The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #2 ***Solution Set 1.0***

Prof. Brian L. Evans

Date: May 8, 2019 Course: EE 445S

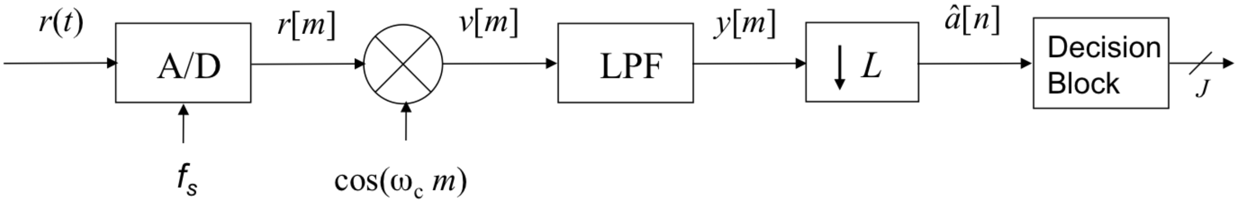
Name: ____
Last, First

- The exam is scheduled to last 75 minutes.
- Open books and open notes. You may refer to your homework assignments and the homework solution sets. You may not share materials with other students.
- Calculators are allowed.
- You may use any standalone computer system, i.e. one that is not connected to a network. **Disable all wireless access from your standalone computer system**.
- Please turn off all smart phones and other personal communication devices.
- Please remove headphones.
- All work should be performed on the quiz itself. If more space is needed, then use the backs of the pages.
- **Fully justify your answers unless instructed otherwise**. When justifying your answers, you may refer to the Johnson, Sethares & Klein (JSK) textbook, the Welch, Wright and Morrow (WWM) lab book, course reader, and course handouts. Please be sure to reference the page/slide number and quote the particular content in your justification.

Problem	Point Value	Your score	Topic
1	24		Bandpass PAM Receiver Tradeoffs
2	30		QAM Communication Performance
3	28		QAM Receiver Design
4	18		Bandpass PAM Receiver Decisions
Total	100		

Problem 2.1. *Bandpass Pulse Amplitude Modulation Receiver Tradeoffs*. *24 points*.

A bandpass pulse amplitude modulation (PAM) receiver is described as



where m is the sampling index and n is the symbol index, and has **system parameters**

$\hat{a}[n]$ received symbol amplitude	f_s sampling rate	f_{sym} symbol rate
$g[m]$ raised cosine pulse	J bits/symbol	L samples/symbol
M number of levels, i.e. $M = 2^J$	ω_c carrier frequency in rad/sample	

The only impairment being considered is additive thermal noise $w(t)$.

Hence, $r(t) = s(t) + w(t)$ where $s(t)$ is the transmitted bandpass PAM signal.

(a) For the additive thermal noise $w(t)$,

i. What is the probability distribution used to model the amplitude values of $w(t)$? *3 points*.
Gaussian distribution $N(0, \sigma^2)$. Each amplitude value is statistically independent. The mean is zero, and the variance σ^2 indicates the average noise power.

ii. What is the justification for using that probability distribution? *3 points*.

Thermal noise is due to the random motion of particles due to temperature. For electromagnetic waves, the particles are electrons. As temperature increases, the random motion is faster and the noise power is higher. Each electron received is statistically independent from the transmitted waveform and other electrons. As the number of electrons tends to infinity, the additive effect would converge to a Gaussian distribution due to the Central Limit Theorem (a.k.a. Law of Large Numbers).

(b) If an optimal matched filter is used for the LPF,

i. Which signal in the receiver is being optimized? *3 points*. **Estimated symbol ampl. $\hat{a}[n]$**

ii. By what measure is the signal in part (b)i optimal? *3 points*. **Signal-to-noise ratio (SNR)**

(c) Give formulas for communication signal quality measures below in terms of system parameters:

i. Bit rate. *3 points*. **$J f_{sym}$. Units: [bits/symbol] x [symbols/s] = [bits/s]**

ii. Probability of symbol error. *3 points*.

$$P_{error}^{PAM} = 2\,\frac{M-1}{M}\,Q\left(\frac{d}{\sigma}\sqrt{T_{sym}}\right) = 2\,\frac{M-1}{M}\,Q\left(\frac{d}{\sigma}\sqrt{L\,T_s}\right)$$

(d) Based on the formulas in (c), what's the impact on bit rate and probability of symbol error if

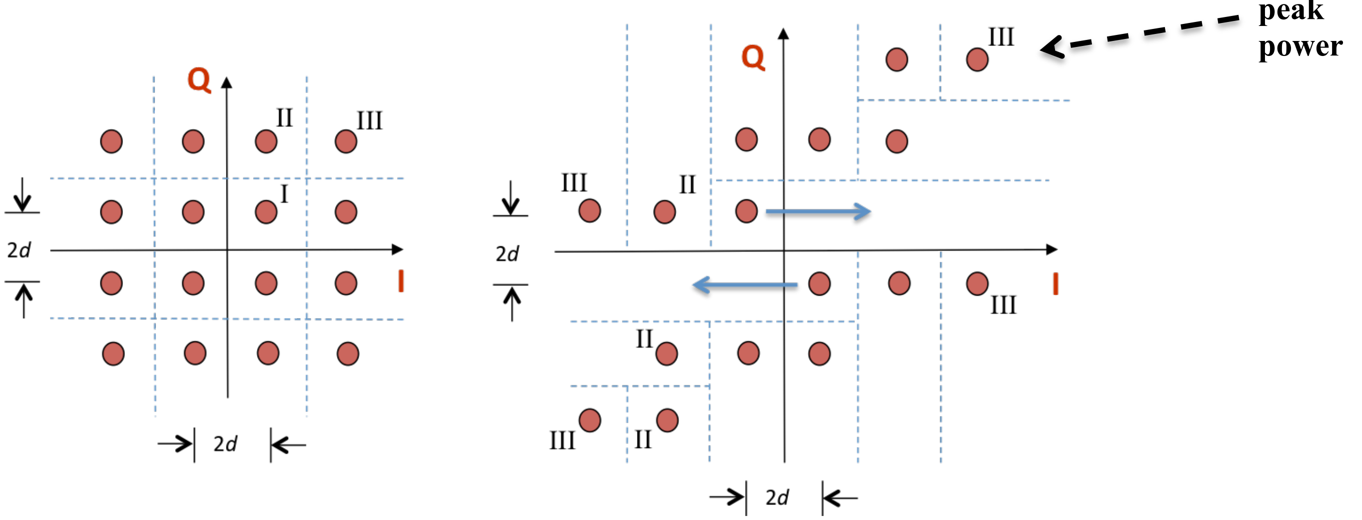
i. Transmit power is increased. *3 points*. **On slide 14-27, transmit power is $\frac{1}{3}(M^2-1)d^2$.**
(1) *M* increases, *d* is same. Increase in bit rate. Mild increase in symbol error prob.
(2) *M* is same, *d* increases. No effect on bit rate. Large decrease in symbol error prob.

ii. Number of samples/symbol, L, is increased. *3 points*. **When L increases, $f_{sym} = f_s / L$, decreases, and hence bit rate decreases. Large decrease in symbol error probability.**

Note: The right constellation is impractical. It consumes too much power and cannot be Gray coded. For lowest symbol error probability in mapping a received symbol amplitude to a symbol of bits, find the closest constellation point in Euclidean distance. For rectangular constellations, such as the one on the left, thresholding would give the same minimum symbol error results as Euclidean distance but leads to a fast divide-and-conquer algorithm using J comparisons for $M = 2^J$ levels (no multiplications).

Problem 2.2 *QAM Communication Performance. 30 points.*

Consider the two 16-QAM constellations below. Constellation spacing is $2d$.



Energy in the pulse shape is 1. Symbol time T_{sym} is 1s. The constellation on the left includes the decision regions with boundaries shown by the in-phase (I) axis, quadrature (Q) axis and dashed lines.

Each part below is worth 3 points. *Please fully justify your answers.*

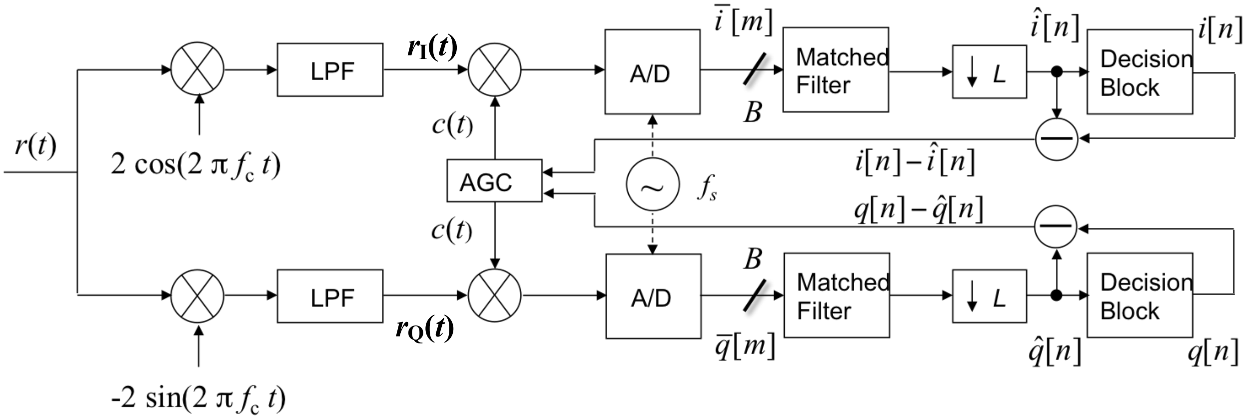
	Left Constellation	**Right Constellation**
(a) Peak transmit power	$18d^2$	$\mathbf{50d^2}$
(b) Average transmit power	$10d^2$	$\mathbf{20d^2}$
(c) Draw the type I, II and/or III decision regions for the right constellation on top of the right constellation *that will minimize the probability of symbol error using such decision regions.* **Avoid Type I regions because they have the highest probability of symbol error. We always have four Type III regions. So, maximize the number of Type II regions.**		
(d) Number of type I regions	4	**0**
(e) Number of type II regions	8	**12**
(f) Number of type III regions	4	**4**
(g) Probability of symbol error for additive Gaussian noise with zero mean & variance σ^2	$3Q\left(\frac{d}{\sigma}\right) - \frac{9}{4}Q^2\left(\frac{d}{\sigma}\right)$	$\frac{11}{4}Q\left(\frac{d}{\sigma}\right) - \frac{7}{4}Q^2\left(\frac{d}{\sigma}\right)$
(h) Express d/σ as a function of the Signal-to-Noise Ratio (SNR) in linear units	$\text{SNR} = \frac{10d^2}{\sigma^2}$ $\frac{d}{\sigma} = \sqrt{\frac{\text{SNR}}{10}}$	$\text{SNR} = \frac{20d^2}{\sigma^2}$ $\frac{d}{\sigma} = \sqrt{\frac{\text{SNR}}{20}}$

(i) In a 16-QAM receiver for the right constellation, an estimated symbol amplitude $-3d - j0.5d$. What is the decoded transmitted constellation point using

- Your constellation regions given above. *3 points.* **$-3d - j\,0.5d$ would map to $d - j\,d$.**
- Smallest Euclidean distance. *3 points.* **The closest constellation point to $-3d - j0.5d$ in Euclidean distance is $-3d + jd$.**

Problem 2.3. *Quadrature Amplitude Modulation (QAM) Receiver Design. 28 points.*

Some QAM receivers have a separate analog-to-digital (A/D) converter for the in-phase component and the quadrature component, as shown below.



System parameters: B bits at A/D output, $2d$ constellation spacing, f_s sampling rate, f_{sym} symbol rate, J bits/symbol, L samples/symbol, and M constellation points (i.e. $M = 2^J$). B is much greater than J.

Assume a rectangular, uniformly spaced, QAM constellation.

(a) If the signal-to-noise ratio (SNR) due to thermal noise in the system increases by 6 dB, and the system is matching the SNR due to thermal noise with the SNR due to quantization noise,

i. How many additional bits are possible for each A/D converter? *3 points.*

Each 6 dB gain in SNR yields one extra A/D bit: $\mathrm{SNR}_{dB} = C_0 + 6B$ for B bits.

ii. What is the overall dB/bit increase in the system? *4 points.*

2 bits have been added, i.e. 3 dB/bit. For QAM systems, $\mathrm{SNR}_{dB} = C_1 + 3B$ for B bits.

(b) What is the largest value of d that prevents clipping in the A/D converter? *3 points.*

In the QAM receiver above, each A/D converter converts a received baseband PAM signal.

Rectangular, uniformly spaced, QAM constellations have a PAM in-phase and quadrature constellations. In Problem 2.2, the left 16-QAM constellation has 4-PAM in-phase and 4-PAM quadrature constellations. 4-PAM symbol amplitudes are $-3d$, $-d$, d and $3d$. Due to raised cosine pulse shaping, the amplitude of the baseband PAM signal lies in $(-6d, 6d)$.

Each A/D converter produces a signed B-bit integer whose values are from -2^{B-1} to $2^{B-1} - 1$.

Set $-6d = -2^{B-1}$ and solve for d. Or, to make d an integer, set $-8d = -2^{B-1}$ and solve for d.

(c) Receiver supports up to 16-QAM. For a 4-QAM training signal, develop an adaptive automatic gain control (AGC) algorithm. Gain $c(t)$ will be applied to the in-phase and quadrature channels. The gain sampled at the symbol time, $c[n] = c(n\,T_{sym})$, will be adapted every symbol period.

For the 16-QAM receiver, signals $i[n]$ and $q[n]$ represent transmitted symbol amplitudes from a 4-PAM constellation. Their possible values are $-3d$, $-d$, d and $3d$. Their values are independent of the gain $c(t)$. Use the value of d computed in part (b).

When using a pulse shape with zero crossings at multiples of the symbol period other than the current symbol, e.g. a raised cosine pulse shape, and without any symbol timing error, $\hat{\imath}[n] = \bar{\imath}[Ln] = Q[\,r_I(n\,T_{sym})\,c(n\,T_{sym})\,]$ and $\hat{q}[n] = \bar{q}[Ln] = Q[\,r_Q(n\,T_{sym})\,c(n\,T_{sym})\,]$. We'll also assume $Q[x] \approx x$ to be able to compute derivatives.

See the explanation in part (c) above, which sets the stage for the following solutions.

i. Give an objective function $J(n)$. *6 points*.

Solution #1: $J(n) = \frac{1}{2}(i[n] - \hat{\imath}[n])^2 + \frac{1}{2}(q[n] - \hat{q}[n])^2$

Solution #2: $J(n) = \frac{1}{2}e^2[n]$ **where** $e[n] = (i[n] - \hat{\imath}[n]) + (q[n] - \hat{q}[n])$

***Drawback*: positive in-phase error can be offset by negative quadrature error, and vice-versa. Solution #1 is better. Both are decision-directed.**

Solution #3: During 4-QAM training, each A/D converter will see a 2-PAM baseband signal. We can modify the "naïve" AGC algorithm from Johnson, Sethares & Klein, e.g. *J*(*n*) in (6.15) and update in (6.16) on page 123. This "naïve" AGC performed the best of the three AGC algorithms evaluated in homework 7.3 w/r to additive noise.

$$J(n) = \text{Average}\left\{|c[n]|\left(\left(\frac{\hat{\imath}^2[n]}{3} - d^2\right) + \left(\frac{\hat{q}^2[n]}{3} - d^2\right)\right)\right\}$$

ii. Derive an update equation for gain $c[n]$. Compute all derivatives. Simplify result. *9 points*

Solution #1: Minimize the objective function:

$$c[n+1] = c[n] - \mu\frac{dJ(n)}{dc[n]} = c[n] - \mu\left((i[n] - \hat{\imath}[n])\left(-\frac{d\hat{\imath}[n]}{dc[n]}\right) + (q[n] - \hat{q}[n])\left(-\frac{d\hat{q}[n]}{dc[n]}\right)\right)$$

$$c[n+1] = c[n] - \mu\left((\hat{\imath}[n] - i[n])\, r_I[n] + (\hat{q}[n] - q[n])\, r_Q[n] \right)$$

Since we do not know either $r_I[n] = r_I(n\,T_{sym})$ **or** $r_Q[n] = r_Q(n\,T_{sym})$**, we'll roll them into the step size parameter μ. That is, we'll treat** $r_I[n] = r_Q[n] =$ **constant.**

$$c[n+1] = c[n] - \mu\left((\hat{\imath}[n] - i[n]) + (\hat{q}[n] - q[n]) \right)$$

Solution #2: Minimize the objective function:

$$c[n+1] = c[n] - \mu\frac{dJ(n)}{dc[n]} = c[n] - \mu\, e[n]\,\frac{de[n]}{dc[n]}$$

$$c[n+1] = c[n] - \mu\, e[n]\left(-r_I[n] - r_Q[n]\right)$$

Since we do not know either $r_I[n] = r_I(n\,T_{sym})$ **or** $r_Q[n] = r_Q(n\,T_{sym})$**, we'll roll them into the step size parameter μ. That is, we'll treat** $r_I[n] = r_Q[n] =$ **constant.**

$$c[n+1] = c[n] - \mu\,(-e[n]) = c[n] - \mu\left((\hat{\imath}[n] - i[n]) + (\hat{q}[n] - q[n]) \right)$$

After the approximations in solutions #1 and #2, they've become the same solution.

Solution #3: We modify (6.16) on page 123 in JSK:

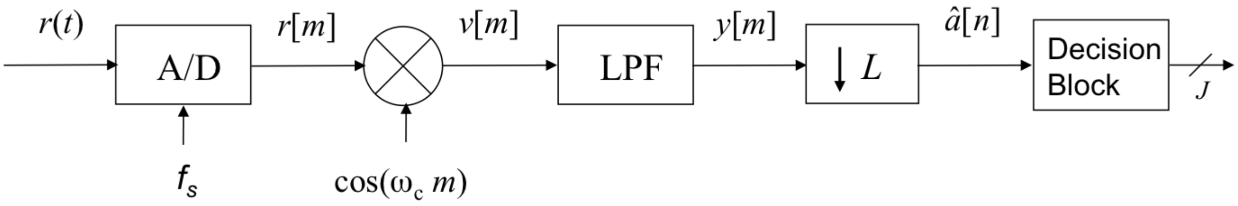
$$c[n+1] = c[n] - \mu\frac{dJ(n)}{dc[n]} = c[n] - \mu\,\text{Average}\left\{\text{sign}(c[n])\left((\hat{\imath}^2[n] - d^2) + (\hat{q}^2[n] - d^2)\right)\right\}$$

iii. What range of values would you recommend for the step size μ? Why? *3 points*.

Use a small positive value for the step size μ, such as 0.01 or 0.001, for convergence of the steepest descent algorithm. A step size of zero will prevent any updates. A negative step size and a large positive step size will cause divergence.

Problem 2.4. *Bandpass Pulse Amplitude Modulation Receiver Decisions*. *18 points*

A bandpass pulse amplitude modulation (PAM) receiver is described as



where m is the sampling index and n is the symbol index, and has system parameters

$\hat{a}[n]$ received symbol amplitude	f_s sampling rate	f_{sym} symbol rate
$g[m]$ raised cosine pulse	J bits/symbol	L samples/symbol
M number of levels, i.e. $M = 2^J$	ω_c carrier frequency in rad/sample	

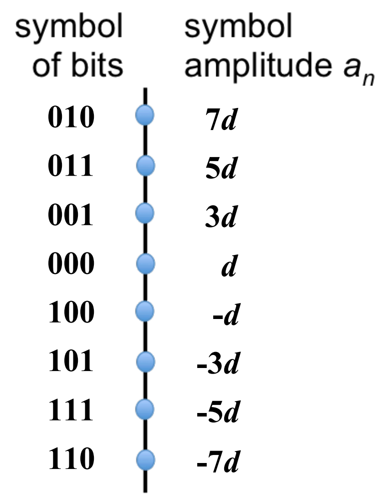
The only impairment being considered is additive thermal noise $w(t)$.

Hence, $r(t) = s(t) + w(t)$ where $s(t)$ is the transmitted bandpass PAM signal.

(a) Consider an 8-PAM bandpass transmitter.

i. Draw an 8-PAM constellation map with Gray coding on the right. *6 points*

Gray code means that the symbols of bits between two adjacent constellation points only differ by one bit. The idea is to minimize the number of bit errors when there is a symbol error so that the receiver can use error correcting codes to try to correct for the flipped bit.



ii. Explain how you would build a lookup table for the constellation map in part (a)i. *3 points*

Use the symbol of bits as an unsigned integer index into a lookup table (array) of symbol amplitudes with entries { *d*, 3*d*, 7*d*, 5*d*, -*d*, -3*d*, -7*d*, -5*d* }.

(b) Consider an M-PAM bandpass receiver. The decision block quantizes the estimated symbol amplitude $\hat{a}[n]$ for M-PAM into a symbol of bits. Give formulas for the computational complexity as a function of M for each decision block quantization algorithm below. *9 points.*

i. Compare $\hat{a}[n]$ against each constellation point in the transmitter constellation map.

M calculations, $O(M)$

ii. Divide-and-conquer to discard half of the candidate constellation points each comparison.

$\log_2 M$ calculations, $O(\log_2 M)$

iii. Determine the index of the closest constellation point using round $\left(\frac{\hat{a}[n]-d}{2d}\right)$, limit the unsigned index to a value between 0 and M-1 inclusive, and then use the unsigned index to find the symbol of bits in the lookup table for the constellation map.

Floating-point subtraction, division, and round operations. Constant time. $O(1)$.