The University of Texas at Austin

Spring 2014 EE 445S Real-Time Digital Signal Processing Laboratory Prof. Evans

Homework #1 Solutions on Z-Transforms and Spectral Analysis

1. Transfer Functions

With $x[n]$ denoting the input signal and $y[n]$ denoting the output signal, give the difference equation relating the input signal to the output signal in the discrete-time domain, give the initial conditions and their values, and find the transfer function in the z-domain and the associated region of convergence for the z-transform function, for the following linear time-invariant discrete-time systems.

Note that (a) and (b) are examples of finite impulse response filters, and (c) and (d) are examples of infinite impulse response filters. More on filters in lectures 5 & 6, and labs 2 & 3.

(a) Causal five-coefficient averaging filter:

The output is the average of the current and previous four input values. The difference equation is

$$y[n] = \frac{1}{5}x[n] + \frac{1}{5}x[n-1] + \frac{1}{5}x[n-2] + \frac{1}{5}x[n-3] + \frac{1}{5}x[n-4]$$

where the initial conditions $x[-1]$, $x[-2]$, $x[-3]$, $x[-4]$ must be zero for the system to be LTI

Transfer function: $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{5}(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4})$

Region of convergence (ROC): all z except $z \neq 0$. Zero is excluded to prevent division by zero.

(b) Causal discrete-time approximation to first-order derivative:

The first-order derivative operation with output $y(t)$ and input $x(t)$ can be defined in terms of a limit:

$$y(t) = x'(t) = \lim_{\Delta t \to 0} \frac{x(t) - x(t - \Delta t)}{\Delta t}$$

We can sample the system. The smallest separation between time-domain samples is the sampling time T_s so that $\Delta t == T_s$. However, we cannot drive T_s to zero in practice:

$$y[n] \approx \frac{1}{T_s}(x[n] - x[n-1])$$

Due to sampling, the approximation is only valid for continuous-time frequencies up to one-half of the sampling rate, i.e. up to ½ f_s, in value.

In discrete-time, we prefer to abstract away the sampling rate when possible. Difference equation is

$$y[n] = x[n] - x[n-1]$$

where the initial condition x [-1] = 0 for the system to be LTI.

Transfer function: $H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-1} = \frac{z-1}{z}$

Region of convergence (ROC): all z except $z \neq 0$. Zero is excluded to prevent division by zero. More generally, the ROC for a finite impulse response filter is the entire z-plane except at zero.

(c) Causal discrete-time approximation to a first-order integrator:

In continuous time, the output of a causal first-order integrator is defined by

$$y(t) = \int_0^t x(t) dt$$

where $x(t)$ is the input. The discrete-time version obtained from sampling is

$$y[n] = \sum_{m=0}^{n} x[m]$$

This is inefficient because it requires unbounded amount of memory to store the previous input values.

Instead, we can use the recursive difference equation, a.k.a. a running summation,

$$y[n] = y[n-1] + x[n]$$

with y [-1] = 0 to make the system LTI.

Transfer function: $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$

ROC: $|z| > 1$ since the system is causal.

(d) Damped bandpass resonator at fixed frequency ω_0 and radius *r*

Difference equation was given in the problem to be

$$y[n] = (2 \cos \omega_0)\, r\, y[n-1] - r^2 y[n-2] + x[n] - (\cos \omega_0)\, x[n-1]$$

and the initial conditions y[-1], y[-2] and x[-1] must be set to zero to make the system LTI.

Taking the z-transform of both sides of the difference equation, we get

$$Y(z) = (2 \cos \omega_0)\, r\, z^{-1} Y(z) - r^2 z^{-2} Y(z) + X(z) - (\cos \omega_0)\, z^{-1} X(z)$$

$$=> \frac{Y(z)}{X(z)} = \frac{1-\cos\omega_0 z^{-1}}{1-2(\cos\omega_0)rz^{-1}+r^2z^{-2}} \quad => \quad H(z) = \frac{1-\cos\omega_0 z^{-1}}{1-2(\cos\omega_0)rz^{-1}+r^2z^{-2}}$$

We need to find the poles of this transfer function by finding the roots of the denominator as follows:

$$1 - (2 \cos \omega_0)\, r\, z^{-1} + r^2 z^{-2} = 0$$

By multiplying each side by z^2 (assuming that $z \neq 0$):

$$z^2 - (2 \cos \omega_0)\, r\, z + r^2 = 0$$

Roots are located at ½ (-b ± sqrt (Δ)). Here,

$$\Delta = 4r^2\cos^2\omega_0 - 4r^2 = -4r^2\sin^2\omega_0$$

Since $\Delta < 0$, there are two complex roots:

$$x_1 = \frac{2r\cos\omega_0 + 2rj\sin\omega_0}{2} = r(\cos\omega_0 + j\sin\omega_0) = re^{j\omega_0}$$
$$x_2 = \frac{2r\cos\omega_0 - 2rj\sin\omega_0}{2} = r(\cos\omega_0 - j\sin\omega_0) = re^{-j\omega_0}$$

Both poles have magnitude of r since $0 < r < 1$. Under the assumption that the system is causal, the region of convergence will be outside of the circle of radius equal to the magnitude of the pole with the greatest magnitude, i.e. $|z| > r$. For stability, $r < 1$ so as to keep the poles inside the unit circle.

2. Spectral analysis

Johnson, Sethares & Klein, Exercise 3.3, but use the following signals (*9 points each*)

a) A rectangular pulse $s(t) = \text{rect}(t/8)$ which has an amplitude of 1 from -4 (inclusive) to 4 (non-inclusive). Plot the signal in the time domain for $-8 < t < 8$. Estimate f_{max}. Plot the spectrum.

Solution:

Approach #1: Use the continuous-time Fourier transform

Using the Fourier transform table in Appendix E.2 in Roberts' *Signals and Systems* book:

$$\text{rect}\left(\frac{t}{\tau}\right) \overset{\mathfrak{I}}{\longleftrightarrow} \tau\frac{\sin(\pi f\tau)}{\pi f\tau} = \tau\,\text{sinc}(f\tau) \qquad \text{rect}\left(\frac{t}{8}\right) \overset{\mathfrak{I}}{\longleftrightarrow} 8\frac{\sin(8\pi f)}{8\pi f} = 8\,\text{sinc}(8f)$$

where $\text{sinc}(x) = \sin(\pi x) / (\pi x)$. This is a lowpass magnitude spectrum with first null at $f = 0.125$ Hz. The magnitude spectrum decays at a rate of $1/f$. We could go with $f_{max} = 0.125$ Hz.

Approach #2: Plot spectrum for increasing sampling rates until the spectral shape no longer changes.

The sampling theorem says that the sampling rate should be greater than twice the maximum frequency, i.e. $f_s > 2 f_{\max}$. By dividing both sides by 2, we see that $f_{\max} < ½ f_s$. Once we pick a sampling rate, the sampling process can only capture frequencies $-½ f_s < f < ½ f_s$. Continuous-time frequencies outside this range will alias and add on top of the continous-time frequencies in the range. (To be precise, the range of continuous-time frequencies should include either $-½ f_s$ or $½ f_s$.)

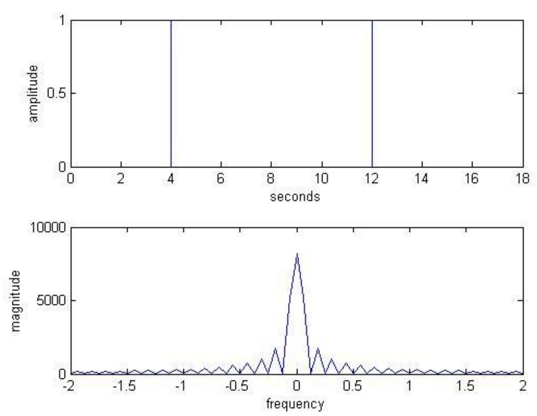
In continuous time, as we saw on homework problem 0.3, we can model the result $y(t)$ of sampling of a signal $x(t)$ as the product of $x(t)$ and an impulse train $p(t)$. Impulses are separated by the sampling time T_s. The Fourier transform is

$$Y(f) = X(f) * P(f) = f_s \sum_{n=-\infty}^{\infty} X(f - n f_s)$$

As the sampling rate f_s changes, the scaling of the sampled spectrum will change proportionally to f_s.

We sample the continuous-time signal using a sampling rate of 1024 Hz and then analyze the magnitude spectrum of the sampled signal. The DC response has a value of 8000. The magnitude spectrum falls below a value of 80 by 8 Hz. A reduction by a factor of 100 corresponds to 80 dB. We could go with $f_{\max} = 8$ Hz.

Note: The plotspec command from *Software Receiver Design* assumes that the time-domain signal starts at the origin. In this problem, the extent of $s(t)$ in the time domain is from -4*s* to 4*s*. A shift in the time domain only affects the phase response— the magnitude response does not change.




```
fs = 1024;
Ts = 1/fs;
t = -8 : Ts : 8;
s = rectpuls(t/8);
plotspec(s,Ts);
xlim([-2, 2]); % Zoom x axis
```

Note: The `plotspec` function is defined in the Matlab files that accompany the *Software Receiver Design* book. The `plotspec` function will plot the spectrum over frequencies from $-½ f_s < f < ½ f_s$ in the bottom plot. In the above Matlab code, we use `xlim` to display the spectrum over -2 Hz < f < 2 Hz.

b) A truncated sinc pulse $s(t) = \mathrm{sinc}(t)\,\mathrm{rect}(t/8)$ where $\mathrm{sinc}(x) = \sin(\pi x) / (\pi x)$. Plot the signal in the time domain for $-8 < t < 8$. Estimate f_{max}. Plot the spectrum.

Solution:

Approach #1: Use the continuous-time Fourier transform

Multiplication in the time domain leads to convolution in the Fourier domain. Using the Fourier transform table in Appendix E.2 in Roberts' *Signals and Systems* book:

$$S(f) = \mathrm{rect}(f) * 8\,\mathrm{sinc}(8f)$$

Here, $sinc(x) = \sin(\pi x) / (\pi x)$.

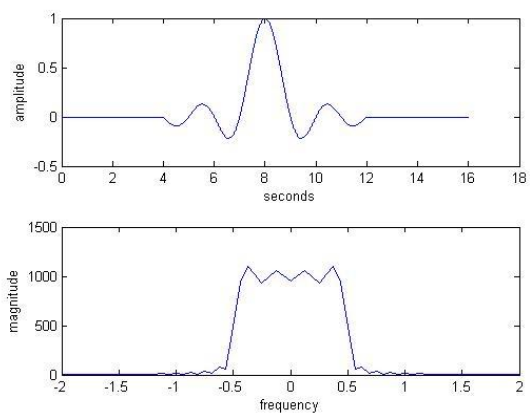
In $S(f)$, the first term has a full width of 1 Hz based on the definition of the rectangle function. The second term has a main lobe width of 1/4 Hz, as shown in part (a).

To estimate the bandwidth of $S(f)$ we proceed with the following logic: The convolution of two finite duration signals, $x(t)$ of duration L_x and $y(t)$ of duration L_y, results in a signal $z(t)$ of duration L_x+L_y.

The area of overlap between the two terms in $S(f)$ will be appreciable when only when the rectangle function overlaps with the main lobe of the sinc function. The main lobe of the sinc function has width 1/4 and the rectangle function has width 1. Thus, we can estimate the full width of $S(f)$ to be 1.25Hz. Accordingly, we estimate the bandwidth of $S(f)$ to be $f_{max} = 0.625$ Hz.

Approach #2: Plot the magnitude spectrum for increasing sampling rates until the magnitude spectrum no longer changes.

```
Ts = 1/1000;
t = -8:Ts:8;
s = sinc(t).*((t >= -4)&(t < 4));
plotspec(s,Ts)
xlim([-2, 2])
```



Note: The `plotspec` function is defined in the Matlab files that accompany the *Software Receiver Design* book. The `plotspec` function will plot the spectrum over frequencies from $-\frac{1}{2} f_s < f < \frac{1}{2} f_s$ in the bottom plot. In the above Matlab code, we use `xlim` to display the spectrum over -2 Hz $< f <$ 2 Hz.

We'll eyeball the spectrum and arbitrarily pick $f_{max} = 0.5$ Hz. Also, $f_{max} = 1$ Hz is a better choice.

Note: The plotspec command from *Software Receiver Design* assumes that the time-domain signal starts at the origin. In this problem, the extent of $s(t)$ in time is from -4*s* to 4*s*. A shift in the time domain only affects the phase response—the magnitude response does not change.

c) A decaying exponential $s(t) = \exp(-t)\ u(t)$. Plot the signal in the time domain for $-8 < t < 8$. Estimate f_{max}. Plot the spectrum.

Solution: This is a commonly occurring signal because it is an impulse response of an LTI system governed by a first-order differential equation, such as an RC circuit.

$$e^{-at}u(t) \overset{\mathfrak{I}}{\leftrightarrow} \frac{1}{a+j\omega}$$

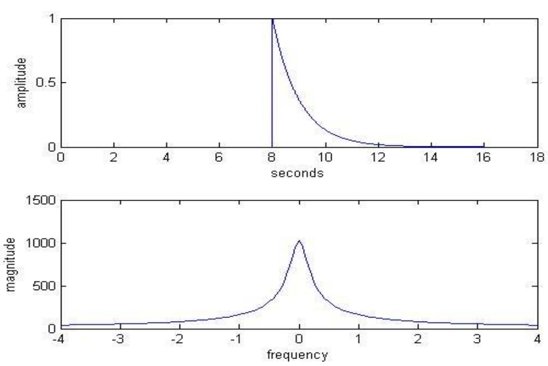
$$e^{-t}u(t) \overset{\mathfrak{I}}{\leftrightarrow} \frac{1}{1+j\omega}$$

where $\omega = 2\pi f$.

$$|S(\omega)| = \left|\frac{1}{1+j\omega}\right| = \frac{1}{\sqrt{1+\omega^2}}$$

When ω =1, the power spectrum $|S(\omega)|^2$, drops to half its value, and $|S(\omega)|$decreases by a factor of $\frac{1}{\sqrt{2}}$, or 3dB. So, the 3 dB bandwidth is $B_{3dB} = 1$ rad/s. In units of Hz, the 3 dB bandwidth is $\frac{1}{2\pi}$, making f_{max} about = $1/(4\pi)$. Plotted in MATLAB:

```
Ts = 1/1000;
t = -8:Ts:8;
s = exp(-t) .* stepfun(t, 0);
plotspec(s,Ts)
xlim([-4, 4])
```

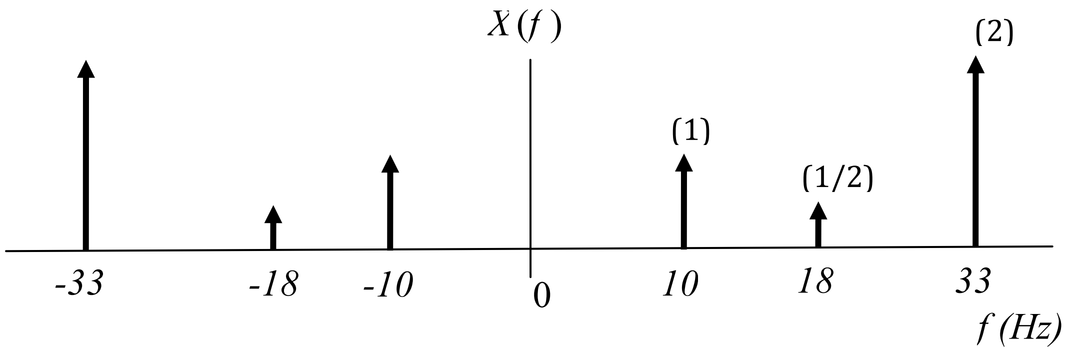


Note: The `plotspec` function is defined in the Matlab files that accompany the *Software Receiver Design* book. The `plotspec` function will plot the spectrum over frequencies from -½ $f_s < f <$ ½ f_s in the bottom plot. In the above Matlab code, we use `xlim` to display the spectrum over -4 Hz < f < 4 Hz.

3. Two-Sided (Everlasting) Sinusoids and Their Finite Length Observations.
Johnson, Sethares & Klein, Exercise 3.7 on page 46.

(a) Give the Fourier transform of $x(t)$. Plot the spectrum of $x(t)$ for $0 < t < 1$. Justify how you determined the sampling rate.

$x(t) = x_1(t) + 0.5\ x_2(t) + 2\ x_3(t)$ where $x_1(t) = \cos(2\ \pi\ 10\ t)$, $x_2(t) = \cos(2\ \pi\ 18\ t)$, $x_3(t) = \cos(2\ \pi\ 33\ t)$



Plotted in MATLAB,

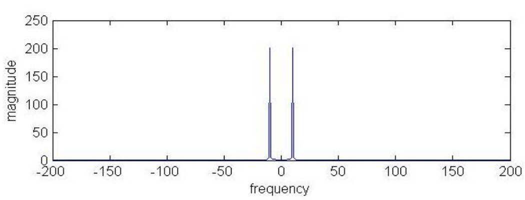
```
fs = 10*40;
Ts = 1/fs;
t = 0:Ts:1;
x1 = cos(2*pi*10*t);
x2 = cos(2*pi*18*t);
x3 = cos(2*pi*33*t);
x = x1 + 0.5*x2 + 2*x3;
figure, plotspec(x1,Ts);
figure, plotspec(x2,Ts);
figure, plotspec(x3,Ts);
figure, plotspec(x,Ts);
```

<u>Note:</u> The `plotspec` function is defined in the Matlab files that accompany the *Software Receiver Design* book. The `plotspec` function will plot the spectrum over frequencies from $-\frac{1}{2} f_s < f < \frac{1}{2} f_s$ in the bottom plot.

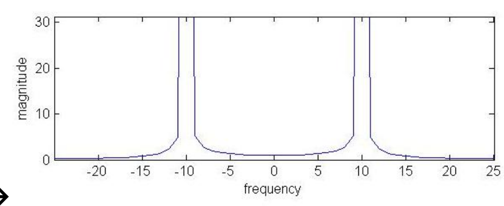
In the Matlab code, the sampling rate f_s is chosen to be 10 f_{max}, as explained later.

Principal frequencies are 10 Hz, 18 Hz and 33 Hz.

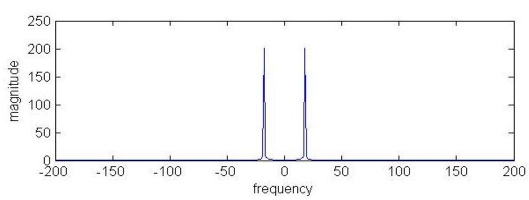
Fourier transform of x_1



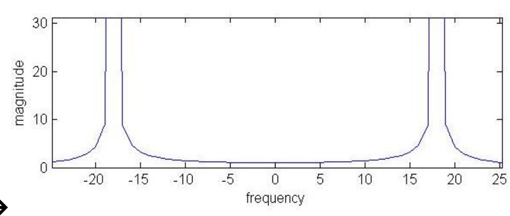
zoomed in→



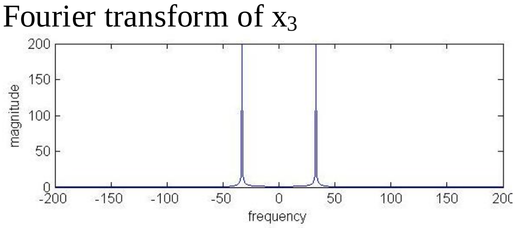
Fourier transform of x_2



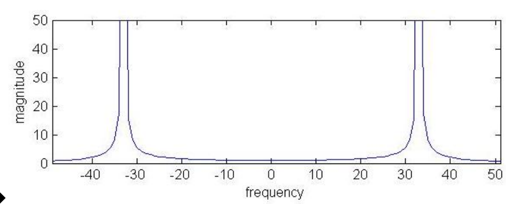
zoomed in→



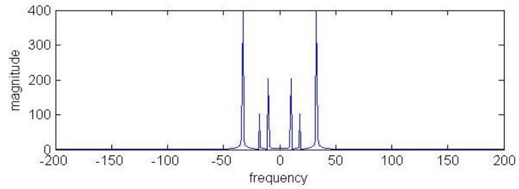
Fourier transform of x_3

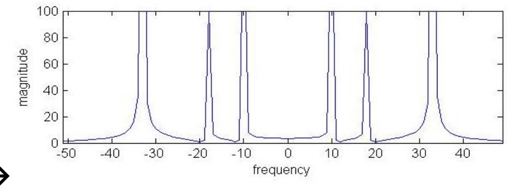


zoomed in→



Fourier transform of $x = x_1 + x_2 + x_3$



zoomed in→ 

This illustrates the linearity property of the Fourier transform, which states that if

$$x(t) = x_1(t) + x_2(t) + x_3(t)$$

Then,

$$X(f) = X_1(f) + X_2(f) + X_3(f)$$

The spectrum of each component adds, and the resulting signal x contains the same frequencies f_1, f_2, and f_3 from x_1, x_2, and x_3 respectively, where $f_1 = 10\ Hz, f_2 = 18\ Hz, f_3 = 33\ Hz$.

Since $\cos(2\pi f_o t) \overset{\mathfrak{I}}{\leftrightarrow} \frac{1}{2}[\delta(f - f_o) + \delta(f + f_o)]$,

$$X(f) = \frac{1}{2}[\delta(f - f_1) + \delta(f + f_1) + \delta(f - f_2) + \delta(f + f_2) + \delta(f - f_3) + \delta(f + f_3)]$$

For the two-sided sinusoidal signal $x(t)$, f_{max} = 33 Hz. In order to plot the signal, we would plot a finite observation. From homework problem 0.3, we know that each sinusoidal component will have a non-zero but narrow bandwidth, as is evident in the earlier spectrum plots. From those spectrum plots, we could pick f_{max} to be 40 Hz, and then safely pick the sampling rate as 10 f_{max}.

(b) In addition, find a formula for the Fourier transform of $y(t) = x_1(t)\ x_2(t)$. What principal frequencies are present in $y(t)$? How do they relate to the principal frequencies in $x_1(t)$ of -10 Hz and 10 Hz and $x_2(t)$ of -18 Hz and 18 Hz? Plot the spectrum of $y(t)$ for $0 < t < 1$.

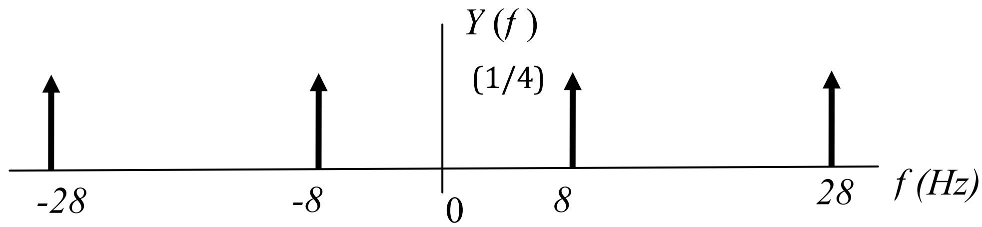
If $y(t) = x_1(t)x_2(t)$, then $Y(f) = X_1(f) * X_2(f)$. So,

$$X_1(f) = \frac{1}{2}[\delta(f - f_1) + \delta(f + f_1)]$$

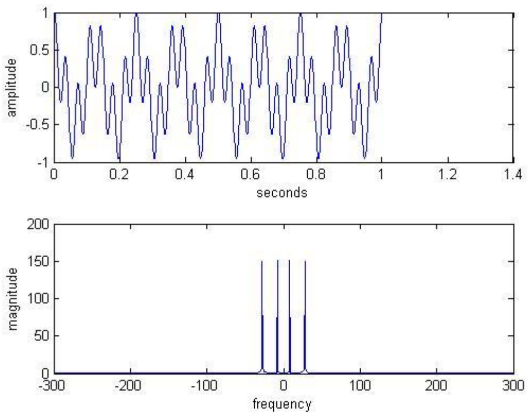
$$X_2(f) = \frac{1}{2}[\delta(f - f_2) + \delta(f + f_2)]$$

$$Y(f) = \frac{1}{2}[\delta(f - f_1) + \delta(f + f_1)] * \frac{1}{2}[\delta(f - f_2) + \delta(f + f_2)]$$

$$Y(f) = \frac{1}{4}[\delta(f - f_1 - f_2) + \delta(f - f_1 + f_2) + \delta(f + f_1 - f_2) + \delta(f + f_1 + f_2)]$$



Or, plotted in MATLAB, with $y(t)$ on the top and $|Y(f)|$ on the bottom:




```
fs = 10*60;
Ts = 1/fs;
t = 0:Ts:1;
x1 = cos(2*pi*10*t);
x2 = cos(2*pi*18*t);
y = x1.*x2;
figure, plotspec(y,Ts);
```

Note: The `plotspec` function is defined in the Matlab files that accompany the *Software Receiver Design* book. The `plotspec` function will plot the spectrum over frequencies from $-½ f_s < f < ½ f_s$ in the bottom plot.

Principal frequencies are 8 Hz and 28 Hz.

These frequencies correspond to the sum and difference of 10 Hz and 18 Hz.

MATLAB Scripts from in Johnson, Sethares and Klein's *Software Receiver Design* textbook

The Matlab scripts should run "as is" in MATLAB or LabVIEW Mathscript facility.

1. Copy the .m files on your computer from the "SRD - MatlabFiles" folder on the CD ROM: http://users.ece.utexas.edu/~bevans/courses/rtdsp/homework/SRD-MatlabFiles.zip
2. Add the folder containing the .m files from the book to the search path.:
   - In MATLAB, use the `addpath` command
   - In LabVIEW, open the Mathscript window in LabVIEW by going to the Tools menu and select "Mathscript Window" (third entry), go the File menu, select "LabVIEW MathScript Properties" an d add the path.

Johnson, Sethares and Klein intentionally chose not to copyright their programs so as to enable their widespread dissemination.

Discussion of this solution set is available at http://www.youtube.com/watch?v=KOTsDFm3ehw