Matching

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Background

Undirected graph $G = (V, E)$ is *bipartite* if V can be partitioned into U and W such that all $e \in E$ have one end point in U and one in W

- easily checked — run DFS, alternately assign red and blue to vertices; conflict iff nonbipartite

$M \subset E$ is a *matching* if no two edges in M have a vertex in common

- vertex v is said to be matched if $\exists e = (v, u) \in M$

Big matchings

Many applications where an assignment is to be computed involve finding a large matching

- men and women at dance party, connected if willing to dance with each other $\Rightarrow$ pair off men and women to maximize number of dance partners
- jobs and repairmen, connected if repairman can perform job $\Rightarrow$ assign jobs to repairmen so as to maximize number of jobs that can be performed

Maximal vs. maximum

Defn. M is *maximal* if $\nexists M' \supset M$ such that M' is a matching (i.e., no edge can be added to M without violating the matching condition).

Defn. M is *maximum* if $\nexists M'$ such that $|M'| > |M|$ (i.e., there is no matching larger than M in sense of cardinality).

- A matching can be maximal without being maximum (the "Z" example)

Algorithm

Alternating path theorem implies can find a maximum matching by starting from any matching, and systematically enlarging it by searching for alternating paths

- find alternating paths by doing DFS from unmatched vertex, alternating between unmatched and matched edges
  - each DFS has complexity $\Theta(|V| + |E|)$, and we perform DFS at most $|V|$ times, i.e., overall complexity is $\Theta(|V| \cdot (|V| + |E|))$. Dense graph $\Rightarrow$ $\Theta(|V|^3)$ complexity
- can improve to $\Theta(\sqrt{|V|} \cdot (|V| + |E|))$ by doing BFS level by level, stopping when hit layer of unmatched nodes (best known algorithm)

Matching for general graphs — 1

Alternating path theorem goes through exactly as before

- problem is in finding the alternating paths:
  - general graph: vertex may appear on alternating path at even or odd distance from start
- allow only one visit to vertex: may miss alternating paths
- allow two visits: may generate path that's not alternating

Computing maximum matchings

Defn. Alternating path for a matching M is a path from $u \in U$ to $w \in W$, both unmatched by M, and along the path, edges alternate between M and $E - M$.

- pretty clear — $\exists$ alternating path for M, then $\exists$ matching M' such that $|M'| = |M| + 1$
  - just flip the edges on the path

Remarkably, the converse also holds: $\nexists$ alternating path for M, then M is maximum (referred to as the "alternating path theorem")

- to see this, it's sufficient to show M is not maximum $\Rightarrow$ $\exists$ alternating path for M

Let P be a maximum matching; consider the graph $G' = (V, P \cup M)$ (P and M not necessarily disjoint)

- in G' each vertex is incident to at most one edge from P, and one edge from M $\Rightarrow$ each vertex in G' is incident to at most two edges
  - consider the connected components of G': singletons, cycles, paths are the only possibilities
    - at least one of the paths must be alternating for M (since $|P| > |M|$) — we're done

Argument: for any matching M and vertex cover V, we have $| M | \leq | V |$

- can build vertex cover from maximal matching N
  - take vertices corresponding to edges in maximal matching N, must be a vertex cover, since remaining vertices have no edges between them, else N not maximal)
  - this cover is of size $2N$, so $2N \geq M$

Hall's theorem

Given a bipartite graph $G = (U \cup W, E)$, a matching exists where all $u \in U$ are matched iff $\forall A \subset U \ \ | \mathcal{N}(A) | \geq | A |$

- proof based on inductively building matching of size $1, 2, \ldots, | U |$
- use condition of Hall's theorem to find alternating paths to increase size

Matching for general graphs — 2

Edmonds showed problem arises only if graph contains certain kinds of cycles

- gave $O(n^4)$ algorithm (1965) which "shrank" these cycles

Current best: Micali & Vazirani (1980)
$\Theta(\sqrt{|V|} \cdot (| V | + | E |))$

Approximating maximum matching

Algo for maximal matching — walk down list of edges, keep adding to matching

- check whether vertices on an edge are already matched is constant time $\Rightarrow$ overall linear time

Maximal matching cannot be less than half the size of a maximum matching

- can be half (Z example)

Applications:

- NOT: find if a perfect matching exists for general bipartite graph
  - exponential number of subsets to check — just run the polytime max matching algo
- any k-regular bipartite graph has perfect matching
- any graph induced by a doubly stochastic matrix has a perfect matching

Just count the number of edges coming out of a subset, and entering the neighboring set