Rectangular M-D Discrete Fourier Transform

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Rectangular Fourier Series

Any rectangularly periodic array with horizontal and vertical periods N_1 and N_2 can be expressed as a finite sum of harmonically related complex sinusoids.

$$\tilde{x}(n_1,n_2) = \frac{1}{N_1 N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}\tilde{X}(k_1,k_2)\exp\left(j\frac{2\pi}{N_1}n_1k_1 + j\frac{2\pi}{N_2}n_2k_2 \right)$$

$$\tilde{X}(k_1,k_2) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\tilde{x}(n_1,n_2)\exp\left(-j\frac{2\pi}{N_1}n_1k_1 - j\frac{2\pi}{N_2}n_2k_2 \right)$$

Each sum involves only a finite number of samples

If the second sum is used to define $\tilde{X}(k_1,k_2)$, that second sequence is seen to be rectangularly periodic with horizontal and vertical periods N_1 and N_2, respectively.

Rectangular DFT

There is a one-to-one relationship between a sequence with periods N_1 and N_2, and a finite extent sequence with support on [0, N_1-1]x[0, N_2-1].

$$x(n_1,n_2) = \begin{cases} \tilde{x}(n_1,n_2) & (n_1,n_2)\in R_{N_1N_2} \\ 0 & \text{otherwise} \end{cases}$$

where $R_{N_1N_2} = \{(n_1,n_2): 0 \le n_1 \le N_1 - 1,\ 0 \le n_2 \le N_2 - 1\}$

$$\tilde{x}(n_1,n_2) = \sum_{k_1}\sum_{k_2} x(n_1 - k_1N_1, n_2 - k_2N_2)$$

$$x(n_1,n_2) \Leftrightarrow \tilde{x}(n_1,n_2) \Leftrightarrow \tilde{X}(k_1,k_2) \Leftrightarrow X(k_1,k_2)$$

The outside sequences are of finite extent. The direct relationship between them is the DFT.

$$x(n_1,n_2) = \frac{1}{N_1 N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}X(k_1,k_2)\exp\left(j\frac{2\pi}{N_1}n_1k_1 + j\frac{2\pi}{N_2}n_2k_2 \right) \quad 0 \le n_1 < N_{1,}\ 0 \le n_2 < N_2$$

$$X(k_1,k_2) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}x(n_1,n_2)\exp\left(-j\frac{2\pi}{N_1}n_1k_1 - j\frac{2\pi}{N_2}n_2k_2 \right) \quad 0 \le k_1 < N_{1,}\ 0 \le k_2 < N_2$$

The DFT corresponds to (rectangular) samples of the discrete-time Fourier transform

$$X(k_1,k_2) = X_a\left(\frac{2\pi k_1}{N_1}, \frac{2\pi k_2}{N_2} \right)$$

Example

$$x(n_1,n_2,n_3) = \delta(n_2, n_3 - 1) \quad (n_1,n_2,n_3) \in R_{N_1N_2N_3}(n_1,n_2,n_3)$$

At location $n_2 = 0$ and $n_3 = 1$, this function is a line impulse in 3-D that varies over integer values of n_1. Recall that $\delta(n_2, n_3-1) = 1\ \delta(n_2)\ \delta(n_3-1)$.

$$X(k_1,k_2,k_3) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1}\delta(n_2,n_3-1)W_{N_1}^{n_1k_1}W_{N_2}^{n_2k_2}W_{N_3}^{n_3k_3}$$

First evaluate the sum with respect to n_3

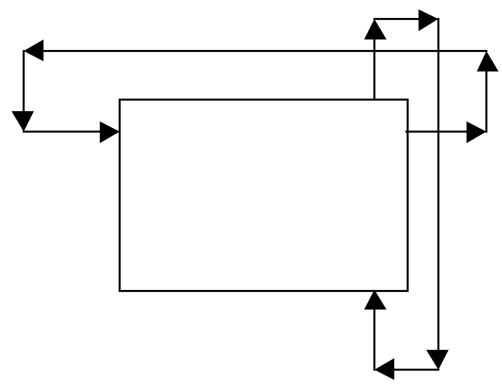
$$X(k_1,k_2,k_3) = W_{N_3}^{k_3}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\delta(n_2)W_{N_1}^{n_1k_1}W_{N_2}^{n_2k_2}$$

Now evaluate with respect to n_2

$$X(k_1,k_2,k_3) = W_{N_3}^{k_3}\sum_{n_1=0}^{N_1-1}W_{N_2}^{n_2k_2} = N_1\ \delta(k_1)W_{N_3}^{k_3} \quad (separable)$$

Properties of the DFT

- Linearity: $a\,x_1(n_1,n_2) + b\,x_2(n_1,n_2) \leftrightarrow a\,X_1(k_1,k_2) + b\,X_2(k_1,k_2)$
- Circular shifts: $x\left(((n_1-m_1))_{N_1},((n_2-m_2))_{N_2}\right) = W_{N_1}^{m_1k_1}\ W_{N_2}^{m_2k_2}\ X(k_1,k_2)$



Circular Shift

- Symmetry: If $x(n_1,n_2) = x^*(n_1,n_2)$, i.e. the sequence is real-valued, then $X^*(k_1,k_2) = X\left(((N_1-k_1))_{N_1},((N_2-k_2))_{N_2}\right)$
- Duality: If $x(n_1,n_2) \leftrightarrow X(k_1,k_2)$, then $X^*(n_1,n_2) \leftrightarrow N_1\ N_2\ x^*(k_1,k_2)$
- Circular Convolution: $x(n_1,n_2)\ \overbrace{N_1 x N_2}^{Circ.Conv.}\ h(n_1,n_2) = X(k_1,k_2)H(k_1,k_2)$
  - The convolution is circular with respect to both variables
  - If the DFT support N_1 x N_2 is large enough to contain the linear convolution, then *circular convolution = linear convolution*
  - If the extent of x is N_1 x N_2 and h is M_1 x M_2, then the linear convolution of x and h is P_1 x P_2 where $P_1 = N_1 + M_1 - 1$ and $P_2 = N_2 + M_2 - 1$.
  - The DFT size must be at least P_1 x P_2