Quick Introduction to the Fourier Transform

by Prof. Brian Evans
and Mr. Jean Farias
The University of Texas
at Austin Spring 2013

Fourier Transform

$$\underbrace{X(f)}_{\text{real variable } f} = \int_{-\infty}^{\infty} \underbrace{x(t)}_{\substack{\text{Amplitude} \\ \text{real or complex}}} e^{-j\overset{\text{radian}}{2\pi} \overset{\text{Hz}}{f} \overset{\text{seconds}}{t}} \, d\underbrace{t}_{\text{Real variable } t}$$

Amplitude is usually complex

$$\underbrace{e^{-j2\pi f t}}_{\substack{\text{complex-valued} \\ \text{sinusoid}}} = \overset{\text{real}}{\cos(-2\pi f t)} + \overset{\text{imag}}{j \sin(-2\pi f t)}$$

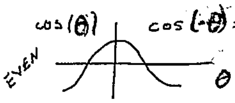
$$= \cos(2\pi f t) - j \sin(2\pi f t)$$

$$x(t) \xrightarrow{\mathcal{F}} X(f)$$

$$x_1(t) + x_2(t) \xrightarrow{\mathcal{F}} X_1(f) + X_2(f)$$

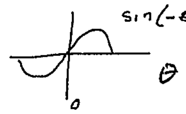
$\cos(\theta)$ $\cos(-\theta) = \cos(\theta)$

EVEN



$\sin(-\theta) = -\sin\theta$

odd



$$\boxed{\begin{array}{l} X(s) = \mathcal{L}\{x(t)\} \\ \text{region of convergence (ROC)} \end{array}}$$

Laplace Transform $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

If the ROC includes the imaginary axis

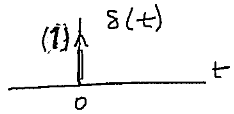
$$X(f) = X(s)\Big|_{s = j2\pi f}$$

Example

Dirac delta functional — Continuous-Time Impulse

unit area

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \quad \text{Unit Area}$$



Mathematically models impulsive event at origin

Amplitude undefined at origin

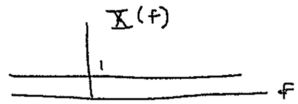
$$\int_{-\infty}^{\infty} g(t)\, \delta(t)\, dt = g(0)$$

Unit area concentrated at origin — area denoted by parentheses

Sifting Property

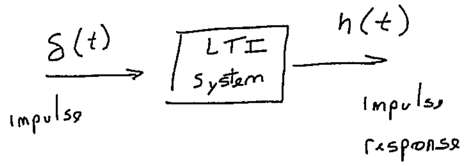
If $g(0)$ is defined at origin

$$\mathbb{X}(f) = \int_{-\infty}^{\infty} \underbrace{\delta(t)\, e^{-j2\pi f t}}_{g(t)}\, dt = 1 \qquad \text{when } t = 0$$



contains all frequencies

How to get direct impulse response

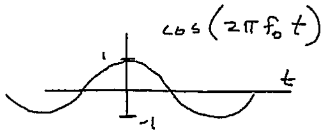


$$\delta(t) \xrightarrow{\mathcal{F}} 1$$

Example

two sided cosine

this does not have laplace
it have fourier transform
can't build this in lab



Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos(2\pi f_0 t) = \frac{1}{2} e^{-j2\pi f_0 t} + \frac{1}{2} e^{j2\pi f_0 t}$$

$$= \frac{1}{2}\left(\cos(-2\pi f_0 t) + j\sin(\cancel{-2\pi f_0 t})\right) + \frac{1}{2}\left(\cos(2\pi f_0 t) + j\sin(\cancel{2\pi f_0 t})\right)$$

$$= \cos(2\pi f_0 t)$$

Inverse Fourier Transform

$$x(t) = \int_{-\infty}^{\infty} \mathbb{X}(f) e^{j2\pi f t} df$$

For $\mathbb{X}(f) = \delta(f)$,

$$x(t) = \int_{-\infty}^{\infty} \delta(f) e^{j2\pi f t} df$$

$$= 1$$

$$1 \xrightarrow{\mathcal{F}} \delta(f)$$

$$\int_{-\infty}^{\infty} e^{-j2\pi f_0 t}\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi(f+f_0)t}\, dt = \delta(f+f_0)$$

example continued on next page

$$X(f+f_0) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(f+f_0)t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f_0 t}\, e^{-j2\pi f t}\, dt$$

$$x(t)\, e^{-j2\pi f_0 t} \xrightarrow{\mathcal{F}} X(f+f_0)$$

$$1 \xrightarrow{\mathcal{F}} \delta(f)$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$ Inverse Fourier

$$X(f) = \delta(f), \quad x(t) = \int_{-\infty}^{\infty} \delta(f)\, \underbrace{e^{j2\pi f t}}_{g(f)}\, df = g(0) = 1$$

always ~~evaluate~~ this ~~at the~~ origin in the above case is when $f = 0$

$$x(t)\, e^{-j2\pi f_0 t} \xrightarrow{\mathcal{F}} X(f+f_0)$$

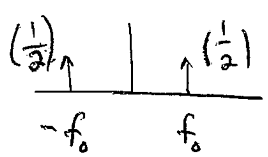
$$1 \xrightarrow{\mathcal{F}} \delta(f)$$

$$1 \cdot e^{-j2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta(f+f_0)$$

$$e^{j2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta(f-f_0)$$

$$\mathcal{F}\{\cos(2\pi f_0 t)\} = \mathcal{F}\left\{\tfrac{1}{2} e^{-j2\pi f_0 t} + \tfrac{1}{2} e^{j2\pi f_0 t}\right\}$$

$$= \underbrace{\tfrac{1}{2}}_{\text{area of } \frac{1}{2}} \delta(f+f_0) + \underbrace{\tfrac{1}{2}}_{\text{area of } \frac{1}{2}} \delta(f-f_0)$$

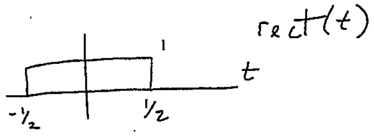


1st find the origin

$f + f_0 = 0$ $\qquad$ $f - f_0 = 0$

$f = -f_0$ $\qquad$ $f = f_0$

Example using the rectangular pulse



Has unit area.

Taking the Fourier transform,

$$\int_{-1/2}^{1/2} \cancelto{1}{x(t)}\, e^{-j2\pi f t}\, dt$$

$$\frac{1}{-j2\pi f}\, e^{-j2\pi f t}\Big|_{-1/2}^{1/2}$$

$$\frac{1}{-j2\pi f}\, e^{-j\pi f} + \frac{1}{j2\pi f}\, e^{j\pi f}$$

$$\frac{j}{j(-j)2\pi f}\, e^{-j\pi f} + \frac{j}{j(j)2\pi f}\, e^{j\pi f}$$

$$\frac{j}{2\pi f}\, e^{-j\pi f} - \frac{j}{2\pi f}\, e^{j\pi f}$$

$$\frac{j}{2\pi f}\left(e^{-j\pi f} - e^{j\pi f}\right)$$

$$\frac{j}{2\pi f}\left(-2j\sin(\pi f)\right) = \frac{\sin(\pi f)}{\pi f} = \mathrm{sinc}(f)$$