Enhancing Vehicle Flow in Random Environments through Dynamic Allocation of Sensing Resources

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***Abstract*—This paper presents a theoretical analysis for a self-driving vehicle's velocity as it navigates through a random environment. We study a stylized environment and vehicle mobility model capturing the essential features of a self-driving vehicle's behavior, and leverage results from stochastic geometry to characterize the distribution of a typical vehicle's safe driving velocity, as a function of key network parameters such as the density of objects in the environment and sensing accuracy. We then consider a setting wherein the sensing accuracy is subject to a sensing/communication rate constraint. We propose a procedure that focuses the vehicle's sensing/communication resources and estimation efforts on the objects that affect its velocity and safety the most so as to optimize its ability to drive faster in uncertain environments. Simulation results show that the proposed methodology achieves considerable gains in the vehicle's safe driving velocity as compared to uniform rate allocation policies.**

***Index Terms*—Autonomous Driving, Safe Driving Speed, Communication Resource Allocation, Situational Awareness**

I. Introduction

The automotive industry is undergoing significant changes tied to realizing the vision for autonomous vehicles, progressively shifting driving and maneuvering decisions from the driver to the vehicle itself. This emerging paradigm is tied with new design challenges with perhaps the most important one being the ability to achieve accurate situational awareness. Clearly, the quality of a vehicle's situational awareness depends on sensor-equipped vehicles and environment, and has a major impact on the safety level that can be guaranteed for vehicles, along with their permissible speed. In this work, we study a stylized vehicle mobility model proposed as a caricature of complex real systems yet capturing their salient features, and we use it to characterize the fundamental safe driving speed of a vehicle evolving in an uncertain environment. We then study the effect that sensing/communication constraints would have on the vehicle's ability to progress safely in random dynamic environments.

Related Work. A considerable number of researchers have investigated systems involving objects moving through a random environment. A set of papers used results from the stochastic geometry literature to study the coverage probability of a vehicle driving through a field of sensors each with a disk coverage around it, thus forming a Boolean process in space [1]–[5]. Some of these works studied the effect of the environment's mobility on the coverage probability, see, e.g., [1], [4], [5]. However, none of these works considered how the environment might impact the speed of vehicles, as well as the effect of constrained communication and sensing resources on the quality of the environment estimation, and hence, the vehicle's velocity.

In [6], the authors investigate traffic rules inspired by particle physics to provide insight on improved collective behavior in vehicular networks. While the authors study the velocity of an *elite particle* (equivalent to the vehicle in our framework) in a densely crowded environment, their focus is to understand and propose a new collective behavior from the objects/particles in the environment to facilitate the motion of the elite particle, instead of the adaptive behavior of the latter.

By contrast, other works have proposed advanced methods to adapt to random and unknown environments, see, e.g., [7], sometimes leveraging tools from Reinforcement Learning [8]. The focus of these works is however on the environment estimation and robot's motion obstacle avoidance algorithms, rather than on examining fundamental properties of the agent's velocity in such environments.

Contributions. The contributions of this paper are threefold. First, we propose an analytical framework based on stochastic geometry to characterize the spatial distribution of the vehicle's safe driving velocity induced by a random environment. We then formalize the idea that vehicles spend more time in dense regions of the network and we relate the vehicle's spatial and temporal velocity distributions.

Second, we present results providing additional insights on the sensitivity of the vehicle's safe driving velocity with respect to key system parameters such as the density of objects in the environment, and the vehicle's sensing capability.

Third, we examine settings wherein the environment estimation accuracy is sensing/communication-dependent, and propose an efficient procedure allocating sensing/communication resources so as to optimize the vehicles' ability to drive faster in uncertain environments.

Paper Organization. The paper is organized as follows. In Section II, we introduce our network and vehicle mobility models. In Section III, we characterize the spatial and temporal distributions of the vehicle's safe driving speed. We then discuss in Section IV the communication-dependent environment estimation model and optimal data polling strategy. Finally, Section V concludes the paper.

II. Network and Mobility Model

A. Network Environment Model

Consider a network wherein a vehicle is driving along a road modeled as an infinite straight line in $\mathbb{R}^2$, say, without loss of generality, in the positive x-axis direction. The road traverses an environment composed of "objects" spatially distributed according to a homogeneous Poisson Point Process (PPP). The vehicle is aware of the existence and tracks all the objects that lie within a circular *sensing region* $\mathcal{S}_t$ of radius r around it at time t. As the situational awareness is enabled by noisy sensor measurements, e.g., cameras or radars equipped on the vehicle, or sensors located on the objects themselves and communicating with it, the vehicle builds an estimate for the position of each object in its sensing region. We denote as Φ_t the resulting PPP of estimated mean object locations at a given time t, and we let $\phi_t = \{(x_{i,t}, y_{i,t}) \; : \; i \in \mathbb{N}\}$ denote a realization of Φ_t, where $x_{i,t}$ and $y_{i,t}$ correspond to the Cartesian coordinates of object i, in the frame of reference of a vehicle located at the origin at time t. The PPP Φ_t has a mean intensity λ objects/m^2, taking into account possible misdetections and false-alarms that might occur in the object tracking process.

In addition, there is an uncertainty region around each estimated object's mean location such that the vehicle has high confidence that the object lies within it. For simplicity, we shall assume that the uncertainty region of any object i is a disk of radius β_i. We shall examine different models for the uncertainty region's radius in the sequel. Figure 1 illustrates a random realization of the described environment.

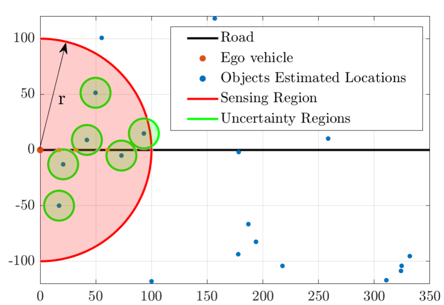


Fig. 1: Figure of a random environment realization, and the vehicle located at the origin driving in the positive direction along the x-axis; for $r = 100$m and $\beta_i = 13$m, for all i. The coordinates are expressed in meters.

B. Vehicle Velocity Adaptation Model

We shall assume that, in general, the vehicle wishes to drive as fast as possible. However, as it progresses through a random environment, it needs to adapt its velocity so as to ensure its own safety, as well as that of its surroundings, given that it is constrained to a maximum deceleration rate constraint of a m/s^2, i.e., it can brake to reduce its velocity by at most a m/s per second. We shall consider the following environment-aware velocity adaptation policy:

- When the vehicle is driving through the uncertainty region of any object in its environment at time t, it sets its velocity to $v_t = v_{\min} > 0$ m/s, assumed to be small enough to ensure its safety and that no harm is done to others, in spite of the object's position uncertainty.
- When the vehicle is **not** driving through the uncertainty region of any object, it sets its velocity so as to be able to decelerate down to $v_{\min}$ m/s prior to the edge of the next object's uncertainty region, i.e., the next intersection point between the road and any uncertainty region's disk ahead of the vehicle (shown in yellow in Figure 1). The deceleration rate constraint induces a braking distance $d_b(v) = \frac{(v^2 - v_{\min}^2)}{2a}$ meters, i.e., by the time the vehicle decelerates from velocity v down to $v_{\min}$, it would have traveled a distance of $d_b(v)$ meters. Using basic geometry, the distance $d_{i,t}$ between the vehicle and the intersection point between the road and the uncertainty disk of object i such that $\phi_t \cap \mathcal{S}_t$ and $|y_{i,t}| \le \beta_i$ at time t is
$$d_{i,t} = x_{i,t} - \sqrt{\beta_i^2 - y_{i,t}^2}. \tag{1}$$
Hence, equating the braking distance to the smallest $d_{i,t}$, the vehicle's velocity v_t m/s at time t in this environment can then be expressed as:
$$v_t = \sqrt{v_{\min}^2 + 2a \min_{i:\ (x_i, y_i) \in \phi_t \cap \mathcal{S}_t} d_{i,t}}. \tag{2}$$
- The vehicle's velocity cannot exceed $v_{\max}$ m/s, modeling, e.g., any internal mechanical constraint or external constraint such as the road's speed limit for instance.

Note that we assume that objects behind the vehicle do not impact its velocity. Thus, we shall restrict our analysis to the region in front of the vehicle, as illustrated in Figure 1.

While other parameters could possibly be integrated into this driving model, e.g., a maximum vehicle acceleration rate, we shall examine the proposed stylized model capturing the salient characteristics of our system, in an effort to keep the analysis insightful, general and tractable.

III. Velocity Characterization

We first aim to characterize the vehicle's ability to make forward progress in the random environment described previously. In this section, we shall assume that for all i in the sensing region, $\beta_i = \beta$, and $r > \frac{(v_{\max}^2 - v_{\min}^2)}{2a}$, i.e., all the objects that can directly impact the vehicle's velocity are observable. To that end, we use tools from stochastic geometry to derive its velocity distribution. In this section, we first discuss the difference between the notions of spatial and temporal velocity, and derive their respective distributions. We then examine the sensitivity to key network parameters such as λ and β, and examine the extent to which they improve the vehicle's ability to progress safely through its environment.

A. Spatial/Temporal Velocity Characterization

The first metric we will use to characterize the vehicle's ability to move forward in a random environment is its *spatial velocity*, defined below.

Definition 1: (Spatial Velocity) *The vehicle's spatial velocity is the velocity at which it is driving at a typical location on the road.*

The spatial velocity, that we denote by a random variable V_S, is a function of the random environment representing the objects' estimated locations and their uncertainty regions.

Interestingly, the spatial velocity metric does not fully capture what the vehicle experiences while progressing through its environment. Indeed, a vehicle might spend most of its time in the "densest" regions of the network, i.e., where several objects are close to the road, as compared to emptier regions where the vehicle can drive at higher speeds, hence spend less time therein. To capture this phenomenon, we define the notion of *temporal velocity*.

Definition 2: (Temporal Velocity) *The vehicle's temporal velocity is the velocity at which it is driving at a typical time.*

Theorem 1 below characterizes the distribution of the vehicle's spatial velocity, and the relation to its temporal velocity. The proof can be found in Appendix A.

Theorem 1: (Velocity Distributions Characterization) *The c.d.f. of the spatial velocity V_S is that of a mixed random variable:*

$$F_{V_S}(v) = \begin{cases} 0, & 0 \leq v < v_{\min}, \\ 1 - e^{-\lambda\left(\frac{\pi\beta^2}{2} + \frac{\beta}{a}(v^2 - v_{\min}^2)\right)}, & v_{\min} \leq v < v_{\max}, \\ 1, & v \geq v_{\max}. \end{cases} \tag{3}$$

The temporal velocity V_T is also mixed with masses as $v_{\min}$ and $v_{\max}$:

$$p_{V_T}(v) = \frac{p_{V_S}(v)}{v \cdot \mathbb{E}[\frac{1}{V_S}]}, \qquad \textit{for } v \in \{v_{\min}, v_{\max}\}, \tag{4}$$

and continuous density for $v \in (v_{\min}, v_{\max})$:

$$f_{V_T}(v) = \frac{f_{V_S}(v)}{v \cdot \mathbb{E}[\frac{1}{V_S}]}, \quad \textit{for } v_{\min} < v < v_{\max}. \tag{5}$$

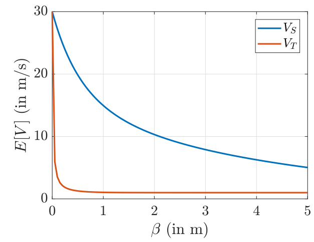
Note that V_S and V_T are mixed random variables with discrete atom masses at $v_{\min}$ and $v_{\max}$.

We emphasize that the temporal velocity is the right metric to use when it comes to evaluating the vehicle's "average velocity" as it captures its progress through its environment.

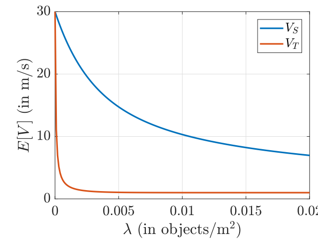
B. Parameter Sensitivity Analysis

Equipped with the distributions for the vehicle's spatial and temporal velocity distributions, we now investigate the role of the two major network parameters impacting the vehicle's ability to progress through its environment, namely the radius of the uncertainty regions β and the density λ of objects in the network. Figure 2 depicts how these parameters impact the mean vehicle velocity.

As can be surmised from the functional form of the previously presented velocity distributions, the mean velocity decreases quickly as either β or λ increase. We also note that the sensitivity to these parameters is much more dramatic for the mean temporal velocity as compared to the mean spatial velocity. This can be explained by the fact that, unlike the vehicle's spatial velocity, its temporal velocity is strongly impacted by objects having uncertainty regions overlapping with the road as the time spent in these regions can be made arbitrarily large for a small enough $v_{\min}$ with the vehicle making little forward progress. These observations suggest that policies controlling these parameters, such as the one we shall study in the sequel, might be effective at improving the vehicle's ability to progress quickly through its environment.



(a) Plot of the mean spatial and temporal velocity as a function of β, for $\lambda = 0.01$ objects/m^2.



(b) Plot of the mean spatial and temporal velocity as a function of λ, for $\beta = 2$m.

Fig. 2: Sensitivity Analysis of the mean spatial and temporal velocity to β and λ, for $v_{\min} = 1m/s$, $v_{\max} = 30m/s$, and $a = 3m/s^2$.

IV. Mobility Analysis under Sensing/Communication Constrained Tracking

In the analysis presented in the previous section, we assumed that the vehicle has access to an estimate of the position of all the objects located in its sensing region with the same accuracy, modelled by an uncertainty region with radius β for all the objects. This model might be reasonable in some scenarios, such as when the vehicle collects location measurements from all the objects with the same accuracy, and at the same sampling rate. However, collecting measurements often comes at a cost, e.g., communication cost if the objects are sending their own position estimates over a wireless channel, or power consumption if the vehicle takes its own measurements. In such a setting, it might be beneficial for the vehicle to focus its available resources on estimating more accurately the position of the objects that have the largest impact on decreasing its velocity. In this section, we propose and evaluate the performance of a general approach to allocating communication/sensing resources so as to optimize the vehicle's ability to drive faster in uncertain environments, subject to a sensing/communication data budget.

A. Problem Formulation

We shall assume that the object i's uncertainty radius $\beta_i = \beta_i(\rho_i)$ is in general a non-increasing function of the object type and the sensing/communication rate ρ_i allocated to it, modeling the fact that an increased sensing/communication rate increases the estimation accuracy. The specific functional form might be, for instance, a function of the measurement quality, and is assumed to be known by the vehicle. It follows from this and from Equation 1 that the distance between the vehicle and object i's uncertainty region $d_{i,t}(\rho_i)$ is a continuous non-negative non-decreasing function of ρ_i. Thus, given a sensing/communication rate budget b available to a vehicle, we aim at optimally allocating the available resources among the objects in the vehicle's sensing region so as to maximize its velocity. Hence, we solve the following optimization problem at any time t:

$$\arg \max_{\boldsymbol{\rho} \in \mathbb{R}^{n_t}} v_t(\rho_i), \qquad \text{s.t.} \sum_i \rho_i \leq b, \qquad (6)$$

$$= \arg \max_{\boldsymbol{\rho} \in \mathbb{R}^{n_t}} \min_{i:(x_i,y_i) \in \phi_t \cap \mathcal{S}_t} d_{i,t}(\rho_i), \qquad \text{s.t.} \sum_i \rho_i \leq b, \qquad (7)$$

where the equality directly follows from the fact that v_t is an increasing function of $\min_{i:(x_i,y_i)\in\phi_t\cap\mathcal{S}_t} d_{i,t}(\rho_i)$ (see Equation 2), and $n_t = |\mathcal{S}_t|$.

B. *Max-Min Algorithm*

To solve the optimization problem in Equation 7, we interpret it as an *utility max-min optimization problem*. It has been proven in [9], [10] that a simple algorithm solving it starts with an all-zeros rate vector and increases the rate to all the objects such that all their respective utilities always remain equal and increase by the same amount until the capacity constraint is reached. Algorithm 1 below proposes a procedure finding an ϵ-optimal rate vector $\boldsymbol{\rho}$ solving the utility max-min fair flow-control problem with non-linear non-decreasing utility functions with complexity $\mathcal{O}(\frac{n_t b}{\epsilon})$, i.e., such that $\|\boldsymbol{\rho} - \boldsymbol{\rho}^*\|_1 \leq \epsilon$ where $\boldsymbol{\rho}^*$ is the true optimal rate vector.

Algorithm 1: UTILITY MAX-MIN FAIR SOLVER

Data: $b \in \mathbb{R}_+$, $1/\epsilon \in \mathbb{N}$, $\{d_{i,t}(\cdot), \forall i\}$.
Result: Solves for $\boldsymbol{\rho}$

```
ρ_i = 0, ∀i
for k=1 to 1/ε do
    m = arg min_i d_{i,t}(ρ_i)
    ρ_m = ρ_m + bε
end
return ρ
```

Alternatively, traditional gradient-based min-max optimization problem solvers could also be used to solve Equation 7. However, such general solvers have proved to execute considerably slower in our experiments for a similar accuracy level ϵ as compared to our algorithm, particularly when n_t is large [11].

Figure 3 shows how the vehicle can optimally allocate its sensing/communication resources so as to maximize its velocity in the same environment as in Figure 1. The objects the furthest from the road are allocated little resources and have the largest uncertainty regions, while the closest ones to the road/vehicle get the most resources and their uncertainty regions are smaller and tangent to the road.

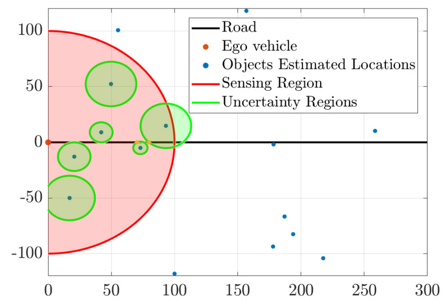
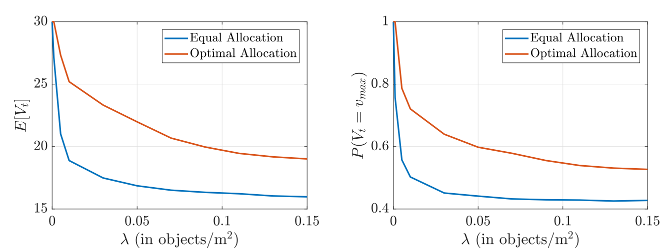


Fig. 3: Figure of a random environment realization, and optimally allocated sensing/communication resources; for $b = 1$, $r = 100$m, and the uncertainty radius model in Equation 8. The coordinates are expressed in meters.

C. *Performance Analysis*

We now evaluate the performance gains associated with the proposed optimal measurement rate allocation policy, and compare it to a simpler algorithm sharing the sensing/communication rate equally to all the objects located in the vehicle's sensing region $\mathcal{S}_t$ at time t. In this section, we augment the object placement and tracking model previously proposed to include object mobility, as opposed to simply studying a snapshot of the environment. Figure 4 below exhibits the gains in terms of the mean temporal velocity experienced by the vehicle, as a function of the density of objects λ. The mobility model adopted in our simulations is an independent bivariate Brownian motion for all objects, with standard deviation ν^2 m^2/s in both the x and y-coordinates. Thus, at every time step of τ seconds, each object moves in an independent random direction characterized by a vector with x and y component being drawn from a normal distribution with mean 0m and variance $\nu^2\tau$ m^2.



(a) Plot of the mean temporal velocity as a function of λ.

(b) Plot of the maximal velocity probability as a function of λ.

Fig. 4: Optimal rate allocation algorithm performance evaluation, for $v_{\min} = 1m/s$, $v_{\max} = 30m/s$, $a = 3m/s^2$, and $\nu^2 = 5m^2/s$.

These results assume that object i's uncertainty radius β_i is related to the measurement rate ρ_i allocated to it as follows:

$$\beta_i(\rho_i) = \min\left[20, \sqrt{20/\rho_i}\right], \forall i. \qquad (8)$$

This functional form captures the essential features of the estimation error as a function of the sampling rate, as the uncertainty region radius is proportional to the error variance, itself inversely proportional to the number of sample (or sampling rate in a given time frame), while the β_i is capped at a large maximum value if no rate is allocated to object i belonging to the sensing region.

Figure 4a exhibits the considerable mean (temporal) velocity gains that can be achieved with the max-min fair sensing/communication rate allocation procedure, particularly in a low object density regime. This algorithm is evidently effective even in dynamic environments, and reduces substantially the mean velocity's sensitivity to the density of objects. In addition, Figure 4b showcases the fact that the proposed algorithm allows the vehicle to drive for a considerably larger fraction of time at maximal velocity compared to the baseline algorithm allocating an equal measurement rate to all objects in the sensing region.

V. Conclusion

In this work, we investigated the effect that a random environment might have on the safe driving velocity of an autonomous vehicle, while estimating its environment in real time. Our stylized system model enabled us to precisely characterize the spatial and temporal velocity experienced by a vehicle driving through its environment by using stochastic geometry. We then examined measurement rate limited scenarios, and argued that the vehicle ought to focus its limited measurement rate budget on estimating with high accuracy the nearby objects impacting its velocity the most. We proposed a formal approach to optimizing sensing/communication resource allocation so as to maximize the vehicle's flow in its environment, and showed that it can achieve considerable velocity gains compared to equal rate allocation policies in dynamic environments.

Appendix A
Theorem 1 Proof

Proof. Let t be the time at which the vehicle hits a typical location on the road, say the origin. Defining the braking distance $D_{i,t}$ at time t induced by object $i \in \Phi_t$ as $D_{i,t} = \left[X_{i,t} - \sqrt{\beta^2 - Y_{i,t}^2}\right]_+$, where $x_+ = \max[x, 0]$, and adopting the convention that $D_{i,t} = \infty$ when $|Y_{j,t}| > \beta$, the spatial velocity V_{S} can be formally expressed as:

$$V_{\text{S}} = \min\{\sqrt{v_{\min}^2 + 2a \min_{i:(x_i,y_i)\in\phi_t\cap\mathcal{S}_t} D_{i,t}}, v_{\max}\} \tag{9}$$

$$= \min_{i:(x_i,y_i)\in\phi_t\cap\mathcal{S}_t} \min\{\sqrt{v_{\min}^2 + 2aD_{i,t}}, v_{\max}\} \tag{10}$$

Additionally, we define the velocity gap $V_{\text{gap}} = v_{\max} - V_{\text{S}}$, which, using Equation 10 can also be expressed as

$$V_{\text{gap}} = \max_{i:(x_i,y_i)\in\phi_t\cap\mathcal{S}_t} \left[v_{\max} - \sqrt{v_{\min}^2 + 2aD_{i,t}}\right]_+, \tag{11}$$

which has the form of the so-called *extremal* shot noise, see [12], [13], with response function

$$h(x,y) = \left[v_{\max} - \sqrt{v_{\min}^2 + 2a\left[x - \sqrt{\beta^2 - y^2}\right]_+}\right]_+. \tag{12}$$

It follows that V_{gap} has the following distribution:

$$\mathbb{P}(V_{\text{gap}} \le v) = e^{-\lambda \iint_{\mathbb{R}^2} \mathbb{1}\{h(x,y)>v\}dxdy}, \forall v \in \mathbb{R}_+. \tag{13}$$

For the response function expressed in Equation 12, we have

$$\iint_{\mathbb{R}^2} \mathbb{1}\{h(x,y) > v\}dxdy = \int_{-\beta}^{\beta}\int_0^{\frac{(v_{\max}-v)^2 - v_{\min}^2}{2a} + \sqrt{\beta^2-y^2}} dxdy \tag{14}$$

$$= \frac{\pi\beta^2}{2} + \frac{\beta}{a}((v_{\max} - v)^2 - v_{\min}^2). \tag{15}$$

Now, we get by the change of variables $V_{\text{gap}} = v_{\max} - V_{\text{S}}$, we have for any $v_{\min} < v < v_{\max}$:

$$\mathbb{P}(v_{\max} - V_{\text{gap}} > v_{\max} - v) = e^{-\lambda(\frac{\pi\beta^2}{2} + \frac{\beta}{a}((v_{\max}-v)^2 - v_{\min}^2))} \tag{16}$$

$$\iff \mathbb{P}(V_{\text{S}} > v) = e^{-\lambda(\frac{\pi\beta^2}{2} + \frac{\beta}{a}(v^2 - v_{\min}^2))} \tag{17}$$

$$\iff \mathbb{P}(V_{\text{S}} \le v) = 1 - e^{-\lambda(\frac{\pi\beta^2}{2} + \frac{\beta}{a}(v^2 - v_{\min}^2))}. \tag{18}$$

The second part of the theorem follows from the observation that the vehicle's temporal velocity is equivalent to its time-weighted spatial velocity. Particularly, we obtain the distribution of the temporal velocity by biasing the spatial velocity's one by a factor of $\frac{1}{v}$ (and renormalizing the distribution accordingly) as the time Δt spent by the vehicle on a road segment of length Δd is inversely proportional to its velocity, i.e., $\Delta t = \frac{\Delta d}{v}$. □

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