{k})\, x(\mathbf{n}-\mathbf{k})$$

- To compute the system frequency response, start with an analog continuous-space complex sinusoid with fixed frequency $\mathbf{\Omega}$, sample it with sampling matrix $\mathbf{V}$, and take the discrete-time Fourier transform:

$$x_a(t) = e^{j\mathbf{\Omega}^T\mathbf{t}}$$

$$x(\mathbf{n}) = e^{j\mathbf{\Omega}^T\mathbf{V}\mathbf{n}}$$

$$y(\mathbf{n}) = \sum_{\mathbf{k}} h(\mathbf{k})\, x(\mathbf{n}-\mathbf{k})$$

$$y(\mathbf{n}) = e^{j\mathbf{\Omega}^T\mathbf{V}\mathbf{n}} \quad \underbrace{\left[\sum_{\mathbf{k}} h(\mathbf{k}) e^{-j\mathbf{\Omega}^T\mathbf{V}\mathbf{k}}\right]}_{\text{Frequency Response } H(\mathbf{V}^T\mathbf{\Omega})}$$

  The relationship between the discrete-time and continuous-time Fourier domains is $\omega^T = \mathbf{\Omega}^T\mathbf{V}$. Hence, $\omega = \mathbf{V}^T\mathbf{\Omega}$.

2 Scanning

- Sometimes it is convenient to map multidimensional signals to 1-D and vice versa.
- Should broadcast TV signals be processed as a 3-D signal or a 1-D signal? See Figure 2

3 Lexicographic Ordering

This topic was covered in Section 7.4 of the first edition of the Dudgeon & Mersereau but it is not available in the Chapter 2 handout. See Fig. 3.

- Consider an $N \times N$ image, $x(n_1, n_2)$
- Now create an N^2-point 1-D sequence by concatenating the columns of x. See Fig. 4

$$g(Nn_1 + n_2) = x(n_1, n_2)$$

$$n_1 = \text{Column Index} \quad n_2 = \text{Row Index}$$

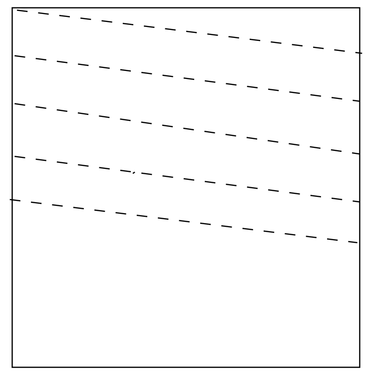

Figure 2: Sampling for broadcast TV signals.

4 Relating the Discrete-Time Fourier Transforms (DTFTs)

- Fact: The 1-D Discrete-Time Fourier Transform of g and the 2-D Discrete-Time Fourier Transform of x are related as follows:

$$G(\omega) = \sum_{n=0}^{N^2-1} g(n) e^{-j\omega n}$$

Let $n = Nn_1 + n_2, 0 \le n_1 < N, 0 \le n_2 < N$,

$$G(\omega) = \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} g(Nn_1 + n_2) e^{-j\omega(Nn_1+n_2)}$$

$$G(\omega) = \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} x(n_1, n_2) e^{-j\omega N n_1} e^{-j\omega n_2} = X(N\omega, \omega)$$

Fig. 5 shows one scan line passing through the origin. Due to the periodicity of the Fourier domain, the scan line replicates every 2π along the ω_2 axis, and every $\frac{2\pi}{N}$ along the ω_1 axis.

- $G(\omega)$ is a scanned version of $X(\omega_1, \omega_2)$.
Lexicographic ordering in $n \overset{F}{\longleftrightarrow}$ Scanning in ω
Scanning in $t_1, t_2 \overset{F}{\longleftrightarrow}$ Lexicographic ordering of 2-D Fourier series coefficients

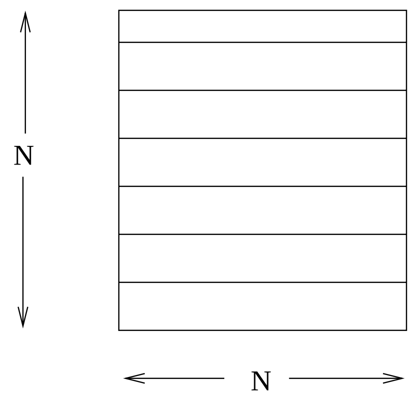

Figure 3: Lexiographic ordering

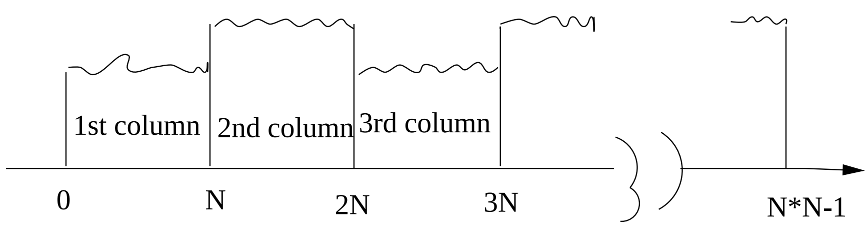

Figure 4: 1-D Sequence

5 Why Might This Be Useful?

- 2-D Filters can be designed using 1-D design algorithms.
- 1-D hardware can be used to implement 2-D Filters.
- 2-D hardware can be used to do 1-D processing.
- Compensation for scan lines, etc.

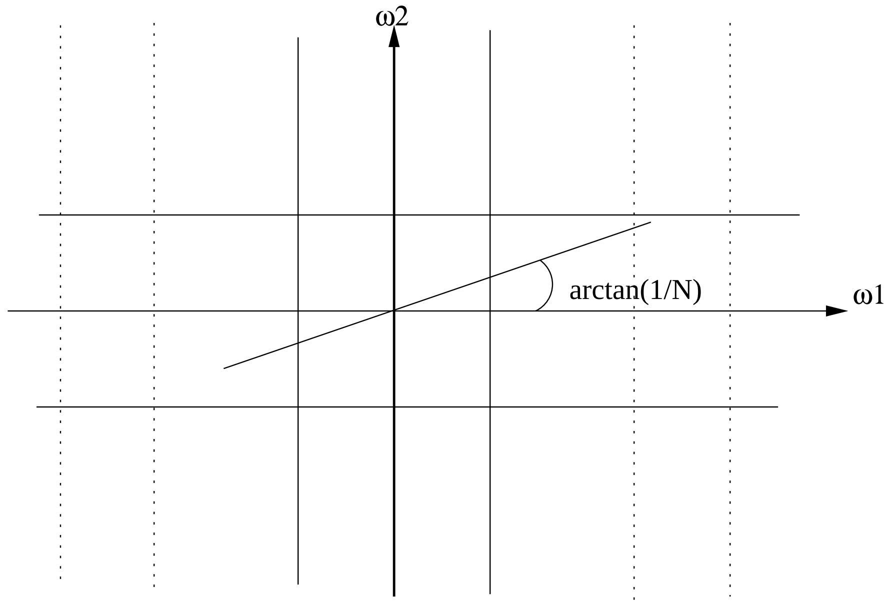

Figure 5: Relationship between the 2-D DTFT and the 1-D DTFT. Replicas of the scan line due to the periodicity of the 2-D Fourier domain are not shown.

6 Periodic Sequences

- ⋄ A sequence is rectangularly periodic if

$$\tilde{x}(n_1, n_2 + N_2) = \tilde{x}(n_1, n_2)$$

$$\tilde{x}(n_1 + N_1, n_2) = \tilde{x}(n_1, n_2)$$

N_1: Horizontal Period
N_2: Vertical Period

- ⋄ More generally, $\tilde{x}(n_1, n_2)$ is periodic with periodicity matrix $\mathbf{N}$ if

$$\tilde{x}(\mathbf{n}) = \tilde{x}(\mathbf{n} + \mathbf{N}\,\mathbf{r}), \forall \mathbf{n} \in \mathcal{I}, \forall \mathbf{r} \in \mathcal{I}$$

where $\mathcal{I}$ is the set of all integer vectors of same dimension as $\mathbf{n}$.

1. $|\det \mathbf{N}| \neq 0$ is the number of samples in one period of $\tilde{x}$.
2. $\mathbf{N}$ is an integer matrix and $|\det \mathbf{N}|$ is a positive integer.
3. The columns of $\mathbf{N}$ represent periodicity vectors.
4. $\mathbf{N}$ diagonal $\implies$ rectangular periodicity.