Chapter 2

Previous Methods for Time Domain Equalizer Design

Communication subsystems should ideally be designed to enable the overall system to achieve data rates that are as close as possible to the channel capacity. In the design of time domain equalizers, optimizing objective functions such as the mean squared error (MSE) or shortening signal-to-noise ratio (SSNR) might increase channel capacity. However, as explained in this chapter, it is possible that a TEQ with worse MSE or SSNR can give better channel capacity than a system with better MSE or SSNR. Among the many TEQ design methods available, however, only one method – maximum geometric SNR (MGSNR) – attempts to maximize channel capacity directly. The MGSNR method has limited success due to inappropriate assumptions and inaccurate approximations.

2.1 Introduction

Time domain equalizer design methods can be categorized into three major approaches: minimizing Mean Squared Error (MSE)Γmaximizing Shortening SNR (SSNR)Γ and maximizing channel capacity. The Minimum MSE (MMSE) approach is the first application of channel shortening to multicarrier systems [9]. Adaptive MMSE design methods [10Γ11] are commonly used in practical systems. Maximizing SSNR is equivalent to minimize the energy of the component of the channel impulse response that cause ISI [12Γ 13Γ 14]MelsaΓ Younce and Rohrs introduced this approach and the optimal solution. Neither the MMSE nor the Maximum SSNR (MSSNR) methods attempt to maximize channel capacity directly. Al-Dhahir and Cioffi [15Γ 16] propose the Maximum Geometric SNR (MGSNR) method to shorten the channel impulse response while maximizing an approximation to the channel capacity.

This chapter summarizes research in the three major approaches to time domain equalizer design. Section 2.2 summarizes MMSE design methods. Section 2.3 derives the MSSNR TEQ design methods and presents two suboptimal divide-and-conquer TEQ design methods. Section 2.4 defines channel capacity for DMT systems and summarizes the MGSNR design method. Section 2.5 concludes this chapter.

2.2 Minimum Mean Squared Error (MMSE) Design

Falconer and Magee [17] introduce the MMSE design method to shorten a channel impulse response for maximum likelihood (ML) receivers. Their goal

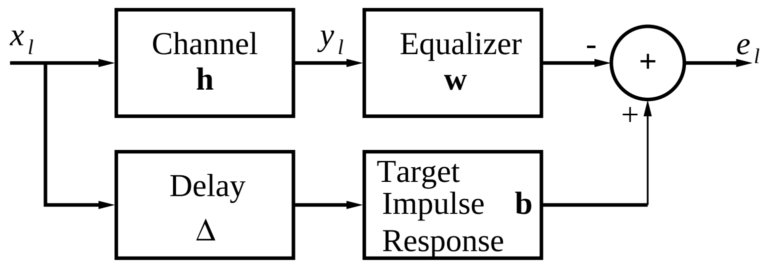

Figure 2.1: Block diagram of the minimum mean-squared error (MMSE) equalizer. The equalizer is an FIR filter with impulse response $\mathbf{w}$. The bottom path does not physically exist, but is part of the design method.

is to design a prefilter to the Viterbi algorithmΓ which is one of the most popular solutions for maximum likelihood data sequence estimation. The prefilter shortens the channel impulse responseΓ which dramatically reduces the computational complexity of the Viterbi algorithm. The Viterbi algorithm requires a number of computations that is exponential in the length of the channel impulse response [18].

Chow and Cioffi [9] are the first to apply channel shortening equalization to multicarrier modulation. They use the MMSE design method to shorten a given channel to the length of the cyclic prefix. Compared to Falconer and Magee's approachΓthey use a training sequence instead of decision directed equalizationΓand apply a unit-tap constraint (UTC) instead of a unit-energy constraint (UEC) to prevent an all-zero trivial solution for the equalizer taps during minimization.

The idea behind the MMSE TEQ design method may be explained by Fig. 2.1. The structure consists of an FIR equalizer in cascade with the channel and a parallel branch that consists of a delay and an FIR filter with a target impulse response (TIR). The goal in the MMSE design of the vector of TEQ taps $\mathbf{w}$ is to minimize the mean square of the error between the output of the equalizer and the output of the TIR. Assume that the error is zero for any given input signal. That means the impulse response of both branches are equal. In other wordsΓ the equalized channel impulse response (upper branch) would be equal to a delayed version of the TIR. Setting the number of taps of the TIR to a desired length forces the equalizer channel impulse response to have the same length.

Al-Dhahir and Cioffi [19] generalize the idea of [9] and [17]. They show that a unit-energy constraint on the target impulse response gives a lower mean squared error than a unit-tap constraint on the target impulse response. I use their derivations to introduce the MMSE design method below.

Assuming an oversampling factor of S at the receiverΓthe L-tap FIR channel output over a block of N_w symbols (each consisting of S samples) can be written as

$$\begin{bmatrix} y_{k+N_w-1} \\ y_{k+N_w-2} \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} h_0 & \cdots & h_L & 0 & \cdots & 0 \\ 0 & h_0 & \cdots & h_L & 0 & \cdots \\ \vdots & \ddots & \ddots & & \ddots & \vdots \\ 0 & \cdots & 0 & h_0 & \cdots & h_L \end{bmatrix} \begin{bmatrix} x_{k+N_w-1} \\ x_{k+N_w-2} \\ \vdots \\ x_{k-L} \end{bmatrix} + \begin{bmatrix} n_{k+N_w-1} \\ n_{k+N_w-2} \\ \vdots \\ n_{k-L} \end{bmatrix} \tag{2.1}$$

where

$$y_k = \begin{bmatrix} y_{k0} \\ \vdots \\ y_{kS} \end{bmatrix}, x_k = \begin{bmatrix} x_{k0} \\ \vdots \\ x_{kS} \end{bmatrix}, n_k = \begin{bmatrix} n_{k0} \\ \vdots \\ n_{kS} \end{bmatrix} \tag{2.2}$$

Writing (2.1) in a more compact formΓ

$$\mathbf{y}_k = \mathbf{H}\mathbf{x}_k + \mathbf{n}_k \tag{2.3}$$

Again the objective is to minimize the MSE which is given as

$$\mathrm{MSE} = \mathcal{E}\{e_k^2\} = \mathbf{b}^T\mathbf{R_{xx}}\mathbf{b} - \mathbf{b}^T\mathbf{R_{xy}}\mathbf{w} - \mathbf{w}^T\mathbf{R_{yx}}\mathbf{b} + \mathbf{w}^T\mathbf{R_{xx}}\mathbf{w} \tag{2.4}$$

where $\mathbf{R_{xx}} = \mathcal{E}\{\mathbf{x}_k\mathbf{x}_k^T\}\Gamma\mathbf{R_{xy}} = \mathcal{E}\{\mathbf{x}_k\mathbf{y}_k^T\}\Gamma\mathbf{R_{yx}} = \mathcal{E}\{\mathbf{y}_k\mathbf{x}_k^T\}\Gamma\mathbf{R_{yy}} = \mathcal{E}\{\mathbf{y}_k\mathbf{y}_k^T\}$. Taking the gradient with respect to $\mathbf{w}$ and setting it to zero yields

$$\mathbf{b}^T\mathbf{R_{xy}} = \mathbf{w}^T\mathbf{R_{yy}} \tag{2.5}$$

By substituting (2.5) into (2.4)Γ

$$\text{MSE} = \mathbf{b}^T\left[\mathbf{R_{xx}} - \mathbf{R_{xy}}\mathbf{R_{yy}}^{-1}\mathbf{R_{yx}}\right]\mathbf{b} = \mathbf{b}^T\mathbf{R_{x|y}}\mathbf{b} \tag{2.6}$$

Define

$$\mathbf{S} = \begin{bmatrix} \mathbf{0}_{(\nu+1)\times\Delta} & \mathbf{I}_{(\nu+1)\times(\nu+1)} & \mathbf{0}_{(\nu+1)\times(N_w+L-\Delta-\nu-1)} \end{bmatrix}^T \tag{2.7}$$

where $\mathbf{0}_{m\times n}$ is a $m \times n$ matrix of zerosΓ$\mathbf{I}_{n\times n}$ is an $n \times n$ identity matrixΓ and $\nu + 1$ is the number of elements in $\mathbf{b}$. By defining

$$\mathbf{R}_\Delta = \mathbf{S}^T\mathbf{R_{x|y}}\mathbf{S} \tag{2.8}$$

the MSE can be written as

$$\text{MSE} = \mathbf{b}^T\mathbf{R_\Delta}\mathbf{b} \tag{2.9}$$

To obtain the unit-tap constraint solutionΓ Al-Dhahir and Cioffi [19] define $\mathbf{e}_i$ as the i^{th} unit vector and form the Lagrangian

$$L^{UTC}(\mathbf{b}, \lambda) = \mathbf{b}^T\mathbf{R_\Delta}\mathbf{b} + \lambda(\mathbf{b}^T\mathbf{e}_i - 1) \tag{2.10}$$

Taking the gradient with respect to $\mathbf{b}$ and setting it to zero

$$\frac{\partial L^{UTC}}{\partial \mathbf{b}} = 2\mathbf{R_\Delta}\mathbf{b} + \lambda\mathbf{e}_i = 0 \tag{2.11}$$

which has the solution

$$\mathbf{b} = \frac{\mathbf{R}_\Delta^{-1}\mathbf{e}_i}{\mathbf{R}_\Delta^{-1}(i,i)} \tag{2.12}$$

where $i \in [0, \nu]$ and $\mathbf{R}_{\Delta}^{-1}(i,i)$ is the i^{th} element in the diagonal of the matrix $\mathbf{R}_{\Delta}^{-1}$. The solution for $\mathbf{b}$ given by (2.12) yields an MSE of

$$\text{MSE} = \frac{1}{\mathbf{R}_{\Delta}^{-1}(i,i)} \tag{2.13}$$

The value of i that minimizes the MSE can be found from

$$i_{opt} = \arg \max_{0 \le i \le \nu} \{\mathbf{R}_{\Delta}^{-1}(i,i)\} \tag{2.14}$$

and the optimal $\mathbf{b}$ is given as

$$\mathbf{b}_{opt} = \frac{\mathbf{R}_{\Delta}^{-1}\mathbf{e}_{i_{opt}}}{\mathbf{R}_{\Delta}^{-1}(i_{opt}, i_{opt})} \tag{2.15}$$

and $\mathbf{w}_{opt}$ can be obtained from (2.5) by using $\mathbf{b} = \mathbf{b}_{opt}$.

If the unit-energy constraint on $\mathbf{b}$ were used instead of the unit-tap constraintΓthen the Lagrangian would become

$$L^{UEC} = \mathbf{b}^T\mathbf{R_{\Delta}}\mathbf{b} + \lambda(\mathbf{b}^T\mathbf{b} - 1) \tag{2.16}$$

After setting the gradient of (2.16) with respect to $\mathbf{b}$ to zeroΓ

$$\mathbf{R_{\Delta}}\mathbf{b} = \lambda\mathbf{b} \tag{2.17}$$

which shows that $\mathbf{b}$ is an eigenvector of $\mathbf{R_{\Delta}}$. Since $\text{MSE} = \mathbf{b}^T\mathbf{R_{\Delta}}\mathbf{b} = \mathbf{b}^T\lambda\mathbf{b} = \lambda$Γ$\mathbf{b}$ should be chosen as the eigenvector corresponding to the minimum eigenvalue of $\mathbf{R_{\Delta}}$ to minimize the MSE. ThusΓ

$$\mathbf{b_{opt}} = \text{ eigenvector of } \mathbf{R_{\Delta}} \text{ corresponding to the minimum eigenvalue} \tag{2.18}$$

Fig. 2.2 shows a TIR and SIR. The MMSE design method formulates the square of the difference between the TIR and SIR as the error and minimizes it. The method minimizes the difference between the TIR and SIR both inside

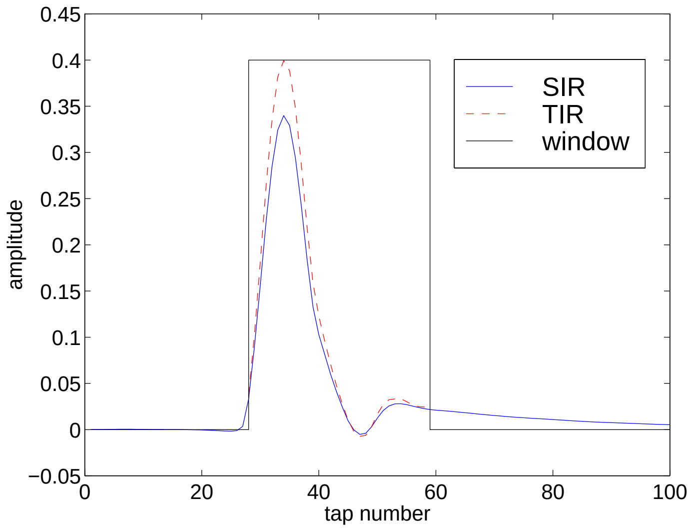

Figure 2.2: A target impulse response (TIR) and shortened impulse response (SIR).

and outside the target window. In factΓthe difference between the TIR and SIR inside the target window does not cause ISI. Both the TIR and SIR inside the target window have higher amplitudesΓ which means that difference inside the target window might contribute more to the MSE than the difference outside.

The MMSE design method maximizes the SNR at the TEQ output. The equalizer frequency responseΓthereforeΓtends to be a narrow bandpass filter placed at a center frequencyΓ which has high SNR. The equalizer increases the output SNR by filtering out the low SNR regions of the channel frequency response. Webster and Roberts [20] mention this problem and suggest to exclude the channel noise from the design procedure. This would ensure that only the ISI is minimized instead of the combination of noise and ISI. HoweverΓ they do not give an algorithm to accomplish this task.

Since the MMSE method in general cannot force the error to become exactly zeroΓsome residual ISI will remain. To maximize channel capacityΓ the residual ISI should be placed in frequency bands with high channel noise. This ensures that the residual ISI would be small compared to the noise and the effect on the SNR would be negligible. The MMSE design method does not have a mechanism to shape the residual ISI in frequency. ThereforeΓit is not optimal in the sense of maximizing channel capacity.

Wang and Adalı [21Γ22Γ23Γ24] propose to weight the error in the frequency domain. They use the weighting function to prevent the optimization of unused subchannels by setting the weight to zero. They do not propose a way to calculate weights so that the residual ISI is shaped to increase channel capacity. WangΓLuΓand Antoniou [25] propose a new constraint for the MMSE design method. Instead of constraining the total energy of the TIRΓ they constrain the energy of the TIR only in the used subchannels to unity. Their method still minimizes an MSE measure which is not directly related to channel capacity.

Kerckhove and Spruyt [26] add the gain of the TIR in the unused frequency bands to the error term. By minimizing the errorΓthe TIR energy in unused bands is also minimized. AckerΓLeusΓMoonenΓWielΓand Pollet [27Γ28] map the TEQ to the frequency domain equalizer. They remove the TEQ from the system and instead add more taps to the frequency domain equalizer. Having an equalizer for each tone makes it is possible to optimize the ISI location in frequency.

The unit-energy constrained MMSE solution requires an eigenvalue decomposition and the unit-tap constrained solution has an additional search direction for the optimal index. Decreasing the computational complexity of

MMSE design methods is an active area of research. Falconer and Magee [17] along with the original MMSE design method propose an adaptive algorithm to calculate the TIR and TEQ. The algorithm is based on the least mean squared (LMS) algorithm. The TIR and the TEQ are separately adapted with standard LMS with an addition to satisfy the unit-energy constraint. After every iterationΓ theTIR is normalized so that its energy is equal to one.

ChowΓ CioffiΓ and Bingham [10Γ 11] propose a different iterative method to calculate the MMSE TIR and TEQ. They propose two methods to update the TIR and TEQ: frequency domain LMS and frequency domain division. To ensure that the updated TIR and TEQ have the desired lengthsΓthey transform the TIR back to time domain and window it. The combination of these methods generates four different MMSE design algorithms. Slow convergence is a major problem with all four methods.

Strait [29] combines the two separate LMS algorithms into a single LMS by forming a new vector with the TEQ and TIR coefficients. Slow convergence is again a major problem. To solve this problemΓ Strait transforms the input signal by using a unitary transform. He shows that transform domain adaptation converges faster but requires higher computational complexity per iteration.

Nafie and Gatherer [30] compute the minimum eigenvalue by using the iterative power method [31]. They also propose to use the LU decomposition and run a pair of iterations that are more stable and faster to implement. In the same paperΓthey also propose an off-line LMS algorithm. Assuming that they have an estimate of the channel impulse responseΓthey use the estimate as the input vector to the TEQ. The TIR is the desired response and the error is the difference between the TIR and the output of the equalizer.

Lashkarian and Kiaei [32] propose an iterative algorithm to solve the MMSE design problem. It is based on the asymptotic equivalence of Toeplitz and circulant matrices to estimate the Hessian of a quadratic form. The proposed algorithm is computationally less complex than the iterative power method and may be parallelized for efficient implementation in hardware.

Another drawback of the MMSE design method is the deep notches in the frequency response of the designed TEQ. The subchannels in which a notch appears cannot used for data transmission because the gain in the subchannel is too small. Farhang-Boroujeny and Ding [33Γ 34] propose an eigen-approach based sub-optimum solution to overcome the this problem. Instead of using only the eigenvector of the minimum eigenvalue as derived in (2.18)Γ theyuse a weighted sum of all eigenvectors as the TEQ. This solution gives higher MSE but equal bit rate to the original MMSE method.

2.3 Maximum Shortening SNR Design

Seeing the TEQ design problem as a channel shortening problem rather than a equalization problemΓ MelsaΓ YounceΓ and Rohrs [12] propose a different solution. The goal is to find a TEQ that minimizes the energy of the SIR outside the target windowΓ while keeping the energy inside constant. They have a reasonable assumption that the channel impulse response is known. In DMT applications such as ADSLΓ the channel FFT coefficients are estimated for bit loading [8]. The channel impulse response can be estimated from the FFT coefficients.

The samples of the SIR inside the target window can be written in matrix form as

$$\mathbf{h}_{win} = \begin{bmatrix} h_{\Delta+1} & h_{\Delta} & \cdots & h_{\Delta-N_w+2} \\ h_{\Delta+2} & h_{\Delta+1} & \cdots & h_{\Delta-N_w+3} \\ \vdots & & \ddots & \vdots \\ h_{\Delta+\nu+1} & h_{\Delta+\nu} & \cdots & h_{\Delta-N_w+\nu+2} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N_w-1} \end{bmatrix} = \mathbf{H}_{win}\mathbf{w}$$

and the samples outside the target window as

$$\mathbf{h}_{wall} = \begin{bmatrix} h_0 & 0 & \cdots & 0 \\ h_1 & h_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ h_{\Delta} & h_{\Delta-1} & \cdots & h_{\Delta-N_w+1} \\ h_{\Delta+\nu+2} & h_{\Delta+\nu+1} & \cdots & h_{\Delta-N_w+\nu+3} \\ \vdots & \vdots & & \vdots \\ h_{L-1} & h_{L-2} & \cdots & h_{L-Nw+1} \\ 0 & h_{L-1} & \cdots & h_{L-Nw+2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & h_{L-1} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N_w-1} \end{bmatrix} = \mathbf{H}_{wall}\mathbf{w}$$

The energy inside and outside the target window is

$$\begin{aligned} \mathbf{h}_{win}^T\mathbf{h}_{win} &= \mathbf{w}^T\mathbf{H}_{win}^T\mathbf{H}_{win}\mathbf{w} = \mathbf{w}^T\mathbf{B}\mathbf{w} \\ \mathbf{h}_{wall}^T\mathbf{h}_{wall} &= \mathbf{w}^T\mathbf{H}_{wall}^T\mathbf{H}_{wall}\mathbf{w} = \mathbf{w}^T\mathbf{A}\mathbf{w} \end{aligned} \quad (2.19)$$

The problem is formulated as

$$\min_{\mathbf{w}} \mathbf{w}^T\mathbf{A}\mathbf{w} \;\text{ s.t. }\; \mathbf{w}^T\mathbf{B}\mathbf{w} = 1 \quad (2.20)$$

This is equivalent to maximizing the SSNR defined as

$$\text{SSNR} = \frac{\mathbf{w}^T\mathbf{B}\mathbf{w}}{\mathbf{w}^T\mathbf{A}\mathbf{w}} \quad (2.21)$$

The solution is

$$\mathbf{w}_{opt} = (\sqrt{\mathbf{B}})^{-1}\mathbf{p}_{min} \tag{2.22}$$

where $\sqrt{\mathbf{B}}$ is the Cholesky decomposition of $\mathbf{B}$ and $\mathbf{p}_{min}$ is the eigenvector corresponding to the minimum eigenvalue of a composite matrix

$$(\sqrt{\mathbf{B}})^{-1}\mathbf{A}(\sqrt{\mathbf{B}^T})^{-1}$$

The matrix $\mathbf{B}$ has to be positive definite in order to have a Cholesky decomposition. It is also assumed that $\mathbf{B}$ is invertible which is true only if $N_w < \nu$. The solution when $\mathbf{B}$ is singular is more complicated [12]. Yin and Yue [35] maximize $\mathbf{w}^T\mathbf{A}\mathbf{w}$ while constraining $\mathbf{w}^T\mathbf{B}\mathbf{w} = 1$. In this case, $\mathbf{A}$ needs to be positive definite and invertible, which is true for most physical channels.

The MSSNR method minimizes the part of the SIR that causes ISI. If the energy outside the target window were zero, then the channel would be perfectly shortened and ISI would be totally eliminated. The solution which gives zero energy outside the target window is optimum also in the sense of maximum channel capacity since this is the case where ISI is totally canceled. In practice, however, this optimum solution cannot be achieved. For this case, the MSSNR solution is not guaranteed to yield maximum channel capacity solution. The reason is similar to that of the MMSE design method; i.e., the residual ISI power cannot be placed in high noise regions in the frequency domain. The method only minimizes the energy outside the target window and does not care where the residual ISI lies in frequency.

Wang, Adalı, Liu, and Vlajnic [36] not only minimize the energy outside the target window but also add another term to be minimized. This term is a frequency weighted energy of the equalizer. Using the weighting function, it is possible to shape the equalizer frequency response. This is useful to prevent large equalizer gains in unused subchannels. HoweverΓit does not optimize the residual ISI.

A second problem with the MSSNR design approach is the computation complexity due to the eigenvalue and Cholesky decompositions. ChiuΓTsaiΓ LiauΓand Troulis [37] propose an inverse power method. This method needs neither Cholesky decomposition nor matrix inversion. It directly iterates on two matrices to obtain the optimal TEQ in the sense of MSSNR.

The divide-and-conquer design method [38] is a faster implementation of the MSSNR method [12]. The idea is to divide the equalizer design problem into smaller problems that are easier to solve and then combine the results together. An FIR filter of length N can be represented as a convolution of $N-1$ two-tap filters. In this TEQ design methodΓthe equalizer is divided into a number of two-tap equalizers. Each two-tap equalizer has only one unknown tap since the first tap is set to one. This can be considered as a unit-tap constraint that is similar to that used in the MMSE design approach. Designing an N_w-tap filter requires the design of $N_w - 1$ two-tap filters. For the i^{th} two-tap filterΓthe method optimizes the two-tap filter $\mathbf{w}_i$Γand convolves the optimized filter with the current channel impulse response to obtain the new channel impulse response to be used at stage $i+1$. Once the $N_w - 1$ two-tap filters have been computedΓthey are convolved together to form one N_w-tap equalizer. Two different versions of the divide-and-conquer (DC) design method are DC TEQ Minimization and DC TEQ Cancellation.

2.3.1 Divide-and-Conquer TEQ Minimization Design

Each TEQ stage in DC TEQ Minimization maximizes the SSNR defined in (2.21) or equivalently minimizes the inverse of it with a two-tap filter defined as

$$\mathbf{w}_i = [1,\ g_i]^T \tag{2.23}$$

Since $\mathbf{w}_i$ consists of two tapsΓ the matrices $\mathbf{A}_i$ and $\mathbf{B}_i$ are 2×2 Toeplitz matrices. For the i^{th} filterΓ theSSNR becomes

$$\frac{\mathbf{w}_i^T \mathbf{A}_i \mathbf{w}_i}{\mathbf{w}_i^T \mathbf{B}_i \mathbf{w}_i} = \frac{\begin{bmatrix} 1 & g_i \end{bmatrix} \begin{bmatrix} a_{1,i} & a_{2,i} \\ a_{2,i} & a_{3,i} \end{bmatrix} \begin{bmatrix} 1 \\ g_i \end{bmatrix}}{\begin{bmatrix} 1 & g_i \end{bmatrix} \begin{bmatrix} b_{1,i} & b_{2,i} \\ b_{2,i} & b_{3,i} \end{bmatrix} \begin{bmatrix} 1 \\ g_i \end{bmatrix}} = \frac{a_{1,i} + 2a_{2,i}g_i + a_{3,i}g_i^2}{b_{1,i} + 2b_{2,i}g_i + b_{3,i}g_i^2} \tag{2.24}$$

The matrices $\mathbf{A}_i$ and $\mathbf{B}_i$ are also indexed with i because at every iterationΓ a new channel impulse response is calculated which changes $\mathbf{A}_i$ and $\mathbf{B}_i$. The denominator in (2.24) does not become zero for any g_i [38].

The optimal g_i is calculated by differentiating (2.24) with respect to g_i and setting the result to zero. The solutions are

$$g_i = \frac{-(a_{3,i}b_{1,i} - a_{1,i}b_{3,i}) \pm c_i}{2(a_{3,i}b_{2,i} - a_{2,i}b_{3,i})} \tag{2.25}$$

where

$$c_i = \sqrt{(a_{3,i}b_{1,i} - a_{1,i}b_{3,i})^2 - 4(a_{3,i}b_{2,i} - a_{2,i}b_{3,i})(a_{2,i}b_{1,i} - a_{1,i}b_{2,i})}$$

From (2.25) the best solution for g_i with respect to (2.24) is chosen. The solution is always real-valued [38].

2.3.2 Divide-and-Conquer TEQ Cancellation Design

The DC TEQ Cancellation method avoids the computationally expensive calculation of $\mathbf{A}_i$ and $\mathbf{B}_i$ at every stage. Instead of maximizing the SSNR onlyΓ the energy outside the target window is minimized. Since unit-tap constraints are used for each two-tap filterΓconstraining the energy of the SIR inside the target window is not necessary. The following derivation is one of the contributions of this thesis.

Define $\mathbf{h}_i$ as the new channel impulse response and $\mathbf{h}_i^{\text{wall}}$ as the new $\mathbf{h}^{\text{wall}}$ at stage iΓ so that $\mathbf{h}_0$ is the channel impulse response and $\mathbf{h}_i$ is the convolution of $\mathbf{h}_{i-1}$ with $\mathbf{w}_i$. Note that the length of $\mathbf{h}_i$ increases with i due to the convolution. At stage iΓ

$$\mathbf{h}_i^{\text{wall}} = \begin{bmatrix} h_{i-1}(1) & 0 \\ h_{i-1}(2) & h_{i-1}(1) \\ \vdots & \vdots \\ h_{i-1}(\Delta) & h_{i-1}(\Delta-1) \\ h_{i-1}(\Delta+\nu+2) & h_{i-1}(\Delta+\nu+1) \\ \vdots & \vdots \\ h_{i-1}(L_{h_{i-1}}) & h_{i-1}(L_{h_{i-1}}-1) \end{bmatrix} \begin{bmatrix} 1 \\ g_i \end{bmatrix} \tag{2.26}$$

$$\mathbf{h}_i^{\text{wall}} = \begin{bmatrix} h_{i-1}(1) + 0 \\ h_{i-1}(2) + h_{i-1}(1)\, g \\ \vdots \\ h_{i-1}(\Delta) + h_{i-1}(\Delta-1)\, g \\ h_{i-1}(\Delta+\nu+2) + h_{i-1}(\Delta+\nu+1)\, g \\ \vdots \\ h_{i-1}(L_{h_{i-1}}) + h_{i-1}(L_{h_{i-1}}-1)\, g \end{bmatrix} \tag{2.27}$$

where $L_{h_{i-1}}$ is the length of $\mathbf{h}_{i-1}$. The energy to be minimized is

$$\mathbf{h}_i^{\mathrm{wall}T}\mathbf{h}_i^{\mathrm{wall}} = \sum_{k \in S} \left(h_{i-1}(k) + g_i\, h_{i-1}(k-1)\right)^2, \tag{2.28}$$

where

$$S = \{1, 2, \ldots, \Delta, \Delta + \nu + 2, \ldots, L_{h_{i-1}}\}$$

The minimum of (2.28) can again be found by taking the derivative with respect to g_i and setting it to zero. The solution is

$$g_i = -\frac{\sum\limits_{k \in S} h_{i-1}(k-1)h_{i-1}(k)}{\sum\limits_{k \in S} h_{i-1}^2(k-1)} \tag{2.29}$$

The TEQ is calculated by convolving $N_w - 1$ two-tap filters. This DC design method does not use any matrix decompositions or matrix inversions; henceΓ it is suitable for real-time implementation. Although it is efficient in terms of computation complexityΓ it retains all of the drawbacks of the MSSNR method.

2.4 Maximum Geometric SNR Design

In a communication systemΓ the ultimate goal is to reach optimum channel capacity. Al-Dhahir and Cioffi [15] introduced the idea of a TEQ design method to optimize channel capacity. Section 2.4.1 gives the channel capacity for multicarrier channels. Section 2.4.2 introduces the maximum geometric SNR method.

2.4.1 Multicarrier Channel Capacity

If the number of subchannels $N/2 + 1$ (i.e. $N/2 - 1$ two-dimensional and two one-dimensional) is largeΓ then it is reasonable to assume that the channel noise power spectrum in the subchannels are flat. In this case each subchannel can be modeled as an independent AWGN channel. The achievable capacity of a multicarrier channel can be written as the sum of the capacities of AWGN channels

$$b_{DMT} = \sum_{i \in \mathcal{S}} \log_2 \left(1 + \frac{\text{SNR}_i^{MFB}}{\Gamma}\right) \quad \text{bits/symbol} \tag{2.30}$$

where i is the subchannel index $\mathcal{S}$ is the set of the indices of the used $\bar{N}$ subchannels out of the $N/2+1$ subchannels SNR_i^{MFB} is the matched filter bound of the SNR in the i^{th} subchannel as defined below in (2.31) and Γ is the SNR gap for achieving Shannon channel capacity and is assumed to be constant over all subchannels. The SNR gap is a function of several factors including the modulation method allowable probability of error P_e coding gain γ_{eff} and desired system margin γ_m.

The system margin accounts for modeling error and is generally 6 dB in ADSL systems [1]. If one needs a channel with an SNR of x dB to transmit a certain amount of bits at the rate of the theoretical bound then in practice an SNR of $x+6$ dB is actually used. The system margin of 6 dB ensures that with the unaccounted errors the desired bit rate can be supported.

The SNR gap can be approximated in the case of QAM as [39]

$$\Gamma \approx \frac{\gamma_m}{3\gamma_{eff}} \left(Q^{-1}\left(\frac{P_e}{2}\right)\right)^2$$

Assuming that the input signal and noise are wide sense stationary the SNR in the i^{th} subchannel can be defined as

$$\text{SNR}_i^{MFB} = \frac{S_{x,i}|H_i|^2}{S_{n,i}} \tag{2.31}$$

where $S_{x,i}$ and $S_{n,i}$ are the transmitted signal and channel noise power respectively and H_i is the gain of the channel spectrum in the i^{th} subchannel. Here the assumption is that the subchannels are narrow enough so that the channel frequency response and transmitted signal power spectrum can be considered constant in each subchannel. The definition in (2.31) does not include the effect of ISI and any equalizers. It is the maximum achievable SNR or the matched filter bound (MFB). If the channel causes ISI or an equalizer has been used then the definition has to be modified.

2.4.2 The maximum geometric SNR method

The maximum geometric SNR (MGSNR) method maximizes a channel capacity cost function that is based on a geometric SNR definition as

$$\mathrm{GSNR} = \Gamma \left(\left[\prod_{i \in \mathcal{S}} \left(1 + \frac{\mathrm{SNR}_i^{EQ}}{\Gamma} \right) \right]^{1/\bar{N}} - 1 \right) \tag{2.32}$$

which is related to channel capacity. By using (2.32) we rewrite (2.30) as

$$b_{\mathrm{DMT}} = \bar{N} \log_2 \left(1 + \frac{\mathrm{GSNR}}{\Gamma} \right) \quad \text{bits/symbol}$$

This means that all of the subchannels act together like $\bar{N}$ AWGN channels with each channel having an SNR equal to the GSNR. Therefore maximizing the GSNR is equivalent to maximizing the channel capacity. In (2.32) the subchannel SNR in (2.31) is modified to include the effect of the equalizer [15]

$$\mathrm{SNR}_i^{EQ} = \frac{S_{x,i} |B_i|^2}{S_{n,i} |W_i|^2} \tag{2.33}$$

where $S_{x,i}$ is signal power $S_{n,i}$ is the noise power andB_i and W_i are the gains of $\mathbf{b}$ and $\mathbf{w}$ in the i^{th} subchannel respectively. This definition is discussed in detail later in this section.

The derivation [15] proceeds with the following approximation of the GSNR, which is obtained by ignoring the $+1$ and -1 terms in (2.32):

$$\text{GSNR} \approx \left[\prod_{i\in\mathcal{S}} \text{SNR}_i^{EQ}\right]^{1/\bar{N}} \tag{2.34}$$

This approximation is valid if the SNR in each subchannel is larger than one, so that the "1" terms can be ignored. This assumption may be reasonable only if bandwidth optimization is used. That is, the channels without sufficient SNR to carry bits are not used [40]. In this case, the problem of maximizing (2.34) can be converted to the maximization of

$$L(\mathbf{b}) = \frac{1}{\bar{N}} \sum_{i\in\mathcal{S}} \ln |B_i|^2 \tag{2.35}$$

which can be obtained by substituting (2.33) into (2.34) and taking the natural logarithm, based on the assumption that $\mathbf{b}$ and $\mathbf{w}$ do not depend on each other. B_i is the i^{th} FFT coefficient of $\mathbf{b}$ defined as

$$B_i = \sum_{k=0}^{N-1} b_k e^{-j\frac{2\pi}{N}ki}$$

The assumption that $\mathbf{b}$ and $\mathbf{w}$ do not depend on each other is not accurate because once $\mathbf{b}_{opt}$ is calculated by maximizing (2.35), the optimum (in the MMSE sense) TIR $\mathbf{w}_{opt}$ is found using

$$\mathbf{w}_{opt}^T = \mathbf{b}_{opt}^T \mathbf{R}_{xy} \mathbf{R}_{yy}^{-1} \tag{2.36}$$

where $\mathbf{R}_{xy}$ and $\mathbf{R}_{yy}$ are the channel input-output cross-correlation and channel output autocorrelation matrices, respectively. This choice of TEQ taps ensures that the MSE is minimum for the given TIR [15].

When maximizing (2.35), a unit-energy constraint is placed on $\mathbf{b}$ to prevent an infinite gain in the TEQ. This constraint maximizes the cost function for $|B_i|^2 = 1\ \forall i$, which implies a zero forcing equalization of the channel.

The goal is not to equalize the channel fully since in fact one of the primary reasons for the application of multicarrier modulation is to avoid full equalization because it requires high-order equalizers. FurthermoreΓfull equalization with a short equalizerΓ as is typical for TEQsΓ would cause large MSE. ThereforeΓan additional constraint is required to keep the MSE below a threshold MSE_{max}. This threshold has to be tuned if the channelΓnoise levelΓor signal power changes. Setting the threshold to the correct value for a given channel is crucial for good performance [37]. Including the above constraintsΓ Al-Dhahir and Cioffi state the optimum TIR problem as

$$\max_{\mathbf{b}} \sum_{i \in \mathcal{S}} \ln |B_i|^2 \text{ s.t. } \|\mathbf{b}\|^2 = 1 \text{ and } \mathbf{b}^T \mathbf{R}_\Delta \mathbf{b} \leq \text{MSE}_{\text{max}} \tag{2.37}$$

where

$$\mathbf{R}_\Delta = \left[\mathbf{0}_{(\nu+1)\times\Delta}\ \mathbf{I}_{\nu+1}\ \mathbf{0}_{(\nu+1)\times P}\right] \left(\frac{1}{S_x}\mathbf{I}_{N_w+L-1} + \mathbf{H}^T\mathbf{R}_{nn}^{-1}\mathbf{H}\right)^{-1} \begin{bmatrix} \mathbf{0}_{\Delta\times(\nu+1)} \\ \mathbf{I}_{\nu+1} \\ \mathbf{0}_{P\times(\nu+1)} \end{bmatrix}$$

HereΓ$P = N_w + L - \Delta - \nu - 2$Γ$\mathbf{0}_{m\times n}$ is an $m \times n$ matrix of zerosΓ$\mathbf{I}_m$ is the $m \times m$ identity matrixΓS_x is the average energy of the input symbolsΓ$\mathbf{R}_{nn}$ is the $N_w \times N_w$ noise correlation matrixΓand $\mathbf{H}$ is the $N_w \times (N_w + L - 1)$ channel convolution matrix. This nonlinear constrained optimization problem does not have a closed-form solution [15]Γbut it may be solved by numerical methods [41].

The MGSNR TEQ design method is based on the maximization of the approximate GSNR. Due to several inaccurate approximationsΓit is not optimum in the sense of maximizing channel capacity. The most important approximation is in the definition of the subchannel SNRΓSNR_i^{EQ}Γin (2.33). This definition includes the equalizer effect but not the ISI effect even though the objective of the TEQ is to minimize ISI. A later modification [40] includes an ISI term:

$$\mathrm{SNR}_i^{ISI} = \frac{S_{x,i}|B_i|^2}{S_{x,i}|B_i - W_iH_i|^2 + S_{n,i}|W_i|^2} \tag{2.38}$$

HoweverΓ this modified definition is only used to evaluate the performance of the MGSNR TEQ methodΓwhich is still based on the definition given in (2.33).

The modified definition in (2.38) represents ISI in the i^{th} subchannel as $S_{x,i}|B_i - H_iW_i|^2$. Assume that the SIR fits perfectly in the target window and there is no energy outside this window. HoweverΓ the SIR differs from the TIR inside the window. Although there is no ISI in the systemΓthe definition measures the difference as ISI. ThereforeΓ this definition is only accurate if the difference between the SIR and TIR is small so that its contribution to ISI is negligible. FurthermoreΓboth the subchannel SNR definition without an ISI term SNR_i^{EQ} in (2.33) and the one with an ISI term SNR_i^{ISI} in (2.38) are only useful if the structure in Fig. 2.1 is used. In generalΓa TIR is not available. ThenΓthese definitions are not suitable and a new definition is necessary.

In summary the drawbacks of the MGSNR TEQ method are that

- its derivation is based on a subchannel SNR definition SNR_i^{EQ} that does not include the effect of ISI;
- it depends on the parameter MSE_{max} which has to be tuned for different channels;
- its objective function (2.37) assumes that $\mathbf{b}$ and $\mathbf{w}$ are independent;
- it requires a constrained nonlinear optimization method; and
- it assumes that the SNR in each subchannel is much greater than one.

Considerable effort has been spent to overcome the last issue. Lashkarian and Kiaei [42] propose a projection onto convex sets method to solve the constrained nonlinear optimization problem iteratively with lower computational complexity. Milisavljević and Verriest [43] propose simulated annealing and genetic algorithms to solve the nonlinear optimization problemΓwhich have high complexity.

2.5 Conclusion

This chapter summarizes several approaches to design DMT TEQs. Table 2.1 summarizes the advantages and disadvantages of all the methods mentioned in this chapter. As shown in Table 2.1 MMSE and MGSNR methods have more disadvantages compared to MSSNR methods. The only method with no disadvantage is the DC TEQ Cancellation method of Section 2.3.2. HoweverΓ none of the MMSE and MSSNR methods optimize channel capacity and the MGSNR methods optimize only an approximation to the channel capacity.

Maximizing channel capacity is the primary goal in designing a TEQ. HoweverΓonly the MGSNR method in Section 2.4 attempts to optimize the channel capacity. As discussed in that sectionΓthe MGSNR method is not optimum in the sense of maximizing channel capacity due to many inaccurate approximations and assumptions. A design method that is not only computationally efficient enough for cost-effective real-time implementations but also truly maximizes the channel capacity is not available. My goal in this dissertation is to write the channel capacity as a function of TEQ taps with minimal assumptions and approximations. By maximizing this function the optimal TEQ coefficients can be calculated.

Advantages	Disadvantages
1. Adaptive or iterative	1. Deep notches in frequency
2. Off-line (initialization)	2. SIR-TIR difference inside window
3. Maximizes channel capacity	3. Slow or uncertain convergence
4. Minimizes directly ISI causing tail	4. Requires eigendecomposition
5. Frequency weighting	5. Requires nonlinear optimization
6. Optimize subchannels	6. Narrowband frequency response
	7. Numerical instabilities possible

	Advantages						Disadvantages						
	1	2	3	4	5	6	1	2	3	4	5	6	7
MMSE methods													
Chow *et al.* [9]		√					√	√				√	√
Chow *et al.* [10, 11]	√						√	√	√			√	√
Al-Dhahir *et al.* [19]		√					√	√		√		√	√
Kerckhove *et al.* [26]		√			√		√	√		√		√	√
Nafie *et al.* [30]	√	√		√			√						
Strait [29]	√						√	√				√	
Wang *et al.* [21, 22]		√			√		√	√		√		√	√
Lashkarian *et al.* [32]	√	√					√	√		√		√	
Acker *et al.* [27, 28]	√	√			√	√			√				
Boroujeny *et al.* [33]		√						√		√		√	√
Wang *et al.* [25]		√			√					√			√
MSSNR methods													
Melsa *et al.*[12]		√		√						√			√
Yin *et al.* [35]		√		√						√			√
Chiu *et al.* [37]	√	√		√					√				
Wang *et al.* [36]		√		√	√					√			√
Lu *et al.* [38]	√	√		√									
MGSNR methods													
Al-Dhahir *et al.* [15]		√	√†			√	√	√			√	√	√
Lashkarian *et al.* [42]	√	√	√†			√	√		√			√	√
Milisavljević *et al.* [43]	√	√	√†			√			√		√		

† Maximizes an approximate GSNR not the true channel capacity.

Table 2.1: Advantage/disadvantages of TEQ design methods mentioned in this chapter.