

EE 3455 Real-Time Digital Signal Processing Lab

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All-Pass Filters

- An all-pass filter has a magnitude response that is constant for all frequencies. The phase response may or may not be linear.
- A simple all-pass filter is the gain, i.e. $y(t) = g x(t)$ or $y[n] = g x[n]$ where x is the input and y is the output. Impulse response is $h(t) = g \delta(t)$ or $h[n] = g \delta[n]$. The frequency response is simply equal to g .
- Another simple all-pass filter is the ideal delay, i.e. $y(t) = x(t - t_0)$ or $y[n] = x[n - n_0]$ where t_0 and n_0 are constants. Impulse response is $h(t) = \delta(t - t_0)$ or $h[n] = \delta[n - n_0]$. The frequency response is $H_{\text{freq}}(f) = e^{-j2\pi f t_0}$ or $H_{\text{freq}}(\omega) = e^{-j\omega n_0}$. The magnitude response is equal to one in either case. Phase response is linear.
- A cascade of a gain and an ideal delay also has an all-pass response.
- A first-order IIR filter with one real-valued pole and one real-valued zero is all-pass if the zero location is equal to the reciprocal of the pole location:

$$H(z) = \frac{z - \frac{1}{r}}{z - r} \longrightarrow H_{\text{freq}}(\omega) = \frac{e^{j\omega} - \frac{1}{r}}{e^{j\omega} - r}$$

assuming that $|r| < 1$ for asymptotic stability. Magnitude response is

$$|H_{\text{freq}}(\omega)| = \left| \frac{e^{j\omega} - \frac{1}{r}}{e^{j\omega} - r} \right| = \frac{|e^{j\omega} - \frac{1}{r}|}{|e^{j\omega} - r|}$$

$$\begin{aligned} \text{Here, } |e^{j\omega} - a| &= |(\cos \omega - a) + j \sin \omega| = \sqrt{(\cos \omega - a)^2 + \sin^2 \omega} \\ &= \sqrt{\cos^2 \omega - 2a \cos \omega + a^2 + \sin^2 \omega} = \sqrt{a^2 - 2a \cos \omega + 1} \end{aligned}$$

$$|H_{\text{freq}}(\omega)| = \frac{\sqrt{\frac{1}{r^2} - \frac{2}{r} \cos \omega + 1}}{\sqrt{r^2 - 2r \cos \omega + 1}} = \frac{\sqrt{\frac{1}{r^2} (r^2 - 2r \cos \omega + 1)}}{\sqrt{r^2 - 2r \cos \omega + 1}} = \frac{1}{|r|}$$

- A first-order IIR filter with one complex-valued pole and one complex-valued zero is all-pass if the zero radius is the reciprocal to the pole radius and if the angles are the same:

$$H(z) = \frac{z - \frac{1}{r_0} e^{j\omega_0}}{z - r_0 e^{j\omega_0}} \Rightarrow H_{\text{freq}}(\omega) = \frac{e^{j\omega} - \frac{1}{r_0} e^{j\omega_0}}{e^{j\omega} - r_0 e^{j\omega_0}}$$

assuming that $r_0 < 1$ for asymptotic stability. The magnitude response is

$$|H_{\text{freq}}(\omega)| = \frac{|e^{j\omega} - \frac{1}{r_0} e^{j\omega_0}|}{|e^{j\omega} - r_0 e^{j\omega_0}|}$$

$$\begin{aligned} \text{Here, } |e^{j\omega} - (a + jb)| &= |(\cos\omega - a) + j(\sin\omega - b)| \\ &= \sqrt{(\cos\omega - a)^2 + (\sin\omega - b)^2} \\ &= \sqrt{\cos^2\omega - 2a\cos\omega + a^2 + \sin^2\omega - 2b\sin\omega + b^2} \\ &= \sqrt{(a^2 + b^2) - 2\sqrt{a^2 + b^2} \cos(\omega + \theta) + 1} \end{aligned}$$

where $\theta = \arctan\left(-\frac{b}{a}\right)$.

$$|H_{\text{freq}}(\omega)| = \frac{\sqrt{\frac{1}{r_0^2} - \frac{2}{r_0} \cos(\omega + \theta) + 1}}{\sqrt{r_0^2 - 2r_0 \cos(\omega + \phi) + 1}}$$

$$\text{where } \theta = \arctan\left(-\frac{\frac{1}{r_0} \sin\omega_0}{\frac{1}{r_0} \cos\omega_0}\right) = -\omega_0$$

$$\phi = \arctan\left(-\frac{r_0 \sin\omega_0}{r_0 \cos\omega_0}\right) = -\omega_0$$

Therefore,

$$\begin{aligned} |H_{\text{freq}}(\omega)| &= \frac{\sqrt{\frac{1}{r_0^2} - \frac{2}{r_0} \cos(\omega - \omega_0) + 1}}{\sqrt{r_0^2 - 2r_0 \cos(\omega - \omega_0) + 1}} \\ &= \frac{\sqrt{\frac{1}{r_0^2} (r_0^2 - 2r_0 \cos(\omega - \omega_0) + 1)}}{\sqrt{r_0^2 - 2r_0 \cos(\omega - \omega_0) + 1}} = \frac{1}{r_0} \end{aligned}$$