The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #1 *Version 4.0*

Date: Oct. 18, 2023   Course: EE 445S Evans

Name: _____**Lupin**_____
                          Last,   First

- **Exam duration**. The exam is scheduled to last 75 minutes.
- **Materials allowed**. You may use books, notes, your laptop/tablet, and a calculator.
- **Disable all networks**. Please disable all network connections on all computer systems. You may not access the Internet or other networks during the exam.
- **Electronics.** Power down phones. No headphones. Mute your computer systems.
- **Fully justify your answers**. When justifying your answers, reference your source and page number as well as quote the particular content in the source for your justification. You could reference homework solutions, test solutions, etc.
- **Matlab**. No question on the test requires you to write or interpret Matlab code. If you base an answer on Matlab code, then please provide the code as part of the justification.
- **Put all work on the test**. All work should be performed on the quiz itself. If more space is needed, then use the backs of the pages.
- **Academic integrity.** By submitting this exam, you affirm that you have not received help directly or indirectly on this test from another human except the proctor for the test, and that you did not provide help, directly or indirectly, to another student taking this exam.

	Problem	*Point Value*	*Your score*	*Topic*
Raoul	1	25		FIR Filter Analysis
Youssef Guedira	2	24		Decreasing the Sampling Rate
Assane Diop	3	27		System Identification
Benjamin Ferel	4	24		Mystery Systems
	Total	104		

Problem 1.1 *FIR Filter Analysis.* 25 points.

Consider a causal linear time-invariant (LTI) discrete-time finite impulse response (FIR) filter with input $x[n]$ and output $y[n]$ observed for $n \geq 0$. The transfer function in the z-domain is

$$H(z) = 1 - z^{-1}$$

(a) Give the equation for output $y[n]$ in terms of the input $x[n]$ in the discrete-time domain. *6 points.*

$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-1}$ which means $Y(z) = (1 - z^{-1}) X(z) = X(z) - z^{-1} X(z)$

Take the inverse z-transform of both sides to obtain $y[n] = x[n] - x[n-1]$ for $n \geq 0$.

(b) What are the initial condition(s) and their value(s)? Why? *6 points.*

Consider $n = 0$: $y[0] = x[0] - x[-1]$ which has one initial condition $x[-1]$.

Consider $n = 1$: $y[1] = x[1] - x[0]$ which does not have any initial conditions.

Our initial condition must equal zero as necessary conditions for LTI to hold: $x[-1] = 0$

(c) Derive a formula for the discrete-time frequency response of the filter. *3 points.*

We first determine the region of convergence for $H(z) = 1 - z^{-1}$ which is $z \neq 0$ because we need to avoid division by zero due to to z^{-1} term. Another way to see this is

$$H(z) = 1 - z^{-1} = \frac{z-1}{z}$$

Because the unit circle is in the region of convergence, we can substitute $z = e^{j\omega}$ into $H(z)$

$$H_{freq}(\omega) = H(z)]_{z=e^{j\omega}} = 1 - e^{-j\omega}$$

(d) Consider implementing a fourth-order version of this filter which would have the transfer function

$$H_4(z) = (1 - z^{-1})^4$$

Assume the input $x[n]$ and output $y[n]$ values are stored in 32-bit IEEE floating-point format. In terms of run-time implementation complexity, which of the following designs would you advocate using? Please fill out the table to justify your answer. *10 points.*

Filter Structure	# multiplications	# additions	# words of memory
Cascade of four first-order sections	0	4	12 or 9 (*see below*)
Cascade of two second-order sections	2	4	10 or 9 (*see below*)
Single fourth-order section	3	4	9

For a cascade of four first-order sections, each section would have input-output relationship $y[n] = x[n] - x[n-1]$ which requires 0 mults, 1 add, and 3 words of memory. If reuse the same memory location for first section output and second section input, etc., 9 total words.

For a cascade of two second-order sections, each second-order section would have an input-output relationship $y[n] = x[n] - 2\,x[n-1] + x[n-2]$ which requires 1 mults, 2 adds, and 5 words of memory to store the coefficient -2 as well as $x[n], x[n-1], x[n-2]$ and $y[n]$. We can obtain the second-order coefficients by conv([1 -1], [1 -1]) or expand $(1 - z^{-1})^2$. If reuse same memory location for first section output and second section input, 9 total words

A single fourth-order section would have the input-output relationship

$$y[n] = x[n] - 4\,x[n-1] + 6\,x[n-2] - 4\,x[n-3] + x[n-4]$$

which requires 3 mults, 4 adds, and 9 words of memory to store the coefficients -4, -6, -4 and input values $x[n], x[n-1], x[n-2], x[n-3]$, and $x[n-4]$ and output $y[n]$.

Reference for 1.1(d): H. G. Brachtendorf and C. Dalpiaz, "Near-optimal Multiplier-Less Broadband Noise Shaping Filters," *Journal Signal Processing Systems*, vol. 95, pp. 991–1002, 2023.

Problem 1.2 *Decreasing the Sampling Rate.* 24 points.

Downsampling by M can be used to decrease the sampling rate of the input signal by a factor of M.

A lowpass finite impulse response (FIR) filter can be placed before the downsampler to reduce the aliasing caused by resampling at the lower sampling rate.

On the right, discrete-time index m is associated with sampling rate f_s and discrete-time index n is associated with sampling rate $\frac{f_s}{M}$.
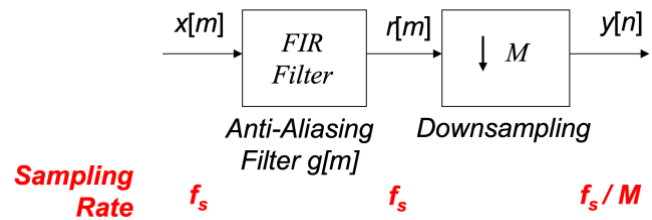
(a) What is the maximum continuous-time frequency f_{max} that is present in $y[n]$? What discrete-time frequency does f_{max} correspond to? *6 points.* **The sampling rate for $y[n]$ is f_s/M .**

Sampling Theorem says to choose the sampling rate $\frac{f_s}{M} > 2\,f_{max}$ so $f_{max} \to \frac{1}{2}\frac{f_s}{M}$.

Let $y(t) = \cos(2\pi f_{max} t)$ for $-\infty < t < \infty$ and sample it at sampling rate $\frac{f_s}{M}$:

$$y[n] = y(t)]_{t=\frac{n}{\frac{f_s}{M}}} = \cos\left(2\pi f_{max}\left(\frac{n}{\frac{f_s}{M}}\right)\right) = \cos\left(2\pi \frac{\frac{1}{2}\frac{f_s}{M}}{\frac{f_s}{M}}n\right) = \cos(\pi n) \text{ for } -\infty < n < \infty$$

The continuous-time frequency f_{max} corresponds to discrete-time frequency π rad/sample.

(b) What discrete-time frequency in $r[m]$ corresponds to the maximum continuous-time frequency f_{max} that is present in $y[n]$? *6 points.*

Since the sampling rate for $r[n]$ is f_s, continuous-time frequency f_{max} in $r[n]$ corresponds to

$$2\pi \frac{\frac{1}{2}\frac{f_s}{M}}{f_s} = \frac{\pi}{M} \text{ rad/sample}$$

(c) Any discrete-time frequencies present in $r[m]$ higher than your answer in part (b) but less than π rad/sample correspond to frequencies that will alias due to downsampling. Give the discrete-time passband frequency ω_{pass} and stopband frequency ω_{stop} you would use for the lowpass filter design. *6 points.* **With 10% rolloff from passband frequency ω_{pass} and stopband frequency ω_{stop}, we could use $\omega_{pass} = 0.95\frac{\pi}{M}$ and $\omega_{stop} = 1.05\frac{\pi}{M}$.**

(d) Here are two possible lowpass FIR filter with linear phase to meet the specifications of a discrete-time passband frequency ω_{pass} and stopband frequency ω_{stop}.

    i.   Averaging filter. How many FIR filter coefficients would you use? Why? What are the filter coefficient values? *3 points.* **Averaging FIR filter with N coefficients has null bandwidth of $\frac{2\pi}{N}$ rad/sample. With $\frac{2\pi}{N} = \frac{\pi}{M}$, we use $N = 2M$ coefficients each with a value of $\frac{1}{N}$.**

    ii.   Impulse response is a truncated sinc pulse. How would you choose the number of samples in the sinc pulse? *3 points.* **An infinite sinc pulse would be an ideal lowpass filter. For a discrete-time cutoff frequency of $\frac{\pi}{M}$, the sinc pulse would have zero crossings every M samples except the origin. To keep even symmetry, we would keep $2\,M\,k + 1$ samples where k is a positive integer. For causality, we'd delay the sinc by $M\,k$ samples. Larger k means better lowpass response but higher run-time complexity and group delay.**

Problem 1.3 *System Identification.* 27 points.

You are given several causal discrete-time linear time-invariant (LTI) systems with unknown impulse responses but you know the response of each system when the input is a unit step function $u[n]$ where

$$u[n] = \begin{cases} 1 & for\ n \geq 0 \\ 0 & otherwise \end{cases}$$

The z-transform of $u[n]$ is $\frac{1}{1-z^{-1}}$ for $|z| > 1$.

(a) When the input is $x[n] = u[n]$, the output $y[n] = \delta[n]$ where $\delta[n]$ is the discrete-time impulse.

$$\delta[n] = \begin{cases} 1 & for\ n = 0 \\ 0 & otherwise \end{cases}$$

Find the impulse response $h[n]$. *9 points.*

Time-domain approach. $y[n] = \delta[n] = u[n] - u[n-1]$. Since $x[n] = u[n]$, we can write $y[n] = x[n] - x[n-1]$ and hence $h[n] = \delta[n] - \delta[n-1]$.

Deconvolution approach. **Assume the LTI system is an FIR filter observed for $n \geq 0$:**
$$y[0] = h[0]\ x[n] + h[1]\ x[n-1] + h[2]\ x[n-2] + \cdots h[N-1]\ x[n-(N-1)]$$
All initial conditions are zero as a necessary condition for LTI properties to hold:
  $y[0] = h[0]\ x[0]$ so $1 = h[0]$ because $y[0] = 1$ and $x[0] = 1$ so $h[0] = 1$
  $y[1] = h[0]\ x[1] + h[1]\ x[0]$ which is $0 = h[0] + h[1]$ so $h[1] = -1$
  $y[2] = h[0]\ x[2] + h[1]\ x[1] + h[2]\ x[0]$ which is $0 = h[0] + h[1] + h[2]$ so $h[2] = 0$
We can check to see that $h[n] = \delta[n] - \delta[n-1]$ convolved with $u[n]$ is $\delta[n]$.

Z-domain approach. **An equalizer problem in disguise. We are trying to find an LTI system $h[n]$ so that $h[n] * u[n] = \delta[n]$. In the z-domain, $H(z)\ U(z) = 1$ which means that**
$$H(z) = \frac{1}{U(z)} = \frac{1}{\frac{1}{1-z^{-1}}} = 1 - z^{-1}\ \text{for } z \neq 0.\ \text{Inverse z-transform is } h[n] = \delta[n] - \delta[n-1].$$

(b) When the input is $x[n] = u[n]$, the output is $y[n] = n\ u[n]$. Find the impulse response $h[n]$. *9 points.* **We rewrite $y[n] = n\ u[n] = (n-1)\ u[n-1]$.**

Deconvolution approach. **Assume the LTI system is an FIR filter observed for $n \geq 0$:**
$$y[0] = h[0]\ x[n] + h[1]\ x[n-1] + h[2]\ x[n-2] + \cdots h[N-1]\ x[n-(N-1)]$$
All initial conditions are zero as a necessary condition for LTI properties to hold:
  $y[0] = h[0]\ x[0]$ so $0 = h[0]$ because $y[0] = 0$ and $x[0] = 1$ so $h[0] = 0$
  $y[1] = h[0]\ x[1] + h[1]\ x[0]$ which is $1 = h[0] + h[1]$ so $h[1] = 1$
  $y[2] = h[0]\ x[2] + h[1]\ x[1] + h[2]\ x[0]$ which is $2 = h[0] + h[1] + h[2]$ so $h[2] = 1$
  $y[3] = h[0]\ x[3] + h[1]\ x[2] + h[2]\ x[1] + h[3]\ x[0]$ which is
    $3 = h[0] + h[1] + h[2] + h[3]$ so $h[3] = 1$
We check to see that $h[n] = u[n-1]$ convolved with $u[n]$ is $n\ u[n]$. So, $h[n] = u[n-1]$.

Z-domain approach. **Rewrite $y[n] = n\ u[n] = (n-1)\ u[n-1]$. We're finding LTI system $h[n]$ so that $h[n] * u[n] = (n-1)\ u[n-1]$. In z-domain, $H(z)\ U(z) = \frac{z^{-1}}{(1-z^{-1})^2}$ for $|z| > 1$:**

$$H(z) = \frac{\frac{z^{-1}}{(1-z^{-1})^2}}{\frac{1}{1-z^{-1}}} = \frac{z^{-1}}{1-z^{-1}}\ \text{for } |z| > 1$$

Taking the inverse z-transform gives $h[n] = u[n-1]$.

(c) When the input is $x[n] = u[n]$, the output is $y[n]$ is a rectangular pulse of L samples in duration:

$$y[n] = \begin{cases} 1 & for\ 0 \le n \le L - 1 \\ 0 & otherwise \end{cases}$$

Find the impulse response $h[n]$. *9 points*.

Time-domain approach. $y[n] = \delta[n] = u[n] - u[n - L]$. Since $x[n] = u[n]$, we can write $y[n] = x[n] - x[n - L]$ and hence $h[n] = \delta[n] - \delta[n - L]$.

Deconvolution approach. **Assume the LTI system is an FIR filter observed for $n \ge 0$:**

$\quad y[0] = h[0]\,x[n] + h[1]\,x[n - 1] + h[2]\,x[n - 2] + \cdots h[N - 1]\,x[n - (N - 1)]$

All initial conditions are zero as a necessary condition for LTI properties to hold:

$\quad y[0] = h[0]\,x[0]$ so $1 = h[0]$ because $y[0] = 1$ and $x[0] = 1$ so $h[0] = 1$
$\quad y[1] = h[0]\,x[1] + h[1]\,x[0]$ which is $1 = h[0] + h[1]$ so $h[1] = 0$
$\quad y[2] = h[0]\,x[2] + h[1]\,x[1] + h[2]\,x[0]$ which is $1 = h[0] + h[1] + h[2]$ so $h[2] = 0$

If $h[n] = \delta[n]$, then $h[n] * u[n] \ne y[n]$. So, we keep computing $h[n]$ values.

$\quad y[L] = h[0]\,x[L] + h[1]\,x[L - 1] + \cdots + h[L]\,x[0]$ which is
$\quad 0 = h[0] + h[1] + \cdots + h[L]$ so $h[L] = 0$

We check to see that $h[n] = \delta[n] - \delta[n - L]$ convolved with $u[n]$ is $y[n]$.

Z-domain approach. **We're finding LTI system $h[n]$ so that $h[n] * u[n]$ is rectangular pulse of L samples in duration. In the z-domain, $H(z)\,U(z) = 1 + z^{-1} + \cdots + z^{-(L-1)}$ which means**

$$H(z) = \frac{1 + z^{-1} + \cdots + z^{-(L-1)}}{\dfrac{1}{1 - z^{-1}}} = (1 + z^{-1} + \cdots + z^{-(L-1)})(1 - z^{-1}) = 1 - z^{-L} \text{ for } z \ne 0$$

Taking the inverse z-transform gives $h[n] = \delta[n] - \delta[n - L]$. In Matlab, polynomial multiplication is computed using the conv command, e.g. conv([1 1 1 1 1 1 1], [1 -1]).

```
%% Deconvolution by Prof. Brian L. Evans.
%% Keep in mind the first element in a
%% MATLAB vector has index 1 and not 0.

%% USAGE
%% FIR filters convolve the input signal
%% and the FIR filter impulse response
%% (which is equal to the filter coeffs).
%% When input signal has finite length,
%% the output is finite length:
%%
%% LengthOfy = LengthOfx + NumCoeffs - 1
%%
%% Given finite-length signals x and y,
%% we can determine how many filter
%% coefficients b there are.
%%
%% If the input signal is infinite in
%% length, then the output could be
%% either infinite or finite in length.
%%
%% Define input and output signals. Give
%% an equal number of x and y values if
%% x is to be considered infinite length.
```

```
x = [ 1 1 1 1 1 1 ];   %% Midterm 1.3(a)
y = [ 1 0 0 0 0 0 ];

%% Determine Nmax based on input signal
%% Finite-length   length(y) - length(x) + 1
%% Infinite-length length(x)
if ( length(x) == length(y)  )
    Nmax = length(x);
else
    Nmax = length(y) - length(x) + 1;
end

b = zeros(1, Nmax);
b(1) = y(1) / x(1);
for k = 2:Nmax
    numer = y(k);
    n = k;
    for m = 1:(k-1)
        if (n >= 1)
            numer = numer - b(m) * x(n);
        end
        n = n - 1;
    end
    b(k) = numer / x(1);
end
```

Problem 1.4. *Mystery Systems.* 24 points.

You're trying to identify unknown discrete-time systems.

You input a discrete-time chirp signal $x[n]$ and look at the output to figure out what the system is.

The discrete-time chirp is formed by sampling a chirp signal that sweeps 0 to 8000 Hz over 0 to 5s

$$x(t) = \cos(2\pi f_1 t + 2\pi\mu t^2)$$

where $f_1 = 0$ Hz, $f_2 = 8000$ Hz, and $\mu = \frac{f_2 - f_1}{2\, t_{max}} = \frac{8000\text{ Hz}}{10\text{ s}} = 800$ Hz^2. Sampling rate f_s is 16000 Hz.

In each part below, identify the unknown system as one of the following **with justification**:
1. filter – give selectivity (lowpass, highpass, bandpass, bandstop) and passband/stopband frequencies
2. upsampler – give upsampling factor
3. downsampler – give downsampling factor
4. pointwise nonlinearity – give the integer exponent k to produce the output $y[n] = x^k[n]$

In the grayscale color map, white has highest magnitude value.

(a) Given spectrograms of the chirp input signal $x[n]$ (left) and output signal $y[n]$ (right). *12 pts.*
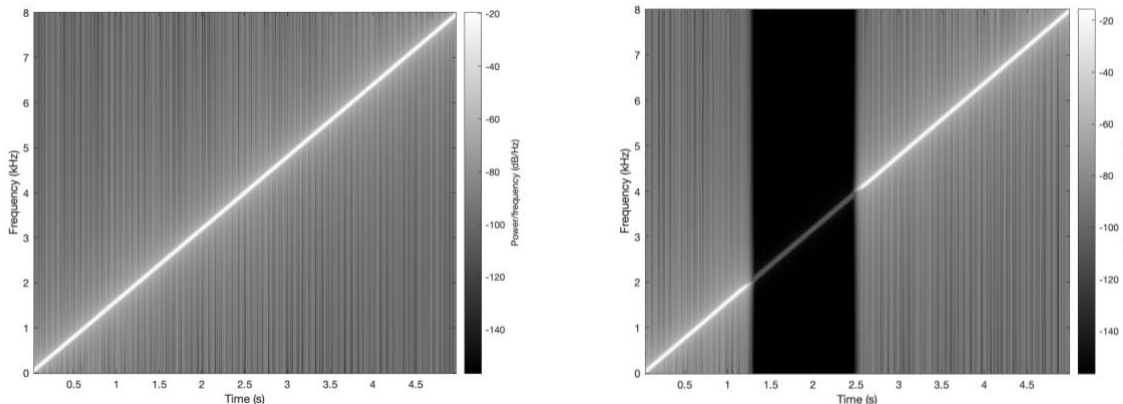
In the output spectrogram, principal frequencies between 2000 and 4000 Hz are severely attenuated and other principal frequencies are passed. No new frequencies are created, so it is likely an LTI filter. This is a bandstop filter similar to HW 2.3. See next page for the Matlab code.

(b) Given spectrograms of the chirp input signal $x[n]$ (left) and output signal $y[n]$ (right). *12 points.*
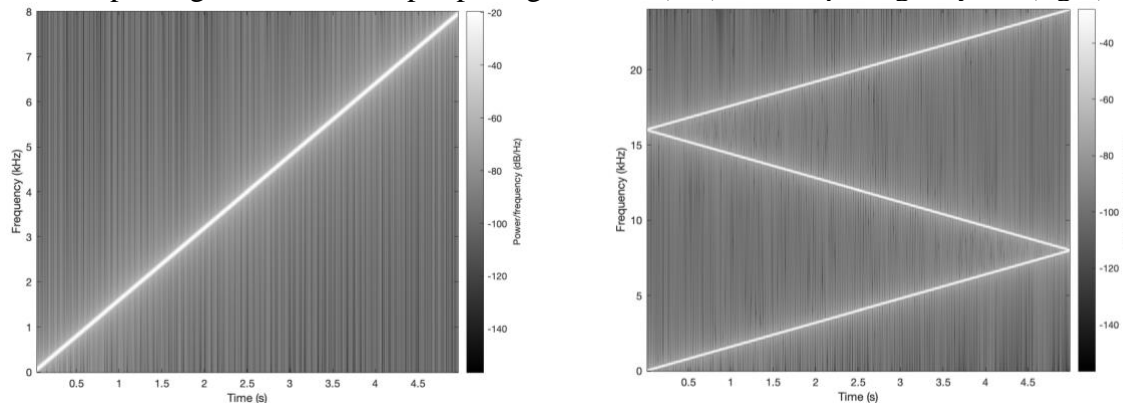
Output spectrogram has about 3x the frequency values on vertical axis over the same time interval vs. input spectrogram. Upsampling by 3 gives input chirp signal plus replicas every 16 kHz. Two replicas are visible -- a replica from 16 to 24 kHz and a replica from -32 to -40 kHz which aliases to 16 to 8 kHz due to a sampling rate of 48 kHz. Upsampling by 3 per HW 2.2(e). See next page.

```matlab
%% Midterm Problem 2.4(a) bandstop filter

fs = 16000;
Ts = 1 / fs;
tmax = 5;
t = 0 : Ts : tmax;

%% Create chirp signal
f1 = 0;
f2 = fs/2;
mu = (f2 - f1) / (2*tmax);
x = cos(2*pi*f1*t + 2*pi*mu*(t.^2));

%% Design lowpass filter
fnyquist = fs/2;
fstop1 = 1800;
fpass1 = 2000;
fpass2 = 4000;
fstop2 = 4200;
ctfrequencies = [0 fstop1 fpass1 fpass2 fstop2 fnyquist];
idealAmplitudes = [1 1 0 0 1 1];
pmfrequencies = ctfrequencies / fnyquist;
filterOrder = 400;
h = firpm( filterOrder, pmfrequencies, idealAmplitudes );
h = h / sum(h .^ 2);

y = conv(x, h);

%%% Spectrogram parameters
blockSize = 1024;
overlap = 1023;
figure;

%%% Plot spectrogram of input signal
figure;
spectrogram(x, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray;

%%% Plot spectrogram of output signal
figure;
spectrogram(y, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray;
```

```matlab
%% Midterm Problem 2.4(b) upsampling by 3


fs = 16000;
Ts = 1 / fs;
tmax = 5;
t = 0 : Ts : tmax;

%% Create chirp signal
f1 = 0;
f2 = fs/2;
mu = (f2 - f1) / (2*tmax);
x = cos(2*pi*f1*t + 2*pi*mu*(t.^2));

%% Upsample by 3
upsampleFactor = 3;
y = zeros(1, upsampleFactor*length(x));
y(1:3:end) = x;

%%% Spectrogram parameters
blockSize = 1024;
overlap = 1023;
figure;

%%% Plot spectrogram of input signal
figure;
spectrogram(x, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray;

%%% Plot spectrogram of output signal
figure;
spectrogram(y, blockSize, overlap, blockSize, upsampleFactor*fs, 'yaxis');
colormap gray;
```