# Goals of the lecture

- Relations

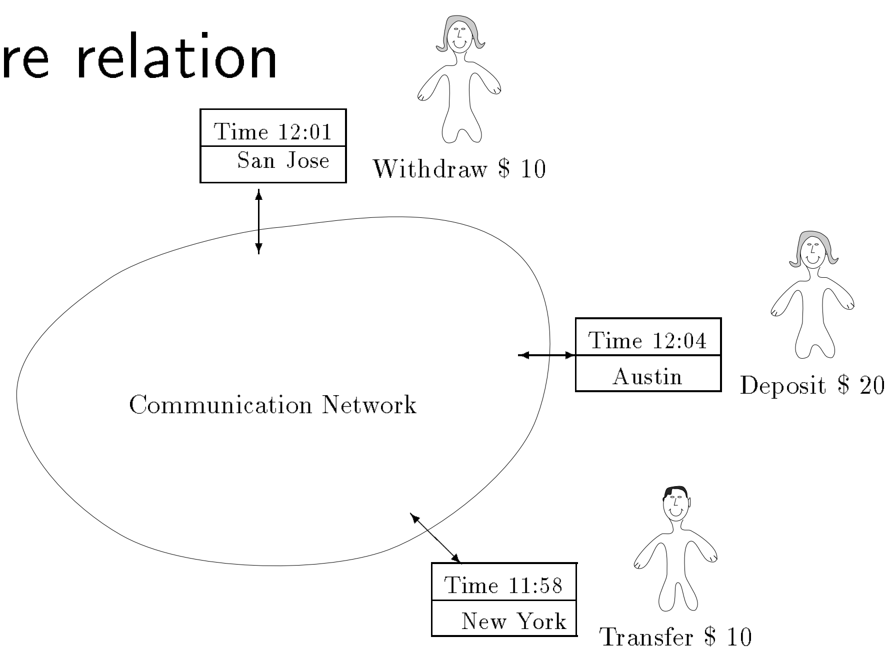
- Posets

- A run or a distributed computation

- Happened-before relation

# Model of Distributed systems

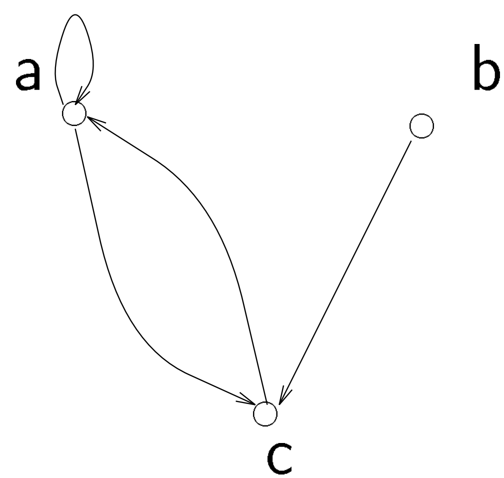
- events
  - beginning of procedure foo
  - termination of bar
  - send of a message
  - receive of a message
  - termination of a process
- happened-before relation



# Relation

- $X$ = any set
  a binary relation R is a subset of $X \times X$.

- Example: $X = \{a, b, c\}$, and
  $R = \{(a, c), (a, a), (b, c), (c, a)\}$.



# Relation [Contd.]

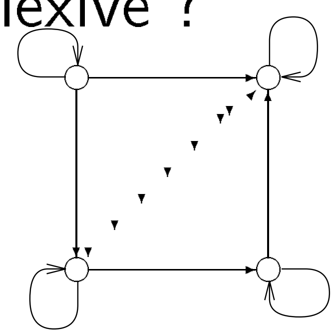
**Reflexive**: If for each $x \in X$, $(x, x) \in R$.

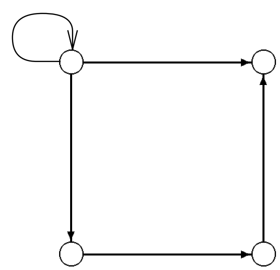
- Example: $X$ is the set of natural numbers, and $R = \{(x, y) \mid x \text{ divides } y\}$.

**Irreflexive**: For each $x \in X$, $(x, x) \notin R$.

- Example: $X$ is the set of natural numbers, and $R = \{(x, y) \mid x \text{ less than } y\}$.

Reflexive or irreflexive ?





# Relation [Contd.]

**Symmetric**: $(x, y) \in R$ implies $(y, x) \in R$.

- Examples: is sibling of, $x$ mod $k = y$ mod $k$.

**Anti-symmetric**: $(x, y) \in R, (y, x) \in R$ inplies $x = y$.

- Examples: $\leq$, divides.

**Asymmetric**: $(x, y) \in R$ implies $(y, x) \notin R$.

- Examples: is child of, $<$.

# Relation [Contd.]

**Transitive**: $(x, y), (y, z) \in R$ implies $(x, z) \in R$.

- Examples: is reachable from, $<$, divides.

Puzzle: Example of a symmetric and transitive but not reflexive relation.

# Partially Ordered Sets [Posets]

Partial Order

| Reflexive | Irreflexive |
|---|---|
| Transitive | Transitive |
| Anti-symmetric | Anti-symmetric |
| Example: $\leq$ | Example: $<$ |

Examples:

- $X$: Ground Set, $(2^X, \subset)$ is a irreflexive partial order
- $(\mathcal{N},$ divides) is a reflexive partial order
- $(\mathcal{R}, \leq)$ is a reflexive partial order (also a total order)
- causality in a distributed system (later ..)

# Posets [Contd.]

Let $Y \subseteq X$, where $(X, \leq)$ is a poset.

**Infimum**: $m = \text{inf}(Y)$ iff

- $\forall y \in Y : m \leq y$
- $\forall x \in X : (\forall y \in Y : x \leq y) \Rightarrow x \leq m$

  $m$ is also called $glb$ of the set $Y$.

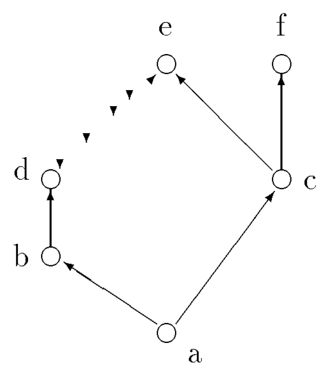
**Supremum**: $s = \text{sup}(Y)$ iff ($s$ is also called $lub$)

- $\forall y \in Y : y \leq s$
- $\forall x \in X : (\forall y \in Y : y \leq s) \Rightarrow s \leq x$

We denote the glb of $\{a, b\}$ by $a \sqcap b$, and lub by $a \sqcup b$.

$$X = \{a, b, c, d, e, f\}$$

$$R = \left\{ \begin{array}{l} (a, b), (a, c), (b, d), \\ (c, f), (c, e), (d, e) \end{array} \right\}$$



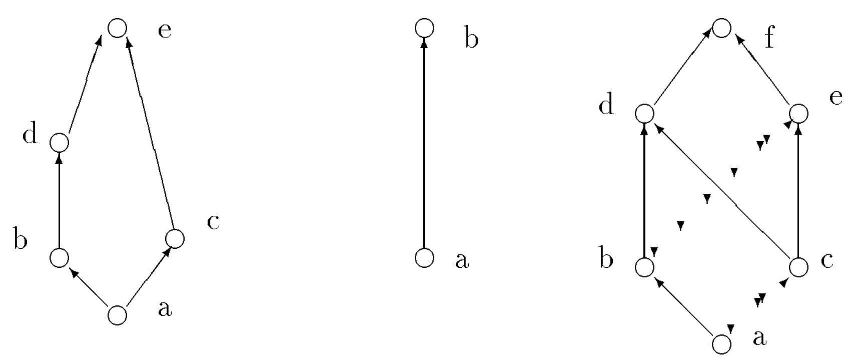
# Lattices

Lattices → sups and infs for finite sets

Lattices → Poset

- Let S be any set, and $2^S$ be its power set. The poset $(2^S, \subseteq)$ is a lattice.
- Set of rationals with usual $\leq$.
- Set of global states

- A lattice is an algebraic system $(L, \sqcup, \sqcap)$ where $\sqcup$ and $\sqcap$ satisfy commutative, associative and absorption laws.

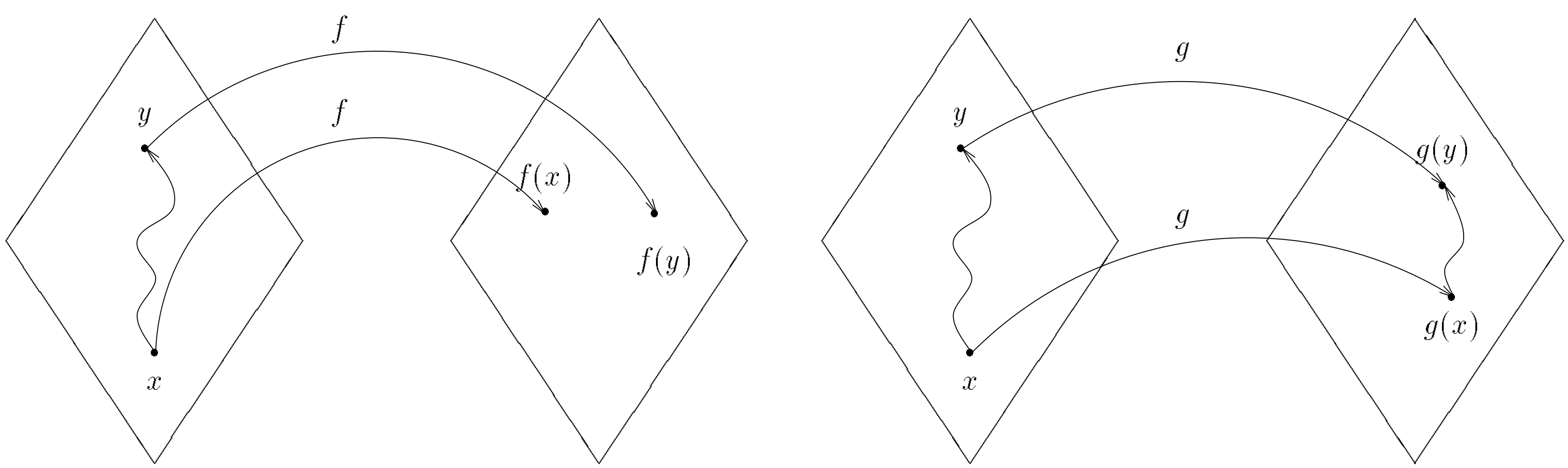


# Monotone functions

A function $f : X \to Y$ is monotone iff

$$\forall\, x, y \in X : x \leq y \Rightarrow f(x) \leq f(y).$$

- Examples
  - union, intersection
  - addition, multiplication with positive number
  - clocks in distributed systems



# Down-Sets and Up-Sets

Let $(X, <)$ be any poset.

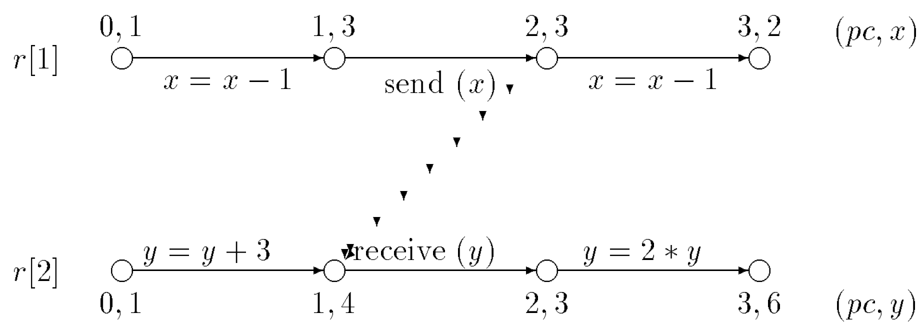
- We call a subset $Y \subseteq X$ a down-set (alternatively, order ideal) if
$$f \in Y \wedge e < f \Rightarrow e \in Y.$$

- Similarly, we call $Y \subseteq X$ an up-set (alternatively, order filter) if
$$e \in Y \wedge e < f \Rightarrow f \in Y.$$

- We use $\mathcal{O}(X)$ to denote the set of all down-sets of $X$. We now show a simple but important lemma.

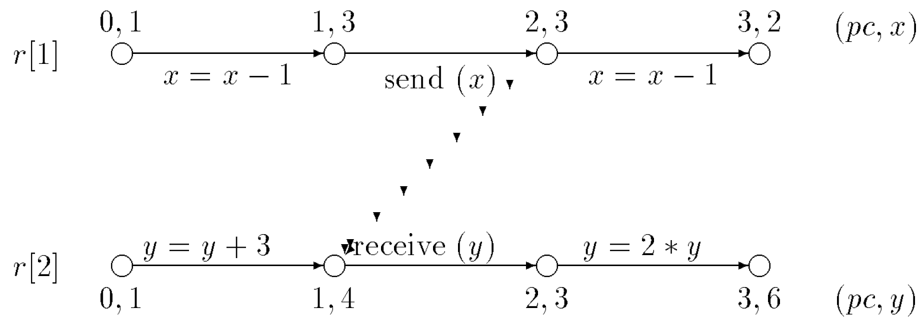
**Lemma 1** *Let $(X, <)$ be any poset. Then, $(\mathcal{O}(X), \subseteq)$ is a lattice.*

# Run



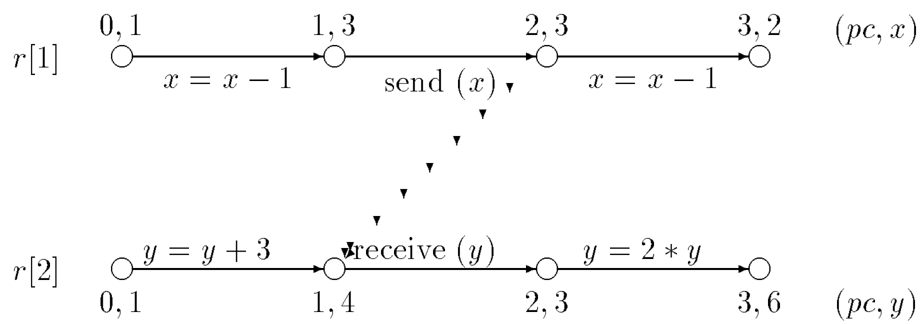

- Each process $P_i$ in a run generates an execution trace $s_{i,0}e_{i,0}s_{i,1}\ldots e_{i,l-1}s_{i,l}$, which is a finite sequence of local $states$ and $events$ in the process $P_i$.
  - state = values of all variables, program counter
  - event = internal, send, receive
- A $run$ $r$ is a vector of traces with $r[i]$ as the trace of the process $P_i$.

# Relations




- $s \prec_1 t$ if and only if $s$ *immediately* precedes $t$ in the trace $r[i]$.
  - $s.next = t$ or $t.prev = s$ whenever $s \prec_1 t$.
  - $\prec$ = irreflexive transitive closure of $\prec_1$.
  - $\preceq$ = reflexive transitive closure of $\prec_1$.
- event $e$ in the trace $r[i] \leadsto$ event $f$ in the trace $r[j]$ if $e$ is the send of a message and $f$ is the receive event of the same message.

# Relations [Contd.]




*causally precedes* relation $\equiv$ the transitive closure of union of $\prec_1$ and $\leadsto$. That is, $s \rightarrow t$ iff

1. $(s \prec_1 t) \vee (s \leadsto t)$, or
2. $\exists u : (s \rightarrow u) \wedge (u \rightarrow t)$

$s$ and $t$ are concurrent (denoted by $s||t$) if $\neg(s \rightarrow t) \wedge \neg(t \rightarrow s)$.