The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #2 ***Solutions 2.0***

Date: May 4, 2022 Course: EE 445S Evans

Name: **Breakfast**, **Japanese** (***Band***)
Last, First

- **Exam duration**. The exam is scheduled to last 75 minutes.
- **Materials allowed**. You may use books, notes, your laptop/tablet, and a calculator.
- **Disable all networks**. Please disable all network connections on all computer systems. You may not access the Internet or other networks during the exam.
- **Electronics.** Power down phones. No headphones. Mute your computer systems.
- **Fully justify your answers**. When justifying your answers, reference your source and page number as well as quote the particular content in the source for your justification. You could reference homework solutions, test solutions, etc.
- **Matlab**. No question on the test requires you to write or interpret Matlab code. If you base an answer on Matlab code, then please provide the code as part of the justification.
- **Put all work on the test**. All work should be performed on the quiz itself. If more space is needed, then use the backs of the pages.
- **Academic integrity.** By submitting this exam, you affirm that you have not received help directly or indirectly on this test from another human except your instructor, Prof. Evans, and that you did not provide help, directly or indirectly, to another student taking this exam.

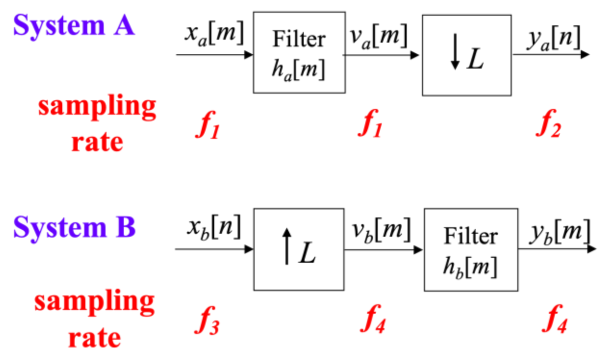
	Problem	**Point Value**	**Your score**	**Topic**
Peter Bradley	1	19		Changing Sampling Rates
Michelle Zauner	2	33		QAM Communication Performance
Craig Hendrix	3	24		Adaptive Spatial Filter
Deven Craige	4	24		Potpourri
	Total	100		

Prologue: *Lectures 13 & 16; JSK Sec. 2.10; JSK Ch. 8 & 11; Lab 5; HW 2.2, 4.2, 4.3, 5.2, 6.1 & 6.2; Midterm 2.1 problems: F17, Sp18, F18, Sp21 & F21*

Problem 2.1. *Changing Sampling Rates. 19 points.*

Consider the two systems to change the sampling rate:

- System A consists of linear time-invariant (LTI) filtering followed by downsampling by L.
- System B consists of upsampling by L followed by LTI filtering.



(a) Give a formula for f_2 in terms of f_1. *2 points.*

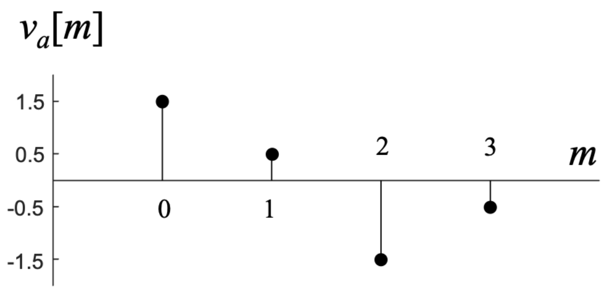
$$\boldsymbol{f_2 = \frac{1}{L} f_1}$$

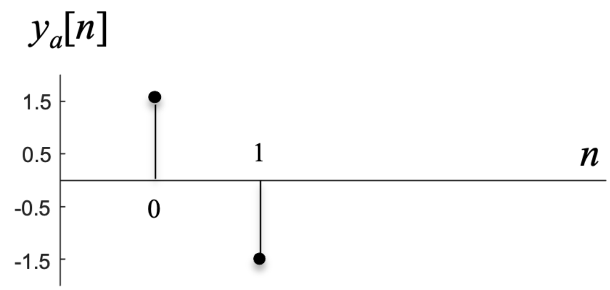
(b) Give a formula for f_4 in terms of f_3. *2 points.*

$$\boldsymbol{f_4 = L\, f_3}$$

(c) Assuming $L = 2$, draw $y_a[n]$ corresponding to the input $v_a[m]$ shown below. *4 points.*

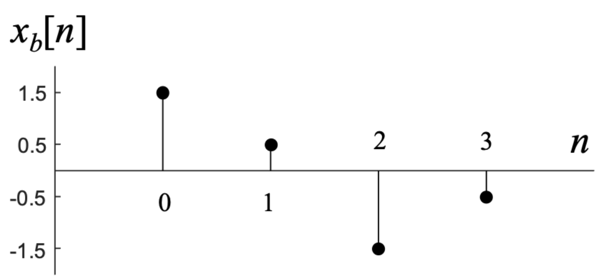
Lecture Slide 26-9; HW 2.2(d)

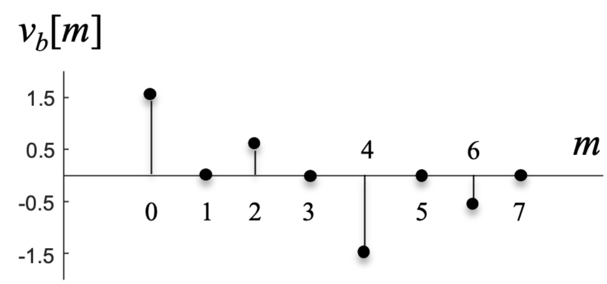




(d) Assuming that $L = 2$, draw $v_b[m]$ corresponding to the input $x_b[n]$ shown below. *4 points.*

Lecture Slides 13-8 and 26-7; HW 2.2(e)





(e) Assume the filter in System B is a finite impulse response (FIR) filter with N coefficients.

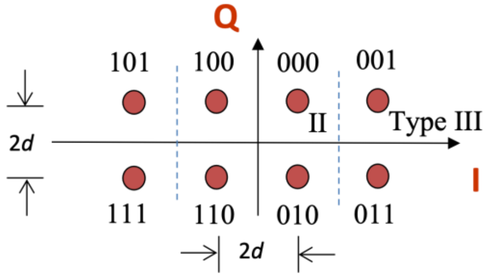
1. How many multiplication operations per second does System B use in the block diagram above? *3 points.* **Upsampling does not require any multiplications. An FIR filter with N coefficients requires N multiplications to compute one output sample in response to an input sample. FIR filter processes f_4 samples per second. Total: $N\, f_4$ mults/sec.**
2. How many multiplication operations per second would System B use if implemented as a polyphase filter bank? *4 points.* **A polyphase filterbank uses a bank of L polyphase FIR filters, each with N/L coefficients, followed by a commutator to take the L parallel outputs and put them in sequential order. Each polyphase FIR filter runs at the lower rate, f_3. Total: $N\, f_3$ mults/sec. Savings in mults/sec by a factor of L because $f_4 = L\, f_3$.**

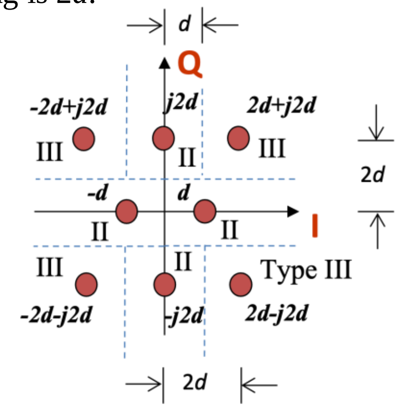
Lecture Slides 13-9, 13-14 to 13-16, and 26-8; HW 5.3

***Prologue*: Lectures 13-16; JSK Ch. 16; Lab 5 & 6; HW 4.2, 4.3, 5.2, 6.3 & 7.3; Handout P PAM vs. QAM; Midterm 2.2 problems: F14, Sp15, F15, Sp16, F16, Sp17, F17, Sp18, F18, Sp19, F19, Sp20, F20, Sp21 & F21**

Problem 2.2 *QAM Communication Performance. 33 points.*

Consider the two 8-QAM constellations below. Constellation spacing is $2d$.





Energy in the pulse shape is 1. Symbol time T_{sym} is 1s.

Each part below is worth 3 points. **Please fully justify your answers.**

	Left Constellation	**Right Constellation**
(a) Peak transmit power	$10d^2$	$\mathbf{8d^2}$
(b) Average transmit power	$6d^2$	$\frac{4(8d^2)+2(4d^2)+2(d^2)}{8} = 5.25d^2$
(c) Peak-to-average power ratio	$\frac{10d^2}{6d^2} = \frac{5}{3} \approx 1.67$	$\frac{8d^2}{5.25d^2} = \frac{32}{21} \approx 1.52$
(d) Draw the type I, II and/or III decision regions for the right constellation on top of the right constellation ***that will minimize the probability of symbol error using such decision regions.***		
(e) Number of type I QAM regions	0	**0**
(f) Number of type II QAM regions	4	**4**
(g) Number of type III QAM regions	4	**4**
(h) Probability of symbol error for additive Gaussian noise with zero mean & variance σ^2.	$P_e = \frac{5}{2} Q\left(\frac{d}{\sigma}\right) - \frac{3}{2} Q^2\left(\frac{d}{\sigma}\right)$	**Same as left constellation due to same number of type I, II and III regions**
(i) Express the argument of the Q function as a function of the Signal-to-Noise Ratio (SNR) in linear units	$\text{SNR} = \frac{6d^2}{\sigma^2}$ $\frac{d}{\sigma} = \sqrt{\frac{\text{SNR}}{6}}$	$\text{SNR} = \frac{5.25d^2}{\sigma^2}$ $\frac{d}{\sigma} = \sqrt{\frac{\text{SNR}}{5.25}}$

(j) Give one advantage of the left constellation vs. the right constellation. *3 points.*

- ***Gray coding* minimizes the number of bit errors when there is a symbol error. The bit pattern for each symbol only differs in one bit with that of the nearest neighbor. Left constellation can be gray coded as shown above; right constellation cannot be gray coded because we only have two bits of freedom at each constellation point and the constellation points at locations d and -d have four nearest neighbors.**
- ***Binary search* fast algorithm can be used for decoding the received symbol amplitude by finding the nearest constellation point. Each iteration rules out half the points with a single**

real-valued comparison. Binary search achieves same accuracy as selecting the constellation point that is closest in Euclidean distance to the received symbol amplitude. For the right constellation, there is no vertical or horizontal line through the complex plane that will eliminate half the points; however, decoding the received symbol constellation using the decision regions will be as accurate as using Euclidean distance.

- ***Rectangular constellation* has 4-PAM (QPSK) in the in-phase component and 2-PAM (BPSK) in the quadrature component. The QAM receiver separates into a 4-PAM receiver in the in-phase direction and 2-PAM receiver in the quadrature direction, which can simplify the design. For example, one could apply a single Costas Loop to only the in-phase component to determine the phase offset for the baseband QAM carrier, instead of having to run eight Costas loops in parallel per homework problem 7.3. The right constellation is not a rectangular constellation.**

(k) Give one advantage of the right constellation vs. the left constellation. *3 points.*

- ***Lower peak transmit power* for same value of d**
- ***Lower average transmit power* for same value of d**
- ***Lower average peak-to-average power ratio* which makes it easier to design the power amplifier for the analog/RF chains in the transmitter and receiver**
- ***Lower symbol error rate vs. SNR*. The Q function is a non-negative monotonically decreasing function of its non-negative argument. The probability of error expressions are identical in terms of $\frac{d}{\sigma}$. Let SNR = 6 in linear units. The value of $\frac{d}{\sigma}$ is 1 for the left constellation and 1.2 for the right constellation, which means the right constellation has lower symbol error rate.**

Prologue: Lectures 12 & 13; JSK Sec. 2.10 & 2.12; JSK Sec. 3.7; JSK Ch. 8 & 11; Labs 5 & 6; HW 4.2, 4.3, 5.1, 5.2, 6.1, 6.2, 7.1, 7.2 & 7.3

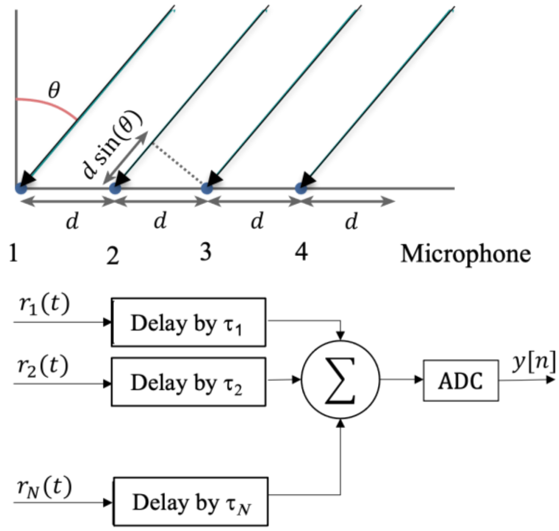
Problem 2.3 *Adaptive Spatial Filter. 24 points.*

A single sound source is recorded by several microphones simultaneously as shown on the right.

The microphones are arranged in a line and separated by distance d. Sound arrives at an unknown angle θ.

Sound arrives at the ith microphone with a different delay t_i based on the distance to the source. The source is far enough away for the propagation to be a plane wave.

The signal recorded by the ith microphone $r_i(t)$ is delayed by τ seconds, and all the signals are added together before being sampled by an analog-to-digital converter:



We would like to design an adaptive spatial filter that amplifies a signal located at an unknown angle θ by adapting the delays $\tau_1, \tau_2, \ldots, \tau_N$.

A known training signal $x[n]$ is sent by the source so that we can find the best values for $\tau_1, \tau_2, \ldots, \tau_N$.

Hence $r_i(t) = x(t - t_i)$ where $x(t)$ is the continuous-time version of the training signal $x[n]$.

(a) **Training**. What training signal $x[n]$ would you send? Why? Describe its parameters. *6 points*.

We want $x[n]$ to be easy to generate at the receiver, have good correlation properties and contain all discrete-time frequencies because the channel will attenuate/reject some frequencies. After digital-to-analog conversion (DAC) of $x[n]$ at rate f_s, continuous-time signal $x(t)$ would have all frequencies $-½ f_s$ to $½ f_s$.

Option #1: **Long maximal-length pseudo-noise sequence. Length is $2^r - 1$ bits where r is the number of states in the PN generator. Map 1 bit to +1 and 0 bit to -1 in amplitude. As an audio signal, a PN sequence sounds like noise.**

Option #2: **Chirp signal that sweeps from 0 Hz to 20 kHz (upper end of audible range) over 0 to t_{max} seconds. $x(t) = \cos(2\pi \gamma t^2)$ where $\gamma = \frac{10 \text{ kHz}}{t_{max}}$. For the DAC, use $f_s > 44$ kHz.**

Since chirp sounds can be startling, we could use low-volume chirps, or chirps in 15-20 kHz range, where human hearing is less sensitive.

(b) **Propagation Delay**. Consider the signal received by the ith microphone $r_i(t)$. Propose and explain an algorithm to find the delay t_i from the source to the ith microphone. *6 points*.

Correlate output of ith microphone $r_i(t)$ against the training sequence, and the location of the first peak will provide an estimate of the delay t_i from the source to the ith microphone.

This method works because the correlation function provides a measure of similarity between $r_i(t)$ and delayed versions of $x(t)$. We had seen this in HW 4.2, 4.3, and 5.2; lab #4 on PN sequences; and labs 5-6 on matched (correlation) filtering.

With the estimates of the t_i values, we can determine the best τ_i values as follows without the need for an adaptive algorithm:

$$t_{max} = \max([\, t_1 \quad t_2 \quad \cdots \quad t_N \,])$$

$$\tau_i = t_{max} - t_i \quad \text{for } i = 1, \ldots, N$$

This will allow all the versions of the received training signal to be aligned in time and add constructively. This assumes the sound source is not moving. One of the advantages of an adaptive algorithm is to be able to track a moving sound source. We can use the above calculation as the initial values for τ_i for the adaptive algorithm.

(c) **Adaptive Spatial Filter**. Develop a discrete-time adaptive algorithm to apply to $y[n]$ to determine the best set of delays

$$\vec{\tau} = [\,\tau_1 \quad \tau_2 \quad \cdots \quad \tau_N]$$

for the microphone array to amplify the sound coming from the source at an unknown angle θ.

1. Give an objective function and explain why you have chosen it. *3 points.*

   $J(\,e[n]\,) = \frac{1}{2}\,e^2[n]$ where $e[n] = y[n] - x[n - n_0]$ and n_0 is a constant average delay. We can choose $n_0 = 0$ for simplicity. Driving $e^2[n]$ to zero will drive $e[n]$ to zero.

2. Give an adaptive steepest descent/ascent algorithm for $\vec{\tau}[i+1]$ in terms of $\vec{\tau}[i]$. *6 points.*

   The adaptive steepest descent algorithm to minimize $J(\,e[n]\,)$ is

$$\vec{\tau}[n+1] = \vec{\tau}[n] - \mu \frac{d}{d\vec{\tau}} J(\,e[n]\,)\Big|_{\vec{\tau}=\vec{\tau}[n]} = \vec{\tau}[n] - \mu\, e[n]\, \frac{d}{d\vec{\tau}}\, y[n]\Big|_{\vec{\tau}=\vec{\tau}[n]}$$

   since $x[n]$ does not depend on $\vec{\tau}$. This would be a sufficient answer on the exam.

   To keep working the problem,

$$y(t) = \sum_{i=1}^{N} r_i(t - \tau_i) = \sum_{i=1}^{N} x(t - t_i - \tau_i)$$

$$y[n] = y(n\,T_s) = \sum_{i=1}^{N} x(n\,T_s - t_i - \tau_i)$$

$$\frac{d}{d\tau_i}\, y[n] = \frac{d}{d\tau_i} x(n\,T_s - t_i - \tau_i)$$

   At this point, we have not included knowledge of the microphone array geometry or the training signal. If a chirp training signal is being used, then $x(t) = \cos(2\pi\,\gamma\, t^2)$ and

$$\frac{d}{d\tau_i} x(n\,T_s - t_i - \tau_i) = \frac{d}{d\tau_i} \cos(2\pi\,\gamma\,(n\,T_s - t_i - \tau_i)^2)$$

$$\frac{d}{d\tau_i} x(n\,T_s - t_i - \tau_i) = 4\pi\,\gamma\,(n\,T_s - t_i - \tau_i)\,\sin(2\pi\,\gamma\,(n\,T_s - t_i - \tau_i)^2)$$

3. What values of the step size would you use? Why? *3 points.*

   We would like to small positive values of the step size μ such as 0.01 to ensure convergence of the algorithm. Using $\mu = 0$ would not allow the iterative algorithm to update. Using a negative μ would convert the steepest descent algorithm to minimize $e[n]$ into a steepest ascent algorithm to maximize $e[n]$. A large positive value would cause the steepest descent algorithm to diverge.

Problem 2.4. *Potpourri*. *24 points*.

(a) Consider 16-QAM system transmitting at a 1200 bps (bits per second) using a 12-sample discrete-time raised cosine pulse shaping filter. What are the possible sampling rates in Hz? *12 points*.

Bit rate is $J\, f_{sym}$ where J is number of bits/symbol and f_{sym} is symbol rate in symbols/s or Hz. With $J = 4$ bits/symbol due to 16-level QAM and a bit rate of 1200 bps, $f_{sym} = 300$ Hz.

Pulse shape has $N = N_g\, L$ samples where N_g is number of symbol periods in the pulse shape and L is number of samples in a symbol period. With $N = 12$ given in the question, possible factorizations are $L = 1$ and $N_g = 12$; $L = 2$ and $N_g = 6$; $L = 3$ and $N_g = 4$; $L = 4$ and $N_g = 3$; $L = 6$ and $N_g = 2$; $L = 12$ and $N_g = 1$.

With the sampling rate $f_s = L\, f_{sym}$, the possible sampling rates are 300, 600, 900, 1200, 1800, and 3600 Hz.

Lecture slide 13-3 for bit rate; slides 13-9 to 13-16 for pulse shape length; slide 13-7 for sampling rate; Labs 5 & 6

(b) Consider a wireless communication system that uses two transmit antennas and two receive antennas. This allows two signals $x_1[n]$ and $x_2[n]$ to be sent at the same time and over the same frequency band as shown below: *12 points*

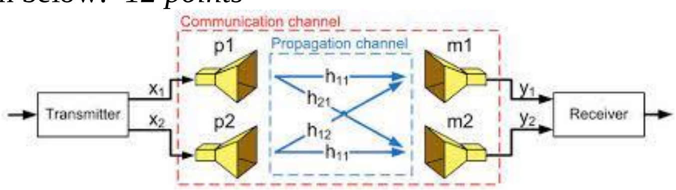


Figure from Lars Reichardt, Juan Pontes, Yoke Leen Sit, and Thomas Zwick, "Antenna Optimization for Time-Variant MIMO Systems", EuCap, 2011.

Each antenna at the receiver receives both transmitted signals.

The communication channel has a **complex-valued** scalar gain between the *i*th transmit antenna and *j*th receive antenna. No other impairments are being modeled.

The received signal is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which can be written as

$$\vec{y} = \boldsymbol{H}\, \vec{x}$$

Lecture 12; JSK Sec. 2.1, 2.10 & 2.12; JSK Ch. 8 & 11; Labs 5 & 6; HW 4.2, 4.3, 5.2, 5.3, 6.1, 6.2, 7.1, 7.2 & 7.3

In the receiver, assume $\boldsymbol{H}$ is known.

Find **two possible values** for matrix $\boldsymbol{G}$ in terms of $\boldsymbol{H}$ that can allow us to estimate $\vec{x}$ from $\vec{y}$ via

$$\vec{x}_{estimated} = \boldsymbol{G}\, \vec{y}$$

Hints: One way could equalize (invert) the channel. Other could use a matched filtering approach.

We want to find $\vec{x}_{estimated} = \boldsymbol{G}\, \vec{y} = \boldsymbol{G}\, \boldsymbol{H}\, \vec{x}$ so that we can recover $\vec{x}$ from $\vec{x}_{estimated}$.

We can equalize (invert) the channel by using $\boldsymbol{G} = \boldsymbol{H}^{-1}$ which would give $\vec{x}_{estimated} = \boldsymbol{G}\, \boldsymbol{H}\, \vec{x} = \boldsymbol{H}^{-1}\, \boldsymbol{H}\, \vec{x} = \vec{x}$. The closed-form formula for $\boldsymbol{H}^{-1}$ is

$$\boldsymbol{H}^{-1} = \frac{1}{det(\boldsymbol{H})} \begin{bmatrix} h_{22} & -h_{12} \\ -h_{21} & h_{11} \end{bmatrix}$$

In general, $\boldsymbol{G} = \alpha\, \boldsymbol{H}^{-1}$ for any non-zero scalar α.

The idea of a matched filter is phase reversal. For example, in pulse amplitude modulation, the matched filter impulse response is $h[m] = k\, g^*[L - m]$ where $g[m]$ is the pulse shape. For this problem, we let $\boldsymbol{G} = \boldsymbol{H}^*$ where $\boldsymbol{H}^*$ means the conjugate transpose of $\boldsymbol{H}^*$:

$$\boldsymbol{H}^* = \begin{bmatrix} h_{11}^* & h_{21}^* \\ h_{12}^* & h_{22}^* \end{bmatrix} \text{ so that } \boldsymbol{H}^*\, \boldsymbol{H} = \begin{bmatrix} h_{11}^2 + h_{21}^2 & h_{11}^* h_{12} + h_{21}^* h_{22} \\ h_{12}^* h_{11} + h_{22}^* h_{21} & h_{12}^2 + h_{22}^2 \end{bmatrix}$$