uesday 29 Aug '84

EE221a LINEAR SYSTEM THEORY - Dr Felix Wu

What is System Theory:

Physical System — Eg. Cct

Models — idealized capacitors, resistors, etc

math representations

Sys. Thry. deals c̄ Math Representation
⇒ develop analysis & design tools based on math tools

This course
- 1 Mostly Linear Systems (to be defined later)
- 2 Attempt to assist you to be able to develop analysis & design tools. (provide nec. training)
- 3 Emphasis on rigorous & precision

Same spirit in exams, HW.

Control System

-1 c̄/o
- simulation
- measurement of I/P or O/P
- numerical soln

study properties of the solution (output)
chc
stability

-2 O/p not satisfactory
- ~~xxxx~~ can we change it
- how " " " "

$$
\begin{aligned}
Ae_1 &= a_{11} b + a_{12}Ab + a_{13} A^2b \quad \cdots \quad a_{1n} A^{n-1}b \\
Ae_2 &= a_{21} b + a_{22} Ab + a_{23} A^2 b + \cdots \quad a_{2n} A^{n-1} b \\
&\vdots \\
Ae_n &= a_{n1} b + \cdots \quad a_{nn} A^{n-1} b
\end{aligned}
$$

we wish to represent this in the form of a single Eqn

complex elements

$$
A \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b \\ Ab \\ A^2 b \\ \vdots \\ A^{n-1} b \end{pmatrix}
$$

$$
\underset{\sim}{A} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = \underset{\sim}{A} \begin{pmatrix} b \\ Ab \\ \vdots \\ A^{n-1} b \end{pmatrix}
$$

$$
\begin{aligned}
\underset{\sim}{A} e_1 &= \underset{\sim}{A}\, b \\
\underset{\sim}{A} e_2 &= \underset{\sim}{A}\, Ab \\
\underset{\sim}{A} e_3 &= \underset{\sim}{A}\, A^2 b
\end{aligned}
\qquad
\underset{\sim}{A} e_1 = Ab \qquad \boxed{\underset{\sim}{A}\, e_m = \underset{\sim}{A}^m b}
$$

$$
\underset{\sim}{A} (e_1\ e_2\ \cdots\ e_n) = \underset{\sim}{A} (b\ \ Ab\ \ A^2 b\ \cdots\ A^{n-1} b)
$$

$$
\underset{\sim}{A}\, I = \underset{\sim}{A}\, (b\ \ Ab\ \cdots\ A^{n-1} b)
$$

reno & sacramento from city 1/2

~~11:00~~ ~~11:00am~~

10:30am

1509 Hearst

#23

Apt 201

1-2: Lin Alg & Diff Eqns
3-4: General Lin. Sys. Fundamentals
5-8: Time Inv.
9: Stability
10-13: Controllability, Observability
14-15: Discrete Event Systems

COPYMAT: 2560 Bancroft → Dr Desoer's Notes

HW: Every Thursday due on Friday next
↳ Seems to be very important (vis a vis your grade)

Discussion: TA will work out examples. Stress discrete time systems.

✓ Kailath: explains more but a little confusing
✓ Chen: more mathematical, all details worked out
Callier & Desoer: for last part
Ogata: //l development in discrete time control

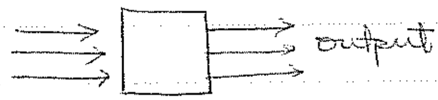
Chapter 1: Math Background

Linear System - define linearity
Two main ~~pa~~ topics in this chapter

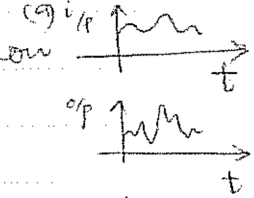
① Linear function
  - representation

② Diff Eqns
  - solution (existence & uniqueness Questions)

Where do we start -
General Setting

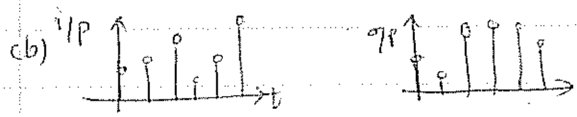


1) Classification



(a): cont. time systems

(b)



(b): Discrete Time

(c) 01001 i/p   00110 o/p   (c): Digital

Leads us to Math. concept of <u>function</u>

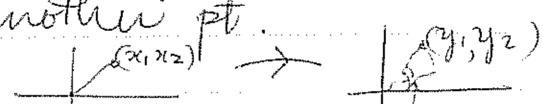
$u(t)$: value of u at t $u(\cdot)$: waveform

Roughly speaking the "system" assigns an o/p $y(\cdot)$ for a given i/p $u(\cdot)$. This is a "function"!

Defn of function $f: X \longrightarrow Y$
$\forall x \in X$ f assigns a unique $f(x)$ in Y

We are familiar with functions ~~such as~~ which move a pt to another pt.

eg. $y = Ax$



$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\underline{Y} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \underline{X}$$

Our system doesn't geometrically move pts in this way. It shift functions (waveforms) to other functions (waveforms)

Q. What do $u(\cdot)$ $y(\cdot)$ have in common?
- They are "vectors" in a "vector space."
  - A vector space is defined over a "field".

1 Vector Space

1) F is a field iff (A) Addition $\forall \alpha, \beta \in F$ $\alpha+\beta \in F$ uniquely defined

(A1) $\alpha+\beta=\beta+\alpha$; commutative

(A2) $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$; associativity $\forall \alpha,\beta,\gamma \in F$

(A3) $\exists\ 0 \in F$ s.t. $\alpha+0=\alpha$ $\forall \alpha \in F$; (additive identity)

(A4) $\forall \alpha \in F\ \exists\ (-\alpha) \in F$ s.t. $\alpha+(-\alpha)=0$; (additive inverse)

(M) multiplication (M1) $\forall \alpha, \beta \in F$ $\alpha\beta \in F$; uniquely defined $\alpha\beta = \beta\alpha$

(M2) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$; commutativity associativity

(M3) $\exists\ 1 \in F$ s.t. $\alpha \cdot 1 = \alpha$; mult identity

(M4) $\forall \alpha \neq 0\ \exists\ \alpha^{-1}$ s.t. $\alpha \cdot \alpha^{-1} = 1$; mult inverse

(D) distribute $\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma$

Eg. ✓$[\mathbb{R},+,\times]$ ✓$[\mathbb{C},+,\times]$ ✗$[\mathbb{Z},+,\times]$ ✓$[\{0,1\},\oplus,\cdot]$
✓[set of rational functions]

2) Vector Space <Linear Space>

(V, F) V is a vector space over the field F iff

(A) $\forall x, y \in V$ $x+y \in V$ uniquely defined

(A1) $x+y=y+x$

(A2) $x+(y+z)=(x+y)+z$

(A3) $\exists\ \theta \in V$ s.t. $x+\theta=x$

(A4) $\forall x \in V\ \exists\ (-x) \in V$ s.t. $x+(-x)=\theta$

(SM) $\forall \alpha \in F\ \forall x \in V$ $\alpha x \in V$ uniquely defined

(SM1) $(\alpha\beta)x = \alpha(\beta x)$

(SM2) $\alpha(x+y)=\alpha x+\alpha y$

(SM3) $(\alpha+\beta)x=\alpha x+\beta x$

$x+(-x)=\theta$
$x+(-x)'=\theta$
$x+(-x)=x+(-x)'$
$1\cdot(-x)=(-x)'$

Eg. Show θ is unique →on any vector space

Suppose both θ & θ' are zeroes
We want to show $\theta=\theta'$.

by defn $\left\{\begin{array}{l} x+\theta=x \\ \forall x\ \exists(-x) \text{ st. } x+(-x)=\theta \end{array}\right\}$ also $\left\{\begin{array}{l} x+\theta'=x \\ \forall x\ \exists(-x)' \ x+(-x)'=\theta' \end{array}\right.$

Is it nece- the same?

$\Rightarrow (x+\theta)+(-x')=\theta'$
$\overset{\shortparallel}{x}$

comm $=(\theta+x)+(-x')$
assoc $=\theta+(x+(-x'))$
$=\theta+\theta'$

sim. $(x+\theta')+(-x)=\theta$
$=\theta+\theta'$
comm $=\theta'+\theta$

$\Rightarrow \theta'=\theta+\theta'=\theta$

Canonical Examples : Pts & functions

31 Aug '89 Thursday

LST-2

Vector Space [Linear Space]
(V,F)
(A) $\forall x,y\in V$ $x+y\in V$ uniquely defined
(A1) $x+y=y+x$
(A2) $x+(y+z)=(x+y)+z$
(A3) $\exists\ \theta\in V$ s.t. $x+\theta=x$
(A4) $\forall x\in V$ $\exists\ (-x)\in V$ s.t. $x+(-x)=\theta$
(SM) $\forall \alpha\in F$ $\forall x\in V$ $\alpha x\in V$ uniquely defined
(SM1) $(\alpha\beta)x=\alpha(\beta x)$
(SM2) $\alpha(x+y)=\alpha x+\alpha y$
(SM3) $(\alpha+\beta)x=\alpha x+\beta x$
(SM4) $1=$ mult identity in F $1x=x$

3) <u>Canonical Examples</u>

(a) (F^n, F) $\quad F^n$ = n tuples of F

eg. $(\mathbb{R}^n, \mathbb{R})$ $\quad x \in \mathbb{R}^n$ if $x = [x_1\, x_2 \ldots x_n]^T$; $x_n \in \mathbb{R}$

$(\mathbb{C}^n, \mathbb{C})$

Exc: Show this is a v.s.

(b) $F_{(D,V)}$: set of all functions $f : \underset{\substack{\uparrow \\ \text{Any Set}}}{\mathbb{D}} \rightarrow \underset{\substack{\uparrow \\ \text{Vector space}}}{\mathbb{V}}$ over $\mathbb{R}$ or $\mathbb{C}$ (or any field)

Can make F into a vector space by defining +, ·

define (A): $f, g \in F(D, V)$

we want to define $f + g \in F(D,V)$

$$\forall d \in D \quad (f+g)(d) \in V$$

$$\therefore \text{define } \underbrace{(f+g)(d) := \underbrace{f(d)}_{\text{in } V} + \underbrace{g(d)}_{\text{in } V}}_{\text{an element in } V!}$$

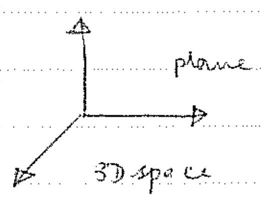
$\leftarrow$ addition is defined in this vect sp.

define (SM): Take $\alpha \in \mathbb{R}$

$$\text{want to define } \underbrace{(\alpha f)}_{\text{in } F(D,V)}(d) = \underbrace{\overbrace{\alpha \underbrace{f(d)}_{\text{in } V}}^{\text{S.M. in } V}}_{\text{in } V}$$

Having defined (A) & (SM) we must ~~define~~ confirm the props.

- - - - - - - - - - -



- plane a subspace of 3D space motiv. for defining subspaces.

W is a subspace of (V, F)

iff (1) $W \subset F$ (That's $W \subset V$)

(2) (W, F) itself qualifies as a vector space

$$\left.\begin{cases} w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W \\ \alpha \in F,\ w \in W \Rightarrow \alpha w \in W \end{cases}\right\} \text{ Don't need to check any further!}$$

Lets examine the set of functions $F(D, V)$

(a) Set of continuous functions $f: \underbrace{[t_0, t_1]}_{\text{in } \mathbb{R}} \to \mathbb{R}$ [$\mathbb{C}$]

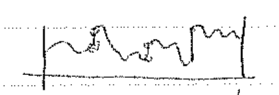
Notation $C[t_0, t_1]$ denotes " " " [",""]

(b) Set of continuous functions $f: [t_0, t_1] \to \mathbb{R}^n$ [$\mathbb{C}^n$]

$C^n[t_0, t_1]$

∴ If infinite interval, could we get countably ∞ pts?

(c) Piecewise continuous functions



(i) discontinuous points strictly finite

(ii) at a discontinuous pt, limits of a fn. from either side exist and are finite.

2. LINEAR TRANSFORMATION

(1) Linear Independence

(V, F) vector space

X set of vectors in V

X is a set of linearly independent vectors iff for any ~~set~~ finite collection of distinct element in X say $\{x_1 \ldots x_N\}$

for any scalars $\alpha_1 \ldots \alpha_N$

if $\alpha_1 x_1 + \alpha_2 x_2 + \cdots \alpha_N x_n = 0 \Rightarrow$ all $\alpha_i = 0$.

If not say set X is lin dep.

then at least one can be written as a lin. comb. of others

$\alpha_1' x_1 + \alpha_2' x_2 + \cdots \alpha_m' x_m = \theta$

say $\alpha_2' = 0$

$x_2' = -\alpha_1' / \alpha_2' \, x_1' - \cdots$

(2) Span of a set of vectors

(V, F)

$X \subset V$

Span of X = set of all l.c. of vectors in X.

$Sp\{X\} = \{ \underset{\text{finite sum}}{\sum_i \alpha_i x_i}, \alpha_i \in F, x_i \in V \}$

(3) BASIS

The set of vectors X is said to be a basis of V iff

(i) X linearly & independant

(ii) $Sp(X) = V$

elements in X are called basis vectors

$\dim(V)$ = cardinality of X   could be ∞ eg. Periodic (F.S.)

(4) COORDINATE SYSTEM

(V, F) $\{b_1, b_2 \ldots b_n\}$ (finite dim. implicit)

an ordered set of basis vectors.

Then there [1]exists a [2]unique set of scalars $(\xi_1, \xi_2, \xi_3, \ldots \xi_n)$ for any vector $x \in V$ s.t.

$x = \sum_{i=1}^{n} \xi_i b_i$; ξ_i called components of x wrt basis $\{b_1 \ldots b_n\}$

Pf(1) (Existence): The fact that $b_1 \ldots b_n$ spans $V \Rightarrow$

$\forall x \in V$, $x = \sum_{i=1}^{n} \xi_i b_i$

Pf(2) (Uniqueness): Suppose not. $\Rightarrow$

$x = \sum_{i=1}^{n} \bar{\xi}_i b_i \Rightarrow \theta = x - x = \sum_{i=1}^{n} (\xi_i - \bar{\xi}_i) b_i$; LI $\Rightarrow \xi_i = \bar{\xi}_i$

Geometrically $V \leftrightarrow F^n$ $\{b_i\}_{i=1}^{n}$ once you pick a basis can view V as $\{\xi\}_{i=1}^{n}$ F^n

(5) LINEAR TRANSFORMATION

vector spaces (U, F) (V, F)

same (over the same field)

$\mathcal{A} : U \to V$ is a linear function (or transformation or map or operator)

iff $\mathcal{A}(\overset{\in F}{\alpha_1} u_1 + \overset{\in F}{\alpha_2} u_2) = \alpha_1 \mathcal{A}(u_1) + \alpha_2 \mathcal{A}(u_2)$

in U   in V   in V

Examples

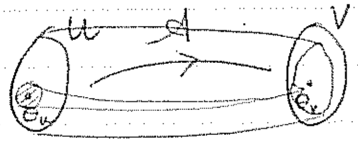
(i) $\mathcal{A} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ over $\mathbb{R}$

$[x_1 \;\; x_2]^T \longmapsto [\alpha x_1 - \beta x_2 \quad \beta x_1 + \alpha x_2]^T$; α, β are fixed

(ii) $\mathcal{A} : P[0,T] \longmapsto P[0,T]$ $\qquad v(t) = (\mathcal{A}u)(t) := \int_0^t e^{-(t-\tau)} u(\tau)\,d\tau$

$u(\cdot) \qquad\qquad v(\cdot) \qquad\qquad t \in [0,T]$

show (i) & (ii) are linear operators.



HW: show $\theta_u \to \theta_v$

It is possible more than one $\in U \to \theta_v$

Nullspace of $\mathcal{A}$: $N(\mathcal{A}) := \{u \in U \text{ s.t. } \mathcal{A}u = \theta_v\}$

Claim: $N(\mathcal{A}) \subset U$ and is a subspace in U.

$\hookrightarrow u_1 \in N(\mathcal{A}) \quad u_2 \in N(\mathcal{A})$

$u_1 + u_2 \in N(\mathcal{A})$ (trivial)

range space of A. $R(\mathcal{A}) := \{v \in V \text{ s.t. } v = \mathcal{A}u \text{ for some } u \in U\}$

Can show $R(\mathcal{A})$ is a subspace of V.

(6) MATRIX REPRESENTATION

$\mathcal{A} : U \longrightarrow V \longleftrightarrow \{v_j\}_{j=1}^m \; F^m$

$\{u_i\}_{i=1}^n$ basis $\quad F^n$

$x = \sum_{i=1}^n \xi_i u_i \xrightarrow{?} y = \sum_{j=1}^m \eta_j v_j$

$[\;]$ coordinates of u_i pulled through $\mathcal{A}$

Let $y = \mathcal{A}x$ (i.e. y is image of x)

How can we directly express $\mathcal{A}$ in terms of the coordinate system? Ans: matrix.

$y = \mathcal{A}x$; $\mathcal{A}x = \mathcal{A}\left(\sum_{i=1}^n \xi_i u_i\right) = \sum_{i=1}^n \xi_i (\mathcal{A}u_i)$; from lin. of $\mathcal{A}$

If we know what $\mathcal{A}$ does to $u_i|_{i=1}^n$, then we know how it acts on any vector x.

Consider $\mathcal{A}(u_i)$

$$\mathcal{A}u_1 = a_{11}v_1 + a_{21}v_2 + \cdots a_{m1}v_m$$
$$\mathcal{A}u_2 = a_{12}v_1 + a_{22}v_2 \quad \cdots \quad a_{m2}v_m$$
$$\vdots$$
$$\mathcal{A}u_n = a_{1n}v_1 + a_{2n}V_2 \quad \cdots \quad a_{mn}V_m$$

$$\begin{aligned}\mathcal{A}x &= a_{11}\xi_1 v_1 + a_{21}\xi_1 v_2 + \cdots a_{m1}\xi_1 v_m \\ &\quad + a_{12}\xi_2 v_1 + a_{22}\xi_2 v_2 + \cdots a_{m2}\xi_2 v_m \\ &\quad \cdots \\ &\quad + a_{1n}\xi_n v_1 + a_{2n}\xi_n v_2 + \cdots + a_{mn}\xi_n v_m \\ &= \eta_1 v_1 + \eta_2 v_2 + \cdots \eta_m v_m\end{aligned}$$

$$\Rightarrow \eta_1 = a_{11}\xi_1 + a_{12}\xi_2 + \cdots a_{1n}\xi_n$$
$$\eta_2 = \cdots$$
$$\eta_m = a_{m1}\xi_1 + a_{m2}\xi_2 + \cdots a_{mn}\xi_n$$

$$\begin{bmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_m\end{bmatrix} = \underbrace{\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}}_{\text{matrix}}\begin{bmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_n\end{bmatrix}$$

i^{th} column of matrix = the coordinate of $\mathcal{A}u_i$ expressed wrt $\{v_j\}_{j=1}^{m}$

This is imp. of generality thm!

Wednesday
Sept 18 '89

LST

Friday 3:00–3:30 TA
3:30–5:00 Lecture

Last time – matrix representation of a linear map.

$$\begin{tikzcd} x\ \mathcal{U} \arrow[r, "\mathcal{A}"] \arrow[d, leftrightarrow, "\{u_i\}_{i=1}^n"'] & \mathcal{V}\ y \arrow[d, leftrightarrow, "\{v_j\}_{j=1}^m"] \\ F^n \arrow[r] & F^m \end{tikzcd} \qquad y = \mathcal{A}x$$

$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_n \end{bmatrix}$ $\quad$ $\begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{bmatrix}$

$\underset{\sim}{A}$ matrix: equiv representation of map

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{\text{matrix}} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$$

i^{th} column $\begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}$ = $\mathcal{A}u_i$ expressed wrt $\{v_j\}$ may be obtained by taking the i^{th} basis vector

$$\mathcal{A}u_i = \sum_{i=1}^{m} a_{ji} v_j$$

Ex. Consider an Identity Map

$$\begin{tikzcd} x\ \mathcal{U} \arrow[r, "I"] \arrow[d, leftrightarrow, "\{\tilde{u}_i\}_{i=1}^n"'] & \mathcal{U} \arrow[d, leftrightarrow, "\{u_i\}_{i=1}^n"] \\ F^n \arrow[r, "P"'] & F^n \end{tikzcd}$$

; clearly one can show this is a linear map.

: can be used to represent any x.

i^{th} column

$$\tilde{u}_i = \sum_{j=1}^{n} p_{ji} u_j$$

Change of Coordinates

$$\begin{tikzcd} \{\tilde{\xi}_i\} & & \{\tilde{\eta}_j\} \\ x\ \mathcal{U} \arrow[u, leftrightarrow, "\{\tilde{u}_i\}"] \arrow[r, "\mathcal{A}"] \arrow[d, leftrightarrow, "\{u_i\}_{i=1}^n"'] & \mathcal{V}\ y \arrow[d, leftrightarrow, "\{v_j\}_{j=1}^m"] \arrow[u, leftrightarrow, "\{\tilde{v}_j\}_{j=1}^m"'] \\ \{\xi_i\}_{i=1}^n\ F^n \arrow[r, "A"'] & F^m\ \{\eta_j\}_{j=1}^m \end{tikzcd}$$

with $\tilde{A}$ between the top spaces, P on the left and Q on the right.

Q: how is A related to $\tilde{A}$?

A, $\tilde{A}$ maps as follows:

$$[\tilde{\eta}_j] = \tilde{A}[\tilde{\xi}_i]$$
$$[\eta_j] = A[\xi_i]$$

$$P[\tilde{\xi}_i] = [\xi_i]$$
$$Q[\eta_j] = [\tilde{\eta}_j]$$

$\Rightarrow \tilde{A} = QAP$ $\quad$ This is not a proof. (See reader for real pf.)

3. DIFFERENTIAL EQUATIONS

1) Normed Linear Space

motivation: notion of length in $(\mathbb{R}^3, \mathbb{R})$

non negative

Δ inequality

independant of the basis ?

x-y symmetric

Properties $\|x\|$, length of vector x, ≥ 0 ($=0 \overset{\Leftrightarrow}{\text{iff}} x = \Theta$) — By consid $(\mathbb{R}^3, \mathbb{R})$

$\|x+y\| \leq \|x\| + \|y\|$

$\|x\| = \|-x\|$

$\|\alpha x\| = |\alpha| \, \|x\|$; (in $\mathbb{R}^3$)

A linear space (V, F) ($\mathbb{R}$ or $\mathbb{C}$) is said to be a normed linear space iff $\exists$ a fn, denoted by $\|\cdot\|$ mapping from $V \longrightarrow \mathbb{R}_+ = \{x \mid x \in \mathbb{R}, x \geq 0\}$ satisfying (1) $\|x\| = 0 \Leftrightarrow x = \Theta$

(2) $\|\alpha x\| = |\alpha| \|x\|$

(3) $\|x+y\| \leq \|x\| + \|y\|$

Ex (a) $(\mathbb{R}^n, \mathbb{R})$ $x = (x_1, x_2 \ldots x_n)$

$(\mathbb{C}^n, \mathbb{C})$ follows

define $\|x\|_\infty := \max\limits_{i} |x_i|$; show it satisfies (i) – (iii)

$\|x\|_1 := \sum\limits_{i=1}^{n} |x_i|$; " " " " " "

$\|x\|_2 := \left(\sum\limits_{i=1}^{n} |x_i|^2\right)^{1/2}$; " " " " " "

in general $\|x\|_p := \left(\sum\limits_{i=1}^{n} |x_i|^p\right)^{1/p}$; " " " " "

Ex (b) $F(D, V)$

(i) $C([t_0, t_1], \mathbb{R})$; real cont. fns. on $[t_0, t_1]$; could do by yourself

$\|f\|_\infty := \max\limits_{t \in [t_0, t_1]} |f(t)|$ cont. fn on a bdd interval $\Rightarrow$ Max. exists

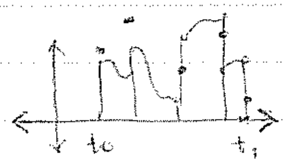
$\|f_1\| := \int_{t_0}^{t_1} |f(t)| \, dt$

(ii) $C([t_0, t_1], \mathbb{C}^n)$

$\|f\|_\infty = \max\limits_{t \in [t_0, t_1]} \underbrace{\|f(t)\|}_{\text{norm in } \mathbb{C}^n}$ — in $\mathbb{C}^n$

In this course $\|\cdot\|_\infty$ taken for norm in $\mathbb{C}^n$

(iii) $P([t_0, t_1], \mathbb{R})$: Piece wise continuous



@ disc. fn may have any value
Req. L & R lims exist (i.e. non ∞)

The right norm is not going to consider pts. of disc. but will consider lim. pts.

$\|f\|_\infty := \text{ess sup} |f(t)|$ (a value attained @ a pt. which has other pts in any nbhood of it.

(2) Linear Maps : Induced Norm

$x \quad U \xrightarrow{\mathcal{A}} V \quad y$

$\|\cdot\|_u \qquad \|\cdot\|_v$

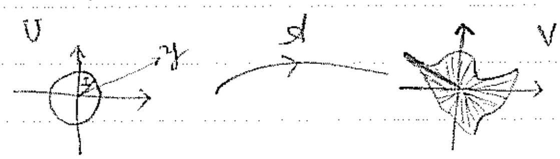
$y = \mathcal{A}x$

$\|y\|_v \qquad \|x\|_u$

Q: any relation between the two? 'max ampli'

Ans: Yes if one introduces norm induced on Lin. Map $\mathcal{A}$

geometrically $\{x \in U : \|x\|_u = 1\}$



$\sup_{\|x\|_u = 1} \|\mathcal{A}x\|_v \qquad \|\mathcal{A}\| := \sup_{\|x\|_u = 1} \|\mathcal{A}x\|_v$

for $\|x\|_u = 1$, Then immediately $\|\mathcal{A}x\|_v \leq \|\mathcal{A}\|$; This is only true on the unit ball.

Excessively Restrictive

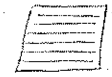
However $\because$ of linearity can scale up & down!
$\therefore$ will hold in general! (Can remove the restriction)

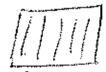
Pick $y \in U$ to be any vector $x := y/\|y\|_u$ then $\|x\|_u = 1$.

$\|\mathcal{A}x\|_v = \|\mathcal{A}\, y/\|y\|_u\|_v = \|\mathcal{A}y\|_v/\|y\|_u$

$$\|\mathcal{A}\| = \sup_{\|x\|_u = 1} \|\mathcal{A}x\|_v = \sup_{y \neq 0} \frac{\|\mathcal{A}y\|_v}{\|y\|_u} \implies \|\mathcal{A}y\|_v \leq \|\mathcal{A}\| \cdot \|y\|_u$$

Ex. $A: \mathbb{C}^n \longrightarrow \mathbb{C}^n$

$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$ (row sum) 

$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$ (column sum) 

$\|A\|_2 = \max_i (\lambda_i (A^* A))^{1/2}$ (eigen values)

How is this connectio to DE?

$\because$ of convergence!

(3) Convergence

A seq. of vectors $v_1, v_2, v_3, \ldots v_K, \ldots$ in (V, F), ($\mathbb{R}$ or $\mathbb{C}$)

conv. to a vector V if

iff $\underbrace{\|v_K - V\|}$ $\to 0$ as $K \to \infty$

(in terms of convergence of seq. of real #s)

(4) Differential Equations

$\cancel{\dot{X}(t) = p(x(t), t) \,;\, x(0) = x_0}$ $\quad \dot{X}(t) = p(x(t), t) \;;\; x(0) = x_0$

$x(t) \in \mathbb{C}^n$

Q: [$\bar{c}$/o solving eqn, we want to know if there is a sol, (ch1) and if a soln exists whether its unique] [if not unique how many, if yes to unique Q. can a basis be picked.] (ch2)

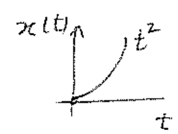
Dont Physical Systems always have existing unique solns

Q - Often modelling is incorrect.

- Chaotic systems: very innocent eqns

Ex 1.) $\dot{x} = -1/x \;;\; x(0) = 0$ no soln exists

Ex 2.) $\dot{x} = 2\sqrt{|x|} \;;\; x(0) = 0$



$dx/2\sqrt{|x|} = dt \Rightarrow \sqrt{|x|} = t \;;\; x(t) = t^2 \quad \forall\ t \geq 0$

claim $\begin{cases} (t-c)^2 = x(t) & \forall\ t \geq c \\ 0 = x(t) & \forall\ t \text{ in } [0, c) \end{cases}$

is also a soln $\forall$ all c! $\Rightarrow \infty$ # of solns, sys. can take off @ anytime.

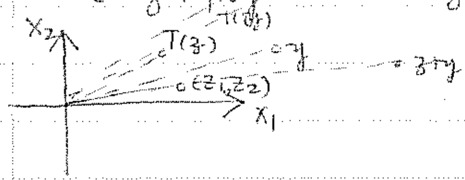
Thursday 15 September '18.

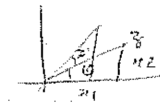
LST: EE221a - Discussion

Ben's office hours
M 11-12   145 Cory
* Th 10-11   642-6954 bbonham@sim

(1) Ex of Linear Transformation : Rotation in 2D
(2) " " " " : Convolution
(3) Matrix rep.
(4) change of coord
(5) induced norms: l_1, l_2, l_∞

(1) $T(\alpha z + \beta y) = \alpha \overset{T(z+y)}{T(z)} + \beta T(y)$





$z = (x_1, x_2)$

$\sqrt{z_1^2+y_2^2} \cos(\theta+30°) \cdot z_1$
$\sqrt{z_1^2+y_2^2} \sin(\theta+30°) \cdot z_2$

$T(z) = (z_1 \sqrt{3}/2 - z_2 1/2,\ z_1 1/2 + z_2 \sqrt{3}/2)$

$\alpha z + \beta y = (\alpha z_1 + \beta y_1,\ \alpha z_2 + \beta y_2)$

$T(\alpha z + \beta y) = [(\alpha z_1 + \beta y_1)\sqrt{3}/2 - (\alpha z_2 + \beta y_2) 1/2,\ (\alpha z_1 + \beta y_2) 1/2 + (\alpha z_2 + \beta y_2)\sqrt{3}/2]$

$= \alpha(z_1 \sqrt{3}/2 - z_2 1/2) + \beta(y_1 \sqrt{3}/2 - y_2 1/2),\ \alpha(z_2 1/2 + z_2 \sqrt{3}/2) + \beta(y_2 1/2 + \sqrt{3}/2\, y_2)$

$= \alpha T(z) + \beta T(y)$

$x(t) = \int_0^t g(t-\tau) u(\tau) d\tau$

$x(t) = \int_0^t g(\tau) u(t-\tau) d\tau$

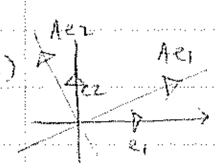
(2) $A : f \to g$

$g(t) = \int_0^t \underbrace{f(t-\tau)}_{\text{kernel}} u(\tau) d\tau$

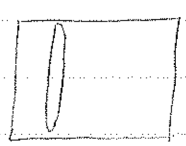
$A : \alpha f_1 + \beta f_2 \longmapsto \int_0^t (\alpha f_1 + \beta f_2) u(t-\tau) d\tau = \int_0^t (\alpha f_1(t-\tau) + \beta f_2(t-\tau)) u(\tau) d\tau$

$= \alpha A f_1 + \beta A f_2$

(3)



A =



i transformed i^{th} basis vector in terms of original basis vectors

$A e_1 = (\cos\theta, \sin\theta) = Ae_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = A \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e_1}$ $Ae_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

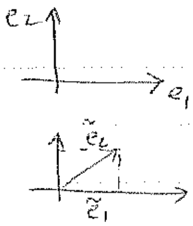
$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(4) change of coordinates

$$\begin{array}{ccc} \tilde{\xi} \overset{\tilde{\pi}}{\leftrightarrows} U & \xrightarrow{\tilde{A}} & V \overset{\tilde{\gamma}}{\leftrightarrows} \tilde{\eta} \\ P \updownarrow \;\; \pi & & \gamma \\ \xi & \xrightarrow{A} & \eta \end{array}$$

change of coordinates

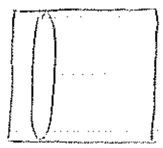
a change of coordinate



$\tilde{e}_2 = e_1 + e_2$

$\tilde{e}_1 = e_1$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \tilde{A} = ?$$



change of coordinates

original i^{th} basis vectors in terms of new basis vectors.

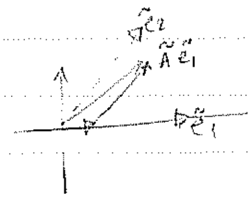
consider conditions under which P is inverse of Q?

eg. from $\tilde{e}$ to e basis

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & -1 \\ 0 & +1 \end{bmatrix}$$

$\eta = A\xi = AP\tilde{\xi}$

$\tilde{\eta} = Q\eta = \underbrace{QAP}_{\tilde{A}}\tilde{\xi}$



$$\tilde{A} = \begin{bmatrix} (\cos\theta - \sin\theta) & -2\sin\theta \\ \sin\theta & (\sin\theta + \cos\theta) \end{bmatrix}$$

$\tilde{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\tilde{A}\tilde{e}_1 = \begin{bmatrix} \frac{1}{2}(\sqrt{3}-1) \\ \frac{1}{2} \end{bmatrix}$ $\tilde{e}_1$ dir, $\tilde{e}_2$ dir

$\Rightarrow$ does work

$A : \mathbb{R}^2 \rightarrow C^2[0,1]$

$(U \rightarrow V)$

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ basis for $\mathbb{R}^2$

$A \cdot (x_1, x_2) \longmapsto t(x_1) + (1-t)x_2 \quad t \in [0,1] \qquad \{(t,0), (0,(1-t))\}$

(5) Induced Norm vector space, ℓ, norm $\|\cdot\|_1$

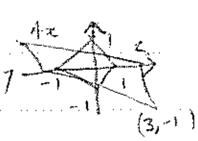
$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$; $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

$\|A\|_1 = \sup_{\|x\|=1} \|Ax\|$

$$A = \begin{bmatrix} 5 & 2 \\ -1 & 0 \end{bmatrix}$$

$\|x\|_1 = 1$

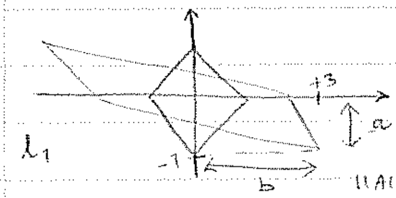
in 2dim $|x_1| + |x_2| = 1$



; suppose the matrix A multiplies this unit sphere : get ▱gm type shape. Then obtain Norm from this ie $(|3| + |-1| = 4$.

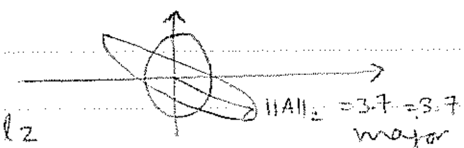
Friday
8 September 1989

LST - Disc + Lect



l_1

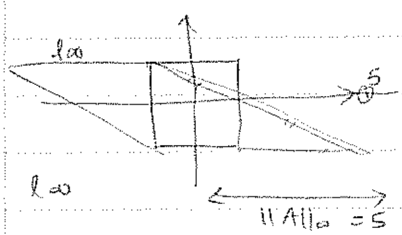
$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$



A as above $\Rightarrow$ ellipse

l_2 major & minor axis determined by e.v.

$\|x\|_2 = \left(\sum_i x_i^2\right)^{1/2}$ $\quad$ $\|A\|_2$ is by max. val. of ~~evalues~~ eigenvalues of $A^H A$



l_∞

l_1 norm : column sum

Proof: $\|A\|_1 = \sup_{\|x\|_1 = 1} \|Ax\|_1 = \sup_{\|x\|_1} \left(\sum_i \left| \sum_j (a_{ij} x_j) \right| \right)$ by defn of $\|\ \|_1$ norm

$(Ax)_i = \left(\sum_j a_{ij} x_j\right)$ $\qquad$ $\sup_{\|y\|=1} \sum_i |y_i|$

$$\sum_i \left| \sum_j a_{ij} x_j \right| \leq \sum_i \sum_j |a_{ij} x_j| = \sum_j \sum_i |a_{ij} x_j|$$

by $\triangle$ ineq.

$$\leq \sum_j \sum_i |a_{ij}|\,|x_j|$$

$$= \sum_j \left(|x_j| \sum_i |a_{ij}| \right)$$

$$\sum_j \left(|x_j| \sum_i |a_{ij}|\right) \leq \sum_j \left(|x_j| \max_j \sum_i |a_{ij}|\right)$$

$$= \max_j \sum_i |a_{ij}| \cdot \underbrace{\sum_j |x_j|}_{=1 \text{ (by constraint}}$$

$$= \max_j \sum_i |a_{ij}|$$

ie $\|A\|_1 \leq \max_j \sum_i |a_{ij}|$; To show equality $\to$ correct x to choose

$\begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix}$ $\leftarrow$ position of max column sum

$$\|A\|_\infty = \sup_{\|y\|_\infty = 1} \Big(\max_k \big| \underbrace{\textstyle\sum_j a_{kj} y_j}_{(Ay)_k} \big| \Big)$$

$\max |y_j| = 1$ (from defn of $\|y\|_\infty = \max_j |y_j|$)

$\therefore \max_k |\sum_j a_{kj} y_j| \le \max_k \sum_j |a_{kj}|$ $\quad (*)$

let m = index of $\max_k \sum_j |a_{kj}| = \sum_j |a_{mj}|$

define $y_j^0 = \operatorname{sgn}(a_{mj})$ $\quad y_0^T = (y_1^0 \; y_2^0 \; \dots \; y_n^0)$

So $\|Ay_0\|_\infty = \max_k |\sum_j a_{kj} \operatorname{sgn}(a_{mj})|$

$\ge |\sum_j a_{mj} \operatorname{sgn}(a_{mj})| = \sum_j |a_{mj}|$ $\quad (**)$

compare $(**)$ $\bar{c}$ $(*)$

$\|Ay\|_\infty \le \max_m \sum_j |a_{mj}| \le \|Ay^0\|_\infty$

$\forall \|y\|_\infty = 1$ $\quad \|A\|_\infty = \max_k \sum_j |a_{kj}|$

LST-lecture

(4) DIFFERENTIAL EQUATIONS

$\dot{x} = p(x,t)$ $\quad x(t_0) = x_0$

Q: Is there a soln ($\bar{c}$/o solving it)

Is the soln unique

dep. on initial condition

Ex. (i) $\dot{x} = -1/x$ $\quad X(0) = 0$ no soln

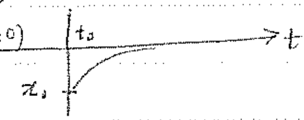
(ii) $\dot{x} = 2\sqrt{|x|}$ $\quad X(0) = 0$ inf soln

(iii) $\dot{x} = x^2$ $\quad X(t_0) = x_0$

$dx/x^2 = -dt \Rightarrow -1/x + 1/x_0 = t - t_0$

$x = 1/(1-(t-t_0)x_0) \cdot x_0$

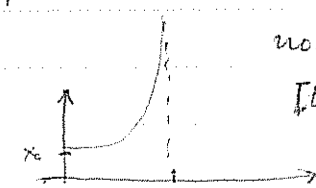
case(1) x_0 is negative (actually ≤ 0)



case(2) x_0 is positive

$1 - (t-t_0)x_0 < 1$ for $t > t_0$

when $(t-t_0)x_0 = 1 \Rightarrow t = t_0 + 1/x_0$



no solution outside $[t_0, t_0 + 1/x_0)$

No soln on $[t_0, \infty)$

(5) Fundamental Theorem of Diff Eqn

- Existence, Uniqueness of soln $\dot{x} = p(x,t) \quad x(t_0) = x_0$

Suppose $p : \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ satisfying

(a) $\forall x \in \mathbb{R}^n \quad t \longmapsto p(x,t)$ piecewise continuous; finite LH & R limits

set of discontinuous points D (maybe empty)

(b) $\forall t \in \mathbb{R}_+ \quad \exists\, K(t)$ piecewise continuous st $\forall \xi, \xi' \in \mathbb{R}^n$

$$\| p(\xi,t) - p(\xi',t) \| \leq K(t) \| \xi - \xi' \| \Leftrightarrow p(\xi,t) \text{ is cont at each } t.$$

(Lipschutz condition)

Then

(i) existence

$\forall x_0 \in \mathbb{R}^n, \forall t \in \mathbb{R}_+$

$\exists$ continuous function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ s.t.

$\dot{\phi}(t) = p(\phi(t), t) \quad \phi(t_0) = x_0$

ϕ is called to soln. of $\dot{x} = p(x,t) \quad x(t_0) = x_0$

(ii) uniqueness

soln is unique

(iii) soln. dep. on x_0 continuously

Note: all ex. dont satisfy Lipschutz cond.

Nice proof in Desoer.

Idea of proof

(i) $\dot{x}(t) = p(x(t), t)$, $x(t_0) = x_0$ (1)

Integrate $\Rightarrow x(t) = x_0 + \int_{t_0}^{t} p(x(t'), t')\, dt'$ (2)

$\updownarrow$ 1:2 by Fund. thm of calc.

RHS is a mapping $F : C^n[0,\infty) \longrightarrow C^n[0,\infty)$

$x(\cdot) \longmapsto (Fx)(\cdot)$

$$(Fx)(t) = x_0 + \int_{t_0}^{t} p(x(t'), t')\, dt'$$

Eqn (2) $\Rightarrow$ $x(\cdot)$ is a solution to (2) $\Leftrightarrow Fx = x$

(iv) consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

contraction mapping fixed point thm

Let $(V, \|\cdot\|)$ complete normed linear space

Let $F : V \longrightarrow V$

then (i) $\exists$ exactly one x^* in V

s.t. $Fx^* = x^*$

(ii) for any $x_0 \in V$ then the sequence $\{x_n\} = \{x_0, Fx_0, F(F(x_0)),$

$x_n \to x^*$

(v) Lipschitz condition $\Rightarrow$ F is a contraction $\Rightarrow F$ has a fixed pt $\Rightarrow$ DE has a solution.

The application of contraction mapping provides a solution technique:

(Picard Method) (IC problems use waveform relaxation techniques which are in essence Picards method)

Consider $\dot{x} = x$; $x(0) = x_0$

$$F : x(\cdot) \longmapsto (Fx)(\cdot)$$

$$(Fx)(t) = x_0 + \int_0^t p(x(t'), t')\,dt'$$

$x_0(\cdot)$ $\qquad = x_0 + \int_0^t x(t')\,dt'$

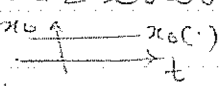
$$F x_0(\cdot) = x_1(\cdot)$$

$$F x_1(\cdot) = x_2(\cdot)$$

$$\vdots \qquad \vdots$$

gives soln to DE eventually : Can start from any fn. eg

const. $x_0(t) = x_0$

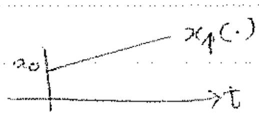


$$x_1(t) = (Fx_0)(t)$$

$$= x_0 + \int_0^t x_0\,dt'$$

$$= x_0 + x_0 t$$

$$= x_0(1+t)$$



$$x_2(t) = (Fx_1)(t)$$

$$= x_0 + \int_0^t x_0(1+t')\,dt'$$

$$= x_0(1 + t + t^2/2!)$$

$$x_n(t) = x_0(1 + t + \cdots \; t^n/n!)$$

$n \to \infty$ $\qquad x_\infty(t) = x_0 e^t$

Computationally very slow.

(6) Cont. dependance on x_0

→ Bellman – Gronwall inequality

$\dot{x} = p(x(t), t)$; $x(t_0) = x_0$

$\dot{y} = p(y(t), t)$ $y(t_0) = y_0$ $\forall T \in [0, \infty)$

if $\|x_0 - y_0\| \to 0$ then $\|x(\cdot) - y(\cdot)\|_\infty \to 0$

norm in $\mathbb{R}^n$ norm in function space defined on $C^n([0,T])$

Derivation of Bellman-Gronwall inequality

Pre analysis

$$x(t) = x_0 + \int_{t_0}^{t} p(x(t'), t')\,dt'$$

$$y(t) = y_0 + \int_{t_0}^{t} p(y(t'), t')\,dt'$$

$$[x(t) - y(t)] = (x_0 - y_0) + \int_{t_0}^{t} [p(x(t'), t') - p(y(t'), t')]\,dt'$$

$$\|x(t) - y(t)\| \leq \|x_0 - y_0\| + \left\|\int_{t_0}^{t} \text{''}\right\|$$

$$\leq \|x_0 - y_0\| + \int_{t_0}^{t} \|\text{''}\|\,dt'$$

use L. Cond.

$$\Rightarrow \leq \underbrace{\|x_0 - y_0\|}_{c_1} + \int_{t_0}^{t} K(t') \underbrace{\|x(t') - y(t')\|}_{u(t')}\,dt'$$

<u>Want</u> $\underbrace{x(t) - y(t)}_{u(t)} \leq \bigodot \|x_0 - y_0\|$

$$u(t) \leq c_1 + \int_{t_0}^{t} K(t')\,u(t')\,dt'$$

want $u(t) \leq \bigodot c_1$

<u>If</u> we had equality

$$u(t) = c_1 + \int_{t_0}^{t} K(t')\,u(t')\,dt'$$

Take derivative $\dot{u}(t) = K(t)\,u(t)$ $u(t_0) = c_1$ Back to DE

we know soln of DE.

$$\Rightarrow u(t) = c_1\, e^{\int_{t_0}^{t} K(t')\,dt'}$$

<u>But</u> we have inequality

<u>Q</u> : (1) $\Rightarrow u(t) \leq c_1\, e^{\int_{t_0}^{t} K(t')\,dt'}$

Result is Bellman – Gronwall inequality

Suppose $K(t) \geq 0 \;\; \forall t \in \mathbb{R}_+$

If $u(t) \leq c_1 + \int_{t_0}^{t} K(t')\,u(t')\,dt' \;\; \forall t \in \mathbb{R}_+$

then $u(t) \leq c_1 e^{\int^{t} K(t')\,dt'} \;\; \forall\, t \in \mathbb{R}$ This would imply ineq.

2 September '0? Tuesday

LST lecture

cont dep on x_0

$\dot{x} = p(x,t) \quad x(t_0) = x_0$

$\dot{y} = p(y,t) \quad y(t_0) = y_0$

want $\|x_0 - y_0\| \to 0 \Rightarrow \|x_0 - y_0\| \to 0$

$x(t) = x_0 + \int_{t_0}^{t} p(x(t'), t')dt'$

$y(t) = y_0 + \int_{t_0}^{t} p(y(t'), t')dt'$

$\Rightarrow \|x(t) - y(t)\| \leq \|x_0 - y_0\| + \int_{t_0}^{t} \kappa(t') \|x(t') - y(t')\| dt'$

want $\|x(t) - y(t)\| \leq \boxed{?} \|x_0 - y_0\|$

Use: Bellman-Gronwall ineq.

suppose $\kappa(t) \geq 0 \;\; \forall t$

If $u(t) \leq c_1 + \int_{t_0}^{t} \kappa(t') u(t') dt' \quad \forall t$

then $u(t) \leq c_1 e^{\int_{t_0}^{t} \kappa(t')dt'} \quad \forall t$

Proof: Desoer's notes very concise; understand each line.

How to prove this?

(2) $\Leftrightarrow$ $u(t) e^{\int_{t_0}^{t} -\kappa(t')dt'} \leq c_1$ (3) more tractable

does it hold for $t = t_0$? yes! (from (1))

To prove LHS of (3) is a decreasing fn of t, look at the derivative.

$\frac{d}{dt}\left[u(t) e^{\int_{t_0}^{t} -\kappa(t')dt'}\right] \underset{\text{we have}}{\leq 0} \quad \forall t \geq t_0$

then (3) would be true if $u(t) \leq 0$ but we don't have it

$\Rightarrow$ No good

Look for bounding value/function

consider (1) $\quad u(t) \leq U(t) \quad$ (4)

If we can show $U(t) \exp \int_{t_0}^{t} -(\kappa(t'))dt' \leq c_1$ (5)

(4) & (5) combined $\Rightarrow$ (3)

consider (5) at $t = t_0$

@ $t = t_0 \quad U(t_0) \leq c_1$ true!

$\frac{d}{dt}\left[U(t) e^{-\int_{t}^{t_0} \kappa(t')dt'}\right] \leq 0$

$LHS = \dot{U}(t) e^{\int_{t_0}^{t} -\kappa(t')dt'} - \kappa(t) U(t) e^{\int_{t_0}^{t} -\kappa(t')dt'}$ (diff in parts)

$= \kappa(t) u(t) \; e^{\cdots} - \kappa(t) U(t) e^{\cdots}$

$= \kappa(t) e^{\cdots} [u(t) - U(t)]$

$\leq 0 \quad$ QED

From BG ineq.

$$\|x(t)-y(t)\| \leq \|x_0-y_0\| + \int_{t_0}^{t} k(t')\|x(t')-y(t')\|dt'$$

$$\Rightarrow \|x(t)-y(t)\| \leq \|x_0-y_0\| \exp\left(\int_{t_0}^{t} k(t')dt'\right)$$

So as $\|x_0-y_0\| \to 0 \Rightarrow \|x(t)-y(t)\| \to 0 \quad \forall t$

ie $\|x-y\|_\infty \to 0$ (∞ norm used here)

In particular $x_0 = y_0 \Rightarrow x(t) = y(t) \quad \forall t$

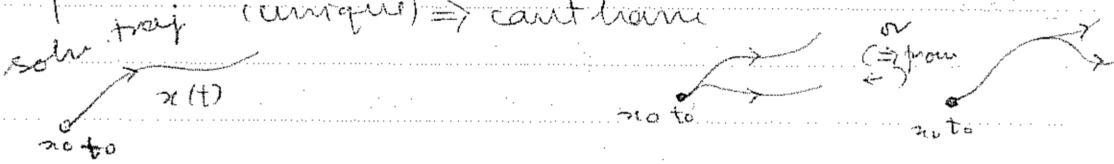
Soln is UNIQUE

(8) Reverse time

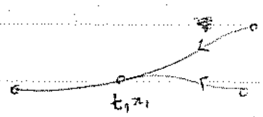
$\dot{x} = p(x,t)$; $x(t_0) = t_0$ (1)

p satisfies the conditions in the Fund Thm then (1) has a unique soln.

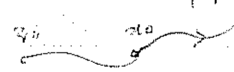
Implication (unique) ⇒ can't have



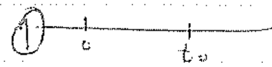
How about



Time reversed in d.e. then what happens



Thm RT $\dot{x} = p(x,t)$ $x(t_0) = x_0$ $t_0 \in [0,\infty)$ (1)



if p satisfies the conds of the Fund. Thm then ∃ unique $z_0 \in \mathbb{R}$ s.t. the unique soln of $\dot{x} = p(x,t)$ $x(0) = z_0$ (2) satisfies $x(t_0) = x_0$ (3)

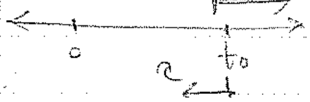
Need to show (i) Find z_0

(ii) Soln to the d.e (2) sat. (3)

(iii) unique

How to find z_0?

Start from x_0, 'reverse' time of the d.e. (1) the soln at $t=0$ is z_0. How to reverse the time?



$\dot{x} = x^2 \Rightarrow x = \frac{x^3}{3}$

$\alpha \frac{x^3}{3} + \beta$

Relation between reverse time & forward time

$\boxed{\tau = t_0 - t}$

Let the soln of the reversed d.e. be $z(\tau)$

$\frac{d}{d\tau} z(\tau) = \bar{p}[z(\tau), \tau]$; $z(0) = x_0$

what $\bar{p}(z,t)$ should be?

For $\tau = t_0 - t$

$x(t) = z(\tau)$

consider $d/d\tau\, z(\tau) = -d/dt\, x(t)$; from $\frac{d}{d\tau} z(\tau) = \frac{dx(t)}{dt}\frac{dt}{d\tau} = -\frac{dx(t)}{dt}$

$= \bar{p}(z,\tau)$

$= -\cancel{d/dt}[p(x(t), t)]$

$= -p(z(\tau), t_0 - \tau)$

So $\bar{p}(z,\tau) = -p(z, t_0 - \tau)$

Proof (not in Desoer!)

Let $\bar{p}(z,\tau) := -p(z, t_0 - \tau)$ $\forall \tau \in [0, t_0]$

p satisfies ~~cond~~ in fund thm.

$\Rightarrow \bar{p}$ " " " "

$d/d\tau\, z(\tau) = \bar{p}(z,\tau)$ $z(0) = x_0$ (6)

has a unique soln. $\tau \in [0, t_0]$

Let us define $z_0 := z(t_0)$ (6½) (well defined from fund thm.)

~~Show~~ Now let $x(t) := z(t_0 - t)$ $t \in [0, t_0]$ (7)

$d/dt\, x(t) = -d/d(t_0-t)\, z(t_0 - t)$ $= \frac{d}{d(t_0-t)} z(t_0-t) \cdot \frac{d(t_0-t)}{dt} = -\frac{dz(t_0-t)}{d(t_0-t)}$

$\overset{(6)}{=} -\bar{p}(z(t_0-t), t_0 - t)$

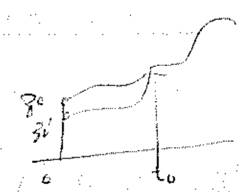
$\overset{(5)}{=} p(z(t_0-t), t)$

$\overset{(7)}{=} p(x(t), t)$

$x(0) \overset{(7)}{=} z(t_0) = z_0$ (6½)

$x(t_0) = z(0) \overset{(6)}{=} x_0$

Uniqueness



Suppose $\exists z_0' \neq z_0$ s.t.

$\begin{cases} d/dt\, y(t) = p(y(t), t) & y(0) = z_0' \\ y(t_0) = x_0 \end{cases}$

We had already shown fn sat. Lip cond & soln exist

$z'(\tau) := y(t_0 - \tau)$ $\frac{d}{d\tau}[z'(\tau)] = \bar{p}(z', \tau)$ $z'(0) = x_0$

Compare ** & (6) $z(\tau), z'(\tau)$ sat. the same i.c.
c̄ initial cons $\bar{p}$ p.c. & Lipschutz.
soln is unique

$$z'(\tau) = z(\tau) \ \forall \tau$$
$$z'(\tau_0) = z(t_0)$$
$$z_0' = z_0$$

4. Systems Fundamentals

$$\dot{x} = A(t)\,x(t) + B(t)\,u(t) \quad ; \ x(t_0) = x_0 \quad (1)$$

$x(t) \in \mathbb{R}^n$ $\quad A(t) \in \mathbb{R}^{n \times n}$
$u(t) \in \mathbb{R}^r$ $\quad B(t) \in \mathbb{R}^{n \times r}$ — components of are p.c. fn of t

For any $u(\cdot)$ the d.e. (1) satisfies the conditions in the fundamental theorem
$\Rightarrow$ solns exists and is unique.

Proof:

$$p(x,t) := A(t)x + B(t)u(t)$$

(i) For fixed x

$$p(x,\cdot) \qquad t \longmapsto A(t)x + B(t)u(t) \ ; \text{ is p.c.}$$

(ii) For fixed t

$$p(\xi,t) - p(\xi',t) = [A(t)\xi + B(t)u(t)] - [A(t)\xi' + B(t)u(t)]$$
$$= A(t)(\xi - \xi')$$

$$\therefore \|p(\xi,t) - p(\xi',t)\| = \|A(t)(\xi - \xi')\|$$
$$\le \underbrace{\|A(t)\|}_{k(t)} \|\xi - \xi'\|$$

Norm the bit about row sums & col sums being Norm. (strictly)
induced norm of A(t) is cont fn. of elements $\Rightarrow$ its p.c. fn of t $\Rightarrow k(t)$ p.c.

$\Rightarrow \exists$ a unique soln to (1) $\Rightarrow \frac{d}{dt} S(t,t_0,x_0,u) = A(t)\,S(t,t_0,x_0,u) + B(t)\,u(t)$

The soln depends on t_0, x_0, u $\quad (S(t_0,t_0,x_0,u) = x_0)$

Lets write explicitly the dependance

soln $= S[(\cdot), t_0, x_0, u]$ at any time t soln is $S(t, t_0, x_0, u)$

time $\quad$ need these because will examine dep. on these.

Soln is defined for $t \geq t_0$ (by fund thm). By thm RT, soln also

Summary
4 Sept 1982

LST-L

Chapter 2 & 3

1 Linear Dynamical Systems

Consider $\dot{x} = A(t)x + B(t)u(t)$; $x(t_0) = x_0$ (1)

(1) <u>Solutions</u>

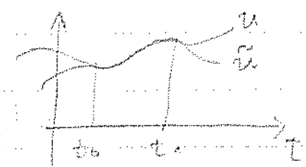
For any $u(\cdot)$, d.e. (1) has a unique solution.

(By the fundamental Theorem)

Let's denote $S(t, t_0, x_0, u)$

$$\begin{cases} \frac{d}{dt} S(t, t_0, x_0, u) = A(t) S(t, t_0, x_0, u) + B(t) u(t) \\ S(t_0, t_0, x_0, u) = x_0 \end{cases}$$

(2) <u>State Transition Property</u>



If $u(t) = \tilde{u}(t) \quad t \in [t_0, t_1]$ $*$

then $S(t_1, t_0, x_0, u) = S(t_1, t_0, x_0, \tilde{u})$

Proof: $S(t, t_0, x_0, u)$

$S(t, t_0, x_0, \tilde{u})$

By defn $\frac{d}{dt} S(t, t_0, x_0, u) = A(t) S(t, t_0, x_0, u) + B(t) u(t)$ (3)

$S(t_0, t_0, x_0, u) = x_0$

$\frac{d}{dt} S(t, t_0, x_0, \tilde{u}) = A(t) S(t, t_0, x_0, \tilde{u}) + B(t) \tilde{u}(t)$ (4)

$S(t_0, t_0, x_0, \tilde{u}) = x_0$

$\because$ of $*$ $t \in [t_0, t_1]$

d. eqn. (3) = d eqn (4)

Fund Thm Sol is unique

$S(t, t_0, x_0, u) = S(t, t_0, x_0, \tilde{u}) \quad t \in [t_0, t_1]$

in particular $S(t_1, t_0, x_0, u) = S(t_1, t_0, x_0, \tilde{u})$

We write $S(t, t_0, x_0, u_{[t_0, t_1]})$

Implication

(1) Given (t_0, x_0)

$x(t) \quad t \in [t_0, t_1]$

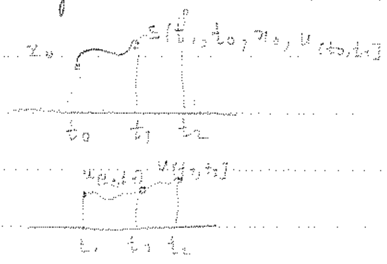
independent of u prior to t_0.

In other words, x_0 summarizes the past history, i.e. everything having happened prior to t_0.

(ii) $x(t) \quad t \in [t_0, t_1]$

is independent of u after t_1, ie system is non-anticipative

(iii) Semigroup property



$S(t_2, t_1, S(t_1, t_0, x_0, u_{[t_0,t_1]}), u_{[t_1,t_2]})$

$= S(t_2, t_0, x_0, u_{[t_0,t_2]})$

$u_{[t_0,t_2]} = u_{[t_0,t_1]} \cup u_{[t_1,t_2]}$

Proof: Fund Thm

(4) Readout function

$\dot{x} = A(t)x + B(t)u$

$y = C(t)x + D(t)u$

$y(t) \in \mathbb{R}^m$ output

$y(t) = r(t, x(t), u(t))$ readout

not nec linear!

Dynamical System

In general we say the model of a physical system is a dynamical system if we can associate with the following entities.

$\mathcal{U}$: a set of fns called input

Σ: a set of fns called state

$\mathcal{Y}$: a set of fns called output

S: State Transition Fn

$x(t) = S(t, t_0, x_0, u)$

r: readout function $\rightarrow t, t_0, x_0, u \Rightarrow y(t) = \rho(t, t_0, x_0, u)$: alternate form

$y(t) = r(t, x(t), u(t))$

The state transition fn. satisfies the following axioms:

(1) State Transition Axiom

(2) Semigroup Axiom

(5) Linearity

Recall $\dot{x} = A(t)x + B(t)u \; ; \; x(t_0) = x_0$

$S(\cdot, t_0, x_0, u)$ For any fixed t_0,

$x_0 \in \Sigma \quad u \in \mathcal{U} \qquad S: \Sigma \times \mathcal{U} \rightarrow \mathcal{Y}$

Claim: S is linear map from $\Sigma \times \mathcal{U} \to \mathcal{Y}$.

Proof: $\alpha(x_0, u)$

take $+ \alpha'(x_0', u')$

$$\underbrace{S(t, t_0, \alpha x_0 + \alpha' x_0', \alpha u + \alpha' u')}_{l(t)} = \underbrace{\alpha S(t, t_0, x_0, u) + \alpha' S(t, t_0, x_0', u')}_{r(t)}$$

show $l(t) = r(t)$; By fundamental thm.

$l(t_0) = \alpha x_0 + \alpha' x_0'$

$\dot{l}(t) = A(t) l(t) + B(t)[\alpha u + \alpha' u']$

$r(t_0) = \alpha x_0 + \alpha' x_0'$

$\dot{r}(t_0) = \alpha[A(t) S(t, t_0, x_0, u) + B(t) u] + \alpha'[A(t) S(t, t_0, x_0', u']$

$= A(t) r(t) + B(t)(\alpha u + \alpha' u')$

$\Rightarrow$ By fund thm $l(t) = r(t) \quad \forall t \geq t_0$

Linear dynamical system

In general we say a dynamical system is linear if the readout map $\rho(t, t_0, x_0, u) = y(t)$ is a linear map from $\Sigma \times \mathcal{U} \to \mathcal{Y}$ i.e.

$$\rho(t, t_0, \alpha x_0 + \alpha' x_0', \alpha u + \alpha' u') = \alpha \rho(t, t_0, x_0, u) + \alpha' \rho(t, t_0, x_0', u') \quad (5)$$

(6) Decomposition

In (5) we take $x_0 = \theta_\Sigma \quad u = u \quad \alpha = 1 \quad x_0' = x_0 \quad u' = \theta_u \quad \alpha' = 1$ (Easy Notation)

Then

$$\underbrace{\rho(t, t_0, x_0, u)}_{\text{total response}} = \underbrace{\rho(t, t_0, \theta_\Sigma, u)}_{\text{zero state resp}} + \underbrace{\rho(t, t_0, x_0, \theta_u)}_{\text{zero i/p response}}$$

(i) Linearity of zero state response

$$\rho(t, t_0, \theta_\Sigma, \alpha u + \alpha' u') = \alpha \rho(t, t_0, \theta_\Sigma, u) + \alpha' \rho(t, t_0, \theta_\Sigma, u')$$

(ii) Linearity of zero i/p response

$$\rho(t, t_0, \alpha x_0 + \alpha' x_0', \theta_u) = \alpha \rho(t, t_0, x_0, \theta_u) + \alpha' \rho(t, t_0, x_0', \theta_u)$$

Corollary: $S(\cdot, t_0, x_0, u) : \Sigma \times \mathcal{U} \to \mathcal{Y}$ (Not linear in general for Σ or $\mathcal{U}$ each unless other is zero vector)

$= ZSR + ZIP$

Z.S.R. linear map of i/p $\qquad$ ZIR linear map of initial states

2. Zero Input Response of $\dot{X} = A(t)X + B(t)u$; $x(t_0) = x_0$

$u(t) \equiv 0$ $\quad \dot{X} = A(t)X$; $X(t_0) = x_0$

Soln is $S(t, t_0, x_0, u)$

$\therefore$ soln is fn of $\phi(t, t_0, x_0)$ $(\because u = 0)$

ie $\dot{\phi}(t,t_0,x_0) = A(t)\,\phi(t,t_0,x_0)$

$\phi(t_0, t_0, x_0) = x_0$

zero i/p response $\phi(t, t_0, x_0)$ is a linear map of x_0

For a fixed t: $\phi(t, t_0, \cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

arb. basis $\{v_i\}_{i=1}^n \longleftrightarrow x_0 \mapsto x(t) = \phi(t, t_0, x_0) \longleftrightarrow$ standard basis $\{e_i\}_{i=1}^n$

Linear Map from $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ has a matrix representation

$x_0 = \sum_{i=1}^{n} \alpha_i v_i$ coord. wrt this basis $[\alpha_1\ \alpha_2\ \ldots\ \alpha_n]^T$

$X(t)$ $\quad \phi(t, t_0, x_0) = [\,\phi_1(t,t_0,x_0);\ \phi_2(t,t_0,x_0), \ldots\,]^T$

$X(t)$ is matrix representation of this linear map w.r.t. $\{v_i\}$ & $\{e_i\}$

ie $\boxed{\underset{\text{vector}}{\phi(t, t_0, x_0)} = \underset{\text{matrix}}{X(t)}\,\underset{\text{vector expressed in }\{v_i\}}{x_0}}$

What is $X(t)$?

algebraic approach

① $\underset{\text{a vector}}{\phi(t, t_0, x_0)} = \underset{\text{a vector}}{\phi}(t, t_0, \sum_{i=1}^{n} \underset{\text{scalars}}{\alpha_i}\, \underset{\text{vectors}}{v_i})$

$$= \sum_{i=1}^{n} \alpha_i\, \phi(t, t_0, v_i) = \underbrace{\left[\,\phi(t,t_0,v_1) \mid \phi(t,t_0,v_2) \mid \ldots\ \phi(t, t_0, v_n)\,\right]}_{\text{MATRIX}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

This must be $X(t)$

conceptual approach

② i^{th} col. of $X(t)$ is $= (v_i) \xrightarrow{T} (\cdot)$ expresses in terms of $\{e_i\}$

$T \to \phi(t, t_0, \cdot)$ soln of d.e.

$= \phi(t, t_0, v_i)$ expressed in terms of $\{e_i\}$

To summarize,

$x_0 \longrightarrow x(t)$: Linear map, has matrix representation

ie know what sys. does to basis, knows what it does for all states

Friday 15 Sept 1989

LST-discussion

1. Uniqueness of soln to $\dot{x} = p(x)$
2. l_2 norm for diagonal matrix
3. existence of soln for cont + discrete time example
4. time reversal for discrete time systems
5. Norms in finite dimension

Q1.  → what if tangents same?

$$\dot{x} = p(x)$$

$$x(t) = \int_0^t p(x)\,dt + x(0)$$

$$\|x-y\| = \left\| \int_0^t [p(x) - p(y)]\,dt' + (x(0) - x(0)) \right\|$$

$$= \left\| \int_0^t (p(x) - p(y))\,dt' \right\|$$

$$\leq \int_0^t \|p(x) - p(y)\|\,dt'$$

$$\|x-y\| \leq \int_0^t k(t)\,\|x-y\|\,dt$$ i.e. $\|x-y\| \leq 0 + \int_0^t k(t)\|x-y\|\,dt' \Rightarrow \|x-y\| \leq \underset{\leq 0}{0 \cdot e^{\int \cdots}}$

Suppose $k(t) \geq 0 \ \forall t'$

$\|x-y\| \leq 0$ by B. G. — if $u(t) \leq c_1 + \int_{t_0}^t k(t')u(t')\,dt' \quad \forall t$

then $u(t) \leq c_1 e^{\int_{t_0}^t k(t')\,dt'} \quad \forall t$

l_2 norm for diagonal matrix $D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$

$$\|A\| = \sup_{\|x\|_2 = 1} \left(\sum_i \left(\sum_j a_{ij} x_j \right)^2 \right)^{1/2}$$

$$= \sup_{\|x\|_2 = 1} \left(\sum (d_i x_i)^2 \right)^{1/2}$$

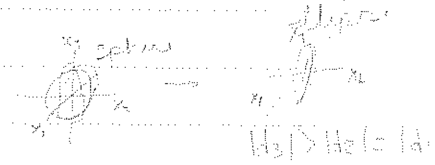
$\max \underbrace{\left(\sum_i (d_i x_i)^2 \right)^{1/2}}_{f(x)}$ s.t. the constraint $\|x\|_2 = 1$

$$g(x) \triangleq \left(\sum x_i^2 \right) - 1 = 0$$

Use Lagrange multipliers

$\exists \lambda$ s.t. $\nabla f = \lambda \nabla g$ @ each extremum

$$\nabla f = \begin{bmatrix} 2d_1^2 x_1 \\ \vdots \\ 2d_n^2 x_n \end{bmatrix} = \lambda \nabla g = \begin{bmatrix} 2x_1 \\ \vdots \\ 2x_n \end{bmatrix}$$



$d_i^2 x_i = \lambda x_i \qquad i = 1..n$

either $x_i = 0$ or $x_i \neq 0 \Rightarrow d_i^2 = \lambda$ several extrema

$$(f)^{1/2} = \left(\sum_i \lambda (x_i)^2\right)^{1/2} = \lambda^{1/2} \underbrace{\left(\sum_i (x_i)^2\right)}_{=1} = \lambda^{1/2} = |d_i|$$

Matrix not diagonal: conjugate transpose

$$\|A\|_2 = [\max \sigma(A^* A)]^{1/2}$$

not diagonal — spectrum (eigenvalues)

Ⓐ $\dot{x} = -1/x$; does it satisfy Lipschitz cond? No (maybe locally)

Ⓑ $\dot{x} = x^2$;

$\exists ?\, K$ s.t. $\|-1/x_1 + 1/x_2\| \leq K \|x_1 - x_2\|$

$\left\| \frac{x_1 - x_2}{x_1 x_2} \right\| \leq ?$

$\|x_1 - x_2\| \, 1/\|x_1 x_2\| \leq ?\, K \|x_1 - x_2\|$

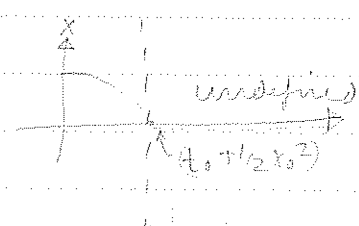
is there a K s.t. $1/\|x_1 x_2\| \leq K$? No

$x\,dx = -dt$

$\frac{1}{2} x^2 = -t + \text{const.}$

$\frac{1}{2} x^2 = -t + \frac{1}{2} x_0^2 + t_0$

$x = \sqrt{x_0^2 - 2(t - t_0)}$



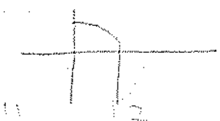
__Discrete Time__ : soln always defined

Approximate

$-1/x_k = \dot{x} = (x_{k+1} - x_k)/\Delta T$; standard approximation

$$x_{k+1} = x_k - \Delta T / x_k$$

$$\Delta T = .25 \; ; \; x_0 = 2$$



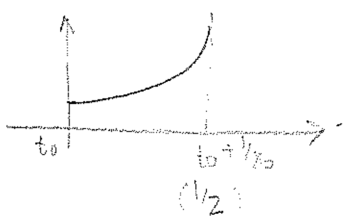
k 0 1 2 3 4 5 6 7 8 9 10 11 12

x_k 2/1.87/1.74/1.59/1.44/1.26/1.07/0.83/0.54/0.07/−3.21 −3.14

← no good →

$$x_{k+1} = x_k - \Delta T / x_k$$

(B) $\dot{x} = x^2$



k	0	1	2	3	4
$x(k)$	2	3	5.2	12.1	480 →

↳ blows up slower.

> Non unique from

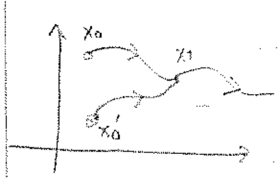
$$\dot{x} = 3|x|^{2/3} \qquad x(0) = 0$$

(i) $x(t) = 0$

(ii) $x(t) = t^3$

(4) Time reversal for discrete time systems

∃?

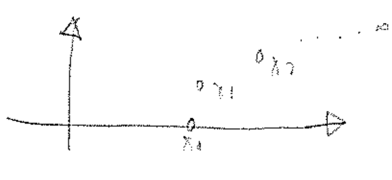


$$x_{k+1} = f(x_k)$$
$$x_{k+2} = f(f(x_k))$$
$$x_{k+n} = \underbrace{f(f(f \cdots f(x_k))) \cdots)}_{\leftarrow n \text{ times} \rightarrow}$$

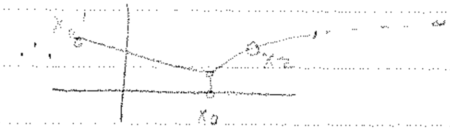
Given x_{k+n} can you find x_k?

ex in $\mathbb{R}^2$ $\quad x_{k+1} = \begin{bmatrix} 1 & 3 \\ 1/6 & 1/2 \end{bmatrix} x_k \quad ; \quad x_{k+1} = A x_k$

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad x_1 = A x_0 = \begin{bmatrix} 1 \\ 1/6 \end{bmatrix} \qquad x_2 = A x_1 = \begin{bmatrix} 3/2 \\ 1/4 \end{bmatrix}$$



$x_0' = \begin{bmatrix}-2\\1\end{bmatrix}$ $\qquad x' = Ax_0' = \begin{bmatrix}1\\1/6\end{bmatrix}$



When the matrix is invertible, you can work backwards.

A: If an inverse exists

$f^{-1}(x_{k+1}) = f^{-1}(f(x_k))$

In general inverse of fn exists $\Rightarrow$ unique trajectory

⑤ Norms in finite dimensions

$\|x\|_p > \|y\|_p \Rightarrow \|x\|_q > \|y\|_q$

$\exists\, c_1, c_2$ s.t. $c_1\|\cdot\|_p \le \|\cdot\|_q \le c_2\|\cdot\|_p$ — schwartz inequality

ex: $\|x\|_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} 1\cdot|x_i| \le \sqrt{\sum_{i=1}^{n} 1}\cdot\sqrt{\sum_{i=1}^{n}|x_i|^2} = n\|x_2\|$

$\|x\|_2 = \sqrt{\sum_{i=1}^{n}|x_i|^2} \le \sqrt{\sum_{i=1}^{n}\|x\|_1^2} = \sqrt{n\|x\|_1^2} = \sqrt{n}\,\|x\|_1$

$\|x\|_1 \le n\|x\|_2$

$\|x\|_2 \le \sqrt{n}\,\|x\|_1$

$1/n\,\|x\|_1 \le \|x\|_2 \le \sqrt{n}\,\|x\|_1$

$1/\sqrt{n}\,\|x\|_2 \le \|x\|_1 \le n\|x\|_2$

Try out for ∞ norm.

EE221a: Lecture

Last time

defined Lin Sys. decomposed to ZIR, ZSR; ZIR lin in i/p, etc

2. Zero input response

$\dot{\underline{x}} = \underset{\sim}{A}(t)\,\underline{x}$; $x(t_0) = x_0$ (vector, vector)

$x(t) = \phi(t, t_0, x_0)$

Can view as a mapping $\mathbb{C}^n \to \mathbb{C}^n$

$\underline{x}_0 \longmapsto \underline{x}(t) = \phi(t, t_0, x_0)$; a linear map

$\{v_i\}$ $\xrightarrow{\underset{\sim}{X}(t)}$ $\{e_i\}$

$x(t) = \phi(t, t_0, x_0)$

$X(t)$ = matrix representation of linear map.

i^{th} col $X(t)$ = v_i after map $= \phi(t, t_0, v_i)$ expressed in terms of $\{e_i\}$

$\therefore \quad \underset{\sim}{X}(t) = [\,\phi(t, t_0, v_1) \mid \phi(t, t_0, v_2) \mid \cdots \mid \phi(t, t_0, v_n)\,]$

Properties of $\underset{\sim}{X}(t)$

(1) $\underset{\sim}{X}(t_0) = [\,\phi(t_0, t_0, v_1) \mid \ldots \mid \phi(t_0, t_0, v_n)\,]$; (By notation used)

$= [\,v_1 \mid v_2 \mid \ldots \mid v_n\,] =: \underset{\sim}{X}_0$

expressed wrt $(e_1 \ldots e_n)$

Since $\{v_i\}$ basis $\Rightarrow$ $\boxed{\det \underset{\sim}{X}_0 \neq 0}$ (else $X_0 \underline{\lambda} = 0$ would have a non trivial soln for $\underline{\lambda}$)

(2) By defn. $\begin{cases} \dot{\phi}(t, t_0, v_i) = A(t)\,\phi(t, t_0, v_i) & i = 1 \ldots n \\ \phi(t_0, t_0, v_i) = v_i \end{cases}$

(vector, matrix, vector)

$$\begin{bmatrix} \dot{\phi}(t, t_0, v_1) & \cdots & \dot{\phi}(t, t_0, v_n) \end{bmatrix} = \begin{bmatrix} A(t)\,\phi(t, t_0, v_1) & \cdots & A(t)\,\phi(t, t_0, v_n) \end{bmatrix}$$

(matrix, matrix)

$$\underbrace{\dot{X}(t)}_{\text{matrix}} = \underbrace{A(t)}_{\text{matrix}} \underbrace{\begin{bmatrix} \phi(t, t_0, v_1) & \cdots & \phi(t, t_0, v_n) \end{bmatrix}}_{\underset{\sim}{X}(t)}$$

$$\boxed{\dot{\underset{\sim}{X}}(t)} = \boxed{\underset{\sim}{A}(t)}\,\boxed{\underset{\sim}{X}(t)}$$

$\underset{\sim}{X}(t_0) = \underset{\sim}{X}_0$ $\qquad$ $X(t)$ is called fund. matrix of the d.e.

i.e. $\underset{\sim}{X}_0$ & $\underset{\sim}{X}(t_0)$ arent $x(t_0)$ & x_0 (vectors!) (there is usually little relation between them)

STATE TRANSITION MATRIX

We pick $\{v_i\}$ to be standard basis $\{e_i\}$

$[\phi(t,t_0,e_1) \mid \phi(t,t_0,e_2) \mid \cdots \mid \phi(t,t_0,e_n)] =: \Phi(t,t_0)$ (s.t.m)

Note: $\Phi(t_0,t_0) = [e_1 \mid e_2 \mid \cdots \mid e_n] = \underline{I}$ NOTE $\frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\,\Phi(t,t_0)$

Properties of $\Phi(t,t_0)$

(1) By defn

$$\boxed{x(t) = \phi(t,t_0,x_0) = \Phi(t,t_0)\,x_0}$$

(2) $\Phi(t,t_0)$ is nonsingular $\forall t$. (if from non zero state to zero ⇒ violation?)

∀ t_0 Look at this ISK

Pf: Suppose $\Phi(t,t_0)$ is singular @ t_1

ie $\exists\, c \neq \theta$. $\Phi(t_1,t_0)\,c = \theta$

Consider $\Phi(t,t_0)\,c =: \Psi(t)$ remember - ($\Psi(t_0,t_0) = I$)

$\Psi(t_1) = 0$ (By construction)

$\dot{\Psi}(t) = A(t)\,\Psi(t)$ ‖ satisfies Lip cond ⇒ $\exists$ unique soln

$\Psi(t_1) = \theta$ ‖

We know $\Psi(t) = \theta\ \forall t$ is a soln.

By Thm soln is unique.

⇒ unique soln is $\Psi(t) = \theta\ \forall t$ ⟨goes both ways⟩

⇒ $\Psi(t_0) = c = \theta$ = contradiction!

(3) Semigroup property $x_0 \quad t_1 \quad \phi(t,t_0,x_0) \quad t_0 \quad \phi(t_1,t_0,x_0) \quad t$

$\phi(t,t_0,x_0) = \phi(t,t_1,\phi(t_1,t_0,x_0))$

$\Phi(t,t_0)\,x_0 = \Phi(t,t_1)\,\phi(t_1,t_0,x_0)$

$\Phi(t,t_0)\,x_0 = \Phi(t,t_1)\,\Phi(t_1,t_0)\,x_0$ (Repeat above step)

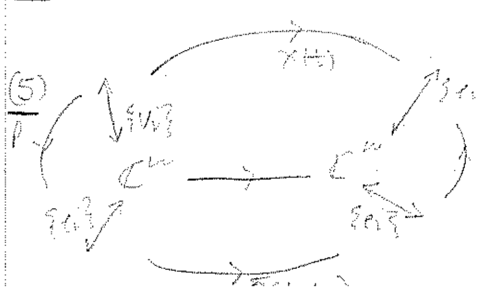
x_0 arb.

Does this mean $\boxed{\Phi(t,t_0) = \Phi(t,t_1)\,\Phi(t_1,t_0)}$

Yes (take $\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \cdots \begin{bmatrix}0\\0\\1\end{bmatrix}$

(4) $\Phi(t_1,t_0) = [\Phi(t_0,t_1)]^{-1}$ (from above)

(5)



$P \to$ ith col. of $P = v_i$ expressed in e_i coords

matrix Matrix matrix matrix

$P = [v_1 \mid v_2 \mid \cdots \mid v_n] = X(t_0) = X_0$ → rep. of $\{v_i\}$ as $\{e_i\}$ in column form

$X(t) = I\,\Phi(t,t_0)\,P$

$\Phi(t,t_0) = X(t)\,[X(t_0)]^{-1}$

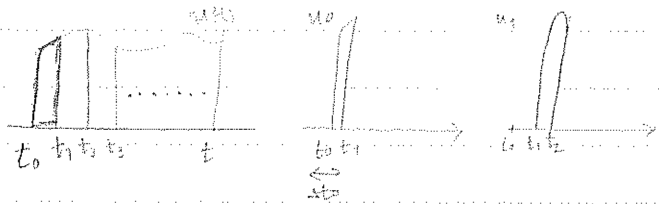
ZERO STATE RESPONSE

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$; $x(t_0) = \theta$

know: Z.S. response is a lin fn of $u(t)$

derive an expression for this linear map.

DERIVATION: consider $u(t)$: break up $u(t)$ to $u_0(t) + u_1(t) + \cdots$



$x_0(t)$ due to u_0

$\dot{x}_0(t) = A(t)x(t) + B(t)u(t)$; ensemidt

$\Rightarrow \dot{x}_0(t) = B(t_0)u(t_0)$

$x_1(t)$ due to u_1

when Δt_0 very small

$x_0(t) \simeq B(t_0)u(t_0)\Delta t_0$

$x(t) = x_0(t) + x_1(t) + x_2(t) + \cdots$

From t_1 on u_0 is equal to zero!

$\therefore$ looking @ zero i/p response, & we know what that is!

$$x_0(t) \simeq \underset{\Phi(t,t_0)}{\Phi(t,t_1)}\, B(t_0)u(t_0)\Delta t_0 = x_0(t_0)$$

so the same for $x_1(t)$ due to u_1

$\dot{x}_1(t_2) = B(t_1)u_1$ integrate $x_1(t_2) \simeq B(t_1)u(t_1)\Delta t_1$

from t_2 on Z.I.R. $x(t) \simeq \Phi(t,t_1)B(t_1)u(t_1)\Delta t_1$

observe pattern in solns.

$$x(t) = x_0(t) + x_1(t) + x_2(t) + \cdots$$

$$\cong \sum_{i=0}^{n} \Phi(t,t_i)B(t_i)u(t_i)\Delta t_i \quad ; \text{ as } \Delta t_i \to 0 \quad \Sigma \to \int$$

$$x(t) \overset{?}{=} \int_{t_0}^{t} \Phi(t,t')B(t')u(t')dt' \quad ; \text{ this is indeed the soln.}$$

Proof of $x(t) = \underbrace{\int_{t_0}^{t} \Phi(t,t')B(t')u(t')dt'}_{x(t)}$: Plug into DE! if works, OK (F.Th.

$x(t_0) = \theta$

$$\dot{x}(t) = \Phi(t,t)B(t)u(t) + \int_{t_0}^{t} \underbrace{\frac{\partial}{\partial t}\Phi(t,t')}_{A(t)\,\Phi(t,t')} B(t')u(t')dt' \quad ; \text{ see calc}$$

$$= I\,B(t)u(t) + A(t)\int_{t_0}^{t} \Phi(t,t')B(t')u(t')dt'$$

i.e. $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ $\therefore x(t)$ is indeed the soln of d.eqn.

$$\frac{\partial}{\partial t}\int_{t_0}^{t} f(t,\tau)\,d\tau = \frac{\partial}{\partial t}$$

(4) Complete response ZIR + ZSR

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$; $x(t_0) = x_0$

$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,t')B(t')u(t')dt'$

Z.S. Response $\underset{\text{fn space}}{u(\cdot)} \longmapsto x(t) = \int_{t_0}^{t} \Phi(t,t')B(t')\underline{\underline{u(t')}}dt'$

$P^c[t_0,t] \longrightarrow \mathbb{C}^{n_i}$

Z.I. Resp. $\mathbb{C}^n \to \mathbb{C}^n$

$x_0 \longmapsto x(t)$ pt to pt

$x_0 \longmapsto x(\cdot)$ pt. to fn.

(5) Impulse Response Matrix (A zero state response!)

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$; $x(t_0) = \theta$

o/p $y(t) = C(t)x(t)$

$y(t) = \int_{t_0}^{t} [C(t)\cdot\Phi(t,t')\cdot B(t')]u(t')dt'$

Let $u(t) = [0 \dots \delta(t-\tau) 0 \dots 0]^T \leftarrow j^{th}$ comp. is δ @ $t = \tau$

output $= [C(t)\Phi(t,\tau)B(\tau)]_{j^{th}\text{ col.}}$

impulse response matrix

$H(t,\tau) = C(t)\Phi(t,\tau)B(\tau)$

[illegible]
548-4750

Thursday
21 Sept 1989

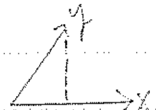
LST-Lecture

Midterm Oct 12 Thursday

(VIII) Adjoint of Linear Map

1 Scalar Product

motivation: inner product in $\mathbb{R}^3$



$\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i$

Properties: (in $\mathbb{R}^3$)

① linear fn of y

② $\langle x,y\rangle = \langle y,x\rangle$

③ $\langle x, x\rangle = 0 \Leftrightarrow x = \theta$

in $\mathbb{C}^3$ we can't define exactly the same $\langle$complex$\rangle$

$x = [1\ j\ 0]^T \Rightarrow \sum_{i=1}^{3} x_i^2 = 0$ but $x \neq 0$

For ③ to hold

$$\langle x, y\rangle = \sum_{i=1}^{n} \bar{x}_i y_i \quad ; \langle x, x\rangle \in \mathbb{R} = 0 \Leftrightarrow x = 0$$

$\hookrightarrow \langle x, y\rangle = \overline{\langle y, x\rangle}$

In a general vector space, $(V, \mathcal{F})$ $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$

The fn. $\langle \cdot, \cdot\rangle : V \times V \longrightarrow \mathcal{F}$ is called an inner product or scalar product iff ① $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

② $\langle x, \alpha y\rangle = \alpha \langle x, y\rangle$

③ $\langle x, y\rangle = \overline{\langle y, x\rangle}$

④ $(\langle x, x\rangle \geq 0) \cap (\langle x, x\rangle = 0 \Leftrightarrow x = \theta)$

(Not really linear in first var $(\because \bar{\alpha})$)

Ex1: $(V, \mathcal{F}) = (\mathbb{C}^n, \mathbb{C})$

$$\langle x, y\rangle := \sum_{i=1}^{n} \bar{x}_i y_i$$

Ex2: $(V, \mathcal{F}) = (P[t_0, t_1], \mathbb{C})$; $P[t_0, t_1] = \{$p.c. fns on $[t_0, t_1]\}$

$$\langle x(\cdot), y(\cdot)\rangle := \int_{t_0}^{t_1} \bar{x}(t) y(t)\, dt$$

Ex3: $(V, \mathcal{F}) = (P^m[t_0, t_1], \mathbb{C})$

$$\langle x(\cdot), y(\cdot)\rangle_{P^m} := \int_{t_0}^{t_1} \langle x(t), y(t)\rangle_{\mathbb{C}^m} dt = \int_{t_0}^{t_1} \sum_{i=1}^{m} \bar{x}_i(t) y_i(t)\, dt$$

Define $\langle x, y\rangle = 0$ we say x & y are orthogonal.

Thm: $\langle x, y\rangle$ inner product $\Rightarrow \|x\| := \sqrt{\langle x, x\rangle}$ is a norm

Proof: Norm ① $\|x\| \geq 0$ $x = \theta \Leftrightarrow \|x\| = 0$

② $\|\alpha x\| = |\alpha| \cdot \|x\|$

③ $\|x+y\| \leq \|x\| + \|y\|$

① follows from ④

② follows from ②

③ $\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2$

so show $\leq (\|x\| + \|y\|)^2$

If we can show $\langle x, y\rangle + \langle y, x\rangle \leq 2\|x\| \|y\|$

we know $\langle x - ty, x - ty \rangle \geq 0 \;\forall t$ (from ⊕)

$$= t^2 \langle y, y \rangle - t(\langle x, y \rangle + \langle y, x \rangle) + \langle x, x \rangle \geq 0 \quad \forall t$$

$$\Rightarrow (\langle x, y \rangle + \langle y, x \rangle)^2 - 4 \langle x, x \rangle \langle y, y \rangle \leq 0$$

$$\Rightarrow |(\langle x, y \rangle + \langle y, x \rangle)| \leq 2 \|x\| \cdot \|y\| \qquad \text{QED}$$

$ax^2 + bx + c \geq 0$; +ve $\Rightarrow b^2 - 4ac \leq 0$ ($\because$ no touching x axis!)

2. ADJOINT

motivation $\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$

$$\begin{array}{cc} & n \\ m & \boxed{A} \end{array} \qquad \langle x, y \rangle = x^T y$$

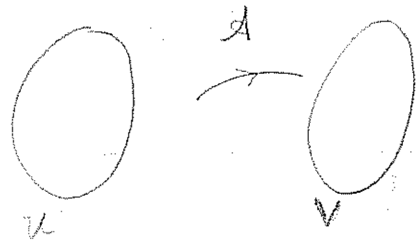
$$\begin{array}{cc} & m \\ n & \boxed{A^T} \end{array}$$

Note: $\langle y, Ax \rangle = y^T A x$; by defn

$= x^T A^T y$ ($\because$ it's a scalar)

$= \langle x, A^T y \rangle$ ← this is well defined!

$= \langle A^T y, x \rangle$; true $\forall x \in \mathbb{R}^n$ $y \in \mathbb{R}^m$



$\mathcal{A} : \mathcal{U} \to \mathcal{V}$ linear

$(\mathcal{U}, \mathbb{F})$ $\langle \cdot, \cdot \rangle_{\mathcal{U}}$

$(\mathcal{V}, \mathbb{F})$ $\langle \cdot, \cdot \rangle_{\mathcal{V}}$

The adjoint of the linear map $\mathcal{A}$: $\mathcal{A}^* : \mathcal{V} \longrightarrow \mathcal{U}$ is defined by

$\forall x \in \mathcal{U}\ \forall y \in \mathcal{V}$ $\langle y, \mathcal{A}x \rangle_{\mathcal{V}} = \langle \mathcal{A}^* y, x \rangle_{\mathcal{U}}$

Ex 1 $A : \mathbb{C}^n \longrightarrow \mathbb{C}^m$

$$\langle y, Ax \rangle_m = \sum_{i=1}^{m} \bar{y}_i \sum_{j=1}^{n} a_{ij} x_j$$

$$\langle A^* y, x \rangle_n = \sum_{j=1}^{n} \Big(\sum_{i=1}^{m} y_i \bar{a}_{ij} \Big) x_j$$

$A^* :=$ conjugate transpose of A

(Ex2.) ZERO INPUT RESPONSE

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) \end{cases}$$

using s instead of f

$ZIR = \Phi(t,t_0)x_0 \qquad \overset{(t,t_0,x_0,0_u)}{=} y(t) = C(t)\,\Phi(t,t_0)x_0$

Two views of ZIR

(i) for any given t,

$$x_0 \longmapsto C(t)\,\Phi(t,t_0)x_0$$

$$\mathbb{C}^n \longrightarrow \mathbb{C}^m$$

adjoint $\Phi^*(t,t_0)\,C^*(t) \qquad (* \Rightarrow \bar{\ }^T$ complex conj & transpose)

(ii) view $y(\cdot)$ as a fn of time

$$x_0 \longmapsto C(\cdot)\,\Phi(\cdot,t_0)x_0$$

$$\mathbb{C}^n \rightarrow \mathcal{Y} = \mathcal{P}[t_0,t_1]$$

Define $L_0 : x_0 \longmapsto C(\cdot)\,\Phi(\cdot,t_0)x_0$

Find the adjoint $\forall x \in \mathbb{C}^n,\ \forall y(\cdot) \in \mathcal{P}^m$

$$\langle y(\cdot), L_0 x_0\rangle_{\mathcal{P}^m} = \langle y(\cdot), C(\cdot)\,\Phi(\cdot,t_0)x_0\rangle_{\mathcal{P}^m}$$

$$= \int_{t_0}^{t_1} y^*(t)\,C(t)\,\Phi(t,t_0)x_0\,dt$$

$$\langle L_0^* y, x_0\rangle_{\mathbb{C}^n} = \left[\int_{t_0}^{t_1} \Phi^*(t,t_0)C^*(t)y(t)\,dt\right]^* x_0$$

$$\Rightarrow L_0^* : y(\cdot) \longmapsto \int_{t_0}^{t_1} \Phi^*(t,t_0)\,C^*(t)\,y(t)\,dt \ ;$$ compare with ZSR

$$ZSR = \int_{t_0}^{t_1} \underset{\Phi^*(t,t_0)}{\underset{\updownarrow}{\Phi(t_1,t)}}\ \underset{C^*(t)}{\underset{\updownarrow}{B(t)}}\ \underset{y(t)}{\underset{\updownarrow}{u(t)}}\,dt$$

; integ. is wrt 2^{nd} argument here!

Q. Can we make this $\Phi^*(t,t_0)$ a state transition matrix of something?

say $\Psi(t_0,t)$ is a s.t.m. from a d.e.

$\Phi^*(t,t_0) = \Psi(t_0,t)$; use τ in place of t_0 i

$\Phi^*(t,\tau) = \Psi(\tau,t)$

In order to find $\Psi(\tau,t)$ is the state transition matrix of what, lets what d.e. it satisfies.

$\partial/\partial\tau\ \Psi(\tau,t) = ?$

$$\partial/\partial\tau\ \Psi(\tau,t) = \partial/\partial\tau\,[\Phi^*(\tau,t)]^{-1}$$

How do you find this deriv? By defn

$\Phi^*(\tau,t)\,[\Phi^*(\tau,t)]^{-1} = I$

$[\partial/\partial\tau\, \Phi^*(\tau,t)]\,[\quad]^{-1} + \Phi^*(\tau,t)\,\partial/\partial\tau\,[\quad]^{-1} = 0$

$\hookrightarrow \partial/\partial t$ & $[\ ]^*$ commute!

$\therefore\ [\partial/\partial\tau\, \Phi(\tau,t)]^*\,[\quad]^{-1} + \Phi^*(\tau,t)\,\frac{\partial}{\partial\tau}\,[\quad]^{-1}$

$[A(\tau)\,\Phi(\tau,t)]^*$ (nonsingular)

$\Phi^*(\tau,t)\,A^*(\tau)\,[\quad]^{-1} + \Phi^*(\tau,t)\,\partial/\partial\tau\,[\quad]^{-1} = 0$

$[\Phi^*(t,t)]^{-1} = I$ $\quad \partial/\partial\tau\,[\Phi^*(\tau,t)]^{-1} = -A^*(\tau)\,[\Phi^*(\tau,t)]^{-1}$

indeed $\Psi(\tau,t)$ is indeed the state transition matrix of $\dot{p}(\tau) = -A^*(\tau)\,p(\tau)$

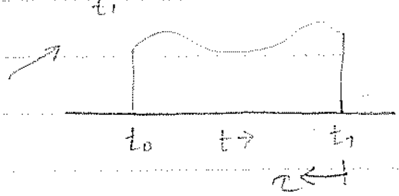
$p(t) = p_0$ i.c.

Back to $L_0^* y = \int_{t_0}^{t_1} \Phi^*(t,t_0)\,C^*(t)\,y(t)\,dt$

$= \int_{t_0}^{t_1} \Psi(t_0,t)\,C^*(t)\,y(t)\,dt$; no problem (just change sign)

$= \int_{t_1}^{t_0} \Psi(t_0,t)\,[-C^*(t)]\,y(t)\,dt$

Looking @



$\uparrow$ it is the ZSR of a.d.e.

$\begin{cases} d/d\tau\, p(\tau) = -A^*(\tau)\,p(\tau) - C^*(\tau) \\ p(t_1) = \theta \end{cases}$

$\hookrightarrow$ is the adjoint of the ZIR

$\frac{d}{dt}X(t) = A(t)X(t) + B(t)U(t)$

$y(t) = C(t)X(t)$

(Ex3) ZSR

$\rho(t,t_0,\theta_\Sigma,u) = \int_{t_0}^{t} C(t)\,\Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau$

Two views: (i) for a given t

$u(\cdot) \longmapsto \int_{t_0}^{t_1} C(t_1)\,\Phi(t_1,\tau)\,B(\tau)\,u(\tau)\,d\tau$

$\mathcal{L}^2[t_0,t_1] \longrightarrow \mathbb{C}^n$

adjoint of this will be from $\mathbb{C}^n \longrightarrow \mathcal{L}^2[t_0,t_1]$

(ii) View $y(\cdot)$ as a fn. $u(\cdot) \longmapsto y(\cdot)$

$\rho^r[t_0,t_1] \longrightarrow \rho^m[t_0,t_1]$

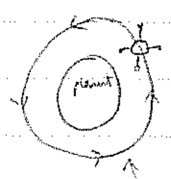
Find its adjoint
(we'll do next time)

Friday
22 Sept 1989

αST - Discussion

$\{x > R\}$ Locally Lipschitz.
Lipschitz in an open set, $\forall x, y \in \mathcal{U}$
what about for $t < t_0$ $\exists K(t)$ p.c.

$\Phi(t,0)$ $\Phi(t,t_0) = \Phi(t,0)\,\Phi^{-1}(0,t_0)$
$\hookrightarrow \Phi^{-1}(t_0,0)$



$\dot{x} = f(x,u,t)$: non linear
Control this $\bar{c}/o$ non linear control

soln is an $\Rightarrow$ point.

(x_0, t_0) with specific control
$\tilde{u}(t)$

what happens to x if we change control from $\tilde{u} \to \tilde{u} + \delta u$

$\tilde{x}$ = soln $\bar{c}$ $\tilde{u}$

$\tilde{x} + \delta x$ = " " $\tilde{u} + \delta u$

$(\dot{\tilde{x}} + \delta x) = f(\tilde{x} + \delta x, \tilde{u} + \delta u, t)$

$f(\tilde{x} + \delta x, \tilde{u} + \delta u, t) \overset{\text{(Taylor)}}{\approx} f(\tilde{x}, \tilde{u}, t) + \partial f/\partial x\Big|_{\tilde{x},\tilde{u}} \delta x + \partial f/\partial u\Big|_{\tilde{x},\tilde{u}} \delta u$

$\dot{x} + \delta \dot{x} = \underbrace{f(\tilde{x}, \bar{u}, t)}_{\dot{x}} + \partial f/\partial x\Big| \delta x + \partial f/\partial u\Big| \delta u$

$\delta \dot{x} = \partial f/\partial x\Big|_{\tilde{x},\tilde{u},t} \delta x + \partial f/\partial u\Big|_{\tilde{x},\tilde{u}} \delta u$

Jacobian

$\delta \dot{x} = A(t)\,\delta x + B(t)\,\delta u$

$f(x,u,t) = \cos x + 3u$

$\dot{x} = \cos x + 3u$

linearize @ $x=1$ $\Rightarrow$ $\partial f/\partial x = -\sin x\Big|_{x=1} = -\sin x\Big|_{x=1} = -\sin 1$

$\partial f/\partial u = 3$ $\delta \dot{x} = -\sin 1\, \delta x + 3\delta u$

$$f(x,u) = \begin{bmatrix} \cos(x_1+x_2) + 3u \\ e^{x_2} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\begin{bmatrix} \delta \dot{x}_1 \\ \delta \dot{x}_0 \end{bmatrix} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}$$

DISCRETE TIME SYSTEMS

(*) is eqn

$$R = \begin{cases} x_{k+1} = A(k)x_k + B(k)u_k \\ y_k = C(k)x_k + D(k)u_k \end{cases} \quad x_k \in \mathbb{R}^n$$

$\forall x_{k_0}$, $\{x_k\}_0^\infty$ & $\{y_k\}_0^\infty$ exist for any $\{u_k\}_0^\infty$

Thm The ZIR of the system represented by R is given by

$x_k = \phi(k, k_0) x_{k_0} \quad \forall k \geq k_0$

where $\Phi(k_0, k_0) = I$

$\Phi(k, k_0) = A(k-1)\, A(k-2) \cdots A(k_0)$

Pf : By induction

assume true for k'

Then $x_{k'} = \phi(k', k_0) x_{k_0}$

$= A(k'-1) \cdots A(k_0)$

From (*)

$$\begin{aligned} x_{k'+1} &= A(k')x_{k'} \\ &= A(k')\,\Phi(k', k_0)x_{k_0} \\ &= \underbrace{A(k')\,A(k'-1) \cdots A(k_0)}_{\Phi(k'+1, k_0)} x_{k_0} \end{aligned}$$

Now for $x_{k_0+1} = A(k_0) x_{k_0}$

$= \Phi(k_0+1, k_0) x_{k_0}$ QED

$\Phi(k, k_0) = A^{k-k_0} = A(k) \cdots (A^{k})$

Complete State Response:

$$x_{k_0+1} = A(k_0)x_{k_0} + B(k_0)u_{k_0}$$

$$x_{k_0+2} = A(k_1)\left[A(k_0)x_{k_0} + B(k_0)u_{k_0}\right] + B(k_1)u_{k_1}$$

$$= \underbrace{A(k_1)\,A(k_0)}_{\Phi(k_0+1,\,k_0)}x_{k_0} + A(k_1)B(k_0)u_{k_0} + B(k_1)u_{k_1}$$

$k_1 = k_0+1$

$x_{k_0+3} =$ (big expansion ...)

$$\Phi(k_0+3, k_0)x_{k_0} + \sum_{k'=k_0}^{k_0+2} \Phi(k_0+3, k'+1)\, B(k')\, u_{k'}$$; Composure by ind.

$$x_k = \Phi(k, k_0)\, x_{k_0} + \sum_{k'=k_0}^{k-1} \Phi(k, k'+1)\, B(k')\, u_{k'}$$

state transition function.

$\boxed{k \geq k_0}$ (∵ can't run discrete time sys backwards

$$y_k = \underbrace{C(k)\,\Phi(k,k_0)\,x_{k_0}}_{ZIR} + \underbrace{\sum_{k'=k_0}^{k-1} C(k)\,\Phi(k,k')\,B(k')u_{k'} + D(k)u_k}_{ZSR}$$

response function

$$y(t) = \left[C(t)\,\phi(t,t_0)\,x_0 + \left[\int_{t_0}^{t} C(t)\,\phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau\right] + D(t)\,u(t)\right] \quad \boxed{\forall t}$$

Adjoint for discrete time systems

$$x_{k+1} = A(k)\,x_k$$

$$x_k = \phi(k, k_0)\, x_{k_0} \quad \forall\, k \geq k_0$$

$$p_k = A(k)^* p_{k+1}$$

$A(k)^*$ = conj. transpose of $A(k)$

$$A = \begin{bmatrix} 1 & 1-j \\ 2+j & 0 \end{bmatrix} \qquad A^* = \begin{bmatrix} 1 & 2-j \\ 1+j & 0 \end{bmatrix}$$

Start at p_{k_0} next step $\rightarrow p_{k_0-1}$ (A could be singular)

$$\mathcal{F} : U[t_0, t_1] \longmapsto X_1$$

fn of time.

$$\mathcal{F}^* : X_1 \longmapsto U[t_0, t_1]$$

$$p_{k_0} = \left[A(k_0)^*\right]^{-1} p_{k_0-1}$$

$$x_{k_0} = A(k_0)\, x_{k_0-1}$$

← p_k isn't in the null space of A^* i.e. $\boxed{p_k \notin R(A^*)}$

examples : (1) $A^* = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ $\quad \underset{p_k}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \underset{A^*}{\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}} \underset{p_{k+1}}{\begin{bmatrix} x \\ y \end{bmatrix}}$

$2y = 0 ; y = 1$ <u>no soln.</u>

(2) $A^* = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ an ∞ of solns.

$\boxed{p_k \in R(A^*)}$

$\Psi(l, k) = A(l)^* A(l+1)^* \ldots A(k)^*$

$\boxed{\Psi(l, k)^* = \Phi(k, l)}$

SAMPLED DATA SYSTEMS

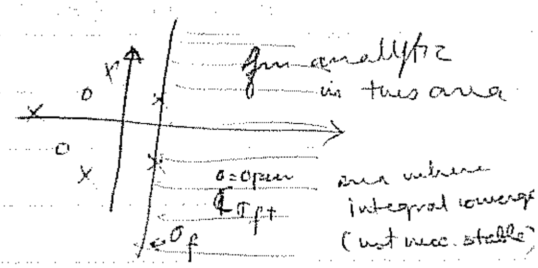
$\hat{f} = $ L.T. $\quad f : \mathbb{R}^+ \to \mathbb{R}$ then $\hat{f} = \int_{0^-}^{\infty} f(\tau) e^{-s\tau} d\tau \quad \forall s$ s.t. integ conv

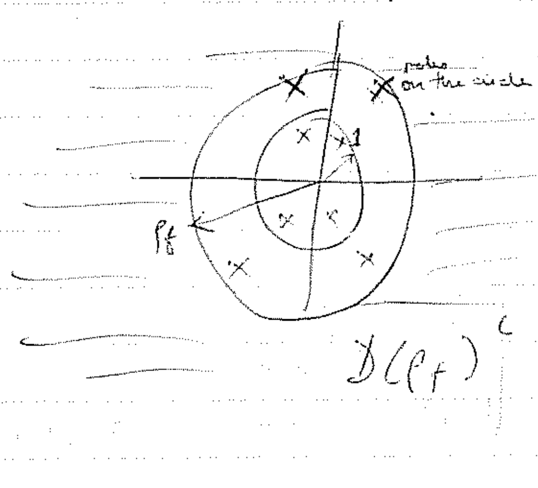
Z.T. $= \tilde{f} \quad \{f_n\}_{n=0}^{\infty} \quad \tilde{f}(z) = \sum_{n=0}^{\infty} z^{-n} f_n$

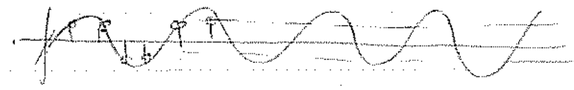
function $f(t)$: $\tilde{f}(t) = \sum_0^{\infty} z^{-n} f(nT)$ $\quad$ defined for all z s.t. Σ conv

Reln between Z.T. & L.T.

$$\begin{tikzcd}
\boxed{\hat{f} : \mathbb{C}_{\sigma_f^+} \to \mathbb{C}} \arrow[r, "T"', "\text{linear}"] & \boxed{} \arrow[r] & \tilde{f} : D(\rho_f)^c \to \mathbb{C} \\
\boxed{f : \mathbb{R}^+ \to \mathbb{R}} \arrow[u, leftrightarrow, "\text{linear}"', "\text{L.T.}"] \arrow[r, "\text{Sampling (linear)}"'] & \boxed{\{f(nT)\}_{n=0}^{\infty}} \arrow[u, leftrightarrow, "\text{Z.T.}", "\text{linear}"']
\end{tikzcd}$$





S is not injective (1-1) 

If f is Bandlimited you end up bijective
ie (1-1) & onto

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin(\pi/_T (t-nT))}{\pi/_T (t-nT)}$$

TUESDAY
16 Sept 1999

L.S.T. lecture

Ex3 Zero State Response

$\mathcal{R}$: $u(\cdot) \longrightarrow y(\cdot)$

$\mathcal{R} : \rho^n[t_0,t_1] \longrightarrow \rho^m[t_0,t_1]$

$$\underbrace{(\mathcal{R}u)(t)}_{\rho^m[t_0,t_1]} = \int_{t_0}^{t} \underbrace{C(t)\,\Phi(t,\tau)B(\tau)}_{H(t,\tau)} u(\tau)\,d\tau$$

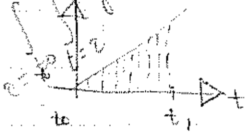
Find adjoint of $\mathcal{R}$

$\forall y \; \forall u(\cdot)$ $\langle y(\cdot), \mathcal{R}u(\cdot)\rangle_{\rho^m} \triangleq \int_{t_0}^{t_1} y^*(t)\,[\mathcal{R}u(t)]\,dt = \int_{t_0}^{t_1}\int_{t_0}^{t} y^*(t)H(t,\tau)u(\tau)\,d\tau\,dt = \int_{t_0}^{t_1}\int_{\tau}^{t_1} y^*(t)H(t,\tau)u(\tau)\,dt\,d\tau$

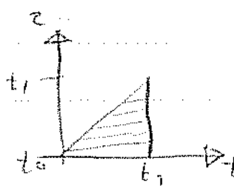
$= \langle \mathcal{R}^* y(\cdot), u(\cdot)\rangle_{\rho^n} = \int_{t_0}^{t_1} y^*(t)\int_{t_0}^{t} H(t,\tau)u(\tau)\,d\tau\,dt$ ← change order of integration is required.

$\parallel$

$\int_{t_0}^{t_1} [\mathcal{R}^* y(\tau)]^* u(\tau)\,d\tau$



$$\Rightarrow \int_{t_0}^{t_1}\int_{\tau}^{t_1} y^*(t)\,H(t,\tau)\,\underline{u(\tau)}\,dt\,d\tau$$



$$= \int_{t_0}^{t_1}\left[\int_{\tau}^{t_1} H^*(t,\tau)\,y(t)\,dt\right]^* u(\tau)\,d\tau$$

$\mathcal{R}^* : \rho^m[t_0,t_1] \longrightarrow \rho^n[t_0,t_1]$

$$\boxed{(\mathcal{R}^* y)(\tau) = \int_{\tau}^{t_1} H^*(t,\tau)\,y(t)\,dt}$$

$$= \int_{\tau}^{t_1} \underbrace{B^*(\tau)\,\Phi^*(t,\tau)}\,C^*(t)\,y(t)\,dt$$

ZSR $\dot{x}(t) = A(t)x(t) + B(t)u(t)$; $x(t_0) = \theta$
$\Rightarrow x(t) = \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau$

$z(t) := B^*(t)\,p(t)$

$$p(\tau) := \int_{\tau}^{t_1} \Phi^*(t,\tau)\,C^*(t)\,y(t)\,dt$$

the same as the previous system!

$p(\tau)$ = ZSR of the "adjoint system"

$\dot{p}(\tau) = -A^*(\tau)p(\tau) - C^*(\tau)y(\tau)$; $p(t_1) = \theta$

output of system
$z(\tau) = B^*(\tau)\,p(\tau)$

Duality

$R : u(\cdot) \to y(\cdot)$ $\qquad$ $R^*: y(\cdot) \to s(t)$

$A(t)$ $\qquad$ $-A^*(\tau)$

$B(t)$ $\qquad$ $-B^*(\tau)$

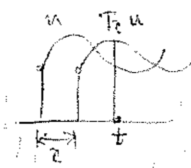
$C(t)$ $\qquad$ $C^*(\tau)$

Chapter IV $\quad [A, B, C]$

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t)$$

1. Time Invariant

translation operator T_τ



$(T_\tau u)(t) = u(t-\tau)$

A dynamical system is time invariant $\Leftrightarrow$

(a) $u \in \mathcal{U}$ $\quad T_\tau u \in \mathcal{U}$ $\quad \forall \tau$

($\mathcal{U}$ is closed under the translation operation)

(b) $\rho(t_1, t_0, x_0, u) = \rho(t_1+\tau, t_0+\tau, x_0, T_\tau u)$ $\quad \forall\, t_0, t_1, \tau \quad \forall\, x_0, u$

- i.e. Response depends only on $(t_1 - t_0)$
- ~~Time invariant systems~~ may always take $t_0 = 0$.

Ex. Show (I) is time invariant.

2. State Transition Matrix

$\dot{x}(t) = A\,x(t)$; $x(0) = x_0$ $\quad$ (take $t_0 = 0$)

$\Phi(t) := \Phi(t, 0)$

$\frac{d}{dt}\Phi(t) = A\,\Phi(t)$; $\Phi(0) = I$ $\quad \Leftrightarrow \quad \Phi(t) - I = \int_0^t A\,\Phi(t')\,dt'$

Picard iteration: $\Phi_0(t) = I$

$$\Phi_1(t) = I + \int_0^t A\,\Phi_0(t')\,dt' = I + At$$
$$\Phi_2(t) = I + \int_0^t A(I + At')\,dt'$$
$$= I + At + A^2 t^2/2!$$

$$\Phi(t) = I + At + A^2t^2/2! + A^3t^3/3! + \cdots \infty$$

$\Phi(t) = \sum_{i=0}^{\infty} A^i t^i / i!$; in the scalar case this would be $\exp(\cdot)$

Here lets define $\Phi = \exp(At)$ for matrices.

$x(t) = [\exp(At)]\, x_0$; different modes of the system response

Properties

(1) If $A = PBP^{-1}$; correspond to change of coord on both domain & range

then $\exp A = P(\exp B)P^{-1}$

$$P\left(\sum_{k=0}^{N} B^k/k!\right)P^{-1} = \sum_{k=0}^{N} \frac{PB^kP^{-1}}{k!}$$; examine $(PBP^{-1})^k = \overbrace{PBP^{-1}PBP^{-1}\ldots P}^{k\ \text{times}} = PB^kP^{-1}$

$$= \sum_{k=0}^{N} \frac{(PBP^{-1})^k}{k!}$$

$N \to \infty \Rightarrow$ Proved

(2) scalars $\exp(a+b) = \exp a \cdot \exp b$

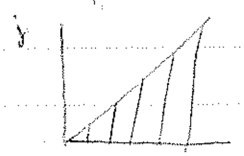
matrix iff $a \cdot b = b \cdot a$ then $\exp(A+B) = \exp A \exp B$

$$\exp(A+B) = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!}$$

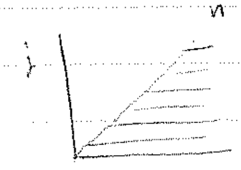
$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$

if commute $(A+B)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} A^j B^{n-j}$

$$\exp(A+B) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{A^j B^{n-j}}{j!\,(n-j)!}$$



$$= \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} \frac{A^j B^{n-j}}{j!\,(n-j)!}$$



$$= \sum_{j=0}^{\infty} \frac{A^j}{j!} \cdot \sum_{k=0}^{\infty} B^k/k!$$

$$= \exp(A) \cdot \exp(B)$$

(3) $\exp A(t_1+t_2) = \exp At_1 \exp At_2$

(4) $(\exp(A))^{-1} = \exp(-A)$

T.3. Cayley - Hamilton Theorem

$$\frac{d}{dt}\Phi(t) = A\Phi(t)$$

$$\Phi(0) = I$$

Take Laplace Transform (can do $\because$ T.I.)

$\Phi(t) \longrightarrow \hat{\Phi}(s)$

$$s\hat{\Phi}(s) - I = A\hat{\Phi}(s)$$

L.T. of state transition matrix

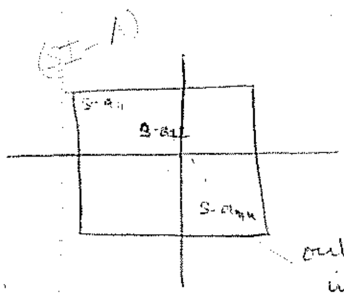
$$\hat{\Phi}(s) = (sI-A)^{-1}$$

$$\hat{\Phi}(s) = (sI-A)^{-1} \quad \text{; obtain by cramers rule}$$

$$= \frac{1}{\det(sI-A)} \cdot [\quad] \leftarrow \text{adjugate of } A$$

$\hookrightarrow$ ij^{th} term $= (-1)^{i+j} \times ij^{th}$ cofactor of A^T

= poly. in s order $= (n-1)$



only diagonal has s in it.

style of many's ...

actually this doesn't matter

$$(sI-A)^{-1} = \frac{[s^{n-1}B_0 + s^{n-2}B_1 + \cdots + B_{n-1}]}{s^n + d_1 s^{n-1} + \cdots d_n}$$

POST MULTIPLY by $(sI-A)$

$$\Rightarrow (s^n + d_1 s^{n-1} + \cdots d_n)I = [s^{n-1}B_0 + s^{n-2}B_1 + \cdots B_{n-1}](sI-A)$$

$$= s^n B_0 + s^{n-1}(B_1 - B_0 A) + s^{n-2}(B_2 - B_1 A) + \cdots + (-B_{n-1}A)$$

recursive formula for $\{B_i\}$ in terms of $\{d_i\}$

$\therefore B_0 = I$

$B_1 - B_0 A = d_1 I \Rightarrow B_1 = B_0 A + d_1 I$

& $B_2 = B_1 A + d_2 I$

$\cdots$

$B_K = B_{K-1}A + d_K I$

$0 = B_{n-1}A + d_n I$

$d(s) = \det(sI - A)$; can do in principal but computationally mean.

what do we get when subs backwards?

$$0 = B_{n-1}A + d_n I$$
$$= (B_{n-2}A + d_{n-1} I)A + d_n I = B_{n-2}A^2 + d_{n-1}A + d_n I$$
$$= B_{n-3}A^3 + d_{n-2}A^2 + d_{n-1}A + d_n I$$
$$\ldots$$

$$0 = A^n + d_1 A^{n-1} + \ldots \ldots d_n I$$

$$d(s) = \det(sI - A) = s^n + d_1 s^{n-1} + \ldots d_n$$

then $\underset{\text{matrix}}{d(A)} = 0$ matrix

alternatively $\det(A - sI)$ called charac. poly of A

$= (-1)^n d(s)$ $\quad \Delta(s) \triangleq \det(A - sI)$

$\Delta(A) = 0$

The above result is refered to as the Cayley Hamilton thm.

implications

(1) $A^n = -d_1 A^{n-1} - \ldots - d_n I \Rightarrow A^n$ is l.c. of $(I, A, \ldots A^{n-1})$

(2) A^{n+p} is an l.c. of (")

(3) $\exp(A)$ is an l.c. of (")

when A is 2×2 $\exp(A) = \alpha_1 I + \alpha_2 A$

Thursday
28 Sept 1989

LST - Lecture

$\dot{X}(t) = A x(t)$

$x(t) = (\exp At) X_0$

4. Spectral Decomposition (distinct values)

(1) Eigenvalues & Eigenvectors

if $\exists$ $e \neq \theta$ s.t. $Ae = \lambda e$ for some λ.

λ eigenvalue of A

e corresponding eigenvector

Note: $(A - \lambda I) e = \theta$

$\Rightarrow e$ lies in null space of $(A - \lambda I)$ i.e. $e \in \mathcal{N}(A - \lambda I$

[eigenvector corresp. to λ lies in null space of $A -$

(ii) $\exists$ $e \neq \theta$ st $(A - \lambda I) e = \theta$

$\Rightarrow$ (matrix) is singular

$\Leftrightarrow$ det $(\lambda I - A) = 0$

$\Leftrightarrow$ $\Delta(\lambda) = 0$

eigen values are roots of ch. poly

(iii) when $A \in \mathbb{R}^{n \times n}$

λ real $\Rightarrow$ e real

complex $\Rightarrow$ complex

(2) Basis of e-values

assume: A has distinct e-values $(\lambda_1, \lambda_2 \ldots \lambda_n)$

the corresponding e-vectors $\{e_1 \ldots e_n\}$ form a basis for (

Proof: Need to show l.i

i.e. $\sum_{i=1}^{n} \alpha_i e_i = 0 \Rightarrow \alpha_i = 0 \quad \forall i$

Motivation: Looking for something to "isolate" a component or to find a "projection" of $\sum \alpha_i e_i$ onto e_k,

$\Delta(A) = (A-\lambda_1 I)(A-\lambda_2 I) \cdots\cdots (A-\lambda_n I) = 0$

Note: $(A - \lambda_n I)\, e_{n-1}$

$= e_{n-1}(\lambda_{n-1} - \lambda_n)$

$P_k := (A-\lambda_1 I) \ldots (A-\cancel{\lambda_k I}) \ldots (A-\lambda_n I)$

$P_k\left[\left(\sum \alpha_i e_i\right)\right] = \alpha_k(\lambda_k - \lambda_1) \ldots (\lambda_k - \lambda_n)\, e_k = P_k[0] = 0$

$e_k \neq 0$ ($\because$ e.v.)

$\alpha_k(\lambda_k-\lambda_1)\ldots(\lambda_k-\lambda_n)$ none of the others can be zero either.

$\Rightarrow \alpha_k = 0$

(3) Change of basis

~~How does~~ A look like wrt $\{e_1, e_2, \ldots e_n\}$

standard basis $\xrightarrow{A}$ standard basis

$\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} \ldots$

$\uparrow P = \begin{bmatrix} | & | & & | & & | \\ e_1 & e_2 & \cdots & e_i & \cdots & e_n \\ | & | & & | & & | \end{bmatrix}$ P expresses each $\{e_1 \ldots e_n\}$ in the new coordinate system ie the standard basis

P^{-1}

$\downarrow$

Now: eigenvectors basis $\{e_1, e_2, \ldots e_n\}$ $\xrightarrow{A = \boxed{?}}$ eigenvectors basis $\{e_1 \ldots e_n\}$

$\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 & 0 \\ 0 & & \lambda_3 & & 0 \\ \vdots & & & \ddots & 0 \\ \vdots & & 0 & & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}$

$A e_i = 0.e_1 + 0.e_2 + \lambda_i e_i + 0.e_n = \lambda_i e_i$

$\boxed{A = P \Lambda P^{-1}}$ $\quad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}$

— Algebraic Derivation

$A\begin{bmatrix} | & | & \\ e_1 & e_2 & \cdots \\ | & | & \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 e_1 & \lambda_2 e_2 & \cdots & \lambda_n e_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}$

$AP = P\Delta$; first do a P on something (ie $\{e_1 \ldots e_n\} \to$ standard) then do an A on it

RHS = do a Δ on something (ie eigen $\to$ eigen) then a P

$A = P\Delta P^{-1}$

let $Q := P^{-1}$

$\Rightarrow A = P\Delta Q$

("LEFT EIGENVECTOR")

4) Let $Q = \begin{bmatrix} — f_1^T — \\ — f_2^T — \\ \vdots \\ — f_n^T — \end{bmatrix}$ $\{f_i^T\}$ rows of Q

$QA = \Lambda Q$

$$\begin{bmatrix} — f_1^T — \\ — f_2^T — \\ \vdots \\ — f_n^T — \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} — f_1^T — \\ — f_2^T — \\ \vdots \\ — f_n^T — \end{bmatrix} = \begin{bmatrix} — \lambda_1 f_1^T \\ — \lambda_2 f_2^T \\ \vdots \\ — \lambda_n f_n^T \end{bmatrix}$$

$f_i^T A = \lambda_i f_i^T$; defining "left eigen vectors."
↳ eigen vectors of A^T.

5) Reln between f_i^T & e_j

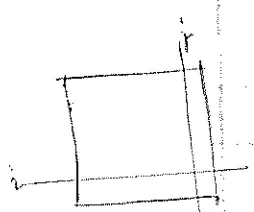
$QP = I$

$$\begin{bmatrix} — f_1^T — \\ — f_2^T — \\ \vdots \\ — f_n^T — \end{bmatrix} \begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix} \qquad f_i^T e_j = \begin{Bmatrix} 1 & i = j \\ 0 & i \neq j \end{Bmatrix} \text{ : orthogonal}$$

A & A^T have same eigenvalues ($\det(A-\lambda I) = \det((A-\lambda I)^T) = \det(A^T-\lambda I)$)

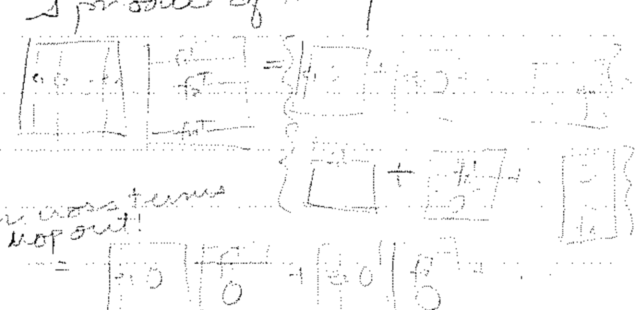
6) $PQ = I$; does this give us anything?

SPECTRAL DECOMPOSITION



$$\begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} — f_1^T — \\ — f_2^T — \\ \vdots \\ — f_n^T — \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

sum product of the form



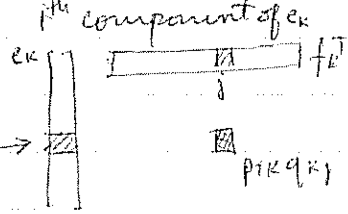
the cross terms drop out!

ij^{th} element of PQ

$$\sum_{k=1}^{n} p_{ik} q_{kj}$$

p_{ik}: i^{th} component of e_k

q_{kj}: j^{th} element of f_k^T



$$\boxed{PQ = \sum_{k=1}^{n} \underbrace{e_k f_k^T}_{\text{matrix}}}$$

let $\boxed{R_K := e_K f_K^T}$

$$\boxed{I = \sum_{k=1}^{n} R_K}$$

$$A = P \Delta Q$$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1 e_1 & \lambda_2 e_2 & \cdots & \lambda_n e_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} — f_1^T — \\ — f_2^T — \\ \vdots \\ — f_n^T — \end{bmatrix} = \sum_{k=1}^{n} \lambda_K e_K f_K^T$$

$$\boxed{A = \sum_{k=1}^{n} \lambda_K R_K}$$

$$\exp(At) = P(\exp \Delta t) Q = \begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} — f_1^T — \\ — f_2^T — \\ \vdots \\ — f_n^T — \end{bmatrix}$$

$$\boxed{\exp(At) = \sum_{k=1}^{n} \underbrace{e^{\lambda_K t}}_{\text{time fns.}} \cdot \underbrace{R_K}_{\text{constants}}}$$

dyadic product

7) Complex eigenvalues

let $A \in \mathbb{R}^{n \times n}$

If λ_i complex $\Rightarrow e_i$ complex $\Rightarrow P, Q$ complex.

Q: Can we pick diff basis s.t. its of all real vectors.

Suppose $\lambda = \alpha + j\beta$ $(\beta \neq 0)$ complex e.value of A

let e be the corresponding eigen vector

$$Ae = \lambda e$$

$$\Rightarrow \overline{Ae} = \overline{\lambda e}$$

$$\Rightarrow \bar{A}\bar{e} = \bar{\lambda}\bar{e} \quad ; A \in \mathbb{R}^{n \times n}$$

$\Rightarrow A\bar{e} = \bar{\lambda}\bar{e}$, ie if $\lambda \overset{e.v.}{\in} \mathbb{C} \Rightarrow \bar{\lambda}$ also eigen value.

corresponding eigen vector is $\bar{e}$.

($\because$ e.v. are from charac. poly. which has n real coefficients.

$\Rightarrow \{ \underset{\text{complex}}{e_1}, \underset{\text{complex}}{\bar{e}_1}, e_3, \ldots \}$ $\quad e_1 = e_1' + j e_1''$ $(e_1', e_1''$ are real$)$

$\lambda_1 = \alpha + j\beta$

claim $\{e_1', e_1''\}$ are l.i.

then $\{e_1', e_1'', e_3, \dots\}$ ← real

Proof of the claim:

suppose not then $c_1 e_1' + c_2 e_1'' = \theta \quad \forall c_1, c_2 \neq 0$

will show $\Rightarrow a_1 e_1 + a_2 \bar{e}_1 = \theta$

non zero

let $a_1 = \frac{1}{2}(c_1 - jc_2) \quad a_2 = \frac{1}{2}(c_1 + jc_2) \quad ; a_1, a_2 \neq 0$

then $a_1 e_1 + a_2 \bar{e}_1 = c_1 e_1' + c_2 e_1'' = 0$

$$\begin{array}{ccc} \text{standard basis} & \xrightarrow{A} & \text{standard basis} \\ \{e_1', e_1'', e_3, \dots\} & \xrightarrow{\Delta'} & \{e_1', e_1'', e_3 \dots\} \end{array}$$

$Q' = (P')^{-1}$ $\quad P' = [e_1' \; e_1'' \; e_3 \; \cdots \; e_n]$

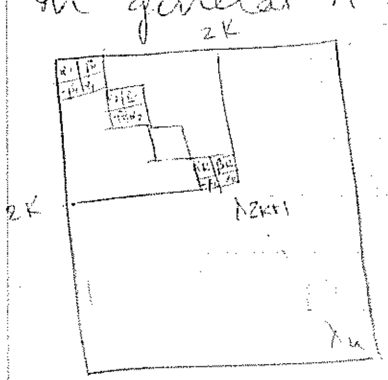
To find Δ' need Ae_1' and Ae_1''

$$A(\underbrace{e_1' + je_1''}_{e_1}) = \underbrace{(\alpha + j\beta)}_{\lambda}(e_1' + je_1'')$$

$$\Rightarrow Ae_1' = \alpha e_1' - \beta e_1''$$
$$Ae_1'' = \beta e_1' + \alpha e_1''$$

$$\Delta' = \begin{bmatrix} \alpha & \beta & & & \\ -\beta & \alpha & & & \\ 0 & 0 & \lambda_3 & & \\ \vdots & 0 & & \lambda_4 & \\ 0 & 0 & & & \lambda_n \end{bmatrix}$$

In general A has k complex eigen values & their conj. and $(n - 2k)$ real



$$A = P' \Delta' Q'$$

all real matrices

5. Modal Decomposition

$$\dot{x}(t) = A x(t) \qquad x(0) = x_0$$

ZIR response

$$x(t) = (\exp At) x_0$$

(1) Algebraic Derivation

(i) $x_0 = \sum_{i=1}^{n} \alpha_i e_i$

$$\Rightarrow \quad x(t) = \sum_{k=1}^{n} e^{\lambda_k t} R_k \left(\sum_{i=1}^{n} \alpha_i e_i \right)$$

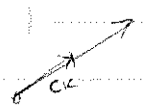
$$R_k e_i = (e_k f_k^T) e_i \quad ; \because R_k \triangleq (e_k f_k^T)$$

$$= e_k \quad \text{if } i = k$$

$$= \theta \quad \text{if } i \neq k$$

$$x(t) = \sum_{k=1}^{n} \alpha_k e^{\lambda_k t} e_k$$

(ii) If $x(0) = c e_k$ in the space spanned by e_k



$x(t) = c e^{\lambda_k t} e_k$ stays in the same space!

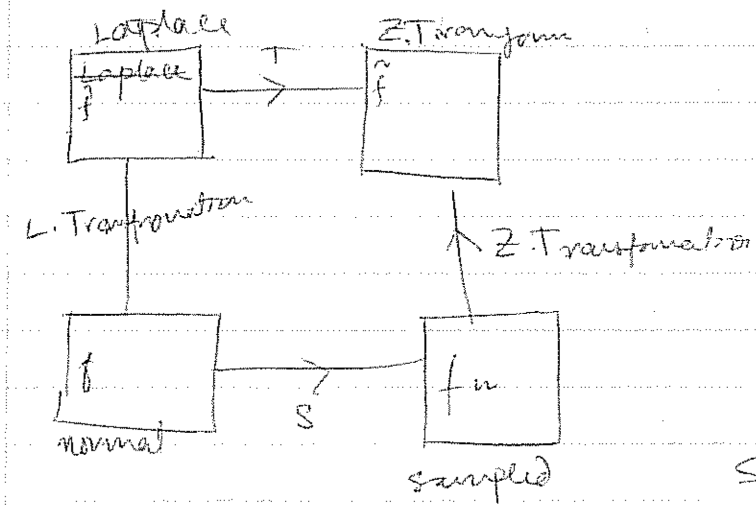
"mode"

Friday
29 Sept 1989

EE221a LST Discussion

- Sampled Data (contd)
- Zeros of a transfer fn
- Cayley Hamilton
- Numerical aspects of exp(At)

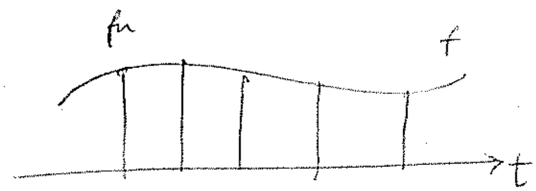
$\phi(t, 0)$



if f is BL and spectrum is supported in $[-\pi/T, \pi/T]$ then

S is injective and

$$f(t) = \sum f_n \left\{ \frac{\sin \frac{\pi}{T}(t - nT)}{\frac{\pi}{T}(t - nT)} \right\}$$



$$f_c = f(t) \sum_{n=0}^{\infty} \delta(\cancel{t}-nT)$$

$$= \sum_{n=0}^{\infty} f(nT)\,\delta(t-nT)$$

$\{f_n\}$ = sequence of samples

$$\mathcal{L}\left(f_c(t)\right) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f(nT)\,\delta(t-nT)\,e^{-st}\,dt$$

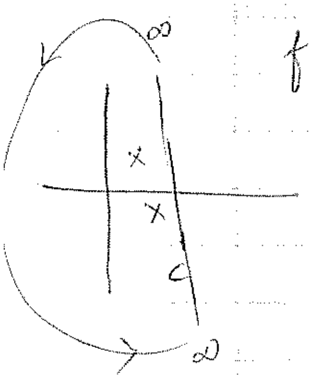
$$\hat{f}(s) = \sum_{n=0}^{\infty} f(nT)\,e^{-snT}$$

$$= \sum_{n=0}^{\infty} f_n\, e^{-nTs}$$

$$\tilde{f}(z) = \sum_{n=0}^{\infty} f(nT)\,z^{-n}$$

$$\tilde{f}_c(s) = \tilde{f}(z)\Big|_{z=e^{sT}}$$

for f_c, Z transform = L.T. evaluation at $z=e^{sT}$



$$f(nT) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \hat{f}(p)\,e^{pnT}\,dp$$

$$= f(n)$$

assume $f(t)$ is continuous

$\therefore f(nT+) = f(nT-) = f(nT)$

$$\tilde{f}(z) = \sum f_n z^{-n}$$

$$= \sum\left[\left(\frac{1}{2\pi j}\right)\int \hat{f}(p)\,e^{pnT}\,dp\right] z^{-n}$$

rtd 221a
29 Sept 1989

EECS 221a: LINEAR SYS.

$$\tilde{f}(e^{sT}) = \sum_{n=0}^{\infty} \tfrac{1}{2\pi j} \int \hat{f}(p) e^{pnT} dp \, e^{-sTn}$$

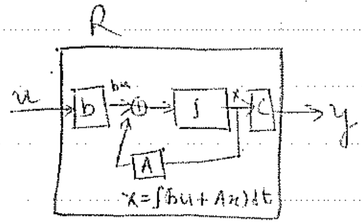
$$= \tfrac{1}{2\pi j} \int \hat{f}(p) \left(\sum_{n=0}^{\infty} e^{(p-s)Tn} \right) dp$$

$$= \tfrac{1}{2\pi j} \int \left\{ \frac{\hat{f}(p)}{1-e^{(p-s)T}} \right\} dp$$

$$\sum_0^{\infty} x^n = 1 + x + x^2 \ldots \quad = \tfrac{1}{1-x}$$; see App. in Reader

---

$$h(s) = \frac{\prod_i (s-z_i)}{\prod_j (s-p_j)} = c(sI-A)^{-1}b$$



$$\dot{x} = Ax + bu$$
$$y = cx$$

z is a zero

$$h(z) = 0 \Rightarrow c(zI-A)^{-1}b = 0 \quad (\text{take } t_0 = 0)$$

$$y(t) = ce^{At}x_0 + \int_0^t ce^{A(t-\tau)} bu(\tau) d\tau$$

Taking Laplace Transform,

$$\hat{y}(s) = c(sI-A)^{-1}x_0 + c(sI-A)^{-1} b\hat{u}(s)$$

$$(sI-A) = (zI-A) + (s-z)I \quad (\text{alg. identity})$$

$$(zI-A)^{-1}(sI-A) = I + (s-z)(zI-A)^{-1}$$

$$(zI-A)^{-1} = (sI-A)^{-1} + (s-z)(zI-A)^{-1}(sI-A)^{-1}$$

$$\hat{y}(s) = c(sI-A)^{-1}x_0 + c\left[\underbrace{(zI-A)^{-1}}_{\to 0} - (s-z)(zI-A)^{-1}(sI-A)^{-1}\right] b\hat{u}(s)$$

$$\hat{y}(s) = c(sI-A)^{-1}x_0 - c(s-z)(zI-A)^{-1}(sI-A)^{-1} b\hat{u}(s)$$

choose $\hat{u}(s) = 1/s-z$ & $\underline{x_0 = (zI-A)^{-1}b}$

$\Rightarrow \quad \hat{y}(s) = 0 \quad \Downarrow$ any initial state won't work (this or a multiple)

$u(t) = 1(t)e^{zt}$

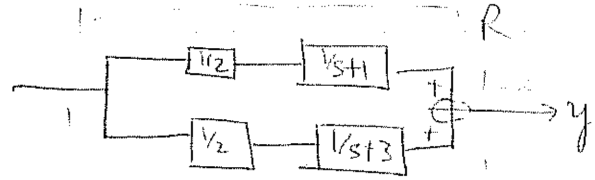
Response fn

$\hat{y}(s) = 0 \overset{\forall t>0}{\Rightarrow} \rho(t, 0, (sI-A)^{-1}b, 1(t)e^{zt}) = 0$ for $t>0$

<u>example</u> $h(s) = \frac{(s+2)}{(s+1)(s+3)} = \frac{1/2}{(s+1)} + \frac{1/2}{(s+3)}$

$\left(h(t) = 1/2\, e^{-t} + 1/2\, e^{-3t}\right)$

$\cancel{\dot{x} = Ax+B}$



no in $y = cx + du$ ← $d = [0]$ $A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$ $b = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ $c = [1 \quad 1]$

$(sI-A) = \begin{bmatrix} s+1 & 0 \\ 0 & s+3 \end{bmatrix} \Rightarrow (sI-A)^{-1} = \begin{bmatrix} 1/s+1 & 0 \\ 0 & 1/s+3 \end{bmatrix}$

find u_p & i.c. s.t. $y_p = 0$

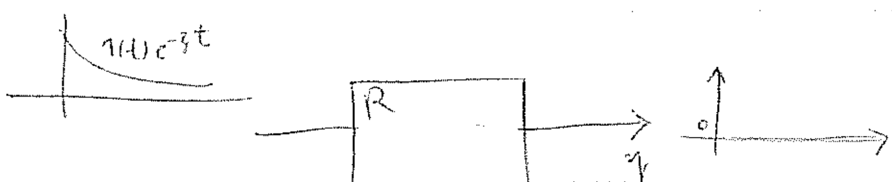
$$X_0 = (zI-A)^{-1}b$$
$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

↑ mit z b(-1)

$u(t) = 1(t)e^{zt}$

$u(t) = 1(t)e^{-2t}$

↑ $z=-2$ (zero) of t.f.



for $x_0 = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$

$C(sI-A)^{-1}x_0 + C(sI-A)^{-1}b\hat{u}(s) = 0$ $\hat{u}(s) = (zI-A)$

$$y(t) = ce^{A(t)}x_0 + \int_0^t ce^{A(t-\tau)}bu(\tau)d\tau$$

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \quad u(t) = e^{-2t}1(t)$$

$$y(t) = [1 \quad 1] \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} + \int_0^t [1 \quad 1] \begin{bmatrix} e^{-(t-\tau)} & 0 \\ 0 & e^{-3(t-\tau)} \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \times e^{-2\tau} d\tau$$

$$= \tfrac{1}{2} e^{-t} + \tfrac{1}{2} e^{-3t} + \tfrac{1}{2} \{ -e^{-2t} + e^{-t} + e^{-2t} - e^{-3t} \}$$

$$= 0 \quad \checkmark$$

Look in Reader ch.

Main thing to remember: ~~for a specification~~ & "If z is any zero of the t-f. there is at least one initial state x_0 s.t. the response of the system to the exponential $u(t) e^{zt}$ starting at $t=0$ from x_0 is identically 0.

$$\hat{\phi}(s) = (sI - A)^{-1} = \frac{s^{n-1} B_0 + s^{n-2} B_1 + \cdots B_{n-1}}{\underbrace{s^n + s^{n-1} d_1 + \cdots d_n}_{= d(s)}}$$

$$d(s) = (sI - A)(s^{n-1} B_0 + \cdots B_{n-1})$$

$B_0 = I$

$B_1 = B_0 A + d_1 I$ $\qquad B_1 = A + d_1 I$

$B_2 = \cdots$ $\qquad B_2 = (A + d_1 I) A + d_2 I = A^2 + d_1 A + d_2 I$

$\vdots$

$B_{n-1} = B_{n-2} A + d_{n-1} I$

$0 = B_{n-1} A + d_n I$ $\qquad \boxed{0 = d(A)}$

$A \in \mathbb{R}^{n \times n}$

$A^m \quad m \geq n = f(I, A, A^2, \ldots A^{n-1})$ ← linear comb.

$\exp(A) = \alpha_0 I + \alpha_1 A + \cdots \alpha_{n-1} A^{n-1}$; finding the α's may not be easy.

$x \in \mathbb{R}^1$

$\exp(x) = \alpha 1$

$= e^x \cdot 1$

example $A = \begin{bmatrix} \frac{59}{5} & \frac{12}{5} \\ \frac{12}{5} & \frac{66}{5} \end{bmatrix}$ $(sI-A)^{-1} = \frac{1}{(s-\frac{59}{5})(s-\frac{66}{5}) - \frac{144}{5}} \begin{bmatrix} s-\frac{66}{5} & \frac{12}{5} \\ \frac{12}{5} & s-\frac{59}{5} \end{bmatrix}$

?

$= (s-10)(s-15)$

Residual matrix

$\hookrightarrow R_i = \lim_{s \to \lambda_i} (s-\lambda_i) \frac{B(s)}{d(s)}$

$\exp(A) = \sum_{i=0}^{n} R_i e^{\lambda_i}$

$R_1 = \lim_{\substack{s \to 15 \\ \lambda_1}} \frac{1}{(s-15)(s-10)} \begin{bmatrix} & \\ & \end{bmatrix}$

$= \frac{1}{5} \begin{bmatrix} 9/5 & 12/5 \\ 12/5 & 16/5 \end{bmatrix}$

$R_2 = \lim_{\substack{s \to 10 \\ \lambda_2}} \frac{1}{(s-15)} \begin{bmatrix} & \\ & \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} -16/5 & 12/5 \\ 12/5 & -9/5 \end{bmatrix}$

Prop.: $R_1 + R_2 = I$ (in general $\sum_{1}^{n} R_i = I$) & $\left(\sum_{1}^{n} \lambda_i R_i = A \right)$

$$\sum_{i=1}^{n} R_i e^{\lambda_i} = e^A \qquad \sum_{i=1}^{n} f(\lambda_i) R_i = f(A)$$

$\phi(t,0) = ? = \exp(At)$

$$e^{A \cdot t} = \sum_{i=1}^{2} R_i e^{\lambda_i t} = \underbrace{\frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}}_{R_1} \underbrace{e^{15t}}_{e^{\lambda_1 t}} + \frac{1}{25} \begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix} e^{10t}$$

$$\therefore \frac{1}{25} \begin{bmatrix} 9e^{15t} + 16e^{10t} & 12e^{15t} - 12e^{10t} \\ 12e^{15t} - 12e^{10t} & 16e^{15t} + 9e^{10t} \end{bmatrix}$$

$t=0 \Rightarrow = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$t=1 \Rightarrow = \frac{1}{25}\begin{bmatrix} 9e^{15}+16e^{10} & 12(e^{15}-e^{10}) \\ 12(e^{15}-e^{10}) & 16e^{15}+9e^{10} \end{bmatrix}$

$= \left(\frac{e^{15}-e^{10}}{5}\right) A + (-2e^{15}+3e^{10}) I$

$\exp(A) = \alpha_1 A + \alpha_0 I$; its obvious that its rather tough to get α_is.

<u>numerical aspects of exp(At)</u>

$\exp(At) = I + \cdots$

$\exp(-6) = 1 - 6 + 36/2! - 216/3! + \frac{1296}{4!} - \frac{7776}{5!} + \cdots$

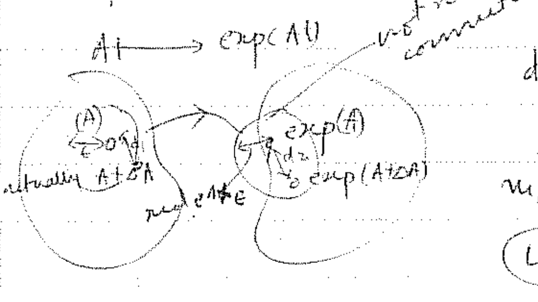
your algo. terminates at S_kth (= partial sum upto k term)
when $[S_{k+1}] = [S_k]$
Say quits @ 8th term $e^{-6} = 17.1143$ (!)

could do by $e^{-6} = (e^{6})^{-1}$
$= \left(1 + 6 + \frac{36}{2!} + \cdots\right)^{-1} = (341.8)^{-1} = 2.926 \times 10^{-3}$
(8 terms) real answer $= 2.4791$

$A = \begin{bmatrix} -49 & 24 \\ -64 & 31 \end{bmatrix}$

dirt method $\Rightarrow \begin{bmatrix} -22.2 & -1.4 \\ -61.5 & -3.5 \end{bmatrix}$ real answer $= \begin{bmatrix} -.73 & 0.55 \\ -1.47 & 1.10 \end{bmatrix}$

exp(At)



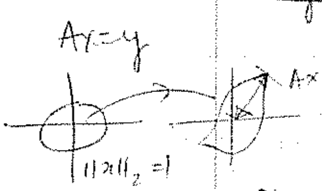
d_2/d_1 = measure of sensitivity

m/c epsilon :

(L.U.B.) $\epsilon : (1+\epsilon) > 1$ (least upper bnd)

Algorithm for $\exp(A)$ <u>stable</u>: ball to ball (w/c e argument)
<u>reliable</u>: tells you if its working error
<u>general</u>: works for broad class of matrices
<u>efficient</u>: fast

<u>Taylor Series</u> $\exp(0.5)$ converges fast (any close to zero)



~~Problems~~
Comments
(1) $\|A\|_2$ isn't always small
(2) good insight
(3) $\exp(A)$ known doesn't guarantee that $\exp(A^n t) = [\exp At]^n$
will be a good answer.

<u>Padé</u> method

$1+x+x^2+\dots = 1/(1-x)$ $|x|<1$; rational fn. expression

valid when $\|A\|_2$ "small"

$$P_{pq}(A) \doteq [D_{pq}(A)]^{-1} N_{pq}(A) \approx \exp(A)$$

$$D_{pq} \doteq \sum_{j=0}^{q} \frac{(p+q-j)!\, q!\, (-A)^j}{(p+q)!\; j!\, \boxed{(q-j)!}}$$

$$N_{pq} \doteq \sum_{j=0}^{p} \frac{(p+q-j)!\, p!\, (A)^j}{(p+q)!\, j!\, (p-j)!}$$

$p=q \gg$ large $\quad N_{pq} \to \exp{.5A}$
$\quad\quad\quad\quad\quad D \to \exp{-.5A}$ $\Big\}$ same trouble as Taylors

$q=p$ optimal Padé expansion

when A isn't small $\Rightarrow$ <u>Scaling & Squaring</u>

$\exp A = [\exp(A/m)]^m$ pick $m=2^k$ (make $\|\frac{A}{m}\|_2$ small then we then compare no)

5/23
#1989

5. Modal Decomposition

$$\dot{x}(t) = Ax(t) \qquad x(0) = x_0$$

$$x(t) = [\exp(At)]x_0$$

(2) Geometric Derivation

(i) $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ expressed in terms of standard basis $\{e_1, e_2, \dots e_n\}$ (if complex $\{e_1', e_1'', e_3 \dots\}$)

Change of coordinates into basis of eigen vectors

Standard basis $x(t) \leftarrow P = \begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix}$

eigen vectors $\{e_1, \dots e_n\}$ coordinates $\xi(t)$

$$x(t) = \xi_1(t)e_1 + \xi_2(t)e_2 + \cdots \xi_n(t)e_n$$

$$= \begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \vdots \\ \xi_n(t) \end{bmatrix}$$

$$= P\xi(t)$$

$\dot{x}(t) = Ax(t)$

$P\dot{\xi}(t) = AP\xi(t)$

$\dot{\xi}(t) = P^{-1}AP\xi(t) = \Lambda\xi(t)$

(ii) If $x(0) = ce_k \Rightarrow \xi(0) = \begin{bmatrix} 0 \\ \vdots \\ c \\ \vdots \\ 0 \end{bmatrix} \leftarrow k^{th}$

$\dot{\xi}_k(t) = \lambda_k \xi_k(t) \quad \xi_k(0) = c$

$\xi_k(t) = ce^{\lambda_k t}$

$x(t) = ce^{\lambda_k t} \cdot e_k$

start in e. vec. dim $\Rightarrow$ stay in it.

(iii) Complex e-value corresp. basis e_k' e_k''

$$\begin{bmatrix} \dot{\xi}_k(t) \\ \dot{\xi}_{k+1}(t) \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} \xi_k(t) \\ \xi_{k+1}(t) \end{bmatrix} \qquad \begin{matrix} \xi_k(0) = c \\ \xi_{k+1}(0) = 0 \end{matrix} \qquad \xi_j(0) = 0$$

$\xi_k(t) = ce^{\alpha t}\cos\beta t$ If initially $x(0)$ is in the subspace

$\xi_{k+1}(t) = ce^{\alpha t}\sin\beta t$ spanned by $\{e_k', e_k''\}$ $x(t)$ will stay

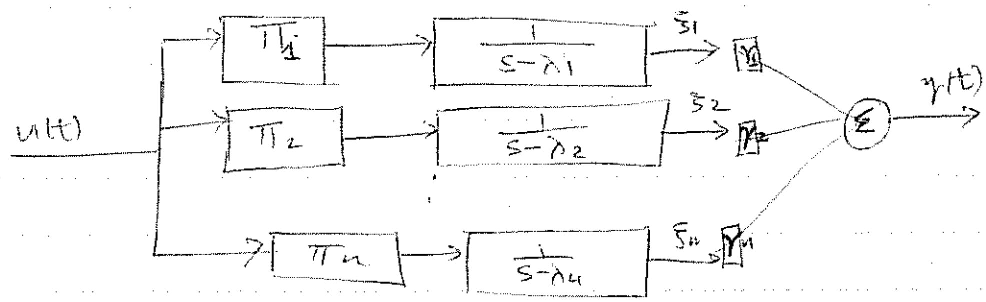
6. BLOCK DIAGRAM

$P\dot{\xi}(t) = P\Lambda P^{-1}\xi(t) + Bu(t)$
$\Rightarrow \dot{\xi}(t) = \Lambda P^{-1}\xi(t) + P^{-1}Bu(t)$

$$\dot{x}(t) = Ax(t) + Bu(t) \quad ; \quad x(0) = x_0$$
$$y(t) = Cx(t)$$
$$x(t) = P\xi(t)$$
$$\dot{\xi}(t) = \Lambda\xi(t) + \Pi u(t) \quad ; \quad \Pi = P^{-1}B$$
$$y(t) = \Gamma\xi(t) \quad ; \quad \Gamma = CP$$

Let's consider SISO (for ease of drawing)

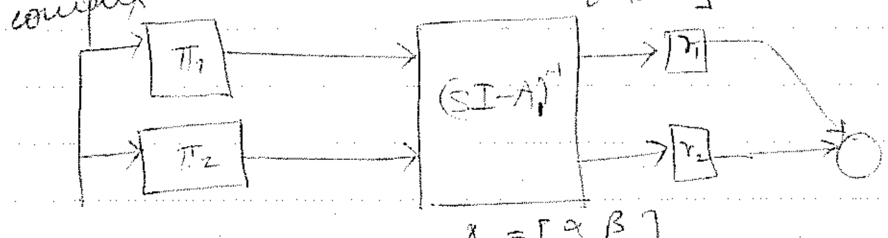
$$\dot{\xi}_1(t) = \lambda_1 \xi_1(t) + \pi_1 u(t)$$
$$\dot{\xi}_2(t) = \lambda_2 \xi_2(t) + \pi_2 u(t)$$
$$\vdots$$
$$y(t) = \gamma_1 \xi_1(t) + \gamma_2 \xi_2(t) + \dots$$



depends on whether input excites that particular mode

$$A_1 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

complex



$$A_1 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

$\pi_k \neq 0 \Rightarrow k^{th}$ mode is excited.

When A doesnt have distinct eigen values, we can still do some sort of decomposition.

on V $R = [A, B, C, D]$

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \qquad x(0) = x_0$$

A has distinct evalues $\Rightarrow A \sim \Lambda \ (P^{-1}AP = \Lambda)$

$\Rightarrow$ modal decomposition

A has multiple e.values $\Rightarrow A \sim ?$

$\Rightarrow ?$

Recall A has distinct e-values

$\Rightarrow \{e_1, e_2, \cdots e_n\}$ form a basis

$\Rightarrow x = \sum_{i=1}^{n} \xi_i e_i$ (decomposition)

where $e_i \in \mathcal{N}(A - \lambda_i I)$ (eigenspace = sub. space spanned by the i^{th} e-vector)

For A multiple e-values need to generalize this decomposition, & eigenspace.

char. of subspace

$e \in \mathcal{N}(A - \lambda_i I) \qquad Ae = \lambda_i e \in \mathcal{N}(A - \lambda_i I)$

(1) Invariant subspace

(V, F) vector space

M subspace of V.

M is said to be an invariant subspace of A iff $x \in M \Rightarrow Ax \in M$

Ex (i) $\mathcal{N}(A)$ $\quad (\because x \in \mathcal{N}(A) \Rightarrow Ax = \theta$

claim $Ax \in \mathcal{N}(A)$

$(\because A(Ax) = \theta)$)

(ii) $\mathcal{R}(A)$

(iii) (Motivating example) $\mathcal{N}(A - \lambda_i I)$

$\quad\;\;$ ↓ polynomial of A (any degree)
(iv) $\mathcal{N}(p(A))$

(2) Direct sum

M_1, M_2 subspaces of (V, F)

V is direct sum of M_1 & M_2 $\iff$ $\forall x \in V$ $\exists$ unique rep. $x = x_1 + x_2$ where $x_1 \in M_1$ & $x_2 \in M_2$

(Expressed as $V = M_1 \oplus M_2$

$M_1, M_2, \ldots, M_K$ subspaces of V

$V = M_1 \oplus M_2 \oplus \cdots \oplus M_K$

$\iff \forall x \in V \;\exists$ unique representation $x = x_1 + x_2 + \cdots + x_K;\; x_i \in M_i$

A has distinct e. values

Ex. $x = \sum_{i=1}^{n} \xi_i e_i \quad \Rightarrow \mathbb{C}^n = \mathcal{N}(A - \lambda_1 I) \oplus \mathcal{N}(A - \lambda_2 I) \oplus \cdots \oplus \mathcal{N}(A - \ldots$

We say x_i is the "projection" of x onto M_i.

(3) Representation Theorem

$F = \mathbb{C}$ or $\mathbb{R}$; $F^n = M_1 \oplus M_2$

$A : F^n \to F^n$ linear

M_1 invariant under A

$\Rightarrow \exists$ a basis for F^n s.t.

A has the matrix representation

$$\begin{array}{c|cc} & K & n-K \\ \hline K & A_{11} & A_{12} \\ n-K & 0 & A_{22} \end{array}$$

$\dim M_1 = K$

pick K lin ind. vectors in M_1

$\{\underbrace{u_1, u_2, \ldots u_K}_{M_1}, u_{K+1} \ldots u_n\}$

augment basis

1^{st} column $= Au_1$ expressed in terms of $\{u_1, \ldots u_n\}$

but $Au_1 \in M_1$ is a l.c. of $u_1 \ldots u_K$.

Corollary M_1, M_2 invariant under A

$$A \sim \left[\begin{array}{c|c} & 0 \\ \hline 0 & \end{array}\right]$$

Ex. A distinct eigen values

$\mathcal{N}(A-\lambda_i I)$ invariant under A

$$\mathbb{C}^n = \mathcal{N}(A-\lambda_1 I) \oplus \mathcal{N}(A-\lambda_2 I) \cdots \oplus \mathcal{N}(A-\lambda_n I)$$

$$A \sim \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$ (applying Thm n times)

2. Decomposition

Recall A distinct e.values

Cayley-Hamilton Theorem ~~A~~ chc. poly $\Delta(s)$

$\Delta(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$

substitute $A \Rightarrow (A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I) = 0$

we used C-H thm to show eigenvectors are lin ind.

We can write $\mathbb{C}^n = \mathcal{N}(A-\lambda_1 I) \oplus \mathcal{N}(A-\lambda_2 I) \cdots \oplus \mathcal{N}(A-\lambda_n I)$

In general A will have multiple eigen values

chc. poly $\Delta(s) = (s-\lambda_1)^{d_1}(s-\lambda_2)^{d_2}\cdots(s-\lambda_\sigma)^{d_\sigma}$; $\sigma \le n$

$[d_1 + d_2 + \cdots d_\sigma = n]$

Cayley Hamilton $(A-\lambda_1 I)^{d_1} \cdots (A-\lambda_\sigma I)^{d_\sigma} = 0$

So we $\mathbb{C}^n \stackrel{?}{=} \mathcal{N}(A-\lambda_1 I)^{d_1} \oplus \mathcal{N}(A-\lambda_2)^{d_2} \cdots \oplus \mathcal{N}(A-\lambda_\sigma)^{d_\sigma}$

Ex $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ ch. poly $= (\lambda-1)^2(\lambda-2)$

CH: $(A-I)^2(A-2I) = 0$

$\mathbb{C}^n \stackrel{?}{=} \mathcal{N}(A-I)^2 \oplus \mathcal{N}(A-2I)$ yes

However, for this example we also have
$(A-I)(A-2I) = 0$ matrix.

$$\mathbb{C}^n \overset{?}{=} \mathcal{N}(A-I) \oplus \mathcal{N}(A-2I)$$

Thursday
5 Oct 1989

LST - Lecture

Midterm - 1hr duration (Friday)

Ch I: except Schwartz Inequality p.22 sec 9, 10

Ch II: except sec 2

Ch III: except sec 7

Ch IV: except sec 6*

Ch V: sec 1-5

VIII A, B, E [Supplementary Notes]

allowed to bring one sheet of paper (one side)

Ex. $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\qquad \lambda = 1, 1, 2$

Cayley Hamilton $(A-I)^2(A-2I) = 0$

But we also have $(A-I)(A-2I) = 0$ (by direct calc. for th

Def: $\psi(\cdot)$ is the min^m polynomial of A

$\Leftrightarrow$ (i) $\psi(A) = 0$

(ii) lowest degree polynomial s.t. $\psi(A) = 0$

(iii) The coefficient of the highest order term $= 1$. (so m

Fact: Roots of the min^m poly $= 0 \Leftrightarrow$ roots of $\Delta(s) = \det(sI - A) =$

$=$ e. values of A.

Pf ($\Leftarrow$) $\quad \lambda_i$ e. value $\Rightarrow \exists\, e_i \neq 0$ s.t.

by dfn $A e_i = \lambda_i e_i$

Want to show $\psi(\lambda_i) = 0$

$A^j e_i = \lambda_i^j e_i$

$\underline{\psi(A)} e_i = \psi(\lambda_i) e_i \quad \Rightarrow \theta = \psi(\lambda_i) e_i \Rightarrow \psi(\lambda_i) = 0$

(note: $Ae_i = \lambda_i e_i \Rightarrow A^2 e_i = \lambda_i A e_i = \lambda^2 e_i$ & $A^n e_i = \lambda_i^n e_i$)

Proof ($\Rightarrow$) $\boxplus$ λ_i is a root of $\psi(s) = 0$

$$\psi(s) = (s - \lambda_i)(q(s))$$

know $\deg[q(s)] < \deg[\psi(s)]$

$$\underset{\displaystyle 0}{\overset{\|}{\psi(A)}} = (A - \lambda_i I)\underbrace{q(A)}_{\neq 0} \quad (\because \text{minimal}) \Rightarrow \exists v \text{ s.t. } v \notin \mathcal{N}[q(A)]$$

$$\theta = (A - \lambda_i I)\underbrace{q(A)v}_{\neq \theta \; = v'}$$

$\therefore Av' = \lambda_i v'$ & v' is an eigen vector

$\Rightarrow \lambda_i$ is an eigen value.

charac polynomial $\Delta(s) = (s - \lambda_1)^{d_1} \cdots (s - \lambda_\sigma)^{d_\sigma}$

Fact $\Rightarrow \psi(s) = (s - \lambda_1)^{m_1}(s - \lambda_2)^{m_2} \cdots (s - \lambda_\sigma)^{m_\sigma}$

where $m_k \le d_k$ for all k. & $\psi(s)$ divides $\Delta(s)$ (if not $\Rightarrow$ could construct lower degree)

$$\psi(A) = (A - \lambda_1 I)^{m_1}(A - \lambda_2 I)^{m_2} \cdots (A - \lambda_\sigma I)^{m_\sigma} = 0$$

Recall in distinct case $\Delta(A) = (A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_n I) = 0$

$$x = \alpha_1 e_1 + \cdots \alpha_n e_n$$

$$\mathbb{C}^n = \mathcal{N}(A - \lambda_1 I) \oplus \mathcal{N}(A - \lambda_2 I) \cdots$$

Now $\mathbb{C}^n \overset{?}{=} \mathcal{N}(A - \lambda_1 I)^{m_1} \oplus \mathcal{N}(A - \lambda_2 I)^{m_2} \oplus \cdots \oplus \mathcal{N}((A - \lambda_\sigma I)^{\sigma}$

Assume C-H : $\Delta(A) = (A - \lambda_1 I)^{d_1} \cdots (A - \lambda_\sigma I)^{d_\sigma} = 0$

$$\mathbb{C}^n \overset{?}{=} \mathcal{N}(A - \lambda_1 I)^{d_1} \oplus \mathcal{N}(A - \lambda_2 I)^{d_2} \cdots \oplus \mathcal{N}(A - \lambda_\sigma I)^{d_\sigma}$$

If $\mathcal{N}(A - \lambda_k I)^{m_k} = \mathcal{N}(A - \lambda_k I)^{d_k}$ Fact (in discussion)

Conjecture: $\boxed{\mathbb{C}^n = \mathcal{N}(A - \lambda_1 I)^{m_1} \oplus \mathcal{N}(A - \lambda_2 I)^{m_2} \oplus \cdots \oplus \mathcal{N}(A - \lambda_\sigma I)^{m_\sigma}}$ Fact

↳ proof: Need to show $\forall x \in \mathbb{C}^n$

$\exists x_1, x_2, \ldots x_\sigma$ s.t. $x_k \in \mathcal{N}(A - \lambda_k I)^{m_k}$

and $x = x_1 + x_2 + \cdots x_\sigma$

representation is unique.

How to find x_k?

= Find the "projection" R_k s.t.

$$x_k = R_k \cdot x \in \mathcal{N}(A - \lambda_k I)^{m_k}$$

and $x = x_1 + x_2 + \cdots x_\sigma$

$$= \sum_{k=1}^{\sigma} R_k x$$

$$\Leftrightarrow \sum_{k=1}^{\sigma} R_k = I$$

(i) $R_k x \in \mathcal{N}(A - \lambda_k I)^{m_k}$

$\Rightarrow (A - \lambda_k I)^{m_k} R_k x = \theta$ is required

but we know $\psi(A) = (A - \lambda_1 I)^{m_1} \cdots (A - \lambda_\sigma I)^{m_\sigma} = 0$

$\Rightarrow$ so pick R_k having factors $(A - \lambda_1 I)^{m_1} \cdots (A - \lambda_k I)^{m_k} \cdots (A - \lambda_\sigma I)$

$$R_k = (A - \lambda_1 I)^{m_1} \cdots \cancel{(A - \lambda_k I)^{m_k}} \cdots (A - \lambda_\sigma I)^{m_\sigma}$$

in general have to multiply by $n_k(A)$ ie

$$R_k = n_k(A)(A - \lambda_1 I)^{m_1} \cdots (\;\nearrow\;) \cdots (A - \lambda_\sigma I)^{m_\sigma}$$

(ii) Want $\sum_{k=1}^{\sigma} R_k = I$

i.e. $n_1(A)(A - \lambda_2 I)^{m_2} \cdots (A - \lambda_\sigma I)^{m_\sigma}$

$+ n_2(A)(A - \lambda_1 I)^{m_1} \cdots (A - \lambda_\sigma I)^{m_\sigma}$

$+ \cdots$

$+ n_\sigma(A)(A - \lambda_1 I)^{m_1} \cdots (A - \lambda_{\sigma-1} I)^{m_{\sigma-1}}$

$= I$

look at in a fashion analogous to polynomials

$n_1(s)(s - \lambda_2)^{m_2} \cdots (s - \lambda_\sigma)^{m_\sigma}$

$+ n_2(s)(s - \lambda_1)^{m_1} \cdots (s - \lambda_\sigma)^{m_\sigma}$

$\cdots$

$+ n_\sigma(s)(s - \lambda_1)^{m_1} \cdots (s - \lambda_{\sigma-1})^{m_{\sigma-1}} = 1$

Divide through by minimal polynomial

$n_1(s)/(s-\lambda_1)^{m_1} + n_2/(s-\lambda_2)^{m_2} + \cdots n_\sigma(s)/(s-\lambda_\sigma)^{m_\sigma} = 1/\psi(s)$

Existence of $n_i(s)$'s is guaranteed by Partial Fraction Expansion Thm!

$\Rightarrow$ Partial Fraction Expansion of $1/\psi(s)$

$$1/\psi(s) = \frac{n_1(s)}{(s-\lambda_1)^{m_1}} + \cdots \frac{n_\sigma(s)}{(s-\lambda_\sigma)^{m_\sigma}} \quad * \quad ; \text{ use } n_1(s), n_2(s) \text{ polys as in partial fraction expansion}$$

obtain $R_k = n_k(A)(A-\lambda_1 I)^{m_1} \cdots (A-\lambda_k I)^{m_k} \cdots (A-\lambda_\sigma I)^{m_\sigma}$

where $n_k(s)$ is obtained from $*$

Then $x_k := R_k x \in \mathcal{N}(A-\lambda_k I)^{m_k}$ well defined

and $x = x_1 + x_2 + \cdots + x_\sigma$

To show uniqueness of decomposition:

Suppose $x = x_1 + x_2 + \cdots x_\sigma$; $x_k = R_k x \in \mathcal{N}(A-\lambda_k I)^{*}$

and also $x = x_1' + x_2' + \cdots x_\sigma'$; $x_k' \in \mathcal{N}(A-\lambda_k I)^{m_k}$

want $x_k' = x_k \quad \forall k$

$\therefore x_1 \in \mathcal{N}(A-\lambda_1 I)$ which is all wrong

$\theta = (x_1 - x_1') + (x_2 - x_2') + \cdots + (x_\sigma - x_\sigma')$

if say $x_1 - x_1' \neq \theta$

$(x_1 - x_1') = (x_2' - x_2) + (x_3' - x_3) + \cdots (x_\sigma' - x_\sigma)$ could consider diff null space

Multiply by $\left[(A-\lambda_2 I)^{m_2} \cdots (A-\lambda_\sigma I)^{m_\sigma}\right]$ vectors of diff null spaces don't have to be l.i.

$\therefore [\qquad](x_1 - x_1') = \theta$?

$\Rightarrow n_1(A)[\qquad](x_1 - x_1') = \theta$ * this is because (will see $R_k(A)$) $(A-\lambda_1 I)^{m_1} \cdots (A-\lambda_\sigma I)^{m_\sigma} x_\rho$ can shuffle!

** i.e. $R_1(x_1 - x_1') = \theta$; but don't know if R_1 is non singular!

scratch work

Consider $(A-\lambda_1 I)^{m_1}$; $(s-\lambda_1)^{m_1}$

$R_1(A)$; $R_1(s)$

Since $R_1(s)$ and $(s-\lambda_1)^{m_1}$ have no common factors (else could lead to a contradiction)

$$\frac{1}{(s-\lambda_1)^{m_1} R_1(s)} = \frac{h_1(s)}{(s-\lambda_1)^{m_1}} + \frac{h_2(s)}{R_1(s)}$$

or $1 = h_1(s) R_1(s) + h_2(s)(s-\lambda_1)^{m_1}$

substitute s by A

$$I = h_1(A)R_1(A) + h_2(A)(A-\lambda_1 I)^{m_1}$$; mult. both sides by $(x_1 - x_1')$

$$\Rightarrow (x_1 - x_1') = \underbrace{h_1(A)R_1(A)(x_1-x_1')}_{=\theta \; **} + h_2(A)\underbrace{(A-\lambda_1 I)^{m_1}(x_1-x_1')}_{\text{wrong} \rightarrow (\text{Eigen value - Eigen vector}) = \theta}$$

zero is ... ∵ $x_1 - x_1'$ in null space of $(A-\lambda_1 I)^{m_1}$

$\Rightarrow x_1 = x_1'$

(3) MATRIX REPRESENTATION

$$\mathbb{C}^n = \mathcal{N}(A-\lambda_1 I)^{m_1} \oplus \mathcal{N}(A-\lambda_2 I)^{m_2} \cdots \oplus \mathcal{N}(A-\lambda_\sigma I)^{m_\sigma}$$

$\Rightarrow \exists$ basis for $\mathbb{C}^n$ such that (union of lin ind. vectors in $\mathcal{N}(A-\lambda_k I)$)

wrt basis A has matrix representation

$V = N_1 \oplus N_2$ $\Rightarrow$ every non zero vectors in N_1 & N_2 are linearly independent

$$\begin{array}{c} p_1 \\ p_2 \\ \\ \\ p_\sigma \end{array}\begin{bmatrix} A_1 & & & \mathcal{O} \\ & A_2 & & \\ & & \ddots & \\ \mathcal{O} & & & A_\sigma \end{bmatrix} \quad n \quad (\because \mathbb{C}^n \xrightarrow{A} \mathbb{C}^n)$$

where $p_k := \dim \mathcal{N}(A-\lambda_k I)^{m_k}$

we know $p_1 + p_2 + \cdots p_\sigma = n$

But we also have

$$\Delta(s) = (s-\lambda_1)^{d_1} \cdots (s-\lambda_\sigma)^{d_\sigma}$$

such that $d_1 + \cdots d_\sigma = n$

<u>Fact</u> dimension $\mathcal{N}(A-\lambda_k I)^{m_k} = d_k$

Pf: $\det(A-sI) = (-1)^n \prod_{k=1}^{\sigma}(s-\lambda_k)^{d_k}$

$$= \prod_{k=1}^{\sigma} \det(A_k - sI_k)$$

$$\det PAP^{-1} = \det P \det A \det P^{-1}$$
$$= \det A$$

want to show $\det(A_k - sI_k) = (-1)^{d_k}(s-\lambda_k)^{d_k}$

$\det(A_k - sI_k)$ has only λ_k as its ~~root~~ zero.

$\det(A_k - sI_k) = (-1)^{p_k}(s-\lambda_k)^{p_k}$

Friday
Oct 1989

EECS 221(a) discussion

- continue exp(A)
- examples s.t. e-value sensitivity
- real/complex basis vectors
- complex basis vectors
- proof of $N(A-\lambda_k I)^{m_k} = N(A+\lambda_k I)^{d_k}$
- sampled data systems
- MidTerm

$$N_{pq}(A) \triangleq \sum_{j=0}^{p} \frac{(p+q-j)!\,p!}{(p+q)!\,j!\,(p-j)!} A^j \quad ; \text{Padé Formula}$$

if $A = PMP^{-1}$ (M diagonal)

$$\exp(A) = I + A + \tfrac{1}{2!}A^2 + \cdots$$
$$= I + PMP^{-1} + \tfrac{1}{2!}(PMP^{-1})(PMP^{-1}) + \tfrac{1}{6}(\ )(\ )(\ ) \cdots$$
$$= PP^{-1} + PMP^{-1} + \tfrac{1}{2!}PM^2P^{-1}$$
$$= P\exp M\, P^{-1}$$

diagonal matrix $\begin{bmatrix} m_1 & 0 & 0 & \cdots \\ 0 & m_2 & 0 & \cdots \\ 0 & 0 & m_3 & \cdots \\ & & & m_n \end{bmatrix}$

$$\exp(M) = \begin{bmatrix} e^{m_1} & & 0 \\ & e^{m_2} & \\ 0 & & e^{m_n} \end{bmatrix}$$

Problem: P is ill conditioned to inversion (near singular)

Schur decomposition $A = QMQ^T$ where M, Q is upper Δular
exp(M) is also upper Δular $\quad Q^T = Q^{-1}$

Set up so that Q is well conditioned

Eigenvalue / Eigenvector Method

If A is non defective (E values are distinct)

$\exists T$ s.t. $A = T D T^{-1}$

D Diagonal

also works for block diagonal.

$$A = \begin{bmatrix} \lambda_1 & & 0 \\ & D_2 & \\ 0 & & \lambda_3 \end{bmatrix} \Rightarrow \exp(A) = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{D_2 t} & \\ 0 & & \; \end{bmatrix}$$

$$A = \begin{bmatrix} 20 & 20 & 0 & \cdots & \cdots & 0 \\ 0 & 19 & 20 & 0 & \cdots & 0 \\ 0 & 0 & 18 & 20 & \cdots & 0 \\ \delta & 0 & \cdots & & & 2\;20 \\ & & & & & 0\;1 \end{bmatrix}$$

$\Rightarrow$ eigenvalues are $1, 2, 3 \ldots 20$ (when no δ)

let $\lambda_i(\delta) = \lambda_i + \xi_i \delta$

turns out $\xi_{20} = -\xi_1 = 4\times10^7$

$\xi_{10} = -\xi_{11} = 4\times10^{12}$

non diagonalization

JORDAN FORM

$\nexists T$ s.t. $A = TDT^{-1}$; D diagonal

$\exists T$ s.t. $A = TJT^{-1}$ J = Jordan Form Matrix (e-values don't have to be distinct)

Jordan Form:
$$\begin{bmatrix} \lambda_1 & 1 & & & 0 \\ & \lambda_2 & 1 & & \\ & & \lambda_3 & 0 & \\ & & & 0 & 1 \\ 0 & & & & \lambda_n \end{bmatrix}$$

Jordan Blocks
$$\begin{bmatrix} \lambda_1 & 1 & & & & & 0 \\ 0 & \lambda_1 & 1 & & & & \\ & & \lambda_1 & & & & \\ & & & \boxed{\begin{matrix}\lambda_2 & 1 \\ 0 & \lambda_2\end{matrix}} & 0 & & \\ 0 & & & & \boxed{\lambda_2} & 0 & \\ & & & & & \boxed{\lambda_3} & 0 \\ & & & & & & \lambda_4 \end{bmatrix}$$

size of largest Jordan block is (m_1 = degree in minimal poly)

$$e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{t^2}{2}e^{t_1 \lambda} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix}$$

The T values are generalized eigenvectors.

n^{th} generalization $\frac{k!}{(k-n)!\,n!} \lambda^{k-n}$

$$J = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_1 & & \\ & & \lambda_1 & \\ 0 & & & 0 \end{bmatrix}, \quad J^4 = \begin{bmatrix} \lambda^4 & 4\lambda^3 & 6\lambda^2 & 4\lambda \\ & \ddots & \ddots & 6\lambda^2 \\ 0 & & & \ddots \end{bmatrix}$$

Singular value decomposition

$A = U \Sigma V$

$UU^T = I$

$VV^T = I$

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \sigma_k \, 0 \, 0 \, 0 \end{bmatrix}$$

σ_i's are singular value

\# of σ's $\neq 0$ tell you the rank of A.

- eigenvalue/eigenvector sensitivity

$$E + A = \begin{bmatrix} 1 & 100 \\ -10^{-5} & 2 \end{bmatrix} \qquad \lambda_1 = 1 \quad \lambda_2 = 2$$

$\lambda_2' = 1.999 \quad \lambda_1' = 1.001$

$$E + A = \begin{bmatrix} 1 & 0 \\ 0 & .99999 \end{bmatrix} \qquad \lambda_1 = 1 \quad \lambda_2 = .99999 \qquad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

10^{-5}

$\lambda_1' = \lambda_2' \qquad e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \; e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (!)

- Residues & Projection Matrices

Residue Matrix: $R_i \doteq e_i f_i^T$ — distinct e-values

e_i's are right e-vectors

f_i^T's are left e-vectors

~~(normalized eigenvectors)~~

$$A = \sum_{i=1}^{n} \lambda_i R_i$$

$R_i x$

$$x = \sum_{j=1}^{n} \alpha_j e_j$$

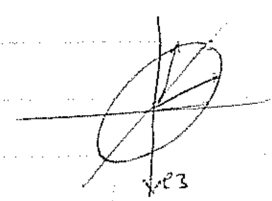
$$R_i x = \sum_{j=1}^{n} R_i \alpha_j e_j$$

$$\sum \alpha_j e_i \underbrace{f_i^T e_j}_{\delta_{ij}}$$ (set up your system so that this holds)

$\boxed{R_i x = \alpha_i e_i}$ component of x in the i^{th} e-vector dir

$$Ax = \sum \lambda_i R_i x$$
$$= \sum \lambda_i \underbrace{\alpha_i e_i}$$ component in i^{th} dim.
$$=$$

$$f_i^T e_j = \delta_{ij} \alpha_i$$



?

$$f_1^T e_1 = \alpha_1$$

$$f_1^T e_2 = 0$$

$$f_1^T e_3 = 0$$

- proof of $N(A-\lambda_k I)^{m_k} = N(A-\lambda_k I)^{d_k}$ by $\supseteq$ $\subseteq$

where m_1 is minimal poly. coeff $\qquad$ d_k charac. poly coeff

$m_k \le d_k$

$\subseteq$ $\qquad (A-\lambda_k I)^{m_k} x = \theta$

$\Rightarrow \quad (A-\lambda_k I)^{m_k+n} x = \theta \quad \forall \varepsilon)$

(⊇): suppose $\exists v$ s.t. $(A-\lambda_k I)^{m_k} v \neq \theta$; $(A-\lambda_k I)^{m_k+1} v = \theta$

Since $(s-\lambda_k)$ and $p_k(s) = (s-\lambda_1)^{m_1} \cdots \cancel{(s-\lambda_k)^{m_k}} \cdots (s-\lambda_\sigma)^{m_\sigma}$
have <u>no</u> common terms
⇒ can do partial fraction expansion

$$1/p_k(s)(s-\lambda_k) = h_1(s)/(s-\lambda_k) + h_2(s)/p_k(s)$$

$$1 = h_1(s)\,p_k(s) + h_2(s)(s-\lambda_k)$$

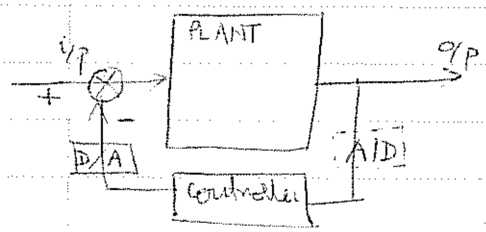
$$\Rightarrow \quad I = h_1(A)\,p_k(A) + h_2(A)(A-\lambda_k I) \quad \text{; an identity}$$

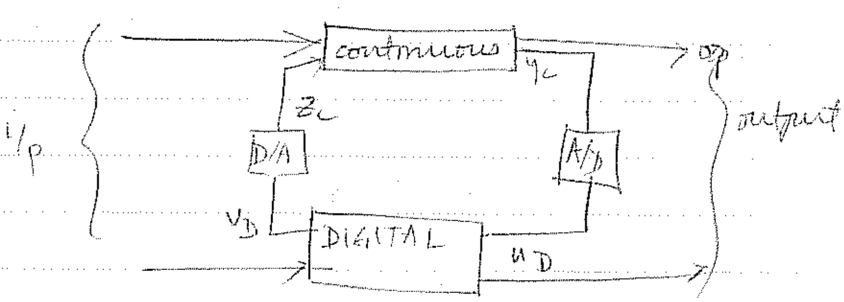
$$\underbrace{[(A-\lambda_k I)^{m_k} v]}_{\neq\ \theta} I = [h_1(A)\,p_k(A) + h_2(A)(A-\lambda_k I)](A-\lambda_k I)^{m_k} v$$

$$= h_1(A)\underbrace{p_k(A)(A-\lambda_k I)^{m_k}}_{\substack{\text{minimal poly}\\ p(A)=\theta}} v + h_2(A)\underbrace{(A-\lambda_k I)^{m_k+1} v}_{\theta \text{ (by hyp.)}} I$$

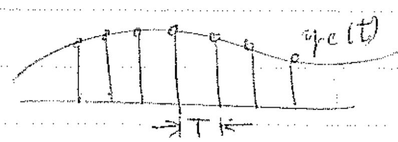
contradiction.

<u>- sampled data systems</u>





A/D converter



$u_D(nT) = y_c(nT)$

$$\tilde{u}_D(z) = \tilde{y}_c(z) = T[\hat{y}_c(s)]$$

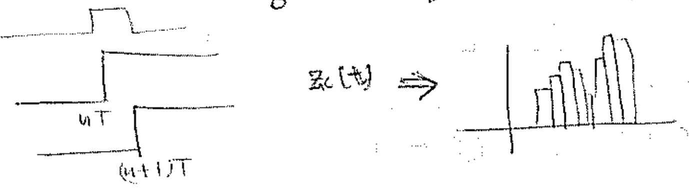
$\mathcal{L}[y_c(t)]$

$$\tilde{u}_D(e^{sT}) = \tfrac{1}{2}\, y_c(0^+) + \tfrac{1}{T} \sum_{k=-\infty}^{\infty} \hat{y}_c\left(s + j\frac{2\pi k}{T}\right)$$

D/A converter:

$$v_D = \{ v_D(nT) \}_{n=0}^{\infty}$$

$$z_c(t) = \sum_{0}^{\infty} v_D(nT)\left[1(t-nT) - 1(t-\overline{n+1}\,T)\right]$$



$$\hat{Z}_c(s) = \underbrace{\sum v_D(nT)\, e^{-nsT}}\left(\frac{1-e^{-sT}}{s}\right)$$

$$= \tilde{V}_D(e^{sT})\,(1-e^{-sT})/s \qquad = \hat{g}(s)\left(\frac{1-e^{-sT}}{s}\right)\hat{z}_c(s)$$

Q: what is the total transfer function

discrete $\longrightarrow \boxed{D/A} \longrightarrow \boxed{\hat{g}(s)} \longrightarrow \boxed{A/D} \longrightarrow$ discrete

$v_D \qquad z_c \qquad y_c \qquad u_D$

$\tilde{v}_D \qquad \hat{z}_c \qquad \hat{y}_c \qquad \tilde{u}_D$

Assuming ① A/D & D/A are synch $\bar{c}$ time T
② $\hat{g}(s)$ is strictly proper / rational ; $\lim_{|s|\to\infty} \hat{g}(s) = 0$
③ $\hat{g}(s)$ zero state at $t=0$

$$\hat{u}_D(z) = T[\hat{y}_c(s)](z) = T[\hat{g}_{eq}(s)\,\tilde{v}_D(e^{sT})](z)$$

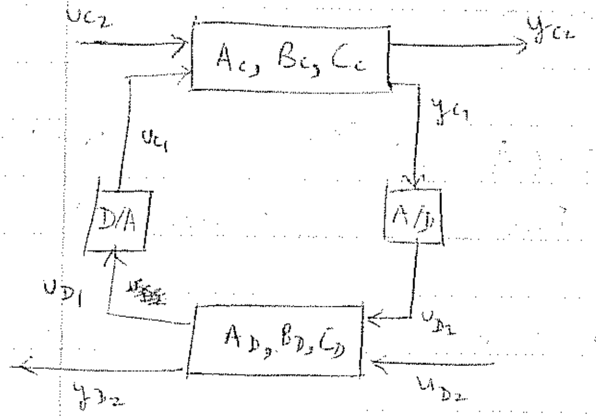
look in back of reader

$$T[\hat{g}_{eq}(s)\,\tilde{V}_D(e^{sT})](z) = \tilde{g}_{eq}(z)\,\tilde{V}_D(z)$$

$$\tilde{u}_D(z) = \tilde{g}_{eq}(z)\cdot\tilde{v}_D(z)$$

$\tilde{v}_D \longrightarrow \boxed{\tilde{g}_{eq}} \longrightarrow \tilde{u}_D$

$\longrightarrow \boxed{D/A} \longrightarrow \boxed{\hat{g}(s)} \longrightarrow \boxed{A/D} \longrightarrow$

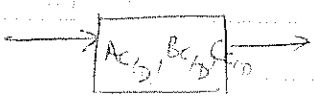


$$\dot{x}_c = A_c x_c + \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} \begin{bmatrix} u_{c1} \\ u_{c2} \end{bmatrix}$$

$$y_c = C_c x_c$$

$$x_D(k+1) = A_D x_D(k) + \begin{bmatrix} B_{D1} \\ B_{D2} \end{bmatrix} \begin{bmatrix} u_{D1} \\ u_{D2} \end{bmatrix}$$

$$y_D(k) = C x_D(k)$$



$$x_c((k+1)T) = \phi((k+1)T, kT)\, x_c(kT) + \int_{kT}^{(k+1)T} \phi((k+1)T, \tau)\, B_c u_c(\tau)\, d\tau$$

$$= e^{A_c((k+1)T - kT)} x_c(kT) + \int_{kT}^{(k+1)T} e^{A_c((k+1)T - \tau)} \left[B_{c1} u_1(\tau) + B_{c2} u_2(\tau) \right] d\tau$$

$$u'_{c2}(kT) \triangleq \int_{kT}^{(k+1)T} e^{A_c((k+1)T - \tau)} B_{c2} u_{c2}(\tau)\, d\tau$$

$$x_c[(k+1)T] = \underbrace{e^{A_c T}}_{A_{c/D}} x_c(kT) + \underbrace{\int_0^T e^{A_c \tau} d\tau}_{B_{c1/D}} \left\{ B_{c1} u_{c1}(kT) \right\} + u'_{c2}(kT)$$

$$x_d[(k+1)T] = A_{c/D}\, x_c(kT) + B_{c1/D}\, u_{c1}(kT) + u'_{c2}(kT)$$

Tuesday
10 Oct 1989

EECS 221a - Lecture

3) Matrix Representation

$$\mathbb{C}^n = \mathcal{N}(A-\lambda_1 I)^{m_1} \oplus \mathcal{N}(A-\lambda_2 I)^{m_2} \cdots \oplus \mathcal{N}(A-\lambda_\sigma I)^{m_\sigma}$$

$\mathcal{N}(A-\lambda_k I)^{m_k}$ is invariant under A.

$\exists$ basis (union of lin ind. vectors in $\mathcal{N}(A-\lambda_k I)^{m_k}$)

A has the matrix representation

$$\begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_\sigma \end{bmatrix}$$

$$\dim \mathcal{N}(A-\lambda_k I)^{m_k} = p_k$$

<u>Fact</u>: $\dim \mathcal{N}(A-\lambda_k I)^{m_k} = d_k$ (from Ch. p...)
where $\det(A-sI) = (-1)^n (s-\lambda_1)^{d_1} \cdots (s-\lambda_\sigma)^{d_\sigma}$

<u>Proof</u>: $\det(A-sI) = \det \begin{bmatrix} A_1 - sI_1 & & 0 \\ & A_2 - sI_2 & \\ 0 & & A_\sigma - sI_\sigma \end{bmatrix}$

$$= \prod_{k=1}^{\sigma} \det(A_k - sI_k)$$

consider $\det(A_k - sI_k)$

<u>claim</u>: Poly in s, $\det(A_k - sI_k)$ has only λ_k as zero
i.e. $\det(A_k - sI_k) = (s-\lambda_k)^{? \to p_k} (-1)^{p_k}$

$$\Rightarrow \det(A-sI) = (-1)^n \prod_{k=1}^{\sigma} (s-\lambda_k)^{p_k}$$

Let μ be a zero of $\det(A_k - sI_k)$
i.e., $\det(A_k - \mu I_k) = 0$; singular matrix
$\exists\, z_k \neq \theta$ s.t. $(A_k - \mu I_k) z_k = \theta$ (1)

but $z = \begin{bmatrix} 0 \\ \vdots \\ z_k \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \updownarrow p_1 \\ \\ \updownarrow p_k \\ \\ \updownarrow p_\sigma \end{matrix}$ $\updownarrow n$ $\in \mathcal{N}(A-\lambda_k I)^{m_k}$

$\Leftrightarrow \mathcal{N}\left(\begin{bmatrix} A_1 - \lambda_1 I & & 0 \\ & \ddots & \\ 0 & & A_\sigma - \lambda_\sigma I \end{bmatrix}^{m_k}\right)$

$z_K \in \mathcal{N}(A_K - \lambda_K I)^{m_K}$

i.e., $(A_K - \lambda_K I)^{m_K} z_K = \theta \qquad (2)$

$\Rightarrow A_k z_k = \mu I_k z_k \Rightarrow \left[A_k^{m_k} - m_k A_k^{m_k-1}(\lambda_k I) \cdots (-1)^{m_k}(\lambda_k I)^{m_k}\right] z_k$

(1) subs. into (2) $\quad (\underbrace{\mu - \lambda_K}_{\text{scalar}})^{m_K} \underbrace{z_K}_{\text{nonzero}} = \theta$

$\Rightarrow (\mu - \lambda_K) = 0$

$\Rightarrow \boxed{\mu = \lambda_K}$

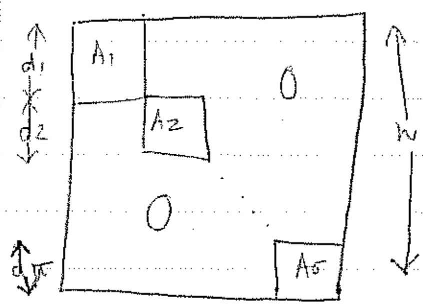
$\Rightarrow \det(A_K - sI_K) = (-1)^{p_K}(s - \lambda_K)^{p_K}$

$\Rightarrow \det(A - sI) = (-1)^n \prod_{K=1}^{\sigma}(s - \lambda_K)^{p_K}$

But $= (-1)^n \prod_{K=1}^{\sigma}(s - \lambda_K)^{d_K}$

$\Rightarrow p_K = d_K$

A has matrix representation:



In order to find out the form of A_K we need to specify a set of lin ind vectors in ~~....~~ $\mathcal{N}(A - \lambda_K I)^{m_K}$

(3) Generalized Eigenvectors

Ex. $A = \begin{bmatrix} 2 & -1 & & & \\ 1 & 0 & -1 & & \\ & & 1 & 0 & \\ & & & 1 & -1 \\ & & & & 1 \end{bmatrix}$ $\quad$ 5×5 matrix

$-\Delta(s) = \det(A - sI) = (s - \lambda)^5 \quad ; \lambda = 1$

$\mathbb{C}^5 = \mathcal{N}(A-\lambda I)^5 \qquad (= \mathcal{N}(A-\lambda I)^3)$

$$(A-\lambda I) = \begin{bmatrix} 1 & -1 & & & \\ 1 & -1 & -1 & & \\ & & 0 & 0 & \\ & & & 0 & -1 \\ & & & & 0 \end{bmatrix} \qquad \text{Rank} = 3$$

$\dim(\mathcal{N}(A-\lambda I)) = 2$

Similarly $(A-\lambda I)^2$ has rank $= 1$

$(A-\lambda I)^3$ has rank 0.

$\psi(s) = (s-\lambda)^3$

$\dim \mathcal{N}(A-\lambda I)^2 = 4$

$\dim \mathcal{N}(A-\lambda I)^3 = 5$

	dim		
$\mathcal{N}(A-\lambda I)^3$	5	u^3	
$\mathcal{N}(A-\lambda I)^2$	4	u^2	v^2
$\mathcal{N}(A-\lambda I)$	2	u^1	v^1

One way of selecting 5 lin ind vectors in $\mathcal{N}(A-\lambda I)$

(i) Select one vector u^3 in $\mathcal{N}(A-\lambda I)^3$ but not in $\mathcal{N}(A-\lambda I)^2$

(ii) select two lin ind vectors u^2, v^2 in $\mathcal{N}(A-\lambda I)^2$ but not $\mathcal{N}(A-\lambda I)$

(iii) select two lin ind vectors u^1, v^1 in $\mathcal{N}(A-\lambda I)$ but not

(i) $(A-\lambda I)^3 u^3 = \theta$ but $(A-\lambda I)^2 u^3 \neq \theta$ (1)

(ii) $(A-\lambda I)^2 u^2 = \theta$ but $(A-\lambda I) u^2 \neq \theta$ (2)

(1) & (2) can be combined if we pick $u^2 = (A-\lambda I) u^3$

Similarly $u^1 = (A-\lambda I)^2 u^3$

In general a vector v is called a <u>generalized</u> eigenvector (of rank K) of A assoc. $\bar{c}$ e-value λ

$\Leftrightarrow \quad (A-\lambda I)^K v = \theta \quad$ but $(A-\lambda I)^{K-1} v \neq \theta$

Let $\quad v_K^K = v_K$
$v_{K-1}^{K-1} = (A-\lambda I) v_K^K$
$\vdots$
$v_1^1 = (A-\lambda I) v_2^2$

The set of vectors $\{v_1^1, v_2^2, \ldots v_K^K\}$ is called a chain of g.e. vectors of length K.
(read two chains in example - u^3, u^2, u^1 ; v^2, v^1)

<u>Properties of generalized eigenvectors</u>:

1) $v^j \in \mathcal{N}(A-\lambda I)^j$ but $v^j \notin \mathcal{N}(A-\lambda I)^{j-1}$
   j is the rank

good

2) $v^j \in \mathcal{N}(A-\lambda I)^K$

3) $v^1 \in \mathcal{N}(A-\lambda I)$
   v^1 is an eigenvector of A.

$v^1 = (A-\lambda I) v^2$
$\quad = (A-\lambda I)^{K-1} v$
$\Rightarrow (A-\lambda I) v^1 = (A-\lambda I)^K v = 0$

4) $\{v^1, v^2, \ldots v^K\}$: these K vectors are lin. ind.

<u>Proof</u> $\alpha_1 v^1 + \alpha_2 v^2 + \ldots \alpha_K v^K = \theta$ Given.
   show $\Rightarrow \alpha_1, \ldots \alpha_K = 0$

i.e. $\alpha_1 (A-\lambda I)^{K-1} v + \alpha_2 (A-\lambda I)^{K-2} v + \ldots + \alpha_{K-1}(A-\lambda I) v + \alpha_K v = \theta$

mult by $(A-\lambda I)^{K-1}$

$$\Rightarrow \alpha_K \underbrace{(A-\lambda I)^{K-1} v}_{\neq \theta} = 0$$

$\Rightarrow \alpha_K = \theta$

Similarly can prove all the coeff must be zero.

(∵ initial pts chosen to be λ ...)

(5) $\{u^1, \dots u^k\}$ and $\{v^1, \dots v^k\}$ are two chains of generalized evectors, then they are linearly independent.
(Pf similar to (4))

(6) The generalized evectors of A assoc. c̄ diff evalue are lin independent.

4. JORDAN BLOCK

$\{u^1, u^2, u^3, v^1, v^2\}$ lin ind
form a basis for $\mathbb{C}^5$.
What is the matrix representation of A wrt this b

$$\begin{array}{ll} Au^1 & \\ Au^2 & m_k \\ Au^3 & \\ Av^1 & \\ Av^2 & \end{array} \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

$Au^1 = \lambda u^1$ (u^1 eigenvector)

$u^1 = (A - \lambda I) u^2$

$Au^2 = u^1 + \lambda u^2$

$Au^3 = u^2 + \lambda u^3$

$Av^1 = \lambda v^1$

$Av^2 = v^1 + \lambda$

$u^1 = (A - \lambda I) u^2$

$u^2 = (A - \lambda I) u^3$

v^1 e. vector

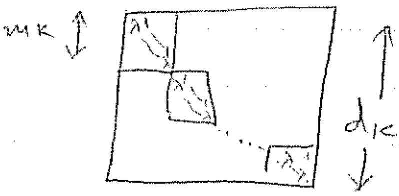
$v^1 = (A - \lambda I) v^2$

$$\begin{tikzcd} \text{Standard Basis} \arrow[r, "A"] \arrow[d, "P^{-1}"'] & \text{Standard basis} \\ \text{gen. evectors} \arrow[r, "J"'] & \text{Gen evectors} \arrow[u, "P = \begin{bmatrix} | & | & | & | & | \\ u_1 & u_2 & u_3 & v_1 & v_2 \\ | & | & | & | & | \end{bmatrix}"'] \end{tikzcd}$$

$A = P J P^{-1}$

$J = P^{-1} A P$

A Ina

In general $\boxed{A_k}$ (d_k) $\sim$ 

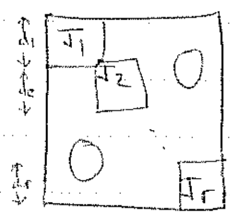
dim of the longest block $= m^k$

ok # of blocks = # of lin ind eigen vectors $= \dim \mathcal{N}(A - \lambda_k I)$

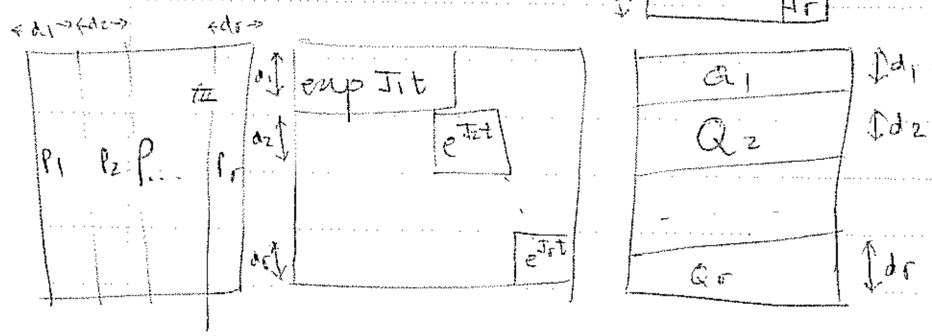
Why do we do all this?

$[\exp(At)]x_0$; A diagonal $\Rightarrow$ modal decomposition

5. STATE TRANSITION MATRIX

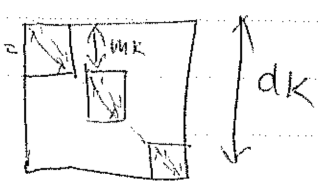
$P^{-1} A P = J =$ 

$\exp At = P(\exp Jt)P^{-1}$

$=$ 

$= P_1(\exp(J_1 t))Q_1 + P_2(\exp J_2 t)Q_2 + \cdots P_r(\exp J_r t)Q_r$

cols of $P \Rightarrow$ corresp. to generalized e-vectors assoc. w/ λ_1

$\exp(J_k t) = ?$ Given $J_k =$ 

Thursday
12 Oct 1989

NST: Disc

Field Axioms -

(A) $(\alpha+\beta) = (\beta+\alpha)$ $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$ $\exists\theta$ s.t. $\forall\alpha$ $\alpha+\theta =$

$\forall\alpha$; $\exists(-\alpha)$ s.t. $\alpha+(-\alpha)=\theta$

(M) $\alpha\beta = \beta\alpha$ $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ $\exists 1$ s.t. $\alpha\cdot 1 = \alpha$

(D) $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$ $\forall\alpha\neq\theta$, $\exists\alpha^{-1}$ s.t. $\alpha(\alpha^{-1}) =$

Vectors Space Axioms -

(A) basically the same

(SM) $\alpha,\beta\in\mathcal{F}$, $x,y\in V$

$\alpha\beta(x) = \alpha(\beta(x))$ $\alpha(x+y) = \alpha x+\alpha y$ $(\alpha+\beta)x = \alpha x+\beta$

$1_F\cdot x = x$

Linearity

$A(\alpha x+\beta y) = \alpha A(x)+\beta A(y)$

transformation

$\alpha,\beta\in\mathcal{F}$

$x,y\in V$

Change of Basis

$A \to \hat{A}$ ← basis $\tilde{u}$

basis u

$\tilde{A} \to Q\,A\,P$

changes from u to $\tilde{u}$

the i^{th} column of Q =

i^{th} basis vector in

$\tilde{u}$ coordinates

P: opposite dim.

Norm

1. $\|x+y\| \leq \|x\|+\|y\|$

2. $\|x\|\geq 0$; $\|x\|=0 \iff x=\theta$

3. $\|\alpha x\| = |\alpha|\,\|x\|$

ℓp norm? $\|X\|_p \triangleq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$; $x\in\mathbb{C}^n$

Induced norm: $\sup_{\|x\|\le 1} \|Ax\|$ $\|A\|_1$ = max of col. sums i.e. $\max_j \sum_{i=1}^{n} |a_{ij}|$

$\|A\|_\infty = \max_i \left[\sum_{j=1}^{n} |a_{ij}| \right]$

$\|\phi(\cdot)\|_\infty = \sup_{t_0 \le t \le t_1} \|\phi(t)\|$ $t \in [t_0, t_1]$

all norms in any finite dim space are equivalent.

Fundamental Thm

$\dot{x} = p(x,t)$

(i) $t \mapsto p(x,t)$ piecewise continuous ē a finite # of disc.

(ii) Lipschitz

Fund thm. given $p(x,t)$ which satisfies (i) & (ii)

(a) $\forall x_0, t_0 \ \exists \phi$ cont. s.t. $\phi(t_0) = x_0$ & $\dot{\phi}(t) = p(\phi(t), t)$

(b) ϕ is unique

Picard Iteration

$x_{m+1}(t) = x(t_0) + \int_{t_0}^{t} p(x_m(t), t)\,dt$

Bellmann Gronwall

Prove something equal to zero. p.39 & 40

Dynamical System

Definition

Need

(i) $(\mathcal{U}, \mathcal{Y}, \Sigma, r, s)$

(ii) s satisfies state transition axiom

(iii) s satisfies Semigroup axiom

(ii) $x(t_1) = s(t_1, t_0, x_0, u_{[t_0, t_1]})$

given x_0, $x(t_1)$ depends only on $u_{[t_0, t_1]}$

not on u outside interval

$s(t_1, t_0, x_0, u_{[t_0,t_1]}) = s(t_1, t_2, \underbrace{s(t_2, t_0, x_0, u_{[t_0,t_2]})}_{s(t_2, \ldots}, u_{[t_2, t_1]})$

$t_0 \le t_2 \le t_1$

Linear Dynamical System

(i) $\rho(t, t_0, x_0, u) = \rho(t, t_0, x_0, \theta_u) + \rho(t, t_0, \theta_\Sigma, u)$

$\rho(t, t_0, x_0, u) = g(t, x(t), u(t))$

$x(t) = s(t, t_0, x_0, u)$

(ii) Superposition property

$\rho(t, t_0, \theta_\Sigma, \alpha u + \beta v) = \alpha \rho(t, t_0, \theta_\Sigma, u) + \beta \rho(\cdot, \cdot, \cdot, v)$

(iii) $\rho(t, t_0, \alpha x_0 + \beta x_1, \theta_u) =$ same idea

$$y(t) = C(t)\phi(t, t_0)x_0 + \int_{t_0}^{t} C(t)\phi(t, \tau)B(\tau)u(\tau)\,d\tau$$

$\underbrace{\rho(t, t_0, x_0, \theta_u)}_{ZIR}$ $\underbrace{\rho(t, t_0, \theta_\Sigma, u)}_{ZSR}$

$$\theta_u \triangleq u(t) \equiv 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \theta_\Sigma = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$ZSR = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

$ZIR = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$ starting from x_0

What is the output starting @ $-x_0$ to i/p $2u(t)$

o/p = 2(ZSR) − (ZIR)

State Transition Matrix

(a) Unique — can be of any time

(b) If $\dot{x} = Ax$, $x(t_0) = x_0$ then trajectory $\phi(t, t_0, x_0) = \Phi(t, t_0)x_0$

(CT) (c) $\Phi(t_1, t_0) = \Phi(t_1, t_2)\Phi(t_2, t_0)$ ∀ all t_0, t_1, t_2 (in DT $\Phi(k_2, k_0) = \Phi(k_2, k_1)\Phi(k_1, k_0)$ when $k_0 \le k_1 \le k_2$; Φ inv for all k_0, k_1, k_2)

(d) $\Phi(t_0, t_0) = I$

(CT) (e) $\frac{\partial}{\partial t}\Phi(t, t_0) = A(t)\Phi(t, t_0)$

(f) For cont time systems $\Phi(t, t_0)$ is even when A is singular **always** non sin

(9) (g) for *any* fundamental matrix X

$\Phi(t,t_0) = X(t)\,X^{-1}(t_0)$

when Φ not singular, true $\Phi(t_1,t_0)=\Phi^{-1}(t_0,t_1)$

Is $\Phi(t,0)=\begin{bmatrix} \frac{1}{4}(3e^{3t}+4) & 2-2e^{-t} \\ 0 & \frac{1}{5}(3e^{t}+2e^{2t}) \end{bmatrix}$ an STM?

(i) non singular for t
(ii) $\Phi(0,0)=I$

Yes it is an STM

What system is it an STM of?

$\partial\Phi/\partial t = \begin{bmatrix} \frac{1}{4}\times 9e^{3t} & 3e^{-t} \\ 0 & \frac{1}{5}(3e^{t}+4e^{2t}) \end{bmatrix}$

must il $\partial\Phi/\partial t\;\Phi^{-1} = A\Phi\,\Phi^{-1}$; solves $\dot{X}=A(t)X$

State Transition Function $\in(\mathbb{Z}/\mathbb{R})+(\mathbb{R}\,S\,\mathbb{R})$ for linear sys.

$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t}\Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau$

response fn $= C(t)\,x(t) + D(t)\,u(t)$

Scalar Product (inner product)

(i) $\langle x|y+z\rangle = \langle x|y\rangle + \langle x|z\rangle$
(ii) $\langle x|\alpha z\rangle = \alpha\langle x|z\rangle$
(iii) $\langle x|x\rangle = 0 \Longleftrightarrow x=\theta$
(iv) $\langle X|Y\rangle = \overline{\langle Y|X\rangle}$

Schwartz Inequality $|\langle X|Y\rangle| \le \|X\|\,\|Y\|$

($\|\,\|$ is norm induced by inner product)

Adjoint

$\langle y|Ax\rangle = \langle A^{*}y|X\rangle$

where A^{*} = adjoint of A

Look over residue matrices (R_k's) (Gives you info on the full)

Cayley Hamilton

conclusion $A^{n} = $ l.c. $\sum_{i=0}^{n-1}\alpha_i A^{i}$

$R_i = e_i f_i^{T}$ (right e.v., left e.v.)

$\sum R_i = I$

$\sum \lambda_i R_i = A$

$\boxed{R_i R_j = \delta_{ij} R_j}$

Zeroes of f. fn. (discussion)

13 Friday Oct '89

LST - Lecture

5. State Transition Matrix

A (may have multiple eigenvalues)

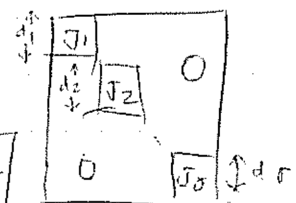
can always get $P^{-1}AP$ ← generalized eigen vectors.

wrt gen. e-vectors A has representation in form of Jordan blocks.

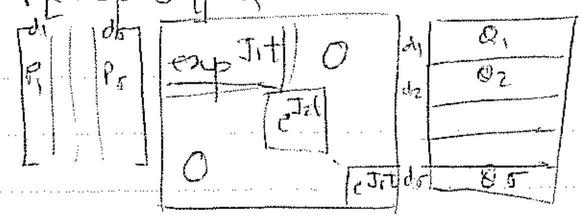
J_k = Jordan bloc assoc. c̄ λ_k

Want e^{At} (STM)

$A = PJP^{-1} = PJQ \qquad J = P^{-1}AP =$

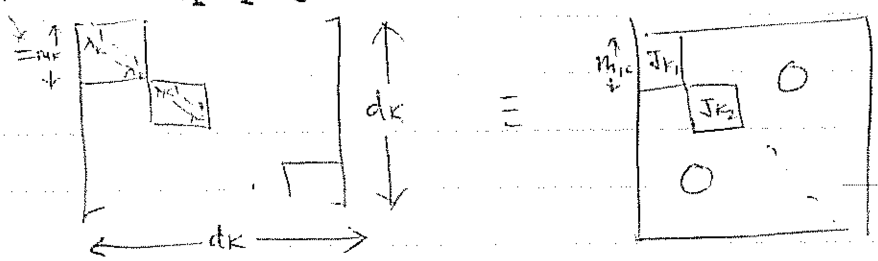


$\exp(At) = P[\exp Jt]\,Q$

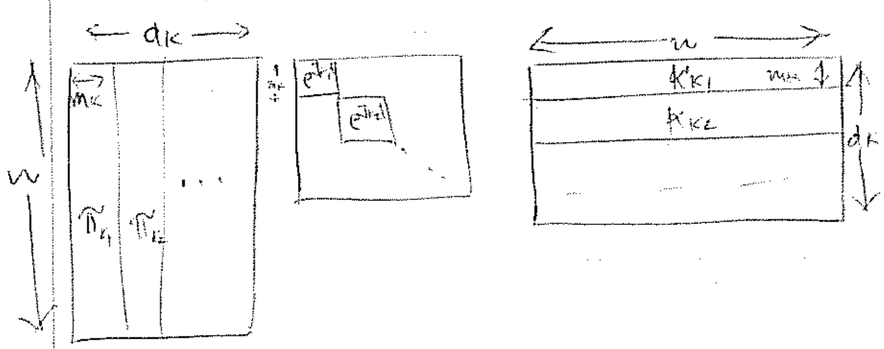
$=$ 

$= P_1(\exp J_1 t)Q_1 + P_2 \exp J_2 t\, Q_2 + \cdots P_\sigma \exp(J_\sigma t)Q_\sigma$

compatible dimension $n \times d_1 \times d_1 \times d_1 \times d_1 \times n$

consider J_K 

$P_K \exp J_K Q_K$; can be further decomposed



$= \Pi_{K_1} \exp J_{K_1}t\, K_{K_1} + \Pi_{K_2}(\exp J_{K_2}t) K_2 + \cdots$

Consider the 1st term $\overset{\leftarrow m_K \rightarrow}{}$ $J_{K_1} = \begin{bmatrix} \lambda_K & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_K \end{bmatrix} m_K$; $\exp(J_{K_1}t) = \begin{bmatrix} e^{\lambda_K t} & te^{\lambda_K t} & \frac{t^2}{2!}e^{\lambda_K t} & \frac{t^{m_K-1}}{(m_K-1)!}e^{\lambda_K t} \\ & \ddots & \ddots & \\ & & & te^{\lambda_K t} \\ & & & e^{\lambda_K t} \end{bmatrix}$

$$= e^{\lambda_k t} \left\{ \underbrace{\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}}_{I} \right\} + t e^{\lambda_k t} \left\{ \underbrace{\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}}_{I_{-1}} \right\}$$

$$+ \cdots t^m e^{\lambda_k t} I_{-m} \quad \cdots \quad t^{m_k - 1} e^{\lambda_k t} I_{-(m_k-1)}$$

$$I_{-k} = 1/k! \begin{bmatrix} & \overbrace{\quad}^{k} 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ & 0 & & \end{bmatrix}$$

matrices could be zero ; rank deficient / n×n matrix

$$\Pi_{k_1} (\exp J_{k_1} t) \alpha_{k_1} = e^{\lambda_k t} (\Pi_{k_1} \alpha_{k_1}) + t e^{\lambda_k t} (\Pi_{k_1} I_{-1} \alpha_{k_1})$$

$$+ \cdots t^{m_k - 1} e^{\lambda_k t} (\Pi_{k_1} I_{(m_k-1)} \alpha_{k_1})$$

$$\Pi_{k_2} (\exp J_{k_2} t) \alpha_{k_2} = e^{\lambda_k t} (\Pi_{k_2} \alpha_{k_2}) + t (e^{\lambda_k t}) (\Pi_{k_2} I_{-1} \alpha_{k_1}) \cdots$$

but no higher than

$$P_k (\exp J_k t) Q_k = e^{\lambda_k t} \overbrace{(\Pi_{k_1} \alpha_{k_1} + \Pi_{k_2} \alpha_{k_2} + \cdots)}^{P_{k0}}$$

$$+ t e^{\lambda_k t} \underbrace{(\Pi_{k_1} I_{-1} \alpha_{k_1} + \cdots)}_{P_{k1}}$$

$$+ \cdots$$

$$= \sum_{l=0}^{m_k - 1} t^l e^{\lambda_k t} P_{kl}$$

$$\therefore \exp(At) = \sum_{k=1}^{\sigma} \left[\sum_{l=0}^{m_k - 1} t^l \exp(\lambda_k t) P_{kl} \right.$$

matrices determined by generalized eigenvectors.

∴ Equivalent Modal Decomposition can be examined

6. Jordan form decomposition

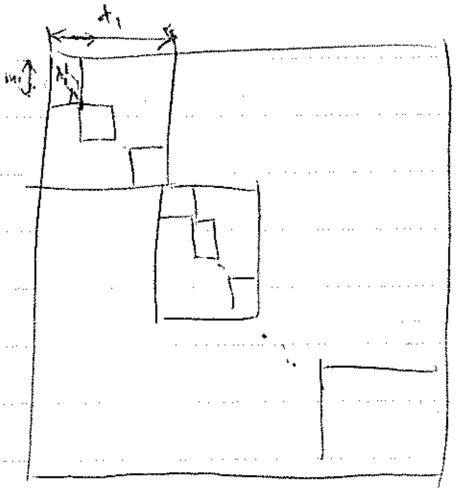
SISO for ease of presentation

$$\begin{cases} \dot{x} = Ax + bu \\ y = Cx \end{cases}$$

A has multiple eigenvalues

change of coordinates $x = P\xi$ columns are gen.

$PAP^{-1} = J =$



$$\dot{\xi} = J\xi + \beta u$$
$$y = \gamma \xi$$

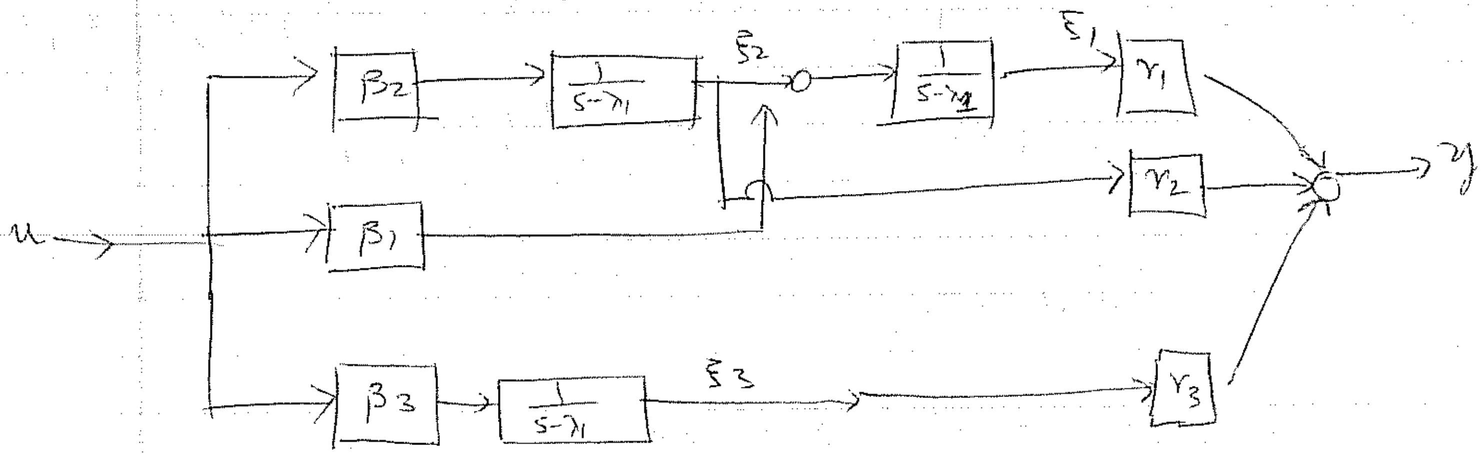
$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u$$

Ex. $J = \begin{bmatrix} \lambda_1 & 1 & \\ & \lambda_1 & \\ & & \lambda_1 \end{bmatrix}$

$\dot{\xi}_1 = \lambda_1 \xi_1 + \xi_2 + \beta_1 u \qquad \xi_1 = \frac{1}{s-\lambda_1}(\xi_2 + \beta_1 u)$

$\dot{\xi}_2 = \lambda_1 \xi_2 + \beta_2 u \qquad \xi_2 = \beta_2 u / (s-\lambda_1)$

$\dot{\xi}_3 = \lambda_1 \xi_3 + \beta_3 u \qquad \xi_3 = \beta_3 u / (s-\lambda_1)$



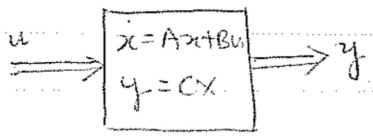
Note which are coupled, which aren't.

day ← probably 17 Oct
Oct 1989

LST-lecture

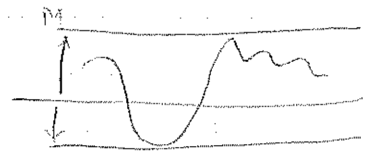
Ch VI Stability

1. Motivation



u, x, y are vectors in some normed vector space

system response to be bdd.



Defn: $f : \mathbb{F} \to V$; $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

f is bdd iff $\exists M$ s.t. $\forall t \in \mathbb{F}$ $\|f(t)\| \le M$

u is bdd; will y be bdd?

will x be bdd?

BIBO stable

$\Rightarrow \forall$ bdd u, y is bdd

LTI system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ $\quad x(0) = x_0$

$$y(t) = C\exp^{At} x_0 + \int_0^t C\exp(A(t-\tau))Bu\,d\tau$$

$$\exp(At) = \sum_{k=1}^{\sigma} \sum_{l=0}^{m_k-1} t^l e^{\lambda_k t} P_{kl}(A)$$

$P_{kl}(A)$ is independent of t.

time behaviour of the resp. y(t) completely charac. by λ_k (location) and m_k

* 1) Conditions under which system is BIBO stable (This Chap)
exp. in terms of location of λ_k.

2) how to control location of eigen values? (Sec XII)

3) when is it possible to do that? (controllability / Obs. a VII Sup 1-5 ...)
condition expressed in terms of rank of a matrix.

4) Numerical Aspects of rank of matrix
other geometric concepts (useful math backgrnd)
(Singular Value Decomposition, Sec VIII)

* Generalize to time varying system

$$\dot{X} = A(t)X + B(t)u$$
$$Y = C(t)X$$

Review

(i) $\|u(t)\|$ for fixed t. $u(t) \in \mathbb{R}$

$$\|u(t)\| := \max_i |u_i(t)|$$

for $u : [0,T] \to \mathbb{R}^r$

$$\|u(\cdot)\| = \max_{t \in [0,T]} \|u(t)\| = \max_{t \in [0,T)} \max_i |u_i(t)|$$

(ii) $H(t, \tau)$

for fixed t, τ $H(t,\tau)$ matrix

induced norm

$$\|H(t,\tau)\| := \max_i \sum_{j=1}^{m} |h_{ij}(t,\tau)| \text{ row sum } (l_\infty \text{ norm})$$

(iii) $\|H(t,\tau)u(\tau)\| \le \|H(t,\tau)\| \cdot \|u(\tau)\|$

2. Zero Input Response

$$\dot{x}(t) = A(t)x(t) \qquad x(t_0) = x_0$$
$$x(t) = \Phi(t, t_0)x_0$$

Fact $\Phi(t,0)$ bdd. $\Leftrightarrow x(t)$ bdd.

Proof: $x(t) = \cancel{\Phi(t,t_0)} \; \Phi(t,0)\Phi(0,t_0)x_0$

$\Phi(t,0)$ bdd $\Rightarrow x(t)$ bdd $\quad \|x(t)\| \le \|\Phi(t,0)\| \cdot \|\Phi(0,t_0)\| \cdot \|x_0\|$

$\Phi(t,0)$ is bdd $\Rightarrow \exists N$ s.t. $\|\Phi(t,0)\| < N \;\; \forall t \Rightarrow x(t)$ is bdd

converse: $\Phi(t,0)$ not bdd $\Rightarrow x(t)$ not bdd [equivalent]

$$\exists M \text{ s.t. } \forall t \;\; \max_i \sum_{j=1}^{m} |\phi_{ij}(t,0)| \le M$$

negate: $\forall M \; \exists t_1$ s.t. " $> M$

$\forall M \exists t_1 \;\; \max_i \sum_{j=1}^{m} |\phi_{ij}(t_1,0)| > M$ say max occurs at k $\;\; \sum_{j=1}^{m} |\phi_{kj}(t_1,0)| > M$

want to show

$\forall K_1 \; \exists t_2 \; \exists \bar{x}_0$ s.t., $\forall K \quad \|x(t_2)\| > K$

i.e. show $\max_i |x_i(t)| > K$

~~$x_i(t) = \Phi\; \Phi_{ij}(t,0)$~~

$$x(t) = \Phi(t,0)\,\underbrace{\Phi(0,t_0)\,x_0}_{x_0'}$$

$$x(t) = \Phi(t,0)\,x_0'$$

i.e. $\max_i \left| \sum \phi_{ij}(t_2,0)\,(\bar{x}_0')_j \right| > K$

~~$\max_i$~~

just change in notation, basically it is just putting $i=k$ → pick $t_2 = t_1$ $\qquad \geq \left| \sum_{k=1}^{n} \phi_{kj}(t_1,0)(\bar{x}_0')_j \right|$

pick $(\bar{x}_0')_j$ s.t. it is $[\operatorname{sgn} \phi_{kj}(t_1,0)]$ ← can do it i.e. $x_0' = \Phi(0,t_0)\,x_0$, Φ non singular, $x_0 = \Phi^{-1}(0,t_0)\,x_0'$

then $\sum |\phi_{kj}(t_1,0)(\bar{x}_0')_j| = \sum |\phi_{kj}(t_1,0)| > K$

But by conjecture can find such a K ($\because \forall K \; \exists t_1$ s.t. ...

LTI

$$\Phi(t,0) = \exp(At) = \sum_{k=1}^{\sigma} \sum_{l=0}^{m_k-1} t^l e^{\lambda_k t} P_{kl}$$

Corollary $x(t)$ bdd $\iff$ (i) $\mathrm{Re}(\lambda_k) \leq 0$ closed LHP.

(ii) $\mathrm{Re}(\lambda_k) = 0$ then $m_k = 1$; corresp. order in min^m poly.

↳ e. value on $j\omega$ axis

3. Zero State Response

$$(*) \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) & x(t_0) = \theta \\ y(t) = C(t)x(t) \end{cases}$$

$$y(t) = \int_{t_0}^{t} \underbrace{C(t)\Phi(t,\tau)B(\tau)}_{H(t,\tau)} u(\tau)\,d\tau$$

$$y(t) = \int_{t_0}^{t} H(t,\tau)\,u(\tau)\,d\tau$$

$*$ is BIBO stable $\iff \left[\forall u \text{ s.t. } \|u(\cdot)\| \leq u_M \Rightarrow \|y(\cdot)\| \leq K u_M \right.$

Claim $*$ is BIBO stable

$$\|y(t)\| \le \int_{t_0}^{t} \|H(t,\tau)\| \, \|u(\tau)\| \, d\tau$$

$$\le \underbrace{\left[\int_{t_0}^{t} \|H(t,\tau)\| d\tau\right]}_{\text{must be bdd}} \|u(\cdot)\|$$

u_M

$\Leftarrow \int_{t_0}^{t} \|H(t,\tau)\| d\tau$ bdd

Proof ($\Leftarrow$) : $\|y(t)\| = \left\| \int_{t_0}^{t} H(t,\tau) \, u(\tau) d\tau \right\|$

$$\le \int_{t_0}^{t} \|H(t,\tau)\| \, \|u(\tau)\| d\tau$$

$$\le \left(\int_{t_0}^{t} \|H(t,\tau)\| d\tau \right) u_M$$

Turns out works in both directions.

($\Rightarrow$) Difficult

So analysis

$(P \Rightarrow Q) \Leftrightarrow (Q' \Rightarrow P')$ $\rightleftarrows$ $\int_{t_0}^{t_L} \max_i \sum_{j=1}^{r} |h_{ij}(t_L,\tau)| d\tau > M$

$\forall M \; \exists \, t_L$ s.t. $\int_{t_0}^{t_L} \|H(t_L,\tau)\| d\tau > M$

$\Rightarrow \forall K \; \exists \, u(\cdot)$ $\|u(\cdot)\| \le u_M$

$\exists \, t_1$ $\|y(t_1)\| > K u_M$

$\{ \exists \, k \; |y_k(t_1)| > K u_M \}$

Set $t_1 = t_L$

consider $y_k(t_L) = \int_{t_0}^{t_L} \sum_{j=1} h_{kj}(t_L,\tau) \, u_j(\tau) d\tau$

$= \int_{t_0}^{t_L} \sum_{j=1}^{r} |h_{kj}(t_L,\tau)| d\tau$ $\quad\quad \leftarrow u_j(\tau) = \operatorname{sgn} h_{kj}(t_L,\tau)$

well defined

pick $k =$ the one which max $\sum_{j=1}^{r} |h_{kj}(t_L,\tau)|$

but the max is a fn of τ.
$\therefore$ Can't do this!

Thursday
19 Oct 1989

LST-Lecture

3. Zero State Response

$$y(t) = \int_{t_0}^{t} H(t,\tau)\, u(\tau)\, d\tau$$

Thm (1) is BIBO stable

$\Leftrightarrow \int_{t_0}^{t} \|H(t,\tau)\| d\tau$ $\forall t_0$

($\Rightarrow$) Analysis

(1) If $\forall M \ \exists t_L$ $\int_{t_0}^{t_L} \max_i \sum_{j=1}^{m} |h_{ij}(t_L,\tau)| d\tau > M$

want $\forall K \ \exists u(\cdot)$ $\|u(\cdot)\| \le u_m$ $\exists t_1, \exists k$ $|y_k(t_1)| > K u_M$

Consider $y_k(t_1) = \int_{t_0}^{t_1} \sum_{j=1}^{m} h_{kj}(t_1,\tau) u_j(\tau) d\tau$

set $t_1 = t_L$, $u_j(\tau) = \mathrm{sgn}\, h_{kj}(t_L,\tau)$

$y_K(t_L) = \int_{t_0}^{t_L} \sum_{j=1}^{m} |h_{kj}(t_L,\tau)| d\tau$, won't work directly

? $> \boxed{?} \alpha$ we know $\int_{t_0}^{t_L} \max_i \sum_j |h_{ij}(t_L,\tau)| d\tau > M$

may be really small

$y_K(t_L) > \alpha M > K$

pick $M > K/\alpha$

What do we know?

$> \int_{t_0}^{t_L} \max_i \sum_j |h_{ij}(t_L,\tau)| d\tau$

① $\int_{t_0}^{t_L} \sum_i \sum_j |h_{ij}(t_L,\tau)| d\tau \geq \int_{t_0}^{t_L} \max_i \sum_j |h_{ij}(t_L,\tau)| d\tau$

② $\exists K$ s.t. $\int_{t_0}^{t_L} \sum_j |h_{Kj}(t_L,\tau)| d\tau \geq \frac{1}{m} \int_{t_0}^{t_L} \sum_{i=1}^{m} \sum_{j=1}^{m} |h_{ij}(t_L,\tau)| d\tau$

pick K s.t. ② want $M/m > K$

$y_K(t_L) \geq \frac{1}{m} M$ pick $M > mK$

(⇒) Formal Proof –

Let $M > mK$

then $\exists t_L$ s.t. $\int_{t_0}^{t_L} \max_i \sum_{j=1}^{m} |h_{ij}(t_L,\tau)| d\tau > mK$

pick K s.t. $\int_{t_0}^{t_L} \sum_j |h_{Kj}(t_L,\tau)| d\tau \geq \frac{1}{m} \int_{t_0}^{t_L} \sum_i \sum_j |h_{ij}(t_L,\tau)| d\tau$

Let $u_j(\tau) = \operatorname{sgn} h_{Kj}(t_L,\tau)$ so $\|u(\cdot)\| \leq 1$

consider $|y_K(t_L)| = \int_{t_0}^{t_L} \sum_j |h_{Kj}(t_L,\tau)| d\tau$ ~~$\geq \frac{1}{m}(mK)$~~

$\geq \frac{1}{m} \int_{t_0}^{t_1} \sum_i \sum_j \cdots d\tau \geq \frac{1}{m} \int_{t_0}^{t_L} \max_i \cdots > \frac{1}{m} mK$

Q.E.D.

$y(t) = \int_{t_0}^{t} H(t,\tau) u(\tau) d\tau \quad ; y(t_0) = 0$

BIBO stable

$\Leftrightarrow \int_{t_0}^{t} \|H(t,\tau)\| d\tau$ bdd

<u>Note</u>: t_0 could be anything even $-\infty$ (this is Desoer's appro

Corollary: ∧TI

$y(t) = \int_{t_0}^{t} H(t-\tau) u(\tau) d\tau$ BIBO stable

$\Leftrightarrow \int_{t_0}^{t} \|H(\tau)\| d\tau$ bdd.

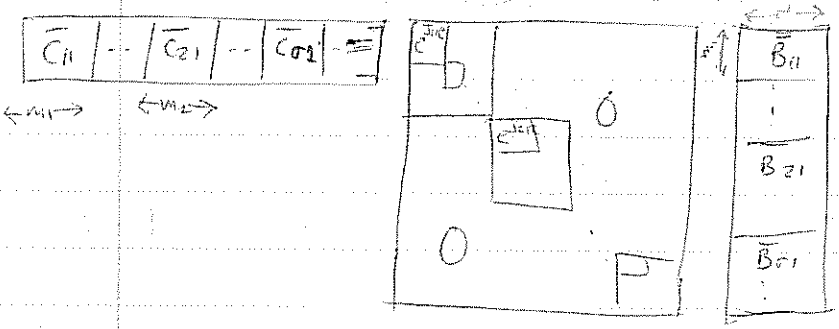
$H(\tau) = C(\exp A\tau) B$; Impulse Response Matrix

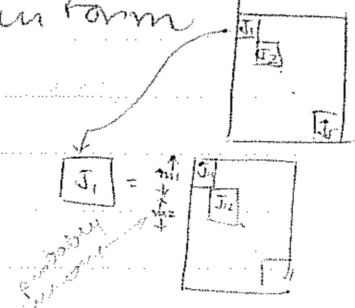
Can do change of coordinates

$\therefore H(\tau) = (CP^{-1}) \cdot e^{J\tau} \cdot (PB)$; $A = P^{-1}JP$

P: col. are gen. e-vectors

J: Jordan form





$$= \sum_{k=1}^{\sigma} \sum_{j} \bar{C}_{kj}\, e^{J_{kj}\tau}\, \bar{B}_{kj}$$

recall $\exp J_{kj}\tau = \begin{bmatrix} e^{\lambda_k \tau} & te^{\lambda_k t} & \\ & \ddots & \ddots \\ & & \ddots \end{bmatrix}$ whatever

Corollary: $\Sigma(T)$ BIBO stable $\Leftrightarrow$ $\bar{C}_{kj} \exp(J_{kj}\tau) \cdot \bar{B}_{kj} = 0$ if $\mathrm{Re}(\lambda_k) \geq 0$

when $=0$ there should have size of Jordan blocks $=1$ [illegible]

4. Stability i.s.L. & a.s.

Often the linear system
$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
comes from linearization of a non linear system.

$\dot{z} = f(z, u, t)$

$\Delta\dot{z} = (\partial f/\partial z)\Delta z + (\partial f/\partial u)\Delta u$

Δz_0, Δu : perturbation

$\Delta z(t)$ deviation

Interested to know

① small perturbation results in small deviation

② Deviation eventually dies out.

1) Stability i.s.L. (in the sense Liapunov)

$\dot{x} = A(t)x \qquad x(t_0) = x_0$ arbitrary

~~(i) $\|x_0\| < \delta \Longrightarrow x(t) = \|\phi(t, t_0, x_0)\|$~~

~~wrong~~ (i) $\forall \delta \; \|x_0\| < \delta \Rightarrow \exists \epsilon > 0$ s.t. $\|x_0\| < \delta \; \|\phi(t, t_0, x_0)\| < \epsilon \quad \forall t$

correct (ii) $\forall \epsilon > 0 \; \exists \delta > 0$ s.t. $\|x_0\| < \delta \Rightarrow \|\phi(t, t_0, x_0)\| < \epsilon$ $\forall t \geq t_0$

Defn: θ of $\dot{x} = A(t)x$ is stable i.s.L.

$\Leftrightarrow \forall t_0, \forall \epsilon > 0 \quad \exists \delta(\epsilon, t_0)$ s.t.

if $\|x_0\| \leq \delta(\epsilon, t_0)$ then $\|\phi(t, t_0, x_0)\| < \epsilon \quad \forall t \geq t_0$

Fact θ is stable i.s.L. $\Leftrightarrow \Phi(t, 0)$ bdd $\boxed{\Leftrightarrow x(t) \text{ bdd}}$

→Proof ($\Leftarrow$) $\|\phi(t, t_0, x_0)\| \leq \|\Phi(t, 0)\| \cdot \|\Phi(0, t_0)\| \cdot \|x_0\|$

$\Phi(t, 0)$ is bdd $\Leftrightarrow \exists M \forall t \quad \|\Phi(t, 0)\| < M$

let $\|\Phi(0, t_0)\| =: \alpha$

So, if we choose $\delta = \epsilon / M\alpha$

then $\|x_0\| < \delta \Rightarrow \|\phi(t, t_0, x_0)\| < \epsilon$

Proof ($\Rightarrow$) (by converse) $\forall M \; \exists t_1 \; \exists K \quad \sum_j |\phi_{kj}(t_1, 0)| > M$

want $\Rightarrow \exists \epsilon > 0 \; \forall \delta \; \exists x_0 \; \|x_0\| < \delta \quad \|\phi(t, t_0, x_0)\| > \epsilon$

↳ this should be $\forall \epsilon > 0 \; \exists t_1 \geq t$. $\forall \delta$ s.t. $\|x_0\| > \delta$ can find t_1 s.t. $\|\phi(t_1, t_0, x_0)\| > \epsilon$

See Desoer Notes

$x_0 = \delta/2 \; \mathrm{sgn} \; \phi_{kj}(t_1, 0)$ evidently $\|x_0\| < \delta$

$M = 2(\epsilon/\delta)$

2) asymptotic stability

Defn θ of $\dot{x} = A(t)x$ is as

iff (i) it is stable in sense Liapunov (i.s.L.)

(ii) $\phi(t, t_0, x_0) \to \theta$ as $t \to \infty$

Fact θ a.s. $\Leftrightarrow$ (i) $\Phi(t, 0)$ is bdd

(ii) $\|\Phi(t, 0)\| \to 0$ as $t \to \infty$

Corollary θ of $\dot{x} = Ax$ a.s. $\Leftrightarrow \mathrm{Re}(\lambda_k) < 0$

5) Exponential Stability

$$\begin{cases} \dot{X} = AX + Bu \\ Y = CX \end{cases}$$

$$y(t) = C\exp(At)x_0 + \int_{t_0}^{t} C\exp(A(t-\tau))\,B\,u(\tau)\,d\tau$$

$$\exp(At) = \sum_{k=1}^{\sigma}\sum_{l=0}^{m_k-1} t^l e^{\lambda_k t} p_{kl}$$

If $Re(\lambda_k) < 0$

$\Rightarrow$ (i) ZIR is bdd $\to \theta$

(ii) ZSR is bdd + bdd u ~~XXX~~

~~moreover the rate @ which $y(t) \to \theta$ is exponential~~

we say (1) is exp stable

(all e-values of A lie in open LHP)

20 Oct 1989

KST - discussion

mean 11.1/20

high 20

0-5 : 7

6-10 : 8

11-15 : 14

16-20 : 9

(5.) Also need property

$$\rho(t,t_0,x_0,u) = \underbrace{\rho(t,t_0,x_0,\theta_u)}_{ZIR} + \underbrace{\rho(t,t_0,\theta_\Sigma,u)}_{ZSR}$$

6. $L : x_0 \longmapsto (y(0), y(1), \cdots y(k))$

$x_0 \in X$

$y \in Y^{k+1}$

$L_0 : X \longrightarrow Y^{k+1}$

$L_0^* : Y^{k+1} \longrightarrow X$

but L_0^* given maps

$L_0^* : Y' \to X$

counter (example): $\langle z, L_0 x\rangle = \sum_{i=0}^{K} z_i^* y(i) x$

$\langle L_0^* z, x\rangle = \Big[\underbrace{\sum_{i=0}^{K} (A^*)^i C^* z_i}_{[L_0^* z]^*} \Big]^* x_0$

we want the form $[L_0^* z]^* x$

$L_0^* = \{ C^*, (A^*) C^*, (A^*)^2 C^*, \dots (A^*)^K C^* \}$

HW prob. For cont. time system is exp stable if $\lambda \in \mathbb{C}_-^o$ (open LHplane)

$\dot{x} = A_c x$; cont sys.

will discretized version be stable

$x(k+1T) = A_{D/c}\, x(kT)$

$A_{D/c} = \exp(A_c T)$
where T is the sampling period

for A_c $\bar{c}$ distinct eigenvalues

$\exp(A_c T) = \sum_{i=1}^{N} \exp^{(\lambda_{c_i} T)} R_i$ when $\lambda_c \in \mathbb{C}_-^o$, then

$|\exp(\lambda_{c_i} T)| < 1$, so

let $\lambda_{D_i} = \exp^{(\lambda_{c_i} T)} = \sum_{i=1}^{N} \lambda_{D_i} R_i$

with $|\lambda_D| < 1$
will be stable!
($\because$ in unit disk)

Non distinct: look @ Jordan form term by term

Jordan Form

$\exp(JT)$ will have ith diagonal element

$\frac{1}{i!} T^i \exp(\lambda_i T)$

$$\begin{bmatrix} e^{\lambda T} & Te^{\lambda T} & \frac{T^2}{2}e^{\lambda T} \\ & \ddots & \\ 0 & & e^{\lambda T} \end{bmatrix}$$

upper triangular

e. values are diag. elem

e.v. are $e^{\lambda T}$

$\Rightarrow$ again will be stable if $Re(\lambda) < 0$

5 Jan
24, 1989

LST - Lecture

Singular Value Decomposition

trivial in finite dimensional case
of matrix representation

Q: - Existence of adjoint
- Uniqueness

1. Review of adjoint

$\mathcal{A} : U \to V$ linear

The adjoint of $\mathcal{A}$

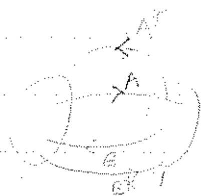
$\mathcal{A}^* : V \to U$,

$$\forall y \in V, \ \forall x \in U \qquad \langle y, \mathcal{A}x \rangle_V = \langle \mathcal{A}^* y, x \rangle_U$$

Properties of $\mathcal{A}^*$

(1) $\mathcal{A}^*$ is unique

(2) $\mathcal{A}^*$ is a linear map also



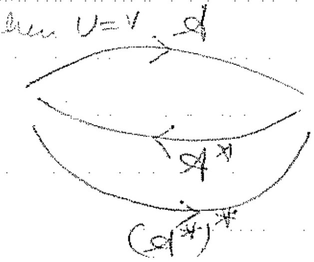
ie $\mathcal{A}^* + \mathcal{B}^* = \mathcal{A}^* + \mathcal{B}^*$

$(\alpha \mathcal{A})^* = \bar{\alpha} \mathcal{A}^*$

A & B both $U \to V$

(3) $(\mathcal{A}\mathcal{B})^* = \mathcal{B}^* \mathcal{A}^*$; makes sense only when $U = V$

(4) $(\mathcal{A}^*)^* = \mathcal{A}$



2. Structure of a linear map.

If $U = M_1 \overset{\perp}{\oplus} M_2$

$\forall x \in M_1 \quad \langle x, y \rangle = 0 \quad \forall y \in M_2$

M_1 and M_2 are orthogonal

then we write $\quad M_1 \perp M_2 \quad U = M_1 \overset{\perp}{\oplus} M_2$

$U = M_1 \oplus M_1^\perp$

$M_2 = M_1^\perp$

$M_1 = M_2^\perp$

Note that for a finite dimensional space we can always write $M_1^\perp$ & and
$V = M_1 \overset{\perp}{\oplus} M_1^\perp$ where M_1 is a subspace of V
for let $\alpha_1 \cdots \alpha_m$ span M_1. Then $(\alpha_1, \cdots \alpha_m, \alpha_{m+1}, \cdots \alpha_n)$ span V [can always choose $\alpha_{m+1}, \cdots \alpha_n$] Now do Gram-Schmidt orthonormalization on $\alpha_1 \cdots \alpha_m, \alpha_{m+1} \cdots \alpha_n$ to get $(e_1 \cdots e_m, e_{m+1}, \cdots e_n)$ Clearly $e_1 \cdots e_m$ span M_1. $M_1^\perp$ is all l.c of $e_{m+1} \cdots e_n \Rightarrow V = M_1 \oplus M_1^\perp$

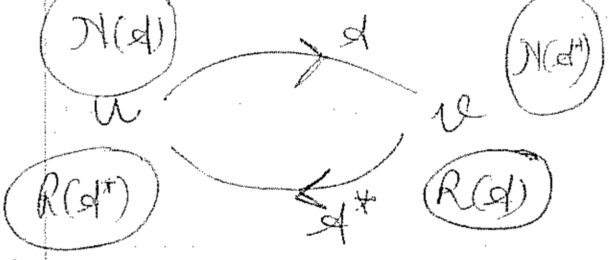
Note: $M^\perp$ = set of vectors that are orthogonal to every vector in M.

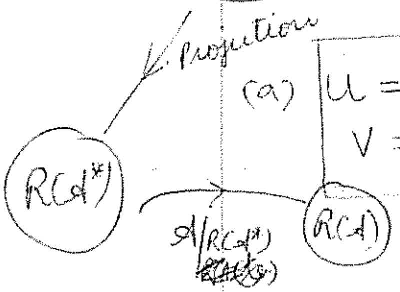
$(M^\perp)^\perp = M$ (for finite dimensional spaces)

1) Let $\mathcal{A} : U \to V$ linear

so $\mathcal{A}^* : V \to U$

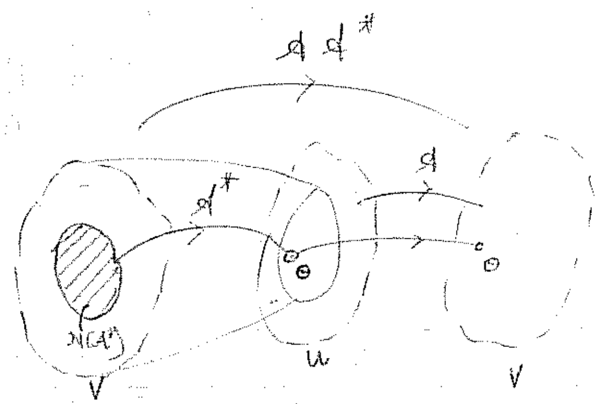
$$\begin{tikzcd} U \arrow[r, bend left, "\mathcal{A}"] & V \arrow[l, bend left, "\mathcal{A}^*"] \end{tikzcd}$$





(a) $\boxed{\begin{aligned} \mathcal{U} &= \mathcal{R}(\mathcal{A}^*) \overset{\perp}{\oplus} \mathcal{N}(\mathcal{A}) \\ \mathcal{V} &= \mathcal{R}(\mathcal{A}) \overset{\perp}{\oplus} \mathcal{N}(\mathcal{A}^*) \end{aligned}}$ || fundamental decomposition! [can obtain matrix repns]

(b) $\boxed{\mathcal{A}\big|_{\mathcal{R}(\mathcal{A}^*)} : \mathcal{R}(\mathcal{A}^*) \longrightarrow \mathcal{R}(\mathcal{A}) \quad \text{one to one, onto}}$



(c) $\boxed{\mathcal{N}(\mathcal{A}^*) = \mathcal{N}(\mathcal{A}\mathcal{A}^*)}$

$\mathcal{N}(\mathcal{A}\mathcal{A}^*) \supseteq \mathcal{N}(\mathcal{A}^*)$ (clear)

also $\mathcal{N}(\mathcal{A}\mathcal{A}^*) = \mathcal{N}(\mathcal{A}^*)$

(d) $\boxed{\mathcal{R}(\mathcal{A}) = \mathcal{R}(\mathcal{A}\mathcal{A}^*)}$

Proof (a) $\mathcal{V} = \mathcal{R}(\mathcal{A}) \overset{\perp}{\oplus} \mathcal{R}(\mathcal{A})^\perp$

$\mathcal{R}(\mathcal{A})^\perp$

$y \in \mathcal{R}(\mathcal{A})^\perp \Leftrightarrow \langle y, \mathcal{A}x \rangle = 0 \ \forall x$

$\Leftrightarrow \langle \mathcal{A}^* y, x \rangle = 0 \ \forall x$; take $x = \mathcal{A}^* y$

Both ways $\Leftrightarrow \mathcal{A}^* y = \Theta$

$\Leftrightarrow y \in \mathcal{N}(\mathcal{A}^*)$

$\Rightarrow \mathcal{R}(\mathcal{A})^\perp = \mathcal{N}(\mathcal{A}^*)$

proof (b) <u>onto</u> $\forall y \in \mathcal{R}(\mathcal{A})$ want to show $\exists x_1 \in \mathcal{R}(\mathcal{A}^*)$ s.t. $\mathcal{A}x_1 = y$

$\forall y \in \mathcal{R}(\mathcal{A})$ by defn $\exists x \in \mathcal{U}$ s.t. $\mathcal{A}x = y$

but a) $\Rightarrow x = x_1 + x_2$, $x_1 \in \mathcal{R}(\mathcal{A}^*)$ $x_2 \in \mathcal{N}(\mathcal{A})$

$\mathcal{A}x = \mathcal{A}x_1 + \mathcal{A}x_2 = \mathcal{A}x_1$

$\Rightarrow \exists x_1 \in \mathcal{R}(\mathcal{A}^*)$ s.t. $\mathcal{A}x_1 = y$ (<u>onto</u>)

<u>(1-1)</u> : suppose $\mathcal{A}x_1 = \mathcal{A}\bar{x}_1 = y$

want to show $x_1 = \bar{x}_1$

$x_1, \bar{x}_1 \in \mathcal{R}(\mathcal{A}^*)$

on the other hand

$\mathcal{A}(x_1 - \bar{x}_1) = \theta$

$\therefore (x_1 - \bar{x}_1) \in \mathcal{N}(\mathcal{A})$

but $(x_1 - \bar{x}_1) \in \mathcal{R}(\mathcal{A}^*)$ (; l.c. of elements of $\mathcal{R}(\mathcal{A}^*)$)

but only element in common is zero.

$\therefore x_1 = \bar{x}_1$

(c) $\mathcal{N}(\mathcal{A}^*) \subseteq \mathcal{N}(\mathcal{A}\mathcal{A}^*)$

(i) $y \in \mathcal{N}(\mathcal{A}^*)$

$\Leftrightarrow \mathcal{A}^* y = \theta$

$\Rightarrow \mathcal{A}\mathcal{A}^* y = \mathcal{A}\theta = \theta \quad \Rightarrow y \in \mathcal{N}(\mathcal{A}\mathcal{A}^*)$

(ii) $\mathcal{N}(\mathcal{A}\mathcal{A}^*) \subseteq \mathcal{N}(\mathcal{A}^*)$

only one way!

$y \in \mathcal{N}(\mathcal{A}\mathcal{A}^*) \Leftrightarrow \mathcal{A}\mathcal{A}^* y = \theta \Rightarrow \langle y, \mathcal{A}\mathcal{A}^* y\rangle_v = 0$

$\Rightarrow \langle \mathcal{A}^* y, \mathcal{A}^* y\rangle_u = 0$

$\Rightarrow \|\mathcal{A}^* y\|_2^2 = 0$

$\Rightarrow \mathcal{A}^* y = \theta$

$\Rightarrow y \in \mathcal{N}(\mathcal{A}^*)$

(d) $\mathcal{R}(\mathcal{A}) = (\mathcal{N}(\mathcal{A}^*))^\perp = (\mathcal{N}(\mathcal{A}\mathcal{A}^*))^\perp$

recall $\mathcal{R}(\mathcal{A})^\perp = \mathcal{N}(\mathcal{A}^*)$

$= (\mathcal{R}(\mathcal{A}\mathcal{A}^*)^\perp)^\perp$

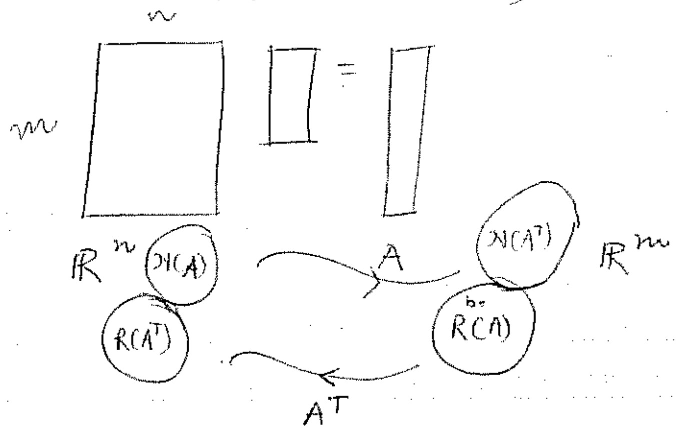
$\mathcal{R}(\mathcal{A}\mathcal{A}^*)^\perp = \mathcal{N}([\mathcal{A}\mathcal{A}^*]^*)$ | $(AB)^* = B^*A^*$

$= \mathcal{N}(\mathcal{A}\mathcal{A}^*)$!

$= \mathcal{R}(\mathcal{A}\mathcal{A}^*)$

3 Least Squares

$Ax = b \quad ; A \in \mathbb{R}^{m \times n} \quad (m > n)$



SCENARIO I ~~Case(1)~~ $\boxed{b \in R(A)}$

$\exists$ unique $x_1 \in R(A^T)$ s.t. $Ax_1 = b$

∵ the map $A|_{R(A^T)} : R(A^T) \to R(A)$ is 1-1 onto

case 1) $\mathcal{N}(A) = \{\theta\}$ i.e. columns are lin ind

i.e. matrix has full (col) rank.

$\therefore \mathbb{R}^n = R(A^T)$ — if A doesn't have full rank then A^T doesn't have full rank

if A^T doesn't have full rank, then there are fewer than n lin ind columns (∵ fewer than n lin ind rows) ∴ can't span $\mathbb{R}^n$

$\Rightarrow \exists$ unique $x \in \mathbb{R}^n$ s.t. $Ax = b$

algebraically,

$Ax = b$

solve this by:

$A^T A x = A^T b$ — $(A^TA)^{-1} \neq A^{-1}(A^T)^{-1}$ in general, ∵ may not be conformable!

$x = (A^T A)^{-1} A^T b$; assume $(\ )^{-1}$ exists

$= (\quad)^{-1} \boxed{A^T} [\ ]$; ~~[illegible]~~ $x \in R(A^T)$

case (2) $\mathcal{N}(A)$ contains non zero elements

$x = x_1 + x_2 \qquad x_2 \in \mathcal{N}(A)$

$Ax = b$; an ∞ of solns, ; but in all of them x_1 is same! (∵ $A: R(A^T) \to R(A)$ is 1-1 onto) & $b \in R(A)$

Elaborate more

but $\|x\|_2^2 = \|x_1\|^2 + \|x_2\|^2$

x_1 is <u>the</u> solution which has minimal norm

Scenario II $\Rightarrow$ $b \notin R(A)$

$b = b_1 + b_2$ $\quad b_1 \in R(A) \quad b_2 \in N(A^T)$

$\exists$ unique $x_1 \in R(A^T)$

s.t. $Ax_1 = b_1$

consider $(Ax - b)$

$= (Ax - b_1 - b_2)$ $\quad \therefore \|Ax - b_1 - b_2\|_2^2 = \|Ax - b_1\|^2 + \|b_2\|^2$

$Ax - b_1 \in R(A)$, $b_2 \in N(A^T) = R(A)^\perp$

$b_2 \in N(A^T) = \{R(A)\}^\perp$

i.e. x_1 is ~~xxxxxxxx~~ a vector which minimizes $\|Ax - b\|_2^2$ (equiv to minimizing $\|Ax - b_1\|^2$)

case 1) $N(A) = \{\theta\}$; A has full rank

~~$\exists!x$~~ $\exists$ a unique x_1 s.t. $\|Ax - b\|$ is minimized (least sq. soln)

$\underbrace{Ax - b}$ $\quad$ Basically to solve $Ax = b_1$ is equivalent to solving $A^T(Ax) = A^Tb = A^T(b_1 + b_2) = A^Tb_1 + 0$ $\quad b_2 \in N(A^T)$

$-b_2 \in N(A^T) \Rightarrow A^T(Ax - b) = \theta$ $\quad$ i.e. $A^T(Ax - b) = \theta$

$(A^TA)x_1 = A^Tb$

A full rank $\Rightarrow$ can solve for x_1 (Any b)

The soln of this eqn gives us the least squares soln

case 2) $N(A) \neq \{\theta\}$ $\Rightarrow$ A hasn't full rank (col. rank that is)

$\Rightarrow \nexists$ a unique x_1 s.t. $Ax - b_1 = \theta$

just solve for $Ax_1 = b_1$ (an ∞ of solns)

& choose that x_1 which has min norm $\Rightarrow$ it will lie in $R(A^T)$

$\hookrightarrow$ Hilbert theory indicates that its unique!

finally we will have x_1 in $R(A^T)$ out of all of these which has min norm

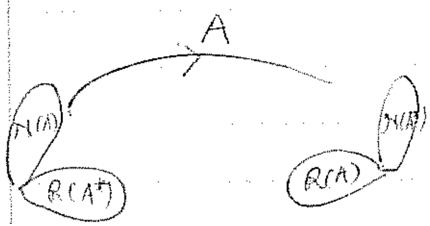
basically $\exists$ unique x in R^n s.t. $Ax = b_1$

for a general x this x will have components in $R(A^T)$ & $N(A)$.

$\hookrightarrow$ No! The minimum norm is x that lies in $R(A^T)$ only (else not min norm)

Thursday
25 Oct 1989

LST- Lecture



wrt bases
A has matrix rep.

$$\begin{array}{r|c|c|}
 & & \\
\mathcal{R}(A)\{ & /\!/\!/ & 0 \\
\mathcal{N}(A^*)\{ & 0 & 0 \\
 & \underbrace{\quad}_{\mathcal{R}(A^*)} & \underbrace{\quad}_{\mathcal{N}(A)}
\end{array}$$

choose basis
$\mathcal{R}(A^*), \mathcal{N}(A)$ $\quad$ $\mathcal{R}(A^*)\ \mathcal{N}(A^*)$

$$\left.\begin{array}{l} u_i \in \mathcal{R}(A^*) \\ A u_i \in \mathcal{R}(AA^*) \\ A u_i \in \mathcal{R}(A) \end{array}\right\} \Rightarrow$$

proof (intuitively clear)

$u_i \in \mathcal{N}(A)$
$A u_i = \theta$

So choose bases to simplify

We know $\mathcal{R}(A^*) = \mathcal{R}(A^*A)$
$\mathcal{N}(A) = \mathcal{N}(A^*A)$

A^*A is better (than AA^*) $\because$ it has property that
$(A^*A)^* = A^*A$

4. Hermitian matrices
defn A Hermitian $\Leftrightarrow (A^* = A)$
(special case: A Real $\Rightarrow A$ symmetric)

Can easily [illegible]

Properties
(1) all eigen values are real
(2) $\lambda_i \neq \lambda_j \Rightarrow \langle e_i, e_j \rangle = 0$ (more than just l.i.) linearly independent
(3) $\mathbb{C}^n = \mathcal{N}(A - \lambda_1 I) \oplus \mathcal{N}(A - \lambda_2 I) \cdots \oplus \mathcal{N}(A - \lambda_p I)$ (previously $\mathcal{N}(A-\lambda_1 I)^{m_1}$ $\mathcal{N}(A-$
(4) $\lambda_{min} \leq \dfrac{\langle x, Ax \rangle}{\langle x, x \rangle} \leq \lambda_{max}$

(3) $\Rightarrow m_i = 1$
$\Rightarrow$ every chain of generalized eigenvector has length 1.
$\Rightarrow \exists$ basis consisting of just eigen vectors.

Can prove by induction. Assume $x_1 \dots x_n$ [illegible] find none orthonormal [illegible] then [illegible] gives you an orthonormal set [illegible] (normal)

[illegible] $= \operatorname{conj}[\langle v_{n+1}, x_{n+1}\rangle - \dots]$ [illegible]

$= 0$

and v_{n+1} is orthonormal

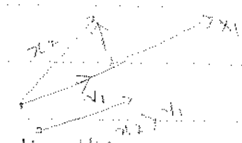
$\Rightarrow$ proved

<u>Claim</u> Given a set of lin ind. eigen vectors $\{x_1, x_2, \dots x_m\}$

$V_1 = x_1 \;/\; \|x_1\|$

$V_2 = x_2 - \langle v_1, x_2\rangle v_1 \;/\; \| x_2 - v_1 \langle v_1, x_2\rangle \|$

$V_3 = (x_3 - v_1\langle v_1, x_3\rangle - v_2\langle v_2, x_3\rangle) \,/\, \| \quad \dots \quad \|$



$\{v_1 \dots v_m\}$ set of orthonormal vectors.

——— x ——— x ——— x ———

$U = \begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix}$; from (3) $n_k = 1$

every chain of gen e-vectors has length 1 ∃ basis consisting of e vectors which is orthonormal

↓ when $\lambda_i \neq \lambda_j$ corresponding eigenvectors are orthogonal. For k eigenvectors which corresp to same λ_i, then any l.c. of them is also eigenvector $\Rightarrow$ can do G-S to get orthonormal set of eigenvectors corresp. to λ_i.

U^{-1} exists (∵ lin ind)

also $U^* U = I$ (∵ orthonormal)

how do you show $AA^{-1} = I \Rightarrow A^{-1}A = I$

$\Rightarrow U^{-1} = U^*$ (unitary matrix) inverse of a matrix is unique

$AU = U\Lambda \quad \Lambda = \operatorname{diag}$ (e.values) (∵ $A[e_1\ e_2 \cdots e_n] = [Ae_1 \mid Ae_2 \cdots \mid Ae_n] = [\lambda_1 e_1 \mid \dots] = [e_1\ e_2 \cdots e_n]\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n \end{bmatrix}$

not nec. distinct!

$\therefore A = U\Lambda U^*$

∴ Hermitian matrix can always be diagonalized.

Pf 1) λ_i e-value $\overset{\text{want}}{\Longrightarrow} \bar{\lambda}_i = \lambda_i$

e_i corresp e-vector $\leftarrow \langle e_i, Ae_i\rangle = \langle e_i, \lambda_i e_i\rangle = \lambda_i \|e_i\|^2$

$= \langle A^* e_i, e_i\rangle$

$= \langle Ae_i, e_i\rangle$ (dfn of Herm)

$= \langle \lambda e_i, e_i\rangle$

$= \bar{\lambda}_i \langle e_i, e_i\rangle$

$= \bar{\lambda}_i \|e_i\|^2$

$\Rightarrow (\lambda_i - \bar{\lambda}_i)\|e_i\|^2 = 0 \quad \Rightarrow \lambda_i = \bar{\lambda}_i$

$\underset{\sim}{A}\underset{\sim}{b} = \underset{\sim}{\theta} \Rightarrow \underset{\sim}{A} = 0$ or $\underset{\sim}{b} = 0$ · if matrix [illegible] singular)

2) $\langle e_i, \lambda e_j \rangle = \lambda_j \langle e_i, e_j \rangle$

$= \langle A e_i, e_j \rangle = \lambda_i \langle e_i, e_j \rangle$

$\therefore (\lambda_j - \lambda_i) \langle e_i, e_j \rangle = 0$; but $\lambda_i \neq \lambda_j \quad \forall\, i \neq j$

$\therefore \langle e_i, e_j \rangle = 0 \quad \forall\, i \neq j$ $\quad e_i^* e_j = 0$ in vector format

3) want to show $m_k = 1$

suppose $m_k \geq 2$

$\Rightarrow \exists\, z \neq \theta$ ($\underline{\text{gen. eigen vector}}$) s.t. $(A - \lambda_k I)^{m_k} z = \theta$

$(A - \lambda_k I)^{m_k - 1} z \neq \theta$

get a contradiction

$= \langle (A - \lambda_k I)^{m_k/2} z, (A - \lambda_k I)^{m_k/2} z \rangle$ ($\because$ Hermitian)

$0 = \langle z, (A - \lambda_k I)^{m_k} z \rangle = \| (A - \lambda_k I)^{m_k/2} z \|^2 = 0$; no problem when you have m_k even.

$\left([(A - \lambda_k I)^{m_k}]^* = [(A^* - \lambda_k I)^{m_k}] = (A - \lambda_k I)^{m_k} \therefore \text{Hermitian} \right)$

if m_k ~~odd~~ even set $m_k = 2\mu_k$

if m_k odd set $m_k + 1 = 2\mu_k$

So $(A - \lambda_k I)^{2\mu_k} z = \theta$

consider

$0 = \langle z, (A - \lambda_k I)^{2\mu_k} z \rangle = \langle (A - \lambda_k I)^{\mu_k} z, (A - \lambda_k I)^{\mu_k} z \rangle$

$\Rightarrow (A - \lambda_k I)^{\mu_k} z = 0$; where $\mu_k < m_k$

$\Rightarrow$ contradiction ($\because (A - \lambda_k I)^{m_k - 1} z \neq \theta$)

TPT.

4) $\lambda_{min} \leq \langle \underbrace{x/\|x\|}_{y}, A \underbrace{x/\|x\|}_{y} \rangle \leq \lambda_{max}$

put $y = x/\|x\| \Rightarrow \|y\| = 1$

TPT $\lambda_{min} \leq \langle y, Ay \rangle \leq \lambda_{max}$

where $\langle y, y \rangle = 1$

ie $\max \langle y, Ay \rangle \leq \lambda_{max}$ when $\|y\| = 1$

$\min \langle y, Ay \rangle \geq \lambda_{min}$ when $\|y\| = 1$

$\max/\min \; y^* A y$ s.t. $y^* y = 1$; max of mult timp under one constrai

Solution $L(y,\lambda) = y^* A y - \lambda(y^* y - 1)$; sort of quadrati (diff by expanding

$\partial L / \partial y = 0 \Rightarrow y^* A - \lambda y^* = 0$

$\Rightarrow A y = \lambda y$ | A hermitian!

ie max or min^m occurs when y is an eigen vec

$\langle y, Ay \rangle = \lambda \langle y, y \rangle = \lambda$

so the maxima is λ_{max}

minima is λ_{min}

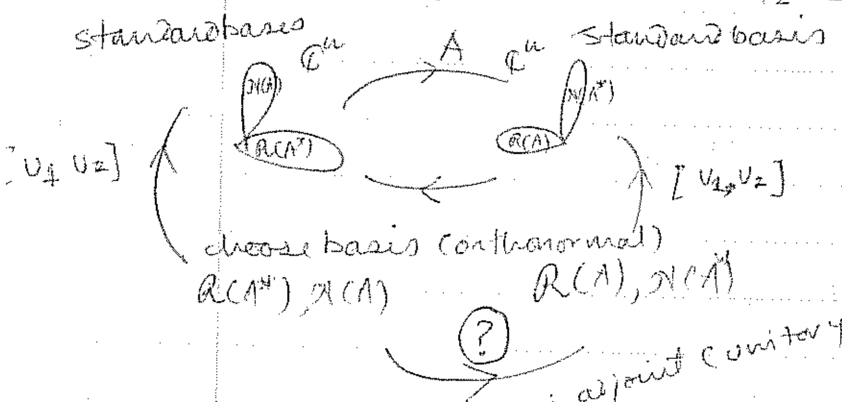
A real, implication 4) $\Rightarrow \langle x, A^T A x \rangle = \|Ax\|_2^2$

$\langle x, A^T A x \rangle \le [\lambda_{max}(A^T A)] \langle x, x \rangle$ (T = * when A is complex)

$= [\lambda_{max}(A^T A)] \|x\|^2$

Define $\|A\|_2 = \sqrt{\lambda_{max}(A^T A)}$

then we have $\|Ax\|_2 \le \|A\|_2 \|x\|_2$; induced norm.



$? = \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} [\,A\,] [\,U_1 \;\; U_2\,]$

$= \begin{bmatrix} V_1^* A U_1 & 0 \\ 0 & 0 \end{bmatrix}$

$V_2^* A U_1$ ($\mathcal{N}(A^*)$) $\Rightarrow V_2^* x (\mathcal{R}(A)) = 0$

How can we simplify $V_1^* A U_1$?

$[U_1\ U_2] \; \mathcal{R}(A^*) = \mathcal{R}(A^* A)$

$\mathcal{N}(A) = \mathcal{N}(A^* A)$

look at $A^* A$: hermitian ∴ can diagonalize!

$\Rightarrow$ Representation simplified

A^*A Hermitian

$\Rightarrow$ (i) e-values are real

(ii) e-vectors orthonormal basis

(iii) e-values are non negative

let (λ, e) be eigenvalue / eigenvector of A^*A

$$\langle e, A^*Ae\rangle = \|Ae\|_2^2 \geq 0$$
$$= \lambda\langle e,e\rangle = \lambda\|e\|^2 \geq 0 \Rightarrow \lambda \geq 0. \quad \underline{QED}$$

Let eigenvalues of A^*A be $\sigma_1^2, \sigma_2^2, \ldots \sigma_r^2, \sigma_{r+1}^2 \ldots \sigma_n^2$ ($\because$ non negative)

(multiple eigenvalues appear $\bar{c}$ their multiplicity)

order them $\rightarrow \sigma_1 \geq \sigma_2 \ldots \geq \sigma_r > 0 \quad \sigma_{r+1} = \cdots \sigma_n = 0$

e-vectors of A^*A

$$\underbrace{u_1 \; u_2 \; \ldots u_r}_{U_1} \; \underbrace{u_{r+1} \; \ldots \; u_n}_{U_2} \quad : \text{basis for } \mathbb{C}^n$$

pick $U_1 \; U_2$ to be orthogonal ie $\begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix}[U_1 \; U_2] = I$

~~and $[A^*A]$~~ and $\begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix}[A^*A][U_1 \; U_2] = \begin{bmatrix} \Sigma_1^2 & 0 \\ 0 & 0 \end{bmatrix}$

where $\Sigma_1^2 = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix}$

columns of U_2 span null space $\mathcal{N}(A^*A) = \mathcal{N}(A)$ $\therefore$ right vectors to use (look at $B \xrightarrow{A}$ diagram)

columns of U_1 span $\mathcal{R}(A^*A) = \mathcal{R}(A^*)$; same.

compare $\bar{c}$ the property of Hermitian matrices:

$$\mathbb{C}^n = \mathcal{N}(A-\lambda_1 I) \overset{\perp}{\oplus} \mathcal{N}(A-\lambda_2 I) \cdots \overset{\perp}{\oplus} \mathcal{N}(A-\lambda_r I)$$

choose v_1 ?

AU_1 spans Range space of $A = \mathcal{R}(A)$ img from figure

in general $V_1 = [AU_1][?]$

each col. results in a l.c. of $[AU_1]$

can form a basis like this

want $V_1^* V_1 = I$ (orthonormal)

$\therefore \; \underbrace{[?]^* U_1^* A^* A U_1 [?]}_{\Sigma_1^2} = I$; looks familiar (recall $\begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$

$(\Sigma^{-1})^* = \Sigma^{-1}$

$\therefore \; [?] = \Sigma_1^{-1}$; $\Sigma_1^2 = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_r^2 \end{bmatrix}$

$AU_1 = \Sigma v_1$

$AU_1 = [\sigma_1 v_1 \; \sigma_2 v_2 \ldots]$

So $V_1 = AU_1 \Sigma_1^{-1}$ will do the job

this even if A isnt square you can move it home!

choose v_2 = orthonormal complement (we arent interested in it)

w/ this choice of $[U_1 \; U_2]$ & $[V_1 \; V_2]$

$V_1^* A U_1 = (\Sigma^{-1} U_1^* A^*)(A U_1) = \Sigma$; a diagonal matrix

$\therefore \; ? = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$; $\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_r \end{bmatrix}$

Final result: choose basis $[U_1 \; U_2]$ $[V_1 \; V_2]$

then matrix rep. is diagonal matrix

$\underbrace{A = [V_1 \; V_2] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix}}$; σ_is are called singular values of A.

This expression is called a singular value decomposition. SVD

$$A = \sum_{i=1}^{r} \sigma_i v_i u_i^*$$

here u_is are e.v. of A^*A

σ_i are $\sqrt{\text{e.v.}}$ of A^*A

Friday
15 Oct 1989

Discussion
KST-~~xxxxx~~

1. Adjoints
2. Stability (motivation)
3. SVD
4. more stability (discrete time system)

Properties of Adjoints
let E & F be Hilbert Spaces
let $A, B : E \to F$

(i) $(A+B)^* = A^* + B^*$
(ii) $(\alpha A)^* = \bar{\alpha} A^*$; $\alpha \in \mathcal{F}$
(iii) $(AB)^* = B^* A^*$; $E = F$
(iv) $A^{**} = A$

Pf(iii) $\forall x \in E$, $y \in F$

$$\langle x | (AB)^* y \rangle = \langle ABx | y \rangle = \langle A(Bx) | y \rangle$$
$$= \langle Bx | A^* y \rangle = \langle x | B^* A^* y \rangle$$

now $\langle x | (B^* A^*) y - (AB)^* y \rangle = 0$ ($\forall$ all x, y)

choose $x = (B^* A^*) y - (AB)^* y$

let $x = B^* A^* - (AB)^* y \Rightarrow \langle B^* A^* y - (AB)^* y | B^* A^* y - (AB)^* y \rangle$

$\therefore \| (B^* A^* - (AB)^* y \|^2 = 0$

$\Rightarrow B^* A^* - (AB)^* = \theta$

$A : E \to F$ $\quad$ $A^* : F \to E$
$B : F \to E$ $\quad$ $B^* : E \to F$
$AB : F \to F$ $\quad$ $B^* A^* : F \to F$

$$\dot{x}_1 = x_1(x_1^2 - 3x_1 + 3) - x_2$$
$$\dot{x}_2 = x_1 - x_2$$

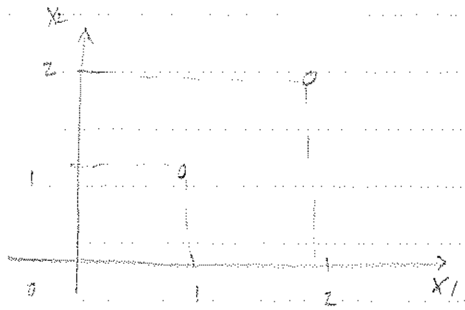
|| Eq pts : places where derivatives are zero

equilibrium pts : $\dot{x} = \theta$

$x_1 = x_2$

$x_1(x_1^2 - 3x_1 + 3) - x_2 = 0$

Eq pts: $\begin{bmatrix}0\\0\end{bmatrix}$ $\begin{bmatrix}1\\1\end{bmatrix}$ $\begin{bmatrix}2\\2\end{bmatrix}$



Linearize around each eq pt (via Jacobian)

$f_1(x_1, x_2) = x_1^3 - 3x_1^2 + 3x_1 - x_2$

$\partial f_1 / \partial x_1 = 3x_1^2 - 6x_1 + 3$

$\partial f_1 / \partial x_2 = -1$

$\partial f_2 / \partial x_1 = 1 \qquad \partial f_2 / \partial x_2 = -1$

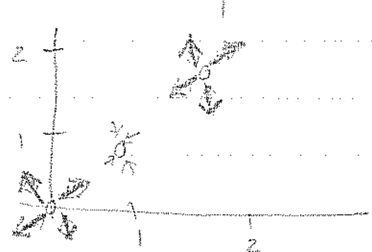
$\dot{x} = Ax \qquad A(x) = \begin{bmatrix} 3x_1^2 - 6x_1 + 3 & -1 \\ 1 & -1 \end{bmatrix}$

$A\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix} = A\left(\begin{bmatrix}2\\2\end{bmatrix}\right)$

$A\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

eigenvalues at $\begin{bmatrix}0\\0\end{bmatrix}$ $\quad \lambda_{1,2} = \frac{1}{2}(2 \pm \sqrt{12})$

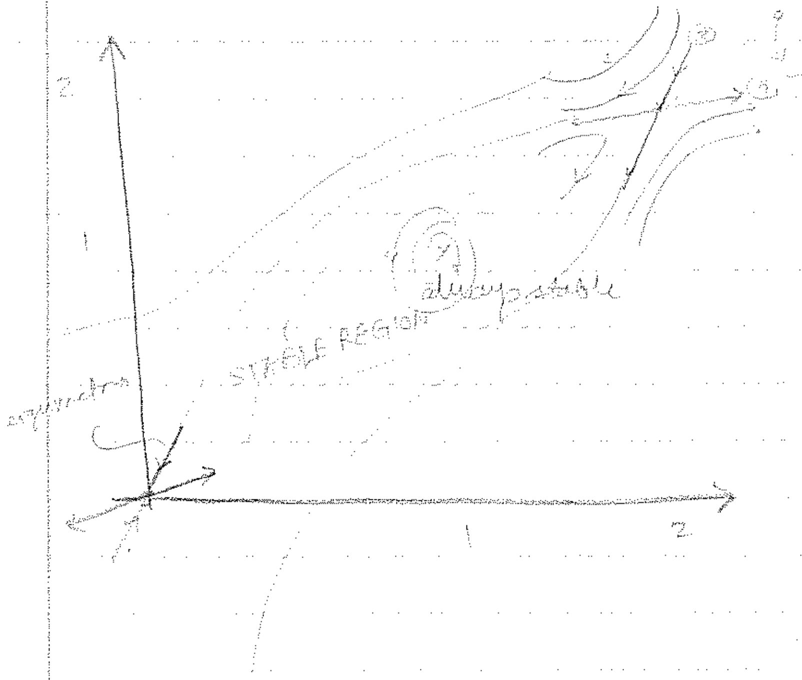
one positive $Re(\lambda) \Rightarrow$ unstable



one e value +ve, one -ve

$\Rightarrow$ saddle pt (not quite top/bottom of hill)

$e_1 = \begin{bmatrix} 1 \\ 2 - \sqrt{3} \end{bmatrix} \qquad e_2 = \begin{bmatrix} 1 \\ 2 + \sqrt{3} \end{bmatrix}$



perturb exactly along eigenvector
① ⇒ goes back
② ⇒ diverges

SVD - why bother?
- controllability & observability issues

OBSERVABILITY - given input $u_{[t_0,t_1]}$ and given the observation of output $y_{[t_0,t_1]}$, the system is called completely observable if the initial state $x(t_0)$ can be uniquely determined.

CONTROLLABILITY - Given x_0, $\overset{\lambda\, t_0, t_1}{x_1}$ $\exists\, u_{[t_0,t_1]}$ s.t $x(t_0)=x_0$; $x(t_1)=$

initial, final states

(example)
$$\begin{bmatrix} -1 & 1 & 1 & 1 & & -1 \\ & -1 & 1 & 1 & \cdot & 1 \\ & & -1 & 1 & 1 & 1 \\ & & & -1 & & \\ 0 & & & & & \\ & & & & & -1 \end{bmatrix}$$

eigenvalues are all -1
non singular ($\det[\cdot] = (-1)^n$)

$$(A+E) = \begin{bmatrix} -1+\frac{1}{2^{n-1}} \cdots & 1 & 1 & \cdots & 1 \\ \frac{1}{2^{n-1}} & -1 & 1 & & \\ \frac{1}{2^{n-1}} & & -1 & & 1 \end{bmatrix}$$ matrix is sing.

perturbation

$$(A+B)\begin{bmatrix}1\\ 1/2\\ \vdots\\ 1/2^{n-1}\end{bmatrix} = \begin{bmatrix}0\\ 0\\ \vdots\\ 0\end{bmatrix}$$

smallest singular value of $(A+B) \sim 1/2^{n-1}$

Example $(n=3)$

$$A = \begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix} \qquad A^*A = \begin{bmatrix}1 & -1 & -1\\ -1 & 2 & 0\\ -1 & 0 & 2\end{bmatrix}$$

$\Delta(\lambda) = \lambda^3 - 6\lambda^2 + 9\lambda - 1 = 0$ for eigenvalues

(all e.v. > 0 (A^*A ~~Hermitian~~))

for λ really small $P(\lambda) \sim -1$

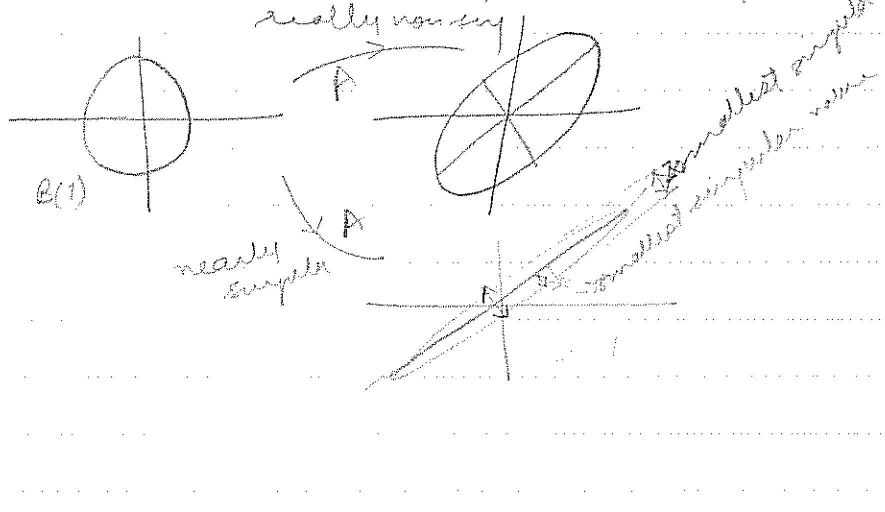
$\lambda \sim 1/6 \Rightarrow (1/6)^3 - 6(1/6)^2 + 9(1/6) - 1 > 0$

$\Rightarrow \exists$ eigenvalue between 0 & $1/6$

say near $1/6 = \lambda_3$

$\sigma_3 \sim 1/\sqrt{6} \sim 1/3$; close to 0.

tells you something about the "condition" of the matrix is (even though e.v. are all 1 & far away)



wrong example
wrong explanation !

example:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 24 \\ 0 & 24 & 32 \end{bmatrix}$ singular

$\lambda_1 = 50$
$\lambda_2 = 1$
$\lambda_3 = 0$

$$\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$$

normalized $\Rightarrow \div$ by 5 $\begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 4/5 \\ -3/5 \end{bmatrix}$

$N(A)$ spanned by e_3

complete the basis

$U_2 = [e_3]$ $\quad U_1 = [e_1 \mid e_2]$

$$\Sigma^2 = U^* A^* A U$$
$$= \begin{bmatrix} 2500 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sigma_1 = 50$
$\sigma_2 = 1$
$\sigma_3 = 0$

same as e.v

Let $V_1 = A U_1 \Sigma^{-1} = \begin{bmatrix} 0 & 1 \\ 3/5 & 0 \\ 4/5 & 0 \end{bmatrix}$

complete the V basis $\rightarrow v_3 = \begin{bmatrix} 0 \\ 4/5 \\ -3/5 \end{bmatrix}$

verify to get the decomposition of $A = V \Sigma U$

$A = V \Sigma U$ $\quad V = [v_1 \mid v_2 \mid v_3]$

$U = [e_1 \; e_2 \; e_3] = [U_1 \mid U_2]$

$Ax = b$ given A, b find x

let $A = V \Sigma U$

$V \Sigma U x = b$; $\sigma_1 \geq \sigma_2 \ldots \geq \sigma_n$; order the singular values now suppose $b = v_1$ = (1st column of V basis)

$\underbrace{V^* V}_{I} \Sigma U^* x = V^* b$ | Recall $U^* = U^{-1}$; $V^* = V^{-1}$

$$\Sigma U^* x = \begin{bmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_n^* \end{bmatrix} \begin{bmatrix} v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Sigma U^* x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} ; \quad \Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \ 0 \end{bmatrix} \quad \Sigma' = \begin{bmatrix} 1/\sigma_1 & & & & & \\ & 1/\sigma_2 & & & & \\ & & \ddots & & & \\ & & & 1/\sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \ 0 \end{bmatrix}$$

$$\Sigma' \Sigma = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$\therefore \ \Sigma' \Sigma U^* x = \Sigma' \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \\ \\ r \\ \\ \end{array} \begin{bmatrix} u_1^* \\ u_2^* \\ \vdots \\ \hline \\ u_n^* \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 1/\sigma_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} U \end{bmatrix} \begin{array}{c} \\ r \\ \end{array} \begin{bmatrix} u_1^* \\ \vdots \\ \hline u_n^* \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} U \end{bmatrix} \begin{bmatrix} 1/\sigma_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x = 1/\sigma_1 \begin{bmatrix} u_1 \end{bmatrix}$$

Now say all σs are non zero

$\Rightarrow x = 1/\sigma_1 [u_1]$

supposing real problem was $A(x + \delta x) = (b + \delta b)$.

$A \delta x = \delta b$ left over

suppose $\delta b = \epsilon v_n$; replace 1s by n's

By same reasoning $\Rightarrow \delta x = \epsilon/\sigma_n \, u_n$ $\quad \dfrac{\|\delta x\|}{\|x\|} = \dfrac{\|\epsilon/\sigma_n \, u_n\|}{\|u_1\|} / \dfrac{\epsilon \, 1/\sigma_1}{1} = \sigma_1/\sigma_n$

$$\|\delta x\| / \|x\| = [\,|\epsilon|/\sigma_n\,]/(1/\sigma_1) = |\epsilon|\,\sigma_1/\sigma_n = \underbrace{\left|\frac{\sigma_1}{\sigma_n}\right|}_{\sigma_1/\sigma_n} \frac{\|\delta b\|}{\|b\|}$$

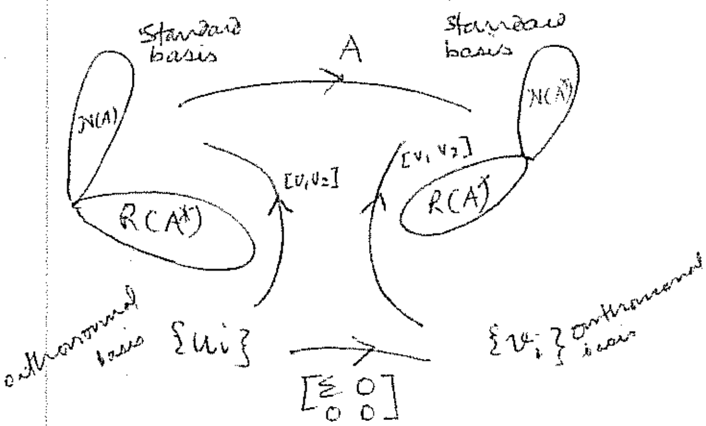
σ_1/σ_n maximum magnification of error in soln

initial vector b in σ_1 dirn
error δb in σ_n dirn

Tuesday
31 October '89

21ST- lecture

SVD Summary



$[U_1 \;\; U_2]$ columns are eigenvectors of A^*A

Σ = diagonal matrix (non zero e-values of A^*A $)^{1/2}$

$V_1 = A U_1 \Sigma^{-1}$

V_2 = orthogonal complement

$$A = [V_1 \quad V_2] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} = [V_1 \; V_2] \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots \; \sigma_n & \\ 0 & & & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{11} & \bar{u}_{21} & \bar{u}_{31} & \cdots & \bar{u}_{n1} \\ \bar{u}_{12} & \bar{u}_{22} & \cdots & & \\ \bar{u}_{1n} & \cdots & & & \bar{u}_{nn} \end{bmatrix}$$

$$= [V_1 \quad V_2] \begin{bmatrix} \sigma_1 \bar{u}_{11} & \sigma_1 \bar{u}_{21} & \cdots & \sigma_1 \bar{u}_{n1} \\ \sigma_2 \bar{u}_{12} & \cdots & & \\ \sigma_3 \bar{u}_{13} & \cdots & & \\ \sigma_r \bar{u}_{1r} & \cdots & & \sigma_r \bar{u}_{nr} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \sum_{k=1}^{r} v_k \sigma_k u_k^*$$

$$A = \sum_{k=1}^{r} \sigma_k v_k u_k^*$$

$$\begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$$

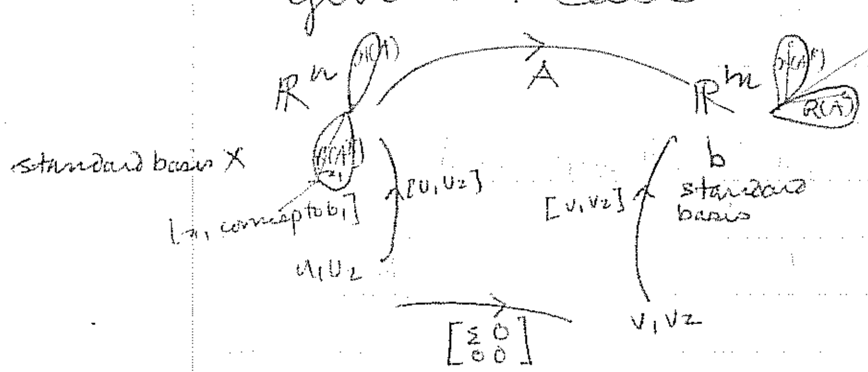
Least Squares Problem

general case

$\min \|Ax-b\|^2$

$b = b_1 + b_2$

$b_1 \in R(A)$



x_1 is the one which minimizes $\|Ax-b\|^2$ & has minimum norm.

$\min \|Ax-b\|^2$

b in SVD coordinates $= \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} b$

b_1 in SVD coordinates $= \begin{bmatrix} V_1^* b \\ 0 \end{bmatrix}$ cm R(A)

x_1 in SVD coordinats $= \begin{bmatrix} \Sigma^{-1} V_1^* b \\ 0 \end{bmatrix} = A^{-1} b_1$ SVD coordinats

x_1 in standard basis $= [U_1\ U_2] \begin{bmatrix} \Sigma^{-1} V_1^* b \\ 0 \end{bmatrix}$

$= U_1 \Sigma^{-1} V_1^* b$

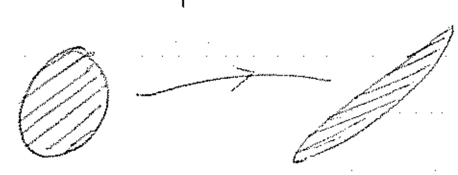
6. Geometric Interpretation of SVD-

Let A be $n \times n$ matrix, non singular

$$\mathbb{C}^n \xrightarrow{A} \mathbb{C}^n$$

Span of A: should be $\mathbb{C}^n$

intuitively: not uniform mapping

ball 

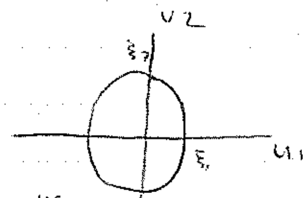
Examining a ball will give us some idea of sp

$B = \{x \in \mathbb{C}^n, \|x\| = 1\}$

$$u \quad \mathbb{C}^n \xrightarrow{A} \mathbb{C}^n \quad v$$

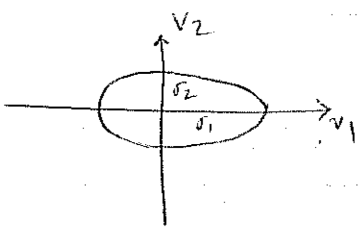
let $X = \sum_{i=1}^{n} \xi_i u_i \qquad \{u_i\}$ orthonormal

$$\|X\| = 1 \iff \sum_{i=1}^{n} \xi_i^2 = 1$$

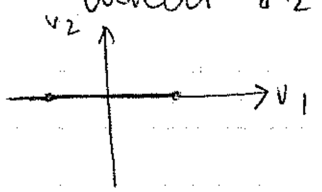


$$y = Ax$$
$$= \sum_{i=1}^{n} \xi_i A u_i = \sum_{i=1}^{n} \xi_i \sigma_i v_i = \sum_{i=1}^{n} \eta_i v_i$$

$$\sum_{i=1}^{n} \xi_i^2 = 1 \iff \sum_{i=1}^{n} (\eta_i / \sigma_i)^2 = 1 \quad \text{: Ellipsoid}$$



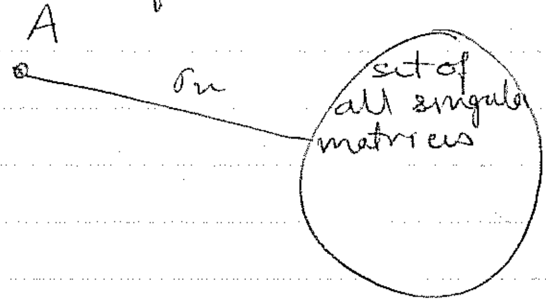
when $\sigma_2 = 0$: matrix is singular



σ_2 small $\Rightarrow$ gives idea of how close we are to singularity.

roughly speaking smallest singular value indication how the matrix is away from being singular.

Set of all $n \times n$ matrices



σ_n is (in some way) the distance between A & the set of all singular matr

Thm: $A \in \mathbb{C}^{m\times n}$
$$A = \sum_{i=1}^{r} \sigma_i v_i u_i^* \qquad \text{rank } A = r$$

Let $A' = \sum_{i=1}^{r-1} \sigma_i v_i u_i^*$ $\therefore$ rank $A' = r-1$

then (i) $\|A - A'\|_2 = \sigma_r$

(ii) $\min_{\text{rank } B = r-1} \|A - B\|_2 = \sigma_r$

Pf (i) $\|A - A'\|_2 = \max_\lambda \sqrt{\lambda (A-A')^* (A-A')}$

$$= \max \sqrt{\lambda\, (U^*(A-A')^*(A-A')\, U}$$

$$= \max \sqrt{\lambda\, (U^* (A-A')^* V V^* (A-A') U}$$

$$= \max \sqrt{\lambda \begin{pmatrix} 0 & & \\ & \sigma_r^2 & \\ & & \ddots 0 \end{pmatrix}}$$

$$= \sigma_r$$

$(A-A')$ has rank 1

$(A-A')^*(A-A')$ has rank 1.

its eigenvalue is σ_r

(ii) Let $B \in \mathbb{C}^{m\times n}$ rank $B = r-1$

dim $N(B) = n - r + 1$

dim $N(A) = n - r$

dim $N(B) >$ dim $N(A)$

$\therefore N(B) \not\subseteq N(A)$

$\exists z \neq \theta$ in the intersection of $N(B) \cap$ span of U_1

$Bz = \theta$

$$Az = \sum_{i=1}^{r} \sigma_i v_i u_i^* z \qquad \|z\| = 1$$

$\|A-B\|_2^2 = \max_{\|X\|=1} \|(A-B)X\|_2^2$

$\geq \|(A-B)z\|_2^2$ ($\because$ we chose $\|z\|=1$)

$= \|Az\|_2^2 = [\sum \sigma_i v_i u_i^* z]^* [\sum \sigma_j v_j u_j^* z]$

$= \sum\sum \sigma_i \sigma_j z^* u_i v_i^* v_j u_j^* z$

$= \sum \sigma_i^2 (u_i^* z)^* (u_i^* z) = \sum \sigma_i^2 \|u_i^* z\|^2$

Examine $\|A-B\|_2^2 := \max_{\|X\|=1} \|(A-B)X\|_2^2 \geq \|(A-B)z\|^2$

$$= \|Az\|_2^2$$

$$= \sum_{i=1}^{r} \sigma_i^2 (u_i^* z)^2 \geq \sigma_r^2$$

$\sum \sigma_i^2 (u_i z)^2$

$z = \sum a_i u_i$

$\therefore \sum \sigma_i^2 a_i^2 \geq \sigma_r^2 \sum a_i^2 \geq \sigma_r^2$

exhibited a lower bound, but we know it's attained (at A') hence proven.

$\therefore$ distance between $\|A-B\|_2 \geq \sigma_r$
but we have already defined one s.t. it has $\|A-A'\|_2 = \sigma_r$ & $\operatorname{rank}(A') = r-1$.

we used only $\dim \mathcal{N}(A) > \dim \mathcal{N}(A)$
if $\operatorname{rank} B = k$, $k \leq r-1$
still true!

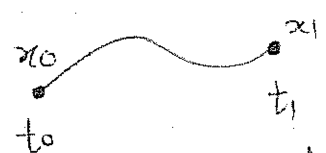
In numerical Analysis SVD imp
(i) error prop.
(ii) numerical stability

Ch VII ② PK (Supp sec I–V)

CONTROLLABILITY & OBSERVABILITY

1. Controllability

① Dynamical System $D = \{\mathcal{U}, \Sigma, \mathcal{Y}, s, r\}$



$t_0, t_1 \in T \quad t_1 > t_0$

$[t_0, t_1]$

dfn D is said to be completely controllable (c.c.) on $[t_0, t_1]$
$\Leftrightarrow \forall x_0, x_1 \in \Sigma \quad \exists u_{[t_0,t_1]}$ steer (x_0, t_0) to (x_1, t_1)
i.e. $x_1 = s(t_1, t_0, x_0, u_{[t_0,t_1]})$

Special Cases

a.) $x_1 = \theta$

D is completely controllable to the origin on $[t_0, t_1]$

$\forall x_0 \in \Sigma \quad \exists u_{[t_0,t_1]}$ s.t. $\theta = s(t_1, t_0, x_0, u_{[t_0, x_0]})$

b.) $x_0 = \theta$ D is completely reachable the origin on $[t_0, t_1]$

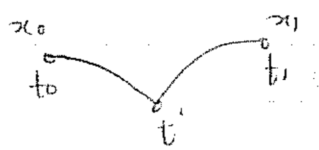
$\forall x_1 \in \Sigma \ \exists u_{[t_0,t_1]}$ s.t. $x_1 = S(t_1, t_0, \theta, u_{[t_0,t_1]})$

(i) c.c. on $[t_0, t_1] \Rightarrow$ c.c. to θ on $[t_0, t_1]$; conv. not

(ii) " " " $\Rightarrow$ c.r. from θ on $[t_0, t_1]$; true in gen.

(iii) cc to θ on $[t_0, t_1']$ + c.r. from θ on $[t_1', t_1]$ $\Big\} \Rightarrow$ c.c. on $[t_0, t_1]$

$t_0 < t' < t_1$



For linear dynamical systems: $\Leftrightarrow$

(2) Linear System $R = [A(\cdot), B(\cdot), C(\cdot)]$

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + \underbrace{D(t)\,u(t)}$$

directly added from i/p to o/p.
doesn't change anything

c.c. on $[t_0, t_1] \Leftrightarrow$ a) c.c. to θ on $[t_0, t_1]$
b) c.r. from θ on $[t_0, t_1]$

<u>Proof</u>: $(\Leftarrow)$ (a) c.c. to θ on $[t_0, t_1]$

$$\Leftrightarrow \forall \xi_0 \ \exists \tilde{u} \text{ s.t. } \theta = \underbrace{\Phi(t_1, t_0)\,\xi_0}_{ZIR} + \underbrace{\int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) \tilde{u}(\tau)\,d\tau}_{ZSR}$$

want $\forall x_0, x_1 \ \exists \bar{u}$ s.t.

$$x_1 = \Phi(t_1, t_0)\,x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)\,B(\tau)\,\bar{u}(\tau)\,d\tau$$

Need to make $\Phi(t_1, t_0)\,x_0 - x_1 = \Phi(t_1, t_0)\,\xi_0$

$$\Phi(t_1, t_0)\overbrace{(x_0 - \Phi^{-1}(t_1, t_0)\,x_1)}^{\xi_0} = \Phi(t_1, t_0)\,\xi_0$$

total response = ZIR + ZSR

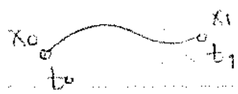
$\Phi(t_1, \tau)$ nonsingular (not so in discrete time sy

Thursday November 2, 1989

XST - lecture

c.c. on $[t_0, t_1]$

$\forall x_0, x_1 \quad \exists u_{[t_0,t_1]} \qquad x_1 = s(t_1, t_0, x_0, u_{[t_0,t_1]})$



Linear System represented by $[A(\cdot), B(\cdot), C(\cdot)]$

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$

$y(t) = C(t)x(t)$

c.c. on $[t_0, t_1] \Leftrightarrow$ (a) c.c. to θ on $[t_0, t_1]$

(total response = ZIR + ZSR & STM is nonsingular $\Phi(t_1, t_0)$)

c.c. on $[t_0, t_1] \Leftrightarrow$ (b) c.r. from θ on $[t_0, t_1]$

($\Rightarrow$ special case)

($\Leftarrow$) c.r. from the origin on $[t_0, t_1]$

$\Leftrightarrow \forall \xi_1 \quad \exists \bar{u}_{[t_0,t_1]}$ s.t. $\xi_1 = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) \bar{u}(\tau) d\tau$

want $\forall x_0, x_1 \quad \exists u_{[t_0,t_1]}$

s.t. $x_1 = \Phi(t_1, t_0) x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) u(\tau) d\tau$

pick $\xi_1 = x_1 - \Phi(t_1, t_0) x_0$

$\therefore \exists \bar{u}_{[t_0,t_1]}$ s.t. $x_1 - \Phi(t_1, t_0) x_0 = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) \bar{u}(\tau) d\tau$

$\Rightarrow x_1 = \Phi(t_1, t_0) x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) \bar{u}(\tau) d\tau$

all we need for this to go through is separability of ZIR & ZSR. (didn't use nonsingular $\Phi(t_1, t_0)$)

consequently this <u>is</u> true for Discrete Time Systems

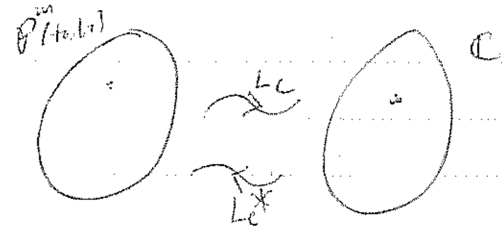
Since linear system

c.c. on $[t_0, t_1] \Leftrightarrow$ c.r. from θ on $[t_0, t_1]$

need only consider ZSR

$$L_c : \mathcal{U}_{[t_0,t_1]} \longrightarrow \int_{t_0}^{t_1} \Phi(t_1,\tau)\, B(\tau)\, u(\tau)\, d\tau$$

$\rho^m[t_0,t_1] \longrightarrow \mathbb{C}^n$; not a friendly map (∵ of lack of [·] fini



$\Rightarrow$ (i) x is reachable from the origin

$\Leftrightarrow x \in R(L_c)$

(ii) $R(L_c)$:= set of reachable states (from 0) on $[t_0,t_1]$

(iii) c.c. on $[t_0,t_1] \Leftrightarrow R(L_c) = \mathbb{C}^n$

But $R(L_c) = R(L_c L_c^*)$

$L_c L_c^* : \mathbb{C}^n \longmapsto \mathbb{C}^n$; a nice map. (matrix)

$L_c^* : \mathbb{C}^n \longmapsto \rho^m[t_0,t_1]$; this is $z \longmapsto B^*(\cdot)\, \Phi^*(t_1,\cdot)\, z$ (ZIR of the adjoint system)

$$L_c L_c^* : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

$$z \longmapsto \int_{t_0}^{t_1} \Phi(t_1,\tau) B(\tau) \left[L_c^*(z) \right](\tau)\, d\tau$$

$$= \underbrace{\int_{t_0}^{t_1} \overset{n\times n}{\Phi(t_1,\tau)}\, \overset{n\times m}{B(\tau)}\, \overset{m\times n}{B^*(\tau)}\, \overset{n\times n}{\Phi^*(t_1,\tau)}\, d\tau}_{n\times n \text{ matrix } = M(t_0,t_1)} \cdot z$$

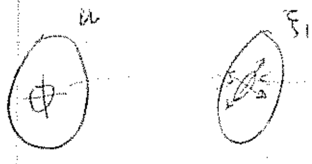
(constant matrix)

(i) $R(L_c) = R(M(t_0,t_1))$

(ii) $R(L_c) = \mathbb{C}^n \Leftrightarrow M(t_0,t_1)$ non singular (!) it's a bijective mapping!

∴ can numerically test com cont.

(eg. det / Sing Values)



↳ practically zero ⇒ practically singular

very small ⇒ need very large u to drive it there. (Practically not possible) which input u would bring it to the desired state?

To find the input $u_{[t_0,t_1]}$ driving $(\overset{x_0}{\otimes}, t_0)$ to (x_1, t_1)
[Given $M(t_0,t_1)$ non singular)

want $$x_1 = \Phi(t_1,t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)\underbrace{u(\tau)}_{\text{unknown}}\,d\tau$$

$$B^*(\tau)\Phi^*(t_1,\tau)\,[M(t_0,t_1)]^{-1}(x_1 - \Phi(t_1,t_0)x_0)$$ ~~multiply by $B^*(\tau)\,\Phi^*(t_1,\tau)\,[M(t_0$~~

think about uniqueness of soln!
its unique ... $\Rightarrow$ ... onto ... unique ... ?

3) Linear Time Invariant $[A, B, C]$

x reachable from the origin on $[t_0,t_1]$

i.e. $x \in R(L_c)$

$$x \in R(L_c) \Rightarrow \exists\, u_{[t_0,t_1]} \text{ s.t. } x = L_c u = \int_{t_0}^{t_1} \exp^{A(t_1-\tau)} B u(\tau)\,d\tau$$

$$\text{C-H}: \quad e^{A(t_1-\tau)} = \sum_{i=0}^{n-1} \alpha_i A^i = \sum_{i=0}^{n-1} \alpha_i(t_1-\tau)A^i \quad \text{(fn of time}$$

$$x = \int_{t_0}^{t_1} \sum_{i=0}^{n-1} \alpha_i(t_1-\tau)A^i\, B\, u(\tau)\,d\tau$$

$$= \sum_{i=0}^{n-1} A^i B \underbrace{\int_{t_0}^{t_1} \alpha_i(t_1-\tau)\,u(\tau)\,d\tau}$$

$$= B\alpha_0(t_1-\tau)u(\tau) + AB\alpha_1(t_1-\tau)u(\tau) + A^2B\,\alpha_2(t_1-\tau)u(\tau)$$

$\therefore \quad x = $ l.c. of columns of $[B \,|\, AB \,|\, A^2B \,|\, \cdots \,|\, A^{n-1}B]$

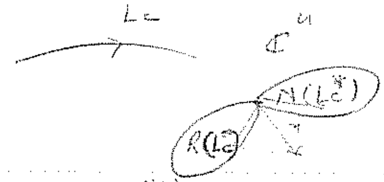
↳ l.c. of cols of B + l.c. of cols of AB ...

let $Q = [B \,|\, AB \,|\, \cdots \,|\, A^{n-1}B] \qquad x \in R(Q)$

$x \in R(L_c)$
$\Rightarrow x \in R(Q)$ where $Q = [B \,|\, AB \,|\, \cdots \, A^{n-1}B]$

now about $(\Leftarrow)$?

do by $x \notin R(L_c) \overset{(?)}{\Rightarrow} x \notin R(Q)$



$X = X_1 + X_2 \quad X_1 \in R(L_c) \quad x_2 \in N(L_c^*)$

& $x_2 \neq \Theta$ (probably $x_2 \neq 0$ for $x \notin R(L_c)$)

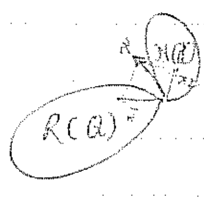
$L_c^* : x_2 \longmapsto B^* e^{A^*(t_1-\tau)} \cdot x_2$

known

$B^* e^{A^*(t_1-\tau)} x_2 = \Theta_{P[t_0,t_1]} \quad \forall \text{ all } \tau \in [t_0, t_1]$

should this be for all $\tau \in [t_0,t_1]$? (so!) $N(L_c^*)$ maps to the zero of $P[t_0,t_1]$ which is the identically zero fn.

want



(??) want $x_2 \in N(Q^*)$

this would imply $x \notin R(Q)$ (for $x_2 \neq \Theta$)

$B^* x_2 = \Theta$
$B^* A^* x_2 = \Theta$
$\vdots$
$B^* (A^*)^{n-1} x_2 = \Theta$

$$B^* \left(I + A^*(t_1-\tau) + (A^*)^2 \frac{(t_1-\tau)^2}{2!} + \cdots \right) X_2 = \Theta \quad \forall \tau \in [t_0, t_1]$$

$\tau = t_1 \Rightarrow B^* x_2 = \Theta$

differentiate wrt τ, set $\tau = t_1$ $\Rightarrow B^* A^* x_2 = \Theta$

$\vdots$

$B^* (A^*)^{n-1} x_2 = \Theta$

Note that the same cond for any $[t_0, t_1]$.

$\therefore$ x reachable from the origin on any interval of finite length $\Leftrightarrow x \in R(Q)$; $Q = [B \mid AB \cdots A^{n-1}B]$
$n \times nm$

$R(Q) :=$ set of reachable states

$[A, B, C]$ c.c. iff $R(Q) = n$ (full row rank)

$[A, B, C]$ c.c. $\iff R(Q) = \mathbb{C}^n$

$\iff \mathcal{N}(Q^*) = \{\theta\}$

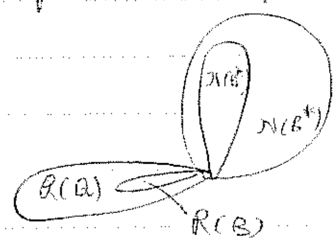
Properties of $R(Q)$ & $\mathcal{N}(Q^*)$

(a) $x \in R(Q) \iff x$ is lin comb of $[B \,|\, AB \,|\, \cdots A^{n-1}B]$

if I have a vector which is l.c. of vectors of B i.e. $x \in R(B$

$\Rightarrow x \in R(Q)$

$R(B) \subseteq R(Q)$



(b) $x \in R(Q)$ $\quad x = \sum_{i=0}^{n-1} A^i B \alpha_i$; α_is are vectors!

$$Ax = \sum_{i=0}^{n-1} A^{i+1} B \alpha_i$$

$$= AB\alpha_0 + A^2B\alpha_1 + \cdots A^{n-1}B\alpha_{n-2} + \underbrace{A^nB\alpha_{n-1}}_{C\text{-}H}$$

$$= B\beta_0 + AB\beta_1 + \cdots A^{n-1}B\beta_{n-1}$$

$\Rightarrow Ax \in R(Q)$

i.e. Range space of Q is invariant under A.

(c) $x \in \mathcal{N}(Q^*) \iff \begin{bmatrix} B^* \\ B^*A^* \\ \vdots \\ B^*(A^*)^{n-1} \end{bmatrix} x = \begin{bmatrix} \theta \\ \\ \\ \end{bmatrix}$

$\Rightarrow B^*x = \theta$

$x \in \mathcal{N}(Q^*) \Rightarrow x \in \mathcal{N}(B^*)$

(d) $x \in \mathcal{N}(Q^*) \overset{(C.H.)}{\Rightarrow} A^*x \in \mathcal{N}(Q^*)$

$\mathcal{N}(Q^*)$ is invariant under A^*.

november '89

LST-lecture

HW#9 11.14.89

Next exam Friday 17 Nov

Starting from Jordan form to end of this week.

$[A, B, C]$ c.c. $\Leftrightarrow \mathcal{R}(Q) = \mathbb{C}^n$

$\Leftrightarrow \mathcal{N}(Q^*) = \{\theta\}$

(a) $\mathcal{R}(B) \subseteq \mathcal{R}(Q)$

(b) $\mathcal{R}(Q)$ invariant under A

(c) $\mathcal{N}(Q^*) \subseteq \mathcal{N}(B^*)$

d) $\mathcal{N}(Q^*)$ invariant under A^*

(e) $A^*: \mathcal{N}(Q^*) \longrightarrow \mathcal{N}(Q^*)$ linear

if dimension $\mathcal{N}(Q^*) \geq 1$

then $\exists$ an eigenvector of A^* in $\mathcal{N}(Q^*)$ fundamental fact

$\nexists$ e.vector of A^* in $\mathcal{N}(Q^*)$ $\Rightarrow \mathcal{N}(Q^*) = \{\theta\} \Rightarrow \mathcal{R}(Q) = \mathbb{C}^n$ $\Rightarrow$ system is c.c.

Further characterization on c.c.

(analysis)

$[A, B, C]$ c.c.

$\Leftrightarrow \mathcal{N}(Q^*) = \{\theta\} \Leftrightarrow \nexists$ e.vector of A^* in $\mathcal{N}(Q^*)$

$\therefore$ c.c. $\Rightarrow \nexists$ eigenvector of A^* in $\mathcal{N}(Q^*)$ c.c. $\Rightarrow \mathcal{N}(Q^*) = \{\theta\} \Rightarrow \nexists$ e.vector of A^* in $\mathcal{N}(Q^*)$

(how?)

c.c. $\Leftarrow \Rightarrow [\nexists$ eigenvector of A^* in $\mathcal{N}(B^*)]$ analysis

$\therefore A^*: \mathcal{N}(B^*) \to \mathcal{N}(B^*)$ (invariant) the two are subspaces $\Rightarrow$ ... eigenvector ... $\mathcal{N}(B^*) = \{\theta\}$ $\Rightarrow$ c.c.

$(\Rightarrow)$ PROOF!

if not true i.e. $\exists$ eigenvector of A^* in $\mathcal{N}(B^*)$

$\hookrightarrow f_K$

couldn't one circumvent this entire argument?

$\therefore$ we have $A^* f_K = \bar{\lambda}_K f_K$ and $B^* f_K = \theta$

$\Rightarrow f_K^* [\lambda_K I - A \mid B] = \theta^T$ is a longer $(n \times n+m)$ matrix

$\Rightarrow$ rank of $[\lambda_K I - A \mid B] < n$: (dim of A)

this is same to show true for rank $[\lambda I - A \mid B] = n$

If rank$[\lambda I - A \mid B] = n \Rightarrow \nexists$ eigenvector of A^* in $\mathcal{N}(B^*)$

$\Rightarrow [A, B, C]$ c.c.

what about $(\Leftarrow)$?

Fact: $\boxed{[A,B,C] \text{ c.c.} \iff \text{rank}[\lambda I - A \mid B] = n \quad \forall \lambda \in \mathbb{C}}$

Proof: $(\Rightarrow)$ Suppose $\text{rank}[\lambda I - A \mid B] < n$

$\Rightarrow \exists\ \alpha^* \neq \theta^T$ s.t. $\alpha^*[\lambda I - A \mid B] = \theta^T$

$$\lambda \alpha^* = \alpha^* A \quad ; \quad \alpha^* B = \theta^T$$

consider: $\alpha^*[B \ AB \ \cdots\cdots \ A^{n-1}B] = [\alpha^* B \mid \underbrace{\alpha^* AB}_{\lambda\alpha^* B} \mid \cdots \mid \alpha^* A^{n-1}B]$

$$= [\alpha^* B \mid \lambda \alpha^* B \mid \cdots \mid \lambda^{n-1}\alpha^* B]$$

$$= \theta^T$$

$\Rightarrow [A,B,C]$ is <u>not</u> c.c.

Q.E.D.

2. <u>Observability</u>

(1) Dynamical System $\mathcal{D} = \{\mathcal{U}, \Sigma, \mathcal{Y}, s, r\}$

$t_0, t_1 \ ; \ t_1 > t_0$

$\mathcal{D}$ is said to be completely observable (c.o.) on $[t_0, t_1$

$\iff \forall\ u_{[t_0,t_1]}$ & corresp. $y_{[t_0,t_1]}$ together $\bar{c}\ \mathcal{D}$ determine uniquely the state of x_0 of $\mathcal{D}$ at t_0

Note: once x_0 is determined

$s(t, t_0, x_0, u_{[t_0,t_1]})$

the state at t

$x(t)$ is uniquely determined

$u \longrightarrow \boxed{\mathcal{D}} \longrightarrow y$

(2) Linear System $[A(\cdot), B(\cdot), C(\cdot)]$

$$y(t) = \underbrace{C(t)\,\Phi(t,t_0)x_0}_{ZIR} + \underbrace{\int_{t_0}^{t} C(t)\,\Phi(t,\tau)\,u(\tau)\,d\tau}_{ZSR}$$

$\leftarrow$ Can be "calculated" from $u_{[t_0,t_1]}$ & $[A(\cdot), B(\cdot), C(\cdot)]$

Can be subtracted from y

For a linear system:

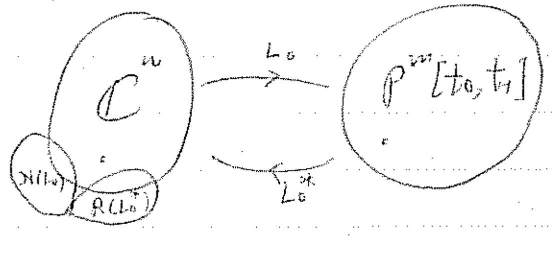
c.o. on $[t_0,t_1] \Leftrightarrow$ c.o. of the ZIR on $[t_0,t_1]$

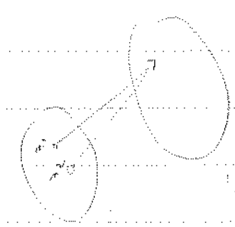
$R(A^*) = R(A^*A)$

$N(A) = N(A^*A)$

$L_0 : x_0 \longmapsto C(\cdot)\,\Phi(\cdot,t_0)\,x_0$

$\mathbb{C}^n \longrightarrow P^m_{[t_0,t_1]}$ (fn space)





(i) System $[A(\cdot), B(\cdot), C(\cdot)]$ is c.o. on $[t_0,t_1] \Leftrightarrow N(L_0) = \{\theta\}$

$\Leftrightarrow R(L_0^*) = \mathbb{C}^n$

(compare $\bar{c}$ c.c. case where it was reqd that $R(L_c) = \mathbb{C}^n$;
here ZSR of adjoint needs to be mapped onto $\mathbb{C}^n$)

$[A(\cdot), B(\cdot), C(\cdot)]$ not c.o.

$N(L_0)$ contains nonzero vectors.

$x \in N(L_0)$ $\quad L_0 x = \theta$ zero function

$\quad L_0\,\theta = \theta$ zero function

we say x is an unobservable state.

The set $N(L_0) \triangleq$ set of unobservable states.

what subset translates that maps to zero function? Just a subspace.

$[A(\cdot), B(\cdot), C(\cdot)]$ c.o. $\Leftrightarrow R(L_0^*) = \mathbb{C}^n$

$R(L_0^*) = R(L_0^* L_0)$

$L_0^* L_0 : \mathbb{C}^n \longrightarrow \mathbb{C}^n$

for onto $\Rightarrow$ matrix representation should be nonsingular

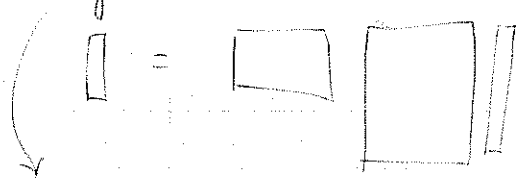
$L_o^* L_o : \mathbb{C}^n \longrightarrow \mathbb{C}^n$

$$x_0 \longmapsto \underbrace{\left[\int_{t_0}^{t_1} \Phi^*(\tau,t_0) C^*(\tau) C(\tau)\, \Phi(\tau, t_0)\, d\tau\right]}_{N(t_0,t_1)} x_0$$

$[A(\cdot),B(\cdot),C(\cdot)]$ c.o. $\iff R(L_o^*) = \mathbb{C}^n \iff N(t_0,t_1)$ non singular

Suppose $[A(\cdot), B(\cdot), C(\cdot)]$ is c.o. $\iff N(t_0,t_1)$ non singu
ie from ZIR should be det x_0 uniquely
how to find x_0.

$y(t) = C(t)\Phi(t,t_0)x_0$



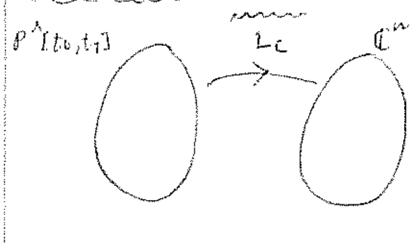
$$\int_{t_0}^{t_1} \Phi^*(t,t_0) C^*(t) y(t)\, dt = \int_{t_0}^{t_1} \Phi^*(t,t_0) C^*(t)\, C(t)\, \Phi(t,t_0)\, dt\; x_0$$

multiply by inverse $\Rightarrow$ $[N(t_0,t_1)]^{-1} \int_{t_0}^{t_1} \Phi^*(t,t_0)\, C^*(t) y(t)\, dt = x_0$

Duality between observability & controllability obvious

Compare c.o. w/ c.c.

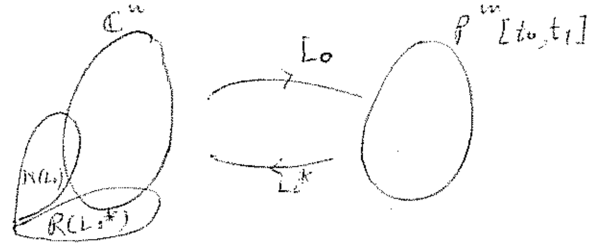
recall c.c.



ZS Response
c.c. $\iff R(L_c) = \mathbb{C}^n$

$L_c : u \longmapsto \int_{t_0}^{t_1} \Phi(t_1,\tau) B(\tau) u(\tau)\, d\tau$

c.o.



c.o. $\iff R(L_o^*) = \mathbb{C}^n$

$L_o^* : y \longmapsto \int_{t_0}^{t_1} \Phi^*(t,t_0)\, C^*(\tau)\, y(\tau)\, d\tau$

duality

c.c.	c.o.
L_c	L_o^*
u	y
$B(\cdot)$	$C^*(\cdot)$
$\Phi(t_1, \tau)$	$\Phi^*(\tau, t_0)$
$A(\cdot)$	$-A^*(\cdot)$ (?)

no further derivations reqd!

(3) Linear time invariant case $[A, B, C]$

$[A, B, C]$ c.o. on any interval of positive length $\Leftrightarrow$ rank $R = n$

← what about C(·) [illegible] cos it means any [illegible] [illegible] this means [illegible] [illegible] interval [illegible]

$$\because \quad Q = [B \mid AB \mid A^2B \mid \cdots \mid A^{n-1}B] \overset{\text{dictionary}}{\Longleftrightarrow} [\underbrace{C^*}_{n \text{ columns}} \mid A^*C^* \mid \cdots \mid (A^*)^{n-1}C^*]$$

$$R = \text{conj transpose of} \nearrow = \begin{array}{c} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \end{array} \begin{array}{l} m \\ m \\ \vdots \\ m \end{array} ;\ mn \text{ rows}$$

$$\Leftrightarrow \text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \quad ; \text{ look @ this intuitively.}$$

$R(Q) \neq \mathbb{C}^n$; some states are reachable some aren't.

9 Nov 1989

LST-lecture

Ch V Sec 1-5, 8-9
Ch VI Sec 1-4
Ch VII Sec 1-4
Suppl I-V, VIII, sec B, C, D
Collier & Desoer Ch I Sec 2 (SVD)

MT: mainly derivations (<5 Questions) 1 hour

3. Decomposition

(1) Seperation of controllable part

Given $[A, B, C]$ not c.c.

not c.c. $\iff$ rank $Q = \dim R(Q) = f < n$

(i) Range Space of $Q = R(Q)$ = set of reachable states
(ii) Range Space of $Q = R(Q)$ invariant under A
(iii) $R(B) \subseteq R(Q)$ Q: why $\subseteq$ and not $R(Q)$?

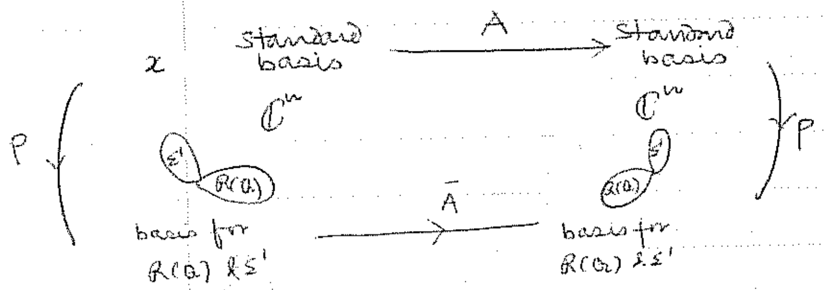
even if B has just one l.i. col, AB may have a diff. lin ind column. (all by only!)

$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $AB = \begin{bmatrix} 7 & 14 \\ 4 & 8 \end{bmatrix}$

L.i. col is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ l.i. col is $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$

l.i. to each other. so $\subseteq$ could be $\subsetneq$ in some cases

$A \in \mathbb{R}^{n \times n}$ $\quad \mathbb{C}^n = R(Q) \oplus \Sigma'$



$\exists P$ s.t. $\bar{x} = Px$

$\bar{x}$ basis for $R(Q)$ & Σ'

$[A, B, C] \leftarrow [\bar{A}, \bar{B}, \bar{C}]$

$\bar{A} = PAP^{-1}$ $\quad P\dot{x} = PAx + PBu$

$\bar{B} = ?\ PB$ $\quad y = Cx$

$\bar{C} = CP^{-1}$ $\quad \bar{x} = Px$

$\dot{\bar{x}} = PAP^{-1}\bar{x} + PBu$

$y = CP^{-1}\bar{x}$

Range space of Q invariant under A $\Rightarrow \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} = \bar{A}$

(Prop (ii))

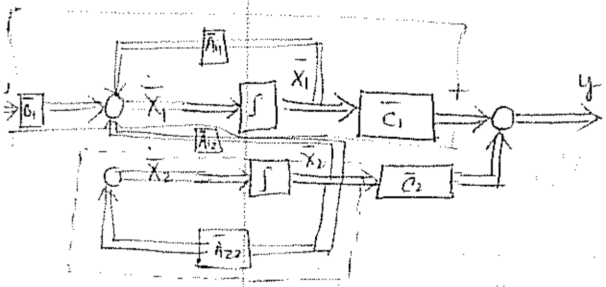
use the range of B coordinates of Q range space representation of PB in representation in terms of R(Q) about → for 1st part zeros

examine $\bar{B} = PB = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$

property (iii)

$\bar{C} = CP^{-1} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}$ (no info on it)

$$\begin{bmatrix} \dot{\bar{X}}_1 \\ \dot{\bar{X}}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} u$$



$[\bar{A}_{11}, \bar{B}_1, \bar{C}_1]$ $\qquad$ $\dot{\bar{x}}_1 = \underset{q\times q}{\bar{A}_{11}} \bar{x}_1 + \bar{B}_1 u$

$\qquad\qquad\qquad\qquad\qquad\qquad y = \bar{C}_1 \bar{x}_1$

$\bar{x}_2(0) = \theta \Rightarrow$ system reduces to above block only.

ZS Response of the top system = ZSR of the total system.
↳ drawing a picture is <u>not</u> a proof.

ZSR of $[A, B, C]$ = ZSR of $[\bar{A}_{11}, \bar{B}_1, \bar{C}_1]$

i.e. Need to show $\int_{t_0}^{t} H(t-\tau)\,v(\tau)\,d\tau = \int_{t_0}^{t} \bar{C}_1 e^{\bar{A}_{11}(t-\tau)} \bar{B}_1 u(\tau)\,d\tau$

$C(t)\, e^{A(t-\tau)} B(\tau)$
constant matrices

Equiv to examine Laplace transform

iff $C(sI-A)^{-1}B = \bar{C}_1 (sI-\bar{A}_{11})^{-1} \bar{B}_1$

Pf $\quad C(sI-A)^{-1}B = \bar{C}P(sI-A)^{-1}P^{-1}\bar{B}$

$\qquad = \bar{C}(PsIP^{-1} - PAP^{-1})^{-1}\bar{B}$

$\qquad = \bar{c}(sI-\bar{A})^{-1}\bar{B}$ $\quad$ t.f. unchanged when you change coordinate systems

$\qquad = [\bar{C}_1\ \bar{C}_2] \begin{bmatrix} sI-\bar{A}_{11} & \bar{A}_{12} \\ 0 & sI-\bar{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{bmatrix}$

inverse of Δgular matrix must be Δgular in same sense

$\therefore [\bar{C}_1 \vdots \bar{C}_2] \begin{bmatrix} (sI-\bar{A}_{11})^{-1} & X \\ 0 & X \end{bmatrix} \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$

$= \bar{C}_1 (sI-\bar{A}_{11})^{-1} \bar{B}_1$ → tf of reduced system

QED

ie t.f. doesnt change if you cut out that portion.

$$Q: \underset{\sim}{C}(s\underset{\sim}{I}-A)^{-1}\underset{\sim}{B} = \frac{1}{\det(sI-A)}\underset{\sim}{C}(\underset{\sim}{\ })\underset{\sim}{B}$$ ⇐ adjugate [as before]

What is the order of this polynomial $\det(sI-A)$? n

$$\bar{C}_1(sI-\bar{A}_{11})^{-1}\bar{B}_1 = \frac{1}{\det(sI-\bar{A}_{11})}\bar{C}_1[X]\bar{B}_1$$

order of this polynomial is q.

what happened to the missing order? Cancellation.

Two systems one with lower order have same t.f after pole zero cancellation.

$\begin{bmatrix}\bar{x}_1\\0\end{bmatrix}$ ← reachable states

$[\bar{A}_{11}, \bar{B}_1, \bar{C}_1]$ c.c. ?

$\dot{\bar{x}}_1 = \bar{A}_{11}\bar{x}_1 + \bar{B}_1 u$

$r = \bar{C}_1\bar{x}_1$

$\text{rank}\,\bar{Q}_1 = \underset{q}{\{}[\bar{B}_1 \mid \bar{A}_{11}\bar{B}_1 \mid \cdots \quad \bar{A}_{11}^{q-1}\bar{B}_1] \overset{?}{=} q$

Given $Q = [B \mid AB \mid \cdots \quad A^{n-1}B]$

$\bar{A} = PAP^{-1}$ $\quad \bar{B} = PB$ $\quad A$ is $n\times n$ $\quad B$ is $n\times m$

$n\times nm$

$PQ = [PB \mid PAB \mid \cdots \mid PA^{n-1}B] = [\bar{B} \mid \bar{A}\bar{B} \mid \cdots \bar{A}^{n-1}\bar{B}]$

$$= \left[\begin{array}{c|c|c|c} \bar{B}_1 & \bar{A}_{11}\bar{B}_1 & \cdots & \bar{A}_{11}^{n-1}\bar{B}_1 \\ 0 & 0 & & 0 \end{array}\right] \}q$$

rank $Q = q < n$ (initial)

$\therefore$ rank $PQ = q$ too ($\because$ P invertible, its rows l.i.)

$\therefore$ rank $\bar{Q}_1 = q$

$\therefore$ The top system is completely controllable.
The other part $\rightarrow$ don't know.

$$\begin{array}{ll} [\bar{A}_{11}, \bar{B}_1, \bar{C}_1] & ZSR \\ & \parallel \\ [A, B, C] & ZSR \end{array} \Rightarrow \text{"Zero State Equivalent}$$

ZS Equiv: H(s) same (if they have the same t.f. matrix)

$[A, B, C]$ state space representation
if $C(SI-A)^{-1}B = H(s)$
then this rep. is a realization of $H(s)$.

(2) Separation of Observable part:
Given $[A, B, C]$ not c.o.

$$R = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

not co $\Rightarrow$ rank $R = r < n$
dim $N(R) = n - r > 0$
Recall (i) $N(R)$ set of unobservable states
(ii) invariant under A
(iii) $N(R) \subset N(C)$

$\mathbb{C}^n = \Sigma'' \oplus \mathcal{N}(R)$ ordering important

in terms of basis for Σ'' & $\mathcal{N}(R)$ — the order is important

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & 0 \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \}\mathcal{N}(R) \qquad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix}$$

$$\hat{x} \in \mathcal{N}(R) \Rightarrow x \in \mathcal{N}(C)$$

$$\updownarrow$$

$$\begin{bmatrix} 0 \\ x_2 \end{bmatrix} \qquad [\hat{C}_1 \ \hat{C}_2]\begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \theta$$

$$\Rightarrow \hat{C}_2 x_2 = \theta$$

$$\Rightarrow \hat{C}_2 = 0$$

$$\text{i.e. } \hat{C} = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix}$$

Verify (i) $[\hat{A}_{11}, \hat{B}_1, \hat{C}_1]$ z.s. eq. to $[A, B, C]$
(straight forward)

(ii) $[\hat{A}_{11}, \hat{B}_1, \hat{C}_1]$ c.o.

combine the above to get c.o. + c.c. subsystem which has same t.f. as orignal system.

(3) Kalman Canonical Structure Theorem

Given $[A, B, C]$ not c.c., not c.o.

$$\mathbb{C}^n = \mathcal{R}(Q) \oplus \Sigma'$$
$$\mathbb{C}^n = \Sigma'' \oplus \mathcal{N}(R)$$
$$\Sigma_2 := \mathcal{R}(Q) \cap \mathcal{N}(R) \quad : \text{is a subspace! (in } \mathbb{C}^n)$$

$$\Sigma_1 \oplus \Sigma_2 \oplus \Sigma_3 \oplus \Sigma_4 = \mathbb{C}^n$$

Σ_1: cc, co; Σ_2: cc, (co)'; Σ_3: (cc)', (co); Σ_4: (co)', (cc)'; $\Sigma_1 \oplus \Sigma_2 = \mathcal{R}(Q)$; $\Sigma_2 \oplus \Sigma_4 = \mathcal{N}(R)$

Span(lin ind vectors in $\mathcal{R}(Q)$ but not in Σ_2.) $=: \Sigma_1$

$\Sigma_4 :=$ span(lin ind vectors in $\mathcal{N}(R)$ but not in Σ_2)

comparison of ... $\mathcal{N}(R)$ - ... states!

$\mathcal{E}_3 :=$ span(lin ind vectors not in $\mathcal{R}(Q)$ nor in $\mathcal{N}(R)$)

$$[A, B, C] \longleftarrow [\tilde{A}, \tilde{B}, \tilde{C}]$$

$$\tilde{A} = \left[\begin{array}{c|c|c|c} \tilde{A}_{11} & 0 & \tilde{A}_{13} & 0 \\ \hline \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ \hline 0 & 0 & \hat{A}_{33} & 0 \\ \hline 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} \end{array}\right] \qquad \tilde{B} = \left[\begin{array}{c} \tilde{B}_1 \\ \hline \tilde{B}_2 \\ \hline 0 \\ \hline 0 \end{array}\right]$$

$\therefore \mathcal{R}(Q)$ invariant under A

$$\tilde{C} = \left[\begin{array}{c|c|c|c} \tilde{C}_1 & 0 & \tilde{C}_3 & 0 \end{array}\right]$$

$\therefore$ (i) $[\tilde{A}_{11} \ \ \tilde{B}_{11} \ \ \tilde{C}_1]$ c.c. & c.o. (easy)

(ii) $[\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1]$ ZSR to $[A, B, C]$
i.e. $[\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1]$ is a realization of $H(s) = C(sI-A)^{-1}B$

(iii) there are many realizations
but this has the lowest dimension

Friday
Nov 10, 1989

LST-~~[illegible]~~ discussion

1 Pole Zero cancellation.
2 PBH test for c.c./c.o.
3 realization of LTI SISO systems.
4 SVD - alternate proof from Wu's notes.

$$\dot{x} = Ax + Bu$$

$\det(sI - A)$ = denominator of transfer fn

$$A = \begin{bmatrix} 1 & 3 \\ 7 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$$

$$g(s) = C(sI - A)^{-1}B = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} (s-5) & 3 \\ 7 & (s-1) \end{bmatrix} \underbrace{\frac{1}{(s+2)(s-8)}}_{\text{scalar}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$g(s) = \frac{(s+2)}{(s+2)(s-8)} = \frac{1}{s-8} \quad \therefore \text{ sys not cc/co} \quad \Leftarrow \text{let see which.}$$

cont. Matrix $Q = [B \mid AB] = \begin{bmatrix} 1 & 4 \\ 1 & 12 \end{bmatrix} \Rightarrow$ is c.c.

$$R = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 4 & 4 \end{bmatrix} \Rightarrow \underline{\text{not}} \text{ c.o}$$

rank $Q = n$ for controllability
when $Q = [B \,\vdots\, AB \,\vdots\, \cdots \, A^{n-1}B]$
PBH test: examine $[\lambda I - A \,\vdots\, B]$ (Popov-Belevitch-Hautus)
for rank

Thm. $\{A, B\}$ is c.c. $\Leftrightarrow$ rank $[\lambda I - A \,\vdots\, B] = n \quad \forall \lambda \in \mathbb{C}$.
if λ not e.v. of $A \Rightarrow \lambda I - A$ nonsingular.
$\Rightarrow$ you only need to check $\lambda \in \sigma(A)$

<u>Proof</u>: ($\Rightarrow$) supposing rank $[\lambda I - A \,|\, B] \neq n$

then $\exists\ \alpha^T$ s.t. $\alpha^T[\lambda I - A \,|\, B] = \theta$; α^T must be e-v else $\lambda I - A$ is non singular

$$\alpha^T B = \theta$$
$$\alpha^T \lambda I = \alpha^T A$$
$$\lambda \alpha^T A = \alpha^T A^2$$
$$\Rightarrow \alpha^T A^j = \lambda^j \alpha^T$$

but we know $\{A, B\}$ c.c. $\Leftrightarrow$ rank $Q = n$; $Q = [B \,|\, AB \,|\, \cdots \,|\, A^{n-1}B]$

$$\alpha^T Q = [\underbrace{\alpha^T B}_{\theta} \,|\, \underbrace{\alpha^T AB}_{\lambda\alpha^T B = \lambda\theta} \,|\, \cdots \,|\, \underbrace{\alpha^T A^{n-1} B}_{\lambda^{n-1}\alpha^T B = \theta}]$$

$\Rightarrow$ contradiction

Proved ($\Rightarrow$)

($\Leftarrow$) suppose $\{A, B\}$ not c.c.

$$\Rightarrow P \text{ s.t. } A \sim \bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} = PAP^{-1}$$

from HW $\rightarrow$ $B \sim \bar{B} = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} = PB$

let λ be an eigenvalue of $\bar{A}_{22}$ (so eigenvalue of $\bar{A}$)
(recall Kailath's formula)

β^T be a corresponding left eigenvector of $\bar{A}_{22}$

let $\alpha^T = [0 \,|\, \beta^T]$

from HW

$$\alpha^T[\lambda I - \bar{A} \,|\, \bar{B}] = [0 \,|\, \beta^T] \left[\begin{array}{cc|c} \lambda I - \bar{A}_{11} & -\bar{A}_{12} & \bar{B}_1 \\ 0 & \lambda I - \bar{A}_{22} & 0 \end{array}\right]$$

$$= [0 \quad \underbrace{\beta^T(\lambda I - \bar{A}_{22})}_{\theta} \quad \theta] \Rightarrow \text{not full rank}$$

$= \alpha^T [P(\lambda I - A)P^{-1} \vdots PB]$

$= \underbrace{\alpha^T P}_{\bar{\alpha}^T} [(\lambda I - A)P^{-1} \vdots B]$

P, P^{-1} non singular $\Rightarrow \bar{\alpha}^T \neq \theta$

$\bar{\alpha}^T (\lambda I - A) = \theta$

$\bar{\alpha}^T [\lambda I -$

<u>LTI system</u> (SISO)

$U(s) \longrightarrow \boxed{h(s)} \longrightarrow Y(s)$

$h(s) = (\beta_1 s^2 + \beta_2 s + \beta_3)/(s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3)$

let $z(s)$ be the response to the same i/p u of the new system ~~input u~~

$y(s)(s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3) = (\beta_1 s^2 + \beta_2 s + \beta_3) u$

$\hookrightarrow h_d(s) = \dfrac{1}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}$

$z(s) = h_d(s) U(s)$

$z(s)(s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3) = U(s)$

$\dddot{z} + \alpha_1 \ddot{z} + \alpha_2 \dot{z} + \alpha_3 z = u$

By superposition

$h(s) = \beta_1 s^2 h_d(s) + \beta_2 s\, h_d(s) + \beta_3 h_d(s)$

$y(s) = \beta_1 s^2 h_d(s) U(s) + \beta_2 s h_d(s) U(s) + \beta_3 h_d(s)$

$y(s) = \beta_1 s^2 z(s) + \beta_2 s z(s) + \beta_3 z(s)$

$\Rightarrow X(s) \Leftrightarrow X(t)$

iff $x'(t) = 0 \quad x(t) = 0$

$y(t) = \beta_1 \ddot{z} + \beta_2 \dot{z} + \beta_3 z \qquad z.i.c.$

Now let $x_1 = z$.

$x_2 = \dot{z} = \dot{x}_1$

$x_3 = \ddot{z} = \dot{x}_2$

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = -\alpha_1 x_3 - \alpha_2 x_2 - \alpha_3 x_1 + u$

$\dddot{z} + \alpha_1 \ddot{z} + \alpha_2 \dot{z} + \alpha_3 z = u$

$\dot{x}_3 + \alpha_1 x_3 + \alpha_2 x_2 + \alpha_3 x_1 = u$

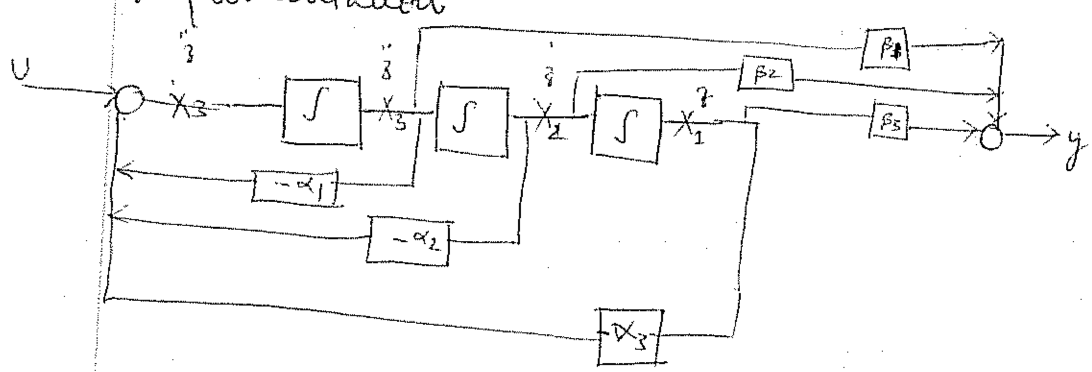
$y = \beta_1 x_3 + \beta_2 x_2 + \beta_1 x_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$y = [\beta_3 \;\; \beta_2 \;\; \beta_1] x \qquad$ c.c. form

controllable canonical form

Implementation



How do we know system is c.c.?

$$Q = [B \mid AB \mid A^2 B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -\alpha_1 \\ 1 & -\alpha_1 & -\alpha_2 + \alpha_1^2 \end{bmatrix} \leftarrow \text{non-singular}$$

observable canonical form:

$$\dddot{y} + \alpha_1 \ddot{y} + \alpha_2 \dot{y} + \alpha_3 y = \beta_1 \ddot{u} + \beta_2 \dot{u} + \beta_3 u$$

$$\dddot{y} + (\alpha_1 \ddot{y} - \beta_1 \ddot{u}) + (\alpha_2 \dot{y} - \beta_2 \dot{u}) + (\alpha_3 y - \beta_3 u) = 0$$

$\dddot{x}_3$

$\ddot{x}_2$

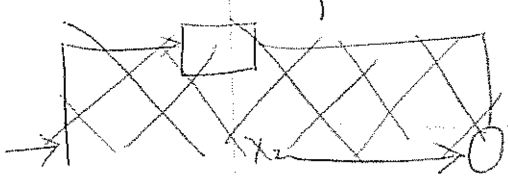
$\dot{x}_1$

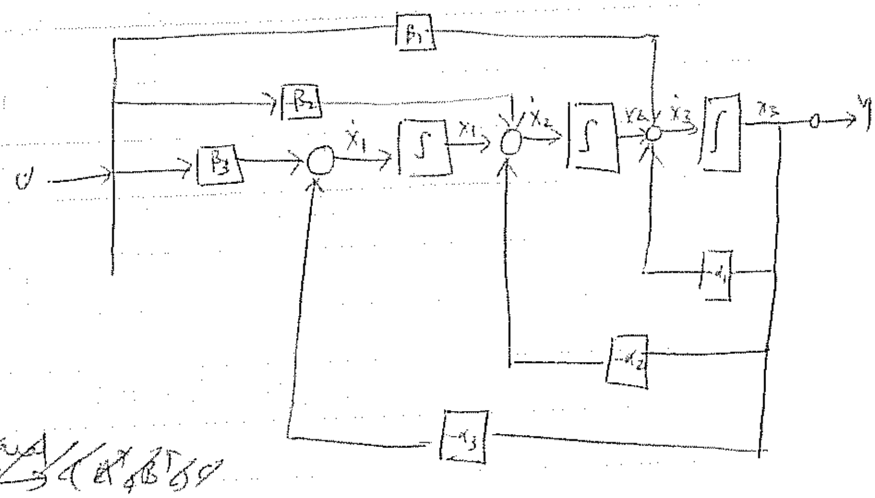
$$\dddot{x}_3 = \dddot{y} \rightarrow x_3 = y$$
$$\ddot{x}_2 = \ddot{y} + (\alpha_1 \ddot{y} - \beta_1 \ddot{u})$$
$$x_2 = \dot{y} + (\alpha_1 y + \beta_1 u)$$
$$x_2 = \dot{x}_3 + (\alpha_1 x_3 + \beta_1 u)$$
$$\dot{x}_3 = x_2 - \alpha_1 x_3 - \beta_1 u$$

$$x_1 = \dot{x}_2 + \alpha_2 x_3 - \beta_2 u$$
$$0 = \dot{x}_1 + \alpha_3 x_3 - \beta_3 u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\alpha_3 \\ 1 & 0 & -\alpha_2 \\ 0 & 1 & -\alpha_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_1 \end{bmatrix} u$$ obs. can. form

$$y = [0 \quad 0 \quad 1]x$$





duals ~~(A, B, C) $\xleftrightarrow{\text{dual}}$ (A^T, B^T, C^T)~~

~~(A, B)~~ $(A, b, c) \xleftrightarrow{\text{dual}} (A^T, c^T, b^T)$

reverse drns on block diagrams

Jordan Form Realization

$$h(s) = \frac{(\beta_1 s^2 + \dots)}{(s^3 + \alpha_1 s^2 + \dots)} = \frac{r_1}{(s-\lambda_1)} + \frac{r_2}{(s-\lambda_2)} + \frac{r_3}{(s-\lambda_2)^2} \text{ ; pf exp}$$

change coordinates to generalized eigvectors

$x = Tz \qquad \dot{x} = Ax + Bu$

$\qquad\qquad\quad Y = CX$

$$\dot{z} = T^{-1}AT\,z + \underbrace{T^{-1}B}_{h}u$$

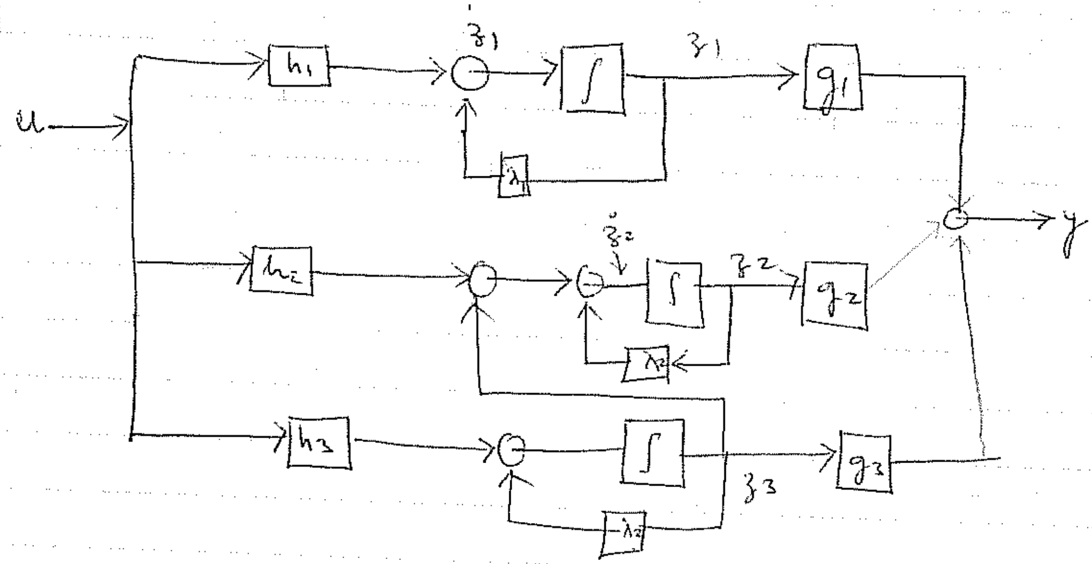
$$Y = \underbrace{CT}_{g}z$$

$T^{-1}B \triangleq h$

$CT \triangleq g$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} u$$

$$y = [g_1 \; g_2 \; g_3]\,z$$



(cf book)

c.c. $\Leftrightarrow h_1, h_2, h_3 \neq 0$

also c.o. $\Leftrightarrow g_1, g_2, g_3 \neq 0$

Proof from Dr Wu's notes of SVD (p. 16 of Class Notes)

$A \in \mathbb{C}^{m \times n}$ rank r

$B \in \mathbb{C}^{m \times n}$ rank $r-1$

$z \in N(B) \cap R(A^*)$

(to show $\min\limits_{\text{rank } B = r-1} \|A - B\|_2 = \sigma_r$)

we know $\mathbb{C}^n = \underset{(n-r+1)}{N(A)} \overset{\perp}{\oplus} \underset{(r)}{R(A^*)}$ $\Rightarrow$ at least one dim can pick z's out of.

$Bz = \theta$ $\quad A = \sum\limits_{i=1}^{r} \sigma_i v_i u_i^*$ $\quad Az = \sum\limits_{i=1}^{r} \sigma_i v_i \underbrace{u_i^* z}_{\text{scalar}}$ scalar vector

$\|Az\|^2 = \sum\limits_{i=1}^{r} \overline{\sigma_i u_i^* z}\, v_i^* \cdot \sum\limits_{j=1}^{r} \sigma_j (u_j^* z) v_j$

v's are orthonormal $\Rightarrow \sum \sum \sigma_i (u_i^* z)\, \sigma_j (u_j^* z)\, v_i^* v_j$

$= \sum\limits_{i=1}^{r} \sigma_i^2 (u_i^* z)^2$

$\|u_i^* z\|^2 \leq \underset{=1}{\|u_i^*\|_2^2} \cdot \underset{=1}{\|z\|_2^2}$ Schwartz inequality

$\therefore \|u_i^* z\| \leq 1$

$\sum\limits_{i=1}^{r} \sigma_i^2 (u_i^* z)^2 \geq \sum\limits_{i=1}^{r} \sigma_r^2 (u_i^* z)^2 = \sigma_r^2 \underbrace{\sum\limits_{i=1}^{r} (\cdot)^2}_{=1 \text{ (Norm)}} \geq \sigma_r^2$

ov '89

LST-lecture

Minimality not in course for MT2.
Exam at 3:00pm

(3) Kalman Decomposition
Given $[A,B,C]$ not c.c. not c.o.
~~xxx~~ $\mathcal{R}(Q)$ charac. controllability
$\mathcal{N}(R)$ not obs states

$\Sigma_2 := \mathcal{R}(Q) \cap \mathcal{N}(R)$ $\quad \mathcal{N}(R)$ c.c.

$$\mathbb{C}^n = \Sigma_1 \oplus \Sigma_2 \oplus \Sigma_3 \oplus \Sigma_4$$

Σ_1: c.c., c.o. $\quad$ Σ_2: c.c., ~~c.o.~~ $\quad$ ($\Sigma_1 \oplus \Sigma_2 = \mathcal{R}(Q)$) $\quad$ Σ_3: ~~c.c.~~, c.o. $\quad$ Σ_4: ~~c.c.~~, ~~c.o.~~

Summary

$$[A,B,C] \longleftarrow [\tilde{A},\tilde{B},\tilde{C}]$$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & 0 & & 0 \\ & & & \\ 0 & 0 & & 0 \\ 0 & 0 & & \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ 0 \\ 0 \end{bmatrix} \qquad \tilde{C} = \begin{bmatrix} \tilde{C}_1 & 0 & \tilde{C}_3 & 0 \end{bmatrix}$$

① $[\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1]$ c.c. & c.o.

② can show $[\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1]$ is equivalent to $[A,B,C]$

③ $[\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1]$ realization $H(s) = C(sI-A)^{-1}B$

↳ this realization has minimal dimension.

Fact: $[A,B,C]$ c.c. & c.o.
$\Leftrightarrow$ $[A,B,C]$ is a minimal realizat

Proof ($\Leftarrow$) Suppose not c.c. or not c.o. want to
∃ minimal realization.

MINIMAL REALIZATION MEANS LOWEST DIMENSION

is this proof?

not c.c. $\Rightarrow$ seperate controllable part
$\Rightarrow$ not minimal realization
not c.o. $\Rightarrow$ seperate observable part
$\Rightarrow$ not minimization realization

Pf ($\Rightarrow$) suppose not min Realization
not min Realization To show $\Rightarrow$ not c.o.
$\Rightarrow \exists\ [\bar{A}, \bar{B}, \bar{C}]$ z.s. eq to $[A, B, C]$ $\quad \bar{A}$ is $\bar{n} \times \bar{n}$

~~dim~~ $\bar{n}$ < ~~dim~~ n

z.s. equivalent $\Rightarrow C e^{At} B = \bar{C} e^{\bar{A}t} \bar{B}$

$C(I + At + A^2 t^2/2! + \cdots) B = \bar{C}(I + \bar{A}t + \cdots) \bar{B}$

$CB = \bar{C}\bar{B}$; take deriv $\Rightarrow CAB = \bar{C}\bar{A}\bar{B}$ in gen. $CA^kB = \bar{C}\bar{A}^k\bar{B}$

c.c. $\quad$ c.o.

$Q = [B \ AB \cdots A^{n-1}B]$ $\quad R = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$

$$RQ = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} CB & CAB & \cdots \\ CAB & CA^2B & \cdots \\ \\ CA^{n-1}B & \cdots & CA^{2n-2}B \end{bmatrix}$$

$= \bar{R}\bar{Q}$

Rank RQ = Rank $\bar{R}\bar{Q}$

ion.

Suppose $[A, B, C]$ is c.c. $\wedge$ c.o.
want to reach a contradiction
rank $R = n$ rank $Q = n$
rank $\bar{R} \leq \bar{n}$ / rank $\bar{Q} \leq \bar{n}$

sh

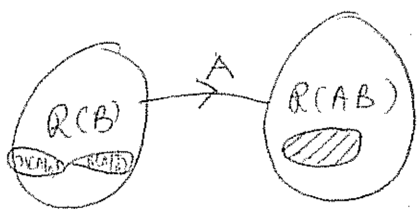
General: $\text{rank}(AB) \le \min(\text{rank } A, \text{rank } B)$

columns of $(AB) = A[\,||||\,] =$ l.c. of columns of A

rows of $(AB) = [\,\equiv\,]B =$ l.c. of rows of B

$\text{rank}(AB) =$ # of l.i. rows/colums
$\le \text{rank } B$
$\le \text{rank } A$
$\le \min(\text{rank } A, \text{rank } B)$

$\text{rank } R = n$
$\text{rank } Q = n$ $\Rightarrow \text{rank } RQ \overset{?}{=} n$



~~$R(A)$~~ $A: R(B) \longrightarrow R(AB)$

$A|_{R(B)}$

$R(A|_{R(B)}) = R(AB)$ obvious

we want onto!!

$\dim R(B) = \underbrace{\dim R[(A|_{R(B)})^*]}_{= \dim R(AB)} + \dim N(A|_{R(B)})$

$\hookrightarrow \le \dim N(A)$

$\dim R(A) + \dim N(A) = n = \dim[\text{domain}(A)]$

$\text{rank}(B) \le \text{rank}(AB) + \dim N(A)$

Sylvester's inequality: $\text{rank } B - \dim N(A) \le \text{rank}(AB) \le \min(\text{rank } A, \text{rank } B)$

RETURN

$\text{rank } \bar{R}\bar{Q} \le \min(\text{rank } \bar{R}, \text{rank } \bar{Q}) \le \bar{n} < n$

$\text{rank } RQ \ge \underset{= n}{\text{rank } Q} - \underset{= 0}{\dim N(R)} = n$

$n \le \text{rank}\, RQ = \text{rank}\, \bar{R}\bar{Q} \le \bar{n} < n$ $\quad$ * contradiction!

$[A, B, C]$ c.c. & c.o.

$\Leftrightarrow$ $[A, B, C]$ is minimal realization of $H(s) = C(sI-A)^{-1}B$

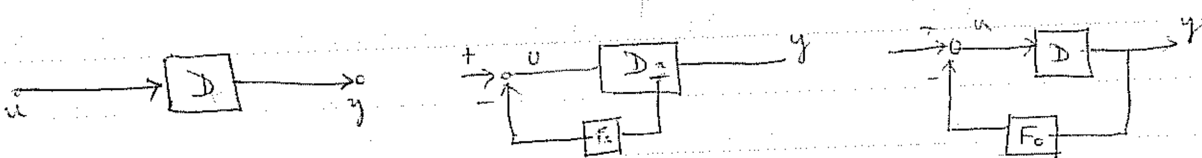
SISO case: $h(s) = c(sI-A)^{-1}b$

$[A, b, c]$ c.c. c.o. $\Rightarrow n^{th}$ order (no pole-zero cancel)

4. Effect of memoryless Feedback on c. & o.

(1) on controllability

$\mathcal{D} = \{ U, \Sigma, Y, s, r \}$



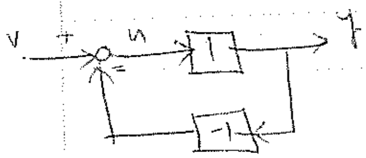
w/o FB $\qquad$ w/ state FB $\qquad$ w/ output FB.

must be instantaneous

should this also be instantaneous

$u(t) = v(t) - F_s(x(t))$

could be very general



$$\begin{cases} y(t) = u(t) \\ u(t) = v(t) + y(t) \end{cases}$$ make sense?

true only if $v(t) \equiv 0$ $\therefore$ $\mathcal{D}_{F_o}$ is not well defined

Assume $\mathcal{D}_{F_s}, \mathcal{D}_{F_o}$ well defined dynamical system

iff $\forall (x_0, t_0)$ $\forall$ $v[t_0, t_1]$

$\exists$ unique $u_{[t_0, t_1]}$ satisfying the system rules

open/closed? intervals?

Fact: If $\mathcal{D}_{F_s}$ & $\mathcal{D}_{F_o}$ well defined

$\mathcal{D}$ is c.c. on $[t_0, t_1]$

$\Leftrightarrow$ (a) $\mathcal{D}_{F_s}$ is c.c. on $[t_0, t_1]$

$\not\Leftarrow\Rightarrow$ (b) $\mathcal{D}_{F_o}$ is c.c. on $[t_0, t_1]$

Pf (a) ($\Rightarrow$) $\forall x_0, x_1 \; \exists \tilde{u}_{[t_0,t_1]}$ drives (x_0,t_0) to (x_1,t_1)

For D_{F_S} want to show $\forall x_0^{t_0}, x_1^{t_1} \exists \bar{v}_{[t_0,t_1]}$ drives (x_0,t_0) to (x_1,t_1)

try $\bar{v}(t) = \tilde{u}(t) + F_S[s(t,t_0,x_0,\tilde{u}_{[t_0,t_1]}]$

By assumption $\exists$ a unique u satisfying the reln.

$$\overset{\parallel}{\tilde{u}} \quad \text{(subtle pt.)}$$

So $\bar{V}[t_0,t_1]$ drives (x_0,t_0) to (x_1,t_1)

($\Leftarrow$) $\forall x_0, x_1 \; \exists v_{[t_0,t_1)}$ drives $[x_0,t_0]$ to $[x_1,t_1]$

Ass. $\Rightarrow \exists$ a unique u s.t. $u(t) = v(t) - F_S[s(t,t_0,x_0,u)]$

$u[t_0,t_1)$ drives x_0,t_0 to (x_1,t_1)

for (b) identical proof

(2) on observability

Fact: If D_{F_O} is well-defined then D is c.o. on $[t_0,t_1]$ $\Leftrightarrow D_{F_O}$ c.o. $[t_0,t_1]$

But no nec. c.o. (w.r.t. D_{F_S})! See Phils Asgn

16 Nov 1989

LST-Lecture Discussion

1. Prior to MT1
2. Calculate SVD

$$A = \begin{bmatrix} -2.896 & -2.292 \\ 1.369 & .738 \end{bmatrix}$$

$$A^*A = \begin{bmatrix} 10.26 & 7.65 \\ 7.65 & 5.8 \end{bmatrix}$$; obtain eigenvalues

eigenvalues of A^*A are σ^2 ; $(0.0625, 16)$

$$\Sigma = \begin{bmatrix} 1/4 & 0 \\ 0 & 4 \end{bmatrix}$$ Rank = 2 ; but nearly singular

$N(A^*A) = \{\theta\}$ $\qquad \mathcal{U} = [U_1 \mid U_2]$ ↳ empty

$U = \begin{bmatrix} .6 & +.8 \\ -.8 & .6 \end{bmatrix}$ eigenvector of A^*A

$$V = A\mathcal{U}\Sigma^{-1} = \begin{bmatrix} .385 & .923 \\ .923 & .385 \end{bmatrix}$$

$A = V\Sigma U^*$ works fine : good to do this to understand.

Gram Schmidt orthonormalization.

(H) redef
Hermitians : If M is Hermitian
(1) all real eigenvalues
(2) Jordan form is diagonal matrix
(3) eigenvectors are orthogonal + diff eigenvalues
(4) $\exists P$ s.t. $P^*P = I$ & PMP^* is a diagonal matrix.

Zeroes of a transfer fn : Require an initial state and an input.

given $h(s) = \cancel{s+1}\,(s-2)/(s+1)(s+3)$
and some realization $\{A, \cancel{B}^{b}, C\}$
find an i/p and an initial state s.t.
the output $\equiv 0$,

input $= \mathcal{L}^{-1}(1/s-2) = 1(t)\,e^{2t}$ (ind of realization)

initial state depends on realization,

$A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $c = [-2\ 1]$

$h(s) = c(sI-A)^{-1}b$

total response $=$ ZIR + ZSR $\neq$
$ZSR = h(s)\ u(s) = (s-2)/(s+1)(s+3) \cdot 1/s-2 = 1/(s-1)(s+3)$

total response $= 0 \Rightarrow ZIR = -ZSR$

$ZIR = -1/(s+1)(s+3)$; controllable Canonical Form

Know what the controllable, Observable, Jordan Canonical Forms look like. (cf discussion)

$$\begin{bmatrix} \lambda_1 & 1 & & & & & & \\ & \lambda_1 & 1 & & & & 0 & \\ & & \lambda_1 & 0 & & & & \\ & & & \lambda_2 & 1 & & & \\ & & & & \lambda_2 & 0 & & \\ & 0 & & & & \lambda_2 & 0 & \\ & & & & & & \lambda_2 & 0 \\ & & & & & & & \lambda_3 \end{bmatrix}$$

$\longleftarrow m_1 \longrightarrow$ $\quad$ $\longleftarrow m_2 \longrightarrow$

ch. poly: $(s-\lambda_1)^3 (s-\lambda_2)^4 (s-\lambda_3) = \Delta(s)$

min poly: $(s-\lambda_1)^3 (s-\lambda_2)^2 (s-\lambda_3) = \psi(s)$

look at power series exp of e^{At} for Jordan Block

least squares

Solve $Ax = b$

(i) A^{-1} exists $\quad x = A^{-1}b$

$b \notin R(A)$ then decompose $b = \underset{\in R(A)}{b_1} + \underset{\in N(A^*)}{b_2}$

non unique soln: decompose x into $x = \underset{\in N(A)}{x_1} + \underset{\in R(A^*)}{x_2}$

$$\begin{aligned} Ax &= A(x_1 + x_2) \\ &= Ax_r = b \\ A^* A x_r &= A^* b \\ x_r &= (A^*A)^{-1} A^* b \end{aligned}$$

$Ax = b_r \qquad A^* b_u = \theta$

$\cancel{A^* A b_r = \theta}$

$A^* b_r = x_r$

BIBO: depends not on state only I/P & O/P

~~...~~ -BIBO $\Leftrightarrow \exists M$ s.t. $\int_{-\infty}^{t} \|H(t,\tau)\| d\tau < M \quad \forall t \in \mathbb{R}_+$

LTI $\quad H(t,\tau) = H(t-\tau)$

Discrete time Systems $W(k_0,k_1) \triangleq [\,B_{k_1-1} \mid \phi(k_1,k_1-1)B_{k_1-2} \mid \cdots \mid \phi(\cdot\,)B_{k_0}\,]$

comp reachable from the origin iff ~~$R(\Phi) \subseteq R(W)$~~ Rank $(W) = n$

c.c. to the origin $\Leftrightarrow R(\Phi) \subseteq R(W)$

for LTI replace W by Q

time varying

iSL $\forall t \geq 0 \quad \epsilon > 0$

$A_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad B_k = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(k \neq 0)}$

$\exists\, \delta(t_0,\epsilon)$ s.t. $\|x_0\| < \delta(t_0,\epsilon)$

$\|x(t)\| < \epsilon \quad \forall t \geq t_0$

Is this c.c. to 0 in 1 step?

no!

$W(0,1) = B_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ rank 1

$W(0,2) = [B_1 \mid AB_0] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ rank $= 2$

Say $X(s) = 1/(s+1)^2 \, X_0$, given ϵ find appropriate δ.

$x(t) = te^{-t} x_0$

max at $t = 1$, max $= 1$

$x(t) \leq \|x_0\| \qquad \delta = \frac{1}{\epsilon}$ wrong

LTI: iSL $\Leftrightarrow$ poles in LHP

or ~~for~~ simple zeros of minimal poly

on $j\omega$ axis

aS stab.: LTI $\Rightarrow$ strictly all poles in LHP

BIBS stable: Reader all poles in LHP, simple ones on $j\omega$ axis, the $j\omega$ poles not controllable from the input.

know tests for c.c.

LTI - Q full rank

M full rank

~~xxxx~~ $\Rightarrow$ $[\lambda I - A \mid B]$ full rank.

rows of $X^{-1}(\tau)\,B(\tau)$ indeps

τ fund matrix.

How do you find c.c. part?

Change of basis to $\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$

this part not controllable

$R \xrightarrow{T} \tilde{R}$

$[\,Q \mid I\,]$ perform elementary row ops on this to

row ops

get $\left[\begin{array}{c} Q_s \\ 0 \end{array} \middle| \; T \right]$ Then $\tilde{A} = TAT^{-1} \quad \tilde{B} = TB$ etc.

Day
12?, 1989

LST - Lecture

New Chapter: XII Eigenvalue Assignment

1. Motivation: LTI system

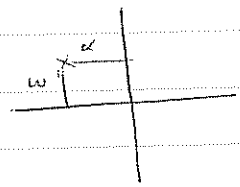
$$\dot{x} = Ax + Bu$$

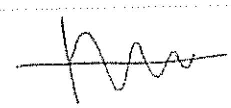
$$y = Cx$$

w/o solving we know soln charac by STM ie

$$\Phi(t,0) = e^{At} = \sum_{k=1}^{\sigma} \sum_{l=0}^{m_k - 1} t^l e^{\lambda_k t} P_{kl}$$

location of the eigenvalues (eigenvalues of A that is)

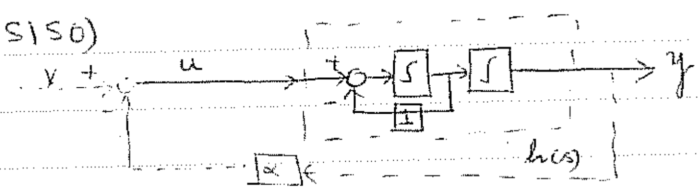


time response will have one component  exp(αt) envelope
time = ω

If time response not good, change location of eigenvalues.

- Can this be done
- How do we do it.

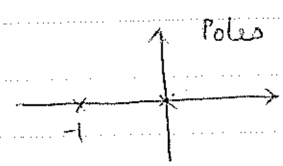
Ex. (SISO)



$\ddot{y} = u - \dot{y}$

$\Rightarrow \ddot{y} + \dot{y} = u$

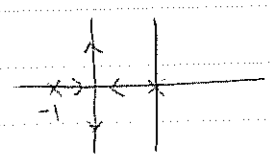
$h(s) = 1/s(s+1)$



$u = v - \alpha y$

$\ddot{y} + \dot{y} = v - \alpha y$

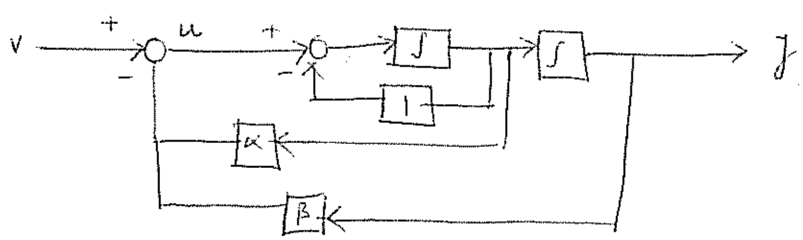
$h'(s) = 1/(s^2 + s + \alpha)$



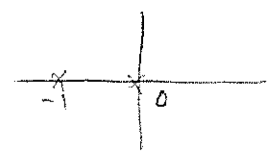
poles of $h'(s)$ changes as α changes
moves along "root locus"

one parameter - one degree of freedom for poles to move.

what happens if we introduce 2 parameters?



$$\ddot{y} + \dot{y} = u = v - \alpha \dot{y} - \beta y$$



$$\therefore G(s) = 1/s^2 + (1+\alpha)s + \beta$$

poles can move anywhere we want.

Q: Can state feedback move the eigenvalues anywhere we want?
(equiv. control the modes)

Will show
① If the system representation for single i/p single o/p is in special form (controllable canonical form) then can be done
② Given arbitrary $[A, b, c]$ transform to cont. canon. form & then see.
→ Q: Can this be transformed into controllable canonical form [$\overset{\text{Turns out}}{\Leftrightarrow}$ $[A, b, c]$ c.c.]
③ Combine ① & ② state f.b. control
④ Generalize to MIMO

Next Chap: If internal states unavailable, the observer design

2. SISO

(1) State F.B. in controllable form.

$$\dot{\xi} = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_1 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$: Controllable Canonical Form

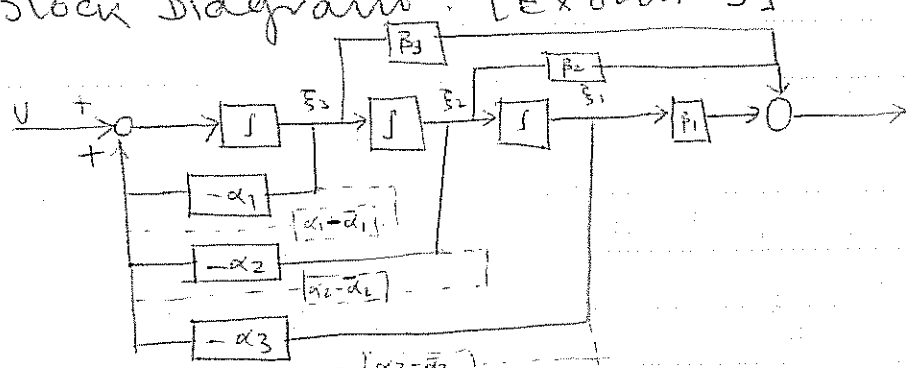
A_c $\qquad$ b_c

eigenvalues : $\lambda_1 \cdots \lambda_n$ (multiple e.v. are allowed)
$\hookrightarrow$ req. complex conj. (for real α_i's)

charac. polynomial $d(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$
$= s^n + \alpha_1 s^{n-1} + \cdots \alpha_n$ [which in cnt can form

Desired eigenvalues are $\tilde{\lambda}_1 \cdots \tilde{\lambda}_n$ [anyplace but conj
$\bar{d}(s) = s^n + \bar{\alpha}_1 s^{n-1} + \cdots \bar{\alpha}_{n-1} s + \bar{\alpha}_n$

Block Diagram : [Ex dim = 3]



changing is trivial [to $\bar{\alpha}_1, \bar{\alpha}_2, \cdots$

Algebraically:
Want $\bar{A}_c = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -\bar{\alpha}_n & -\bar{\alpha}_{n-1} & \cdots & -\bar{\alpha}_1 \end{bmatrix}$ $\qquad$ $\bar{A}_c - A_c = \begin{bmatrix} 0 & 0 & 0 \\ & 0 & \\ 0 & 0 & 0 \\ \alpha_n - \bar{\alpha}_n & \alpha_{n-1} - \bar{\alpha}_{n-1} & \alpha_1 - \bar{\alpha}_1 \end{bmatrix}$

i.e. $\bar{A}_c - A_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [\alpha_n - \bar{\alpha}_n \ \cdots \ \alpha_1 - \bar{\alpha}_1] = b_c k_c$

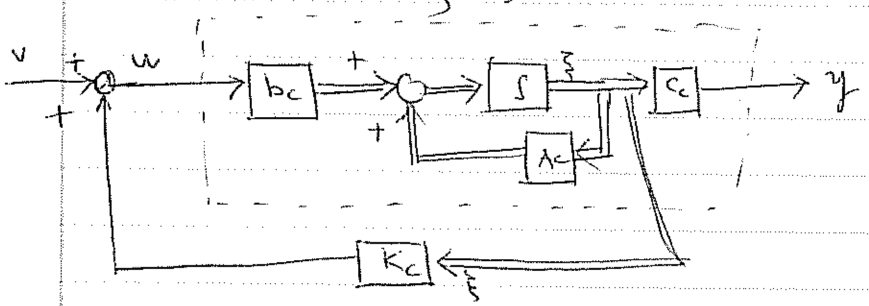
$\therefore \bar{A}_c = A_c + b_c k_c$

$\dot{\xi} = A_c \xi + b_c u$

want $\dot{\xi} = (A_c + b_c K_c)\xi + b_c v$

(state feedback)

Let $u = v + K_c \xi$; this works!



(2) Controllable Form

Given $[A, b, c]$

$$\begin{cases} \dot{x} = Ax + bu \\ y = cx \end{cases}$$

$$d(s) = s^n + \alpha_1 s^{n-1} + \cdots \alpha_n$$

Note: basis $b\ Ab \cdots A^{n-1}b$ $\Rightarrow$ [illegible] $= A_c^T$

$A[T] = T A_c^T$

$T^{-1}A = A_c T^T$

So basis [illegible] transformation [illegible]

$A \xrightarrow{T_1} A_c^T$

$A_c \xleftarrow{T_2}$

- Dont know how to change e-values.

But if in Cont Canon Form: trivial. $\dot{\xi} = A_c \xi + b_c u$, $y = c_c \xi$

where $A_c = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -\alpha_n & \cdots & & -\alpha_1 \end{bmatrix}$ $b_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Is there a coordinate transformation expressing this?

$$x = T\xi$$

$$\begin{tikzcd} x \ \text{std basis} \arrow[r, "A"] & \text{std basis} \\ \xi \ \{t_1, t_2, \ldots t_n\} \arrow[u, "T"] \arrow[r, "A_c"] & \{t_1, t_2, \ldots t_n\} \arrow[u, "T"'] \end{tikzcd}$$

$A = T A_c T^{-1}$

$b = T b_c$

Let try and find T.

$T = [t_1 \ t_2 \ \cdots \ t_n]$

$b = T b_c$

$$AT = TA_c$$

$\therefore \quad \underline{b} = T \underline{b}_c$

$\Rightarrow \underline{t}_n = \underline{b}_c$

$AT = TA_c$

$$[t_1 | t_2 \cdots | t_n] \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -\alpha_n & \cdots & \cdots & -\alpha_1 \end{bmatrix} = \boxed{A} \; \boxed{t_1 \, t_2 \cdots t_n}$$

$t_{n-1} - \alpha_1 t_n = A t_n$; from last column.

$\Rightarrow t_{n-1} = \alpha_1 t_n + A t_n$

$t_{n-2} - \alpha_2 t_n = A t_{n-1}$; from 2^{nd} last column

$\Rightarrow t_{n-2} = \alpha_2 t_n + A t_{n-1}$

Recursion allows us to obtain all upto

$t_1 = \alpha_{n-1} t_n + A t_2$; from 2^{nd} column

Provides Recursive Formula for computing columns of T.

First col : $0 = \alpha_n t_n + A t_1$: an identity (ie redundant)

To qualify as a basis $\{t_1 \ldots t_n\}$ must be l.i.

$\therefore$ check T for singularity!

$t_{n-1} = \alpha_1 b + Ab$

$t_{n-2} = \alpha_2 b + A(\alpha_1 b + Ab) = \alpha_2 b + \alpha_1 Ab + A^2 b$

$\vdots \qquad \vdots$

nonsingular iff Q nonsingular

$$\underbrace{[t_1 \, t_2 \cdots t_n]}_{T} = \underbrace{[b \;\; Ab \; \cdots \; A^{n-1} b]}_{Q} \begin{bmatrix} \alpha_{n-1} & \cdots & \alpha_2 & \alpha_1 & 1 \\ \vdots & & \alpha_1 & 1 & 0 \\ \vdots & & 1 & 0 & \vdots \\ \vdots & & 0 & \vdots & 0 \\ 1 & & \vdots & 0 & \end{bmatrix}$$

T nonsingular $\Leftrightarrow$ Q non singular

If $[A, b, c]$ c.c.

$\Rightarrow \exists$ T nonsingular s.t. $x = T\xi$

transforms $[A, b, c] \leftrightarrow [A_c, b_c, c_c]$

[Ex: show $\Leftarrow$ true]

(3) State feedback $\bar{c}$/arbitrary $[A, b, c]$

Given $[A, b, c]$

use state feedback to move eigenvalues anywhere we want.

$\Leftrightarrow [A, b, c]$ c.c.

Proof: ($\Leftarrow$) $[A, b, c]$ c.c. $\Rightarrow \exists$ T non singular

$X = T\xi$

$[A, b, c] \leftarrow [A_c, b_c, c_c]$

$A_c = T^{-1}AT$ same eigenvalues

$\Rightarrow$ S.F. moves eigenvalues of A_c. (hence A)

Proof ($\Rightarrow$) suppose not c.c.

seperation of controllable part

$$\begin{bmatrix} \dot{\bar{X}}_1 \\ \dot{\bar{X}}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} u$$

e-values of A = (eigenvalues of $\bar{A}_{11}$) $\cup$ (e.v. of $\bar{A}_{22}$)

state feedback $u = v + K\bar{X}$

$$= v + [K_1 \ K_2] \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\bar{X}}_1 \\ \dot{\bar{X}}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} + \bar{B}_1 K_1 & \bar{A}_{12} + \bar{B}_1 K_2 \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} v$$

unchanged!

eigenvalues = (eigenvalues of $\bar{A}_{11} + \bar{B}_1 K_1$) $\cup$ (e-values of Λ_{22})

i.e. there are portions of the eigenvalues of A which will not change!

Procedure [Explicit]

$$\dot{x} = Ax + bu$$
$$y = Cx \qquad \text{c.c.}$$

$$d(s) = s^n + \alpha_1 s^{n-1} + \dots \alpha_n$$

want $\bar{d}(s) = s^n + \bar{\alpha}_1 s^{n-1} + \dots \bar{\alpha}_n$

State Feedback $\quad u = v + kx$

$$\begin{aligned} \dot{x} &= Ax + bu \\ &= Ax + b(v + kx) \\ &= (A + bk)x + bv \end{aligned}$$

$$x = T\xi$$

$$\begin{aligned} \dot{\xi} &= T^{-1}(A + bk)T\xi + (T^{-1}b)v \\ &\cancel{= \bar{A}_c} \\ &= \bar{A}_c \xi + b_c v \\ &= (A_c + b_c k_c)\xi + b_c v \end{aligned}$$

$$\Rightarrow \quad k_c = kT$$

$$k_c = [\alpha_n - \bar{\alpha}_n, \alpha_{n-1} - \bar{\alpha}_{n-1}, \dots \alpha_1 - \bar{\alpha}_1]$$

T obtained from recursive formula.

$$[T^T][k^T] = [k_c^T]$$

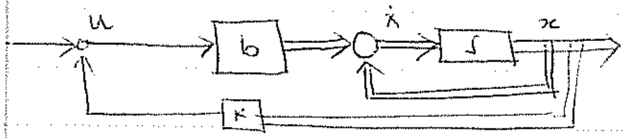
$$\square \; [\,] = [\,] \quad ; \text{ can solve }$$

? { To obtain K! from which we get directly ... feedback!

Tuesday
28 Nov 1989

LST-lecture

Single input



$\dot{x} = Ax + bu$; original

$[A, b]$ c.c.

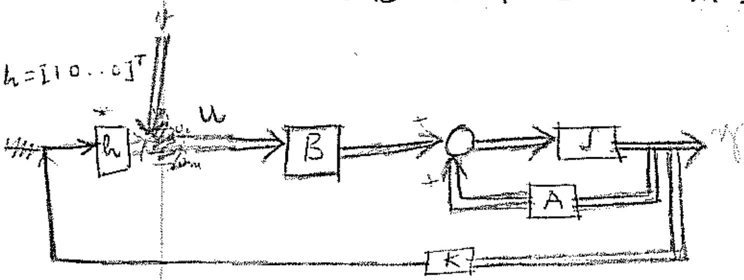
$\dot{x} = (A + bk)x + bv$

3. Multiple input
$[A, B, C]$ $\quad \dot{x} = Ax + Bu$
$\quad y = Cx$

assume $[A, B]$ c.c. $\Leftrightarrow$ rank $Q = [B \; AB \; \dots \; A^{n-1}B] = n$
let $B = [b_1 \; b_2 \; \dots \; b_m]$

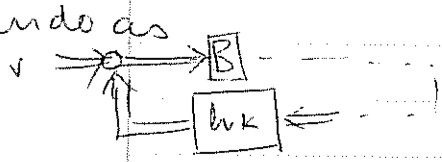
$\Leftrightarrow [b_1, b_2 .. b_m \;\; Ab_1 \, Ab_2 .. Ab_m \; A^n$
$\exists$ n l.i. vectors among the columns of ..

$h = [1 \, 0 \dots 0]^T$



* (i) Sp. case $[b_1 \; Ab_1 \dots A^{n-1}b_1]$ non sing
Then c.c. from just first input!
so use single input state f.b.
design $\bar{c}$ u_1
K overall state feedback.

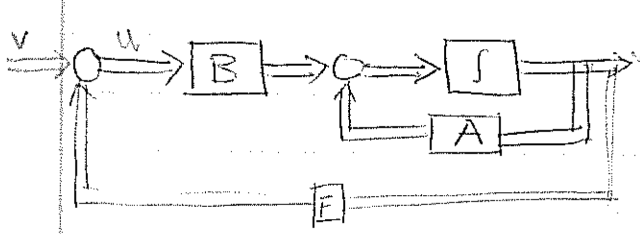
can do as



2) General Case
$[A, B]$ c.c.

However $[b_1 \; Ab_1 \cdots A^{n-1}b_1]$ not nonsingular
(of course $[b_k \; Ab_k \cdot \; A^{n-1}b_k]$ all not nonsingular else shuffle?)

state feedback changes c.c.

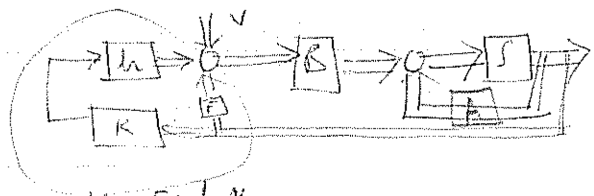


basically $[A, B]$ c.c. $\Leftrightarrow$ $[G, B]$ c.c. $\quad G = A + BF$
$[A, B]$ c.c. $\Leftrightarrow$ sys $\bar{c}$ FB also c.c.

$\dot{X} = Ax + Bu$ originally
$\dot{X} = Ax + Bv + BFx$
$= (A + BF)X + Bu$

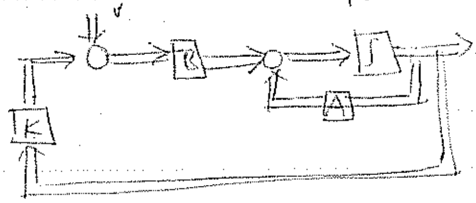
maybe $[b_1 \; Gb_1 \; \dots \; G^{n-1}b_1]$ can be made nonsingular

If we can do that,



$K = F + hK$

becomes



<u>Two step design</u>

1. find $G = A + BF$
   so that $[b_1 \,|\, Gb_1 \,|\, \dots \, G^{n-1}b_1]$ non singular
2. Design k to move eigenvalue of G using single input case.
   $K = F + hk$

How do we find G.

① Select n lin ind columns from B

$M = [\, b_1 \; Ab_1 \; \dots \; A^{n_1-1}b_1, \; b_2, \; \dots \; A^{n_2-1}b_2 \; \dots b_k \; \dots \; A^{k-1}b_k \,]$

stop li. I/prev. col

M nonsingular

② want to generate
$[b_1 \; Gb_1 \; \dots G^{n-1}b_1]$ from M
nonsingular

Let $[b_1 \; Gb_1 \; \dots \; G^{n-1}b_1] = [u_1 \; \dots \; u_n]$ ⇐ has nothing to do $\bar{u}_i$

on the one hand

$u_1 = b_1$

$u_{k+1} = (A+BF)u_k$

on the other hand

$u_1, u_2 \dots u_n$ derived from columns of M

$u_1 = b_1$ call it n_1

$u_2 = \alpha A b_1 + \beta b_1$; obviously $\alpha = 1, \beta = 1$ (from $u_{k+1} = (A+BF)u_k$ typical rule)

~~$u_3 = A^2 b_1 +$~~

$u_3 = Au_2 + \boxed{\text{lin comb of } b_1, b_2 \ldots b_m}$

$\rightarrow b_1$ should be chosen

$u_{n_1+1} = Au_{n_1} + b_2$; irrelevant ~~Au_{n_1}~~ $v_{n_1+1} = Av_{n_1} + b_1$ redundant to put $v_{n_1+1} = Av_{n_1} + b_2 + b_1$

now we have a pattern $u_1 = b_1$, $u_2 = Ab_1 + b_1$, $u_3 = Au_2 + b_1$, $\ldots$ $u_{n_1} = Au_{n_1-1} + b_1$ } already there n_1

$u_{n_1+1} = Au_{n_1} + b_2$

In general $u_1 = b_1$

$u_{k+1} = Au_k + \tilde{b}_k$ ($\tilde{b}_k \in \mathcal{R}(B)$, $\tilde{b}_k = B\tilde{h}_k$)

$\tilde{h}_k = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $k = 1 \ldots n_1$; $\tilde{h}_k = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $k = n_1+1, \ldots n_1+n_2$ if b_2 indep. w/ previous ones

this procedure guarantees linear indep. of $\{u_1, u_2, \ldots u_n\}$ can also easily with condition $u_{k+1} = Au_k + B\tilde{h}_k$

w/ this defn $u_{k+1} = Au_k + B\tilde{h}_k$

Compare w/ $u_{k+1} = Au_k + BFu_k$

$Fu_k = \tilde{h}_k$

$$F[u_1 u_2 \ldots u_n] = [\tilde{h}_1 \tilde{h}_2 \ldots \tilde{h}_n]$$

$$FU = H$$

$F = HU^{-1}$; $\bar{c}$ this choice of F, $[b_1 | Gb_1 | \ldots | G^{n-1} b_1]$ is non si

(If we had taken $b_1 + b_2$ there, diff F would result, which would still do the job)

$K = \underset{\sim}{F} + hk$: not unique

$m \boxed{K}$ ← $m \times n$ parameters n eigenvalues ⇒ lots of degrees of freedom.

There will be a problem of rank of $b_1, Ab_1, \ldots A^{n-1} b_1$

$b_1, Ab_1, \ldots A^{n-1} b_1$

If $b \ldots A^n b$ le $A^{n-1} b$ not li then choose independent vectors?

PROCEDURE

1) Find $b_1 \dots b_K$ from B $n_1 \dots n_K$
   s.t. $M = [\, b_1, Ab_1, \dots A^{n_1-1} b_1 \dots b_K \dots A^{n_K-1} b_K \,]$ is nonsingular

2) $\tilde{h}_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $i = 1 \dots n_1 - 1$

$\tilde{h}_i = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ $i = n_1+1 \dots n_1+n_2-1$

3) $u_1 = b_1$
   $u_{k+1} = Au_k + B\tilde{h}_k$ $\qquad U = [\, v_1 v_2 \dots v_n \,]$

4) $F = HU^{-1}$

5) Apply single ~~variable~~ input Eigen Value Assignment
   TO $\dot{x} = (A+BF)x + b_1 v_1$
   find k

6) The reqd. state fdback is
   $K = F + hk$ where $h = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

The states are not neccessarily available

$[A, B, C]$ e.c.
"Can use state fdbk to move eigenvalues"
(i.e. control modes) X| OBSERVERS

What if the states are inaccessible?
Try to "observe" the ~~states~~ "estimate".
device which estimates states is called the ESTIMATOR or OBSERVER
① devise (design) an ^ observer
SISO; MIMO Asymptotically (Gets true ō time)
Design how fast the ~~errors~~ error dies out. DUAL TO state fb design

② State fdbk c̄ observer (use of observer doesn't change effect performance of state FB)

③ improvement: reduced order observer

30 Nov '89

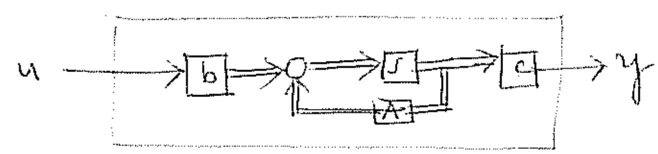
LST - Lecture

correction: $\hat{e}_{n_i} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $i = 1, 2, \cdots n_1 - 1$

(1) OPEN LOOP OBSERVER -

XI OBSERVERS

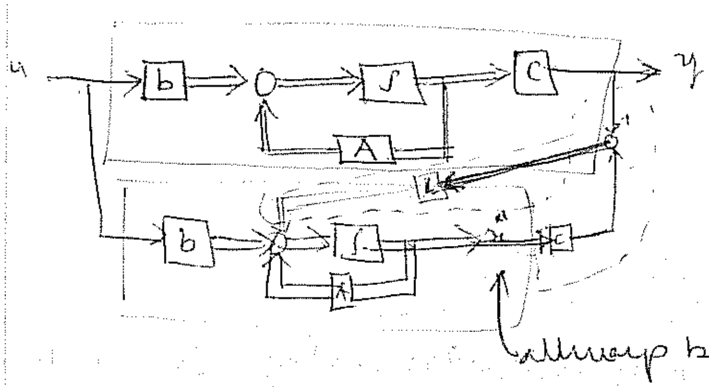
SISO $\dot{x} = Ax + bu$
$y = cx$



states not accessible.

2. full order observer - SISO
(1) open loop observer
$[A, b, c]$ c.o.

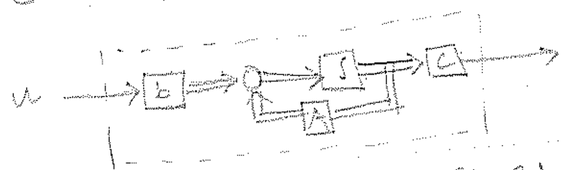
$u[0,t]$ & $y[0,t]$ det uniquely $x(0)$
thus have all the states, ie $x(t)$



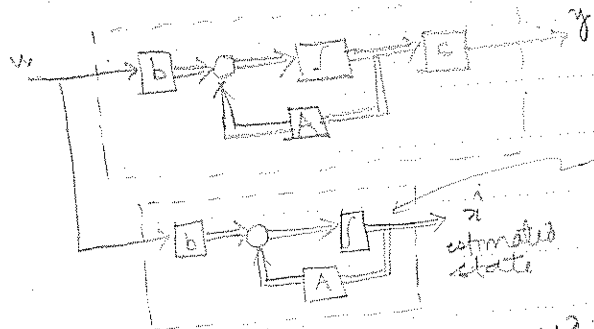
This page is rewritten of class notes!

2. FULL ORDER OBSERVER – SISO

$$\begin{cases} \dot{x} = Ax + bu \\ y = cx \end{cases}$$



(1) Open loop observer: if (A, b, c) c.o. $y[0,t]$ & $u[0,t]$ determine $x(0)$ (hence $x(t)$)



initial state $\hat{x}(0)$ computed from $y[0,t]$ and $u[0,t]$ hence $\hat{x}(t)$

$\dot{\hat{x}} = A\hat{x} + bu$ need to adjust $\hat{x}(t)$ based on observed $y[0,t]$ & $u[0,t]$ $\hat{x}$ and x are identical afterwards

iff computation requires zero time

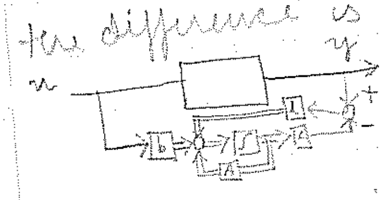
(2) Asymptotic Observer

Idea of using feedback (servo)

the output $y = cx$ is compared w/ $\hat{y} = c\hat{x}$

the difference is used to serve as a "correction" term.

$\dot{\hat{x}} = A\hat{x} + bu + l(y - \hat{y})$ Feedback correction term.



what about error in estimation $e = \hat{x} - x$?

$$\begin{cases} \dot{x} = Ax + bu \\ y = cx \end{cases}$$

observer $\{ \dot{\hat{x}} = A\hat{x} + bu + l(cx - c\hat{x})$

subtracting, we obtain $\dot{e} = (A - lc)e$

(i) the dynamics of the error $e(t)$ are decoupled from $x(t), \hat{x}$

(ii) $e(t) = [\exp(A - lc)t]$

Can we control the eigenvalues of $(A - lc)$

(a) (A, b, c) in observable Canonical Form

$$A_0 = \begin{bmatrix} 0 & \cdots & -\alpha_n \\ 1 & & -\alpha_{n-1} \\ & \ddots & \vdots \\ & 1 & -\alpha_1 \end{bmatrix} \qquad c_0 = [0\,0 \ldots 0\,1]$$

desired eigenvalues of $(A - lc) \Rightarrow \bar{\alpha}_1, \bar{\alpha}_2, \ldots \bar{\alpha}_n$

$$\bar{A}_0 = \begin{bmatrix} 0 & & -\bar{\alpha}_n \\ 1 & & -\bar{\alpha}_{n-1} \\ & \ddots & \vdots \\ & 1 & -\bar{\alpha}_1 \end{bmatrix} = A_0 - l c_0 = \begin{bmatrix} 0 & & -\alpha_n \\ 1 & & \vdots \\ & \ddots & \\ & 1 & -\alpha_1 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix} [0 \ldots 0\,1] \Rightarrow \begin{matrix} l_1 = \alpha_n - \bar{\alpha}_n \\ \vdots \\ l_n = \alpha_1 - \bar{\alpha}_1 \end{matrix}$$

i.e. can select l to move e-values of $(A - lc)$ can make $e(t) \to 0$ exponentially as $t \to \infty$ according e-values of $(A - lc)$ i.e. $\hat{x}(t) \to x(t)$ expo

coordinate transform

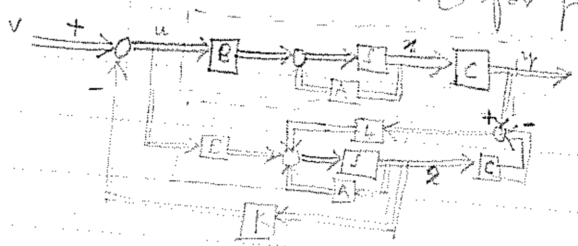
— ... — Full FB

k	OBSERVER l	$A_o = A_c^T$
$A-bk$	$A-lc$	$c_o = b_c^T$
A_c	A_o	$l = k^T$
b_c	c_o	

Problem: Given (A,b,c) find l s.t. the e-values of $(A-lc)$ are desired locat
1. Find the state F.B. k which moves the evalues of (A^T-c^Tk) to the given location
2. $l = k^T$

3. FULL ORDER OBSERVER–MIMO
multiple output (A,C) c.o.
Find L s.t. $(A-LC)$ given eigenvalues
1) Find the state feedback K (MIMO case) which moves e-values of (A^T-C^TK) to desired location.
2.) $L = K^T$

4. Observer & State Feedback $\begin{cases}\dot{x} = Ax+Bu \\ y = Cx\end{cases}$ c.c. & c.o.
use estimated state $\hat{x}(t)$ for feedback to place eigenvalues



Q. ① is $\hat{x}(t)\to x(t)$ still?
② overall system response according to e-values of $A+BK$?

1) $\dot{x} = Ax - BK\hat{x} + BV$
$\dot{\hat{x}} = LCx + (A-LC-BK)\hat{x} + BV$

error in estimation $\epsilon = \hat{x} - x$
$\Rightarrow \dot{\epsilon} = (A-LC)\epsilon$ (**)

(i) dynamics of the error $t \mapsto \epsilon(t)$ decoupled from $x(\cdot)$ & $u(\cdot)$
(ii) $\epsilon(t) = [\exp(A-LC)t]\,\epsilon(0)$ determined by eigenvalues of $(A-LC)$ which can be assigned arbitrarily if (A,C) c.o. by selecting of L
(iii) $\epsilon(t) = \epsilon(t)$ state F.B. doesn't affect the estimation

2) $\dot{x} = (A-BK)x - BK\epsilon + BV$ (*)
(i) the state $x(t)$ is affected by the estimation error $\epsilon(t)$ initially, but as $t\to\infty$, (*) $\approx \dot{x} = (A+BK)x + BV$
(*) & (**) taken together characterize the total system
$\dot{x} = (A-BK)x + BK\epsilon + BV$
$\dot{\epsilon} = (A-LC)\epsilon$
$y = [C \;\; 0]\begin{bmatrix} x \\ \epsilon \end{bmatrix}$

transfer function $v \to y$ $\quad H(s) = [C \;\; 0]\begin{bmatrix} sI-(A-BK) & BK \\ 0 & sI-(A-LC)\end{bmatrix}^{-1}\begin{bmatrix} B \\ 0\end{bmatrix}$
$= C(sI-(A-BK))^{-1}B$
same as $\bar{c}$ state feedback only
Note that TRANSFER FN is associated $\bar{c}$ zero state

3) charac polynomial: $d(s) = \det(sI-(A-BK))\,\det[sI-(A-LC)]$
(due to dynamics of observer not controllable. $\dot{\epsilon} = (A-LC)\epsilon$ ...)

∴ NIO result. One can design the observer and the state feedback separately. (Actually sort of pseudo-state feedback, as we feedback the estimated state)

(2) Asymptotic observer

idea of using ~~fdbk~~ compare o/p $y = cx$ & estimate $\hat{y} = c\hat{x}$

diff used as a correction term.

$$\dot{\hat{x}} = A\hat{x} + bu + (\text{fdbk correction term})\ l(y - \hat{y})$$

Q: (i) ~~whether~~ does it work? i.e. does $\hat{x}(t) \to x(t)$

(ii) How do we pick selection of ~~the~~ l

consider estimation error

$e \triangleq \hat{x} - x$

system has to satisfy $\dot{x} = Ax + bu$ — (1)

observer has to satisfy $\dot{\hat{x}} = A\hat{x} + bu + l(cx - c\hat{x})$ — (2)

(2) − (1) ⇒ $\dot{e} = (A - lc)e$

$$e(t) = \left[\exp(A - lc)t\right] e(0)$$

independent of ~~u(t)~~ $u(t)$

" " $x(t)$

$e(t) \to 0$ determined by eigenvalue of $(A - lc)$

∴ if we can move the e.v. of this matrix we can make the error go to zero!

Can we select l to move eigenvalues of $(A - lc)$

(a) Special case: (A, b, c) in observable canonical form.

$$A_0 = \begin{bmatrix} 0 & \cdots & \cdots & -\alpha_n \\ 1 & \cdots & \cdots & -\alpha_{n-1} \\ & \ddots & & \vdots \\ & & 1 & -\alpha_1 \end{bmatrix} \qquad c_0 = [0 \ \cdots \ 0 \ 1]$$

$$d(s) = s^n + \alpha_1 s^{n-1} + \cdots \ \alpha_n$$

desired eigenvalues give rise to $\bar{d}(s), \bar{\alpha}_1, \bar{\alpha}_2, \ldots$

i.e. $\bar{A}_o = \begin{bmatrix} 0 & \cdots & -\bar{\alpha}_n \\ 1 & & \vdots \\ & \ddots & \vdots \\ & & 1 \; -\bar{\alpha}_1 \end{bmatrix} = \begin{bmatrix} 0 & & -\alpha_n \\ 1 & & \vdots \\ & \ddots & \\ & & 1 \; -\alpha_1 \end{bmatrix} - \begin{bmatrix} & -\alpha_n+\bar{\alpha}_n \\ 0 & \vdots \\ & -\alpha_1+\bar{\alpha}_1 \end{bmatrix} = A_o - l c_o$ $\quad l = \begin{bmatrix} \bar{\alpha}_n - \alpha_n \\ \vdots \\ \bar{\alpha}_1 - \alpha_1 \end{bmatrix}$

answer: yes if $[A,b,c]$ in observable canonical form

Can move eigenvalues of $[A - lc]$ wherever we want

(d) $[A,b,c]$ general

coord transformation

into observable canonical form

Can show if (A,b,c) c.o.

can do

Duality

State FB	Observer
k (row)	l (column)
$A+bk$	$A-lc$
A_c	A_o
b_c	c_o

$A_o = A_c^T$

$c_o = b_c^T$

Given $[A,B,c]$ c.o.

find l s.t. $[A-lc]$ has desired eigenvalues

(1) find state feedback k s.t. $(A^T - c^T k)$ has desired eigenvalues.

$[(A,c)$ c.o. $\Rightarrow (A^T, c^T)$ c.c.$]$

(2) $l = k^T$

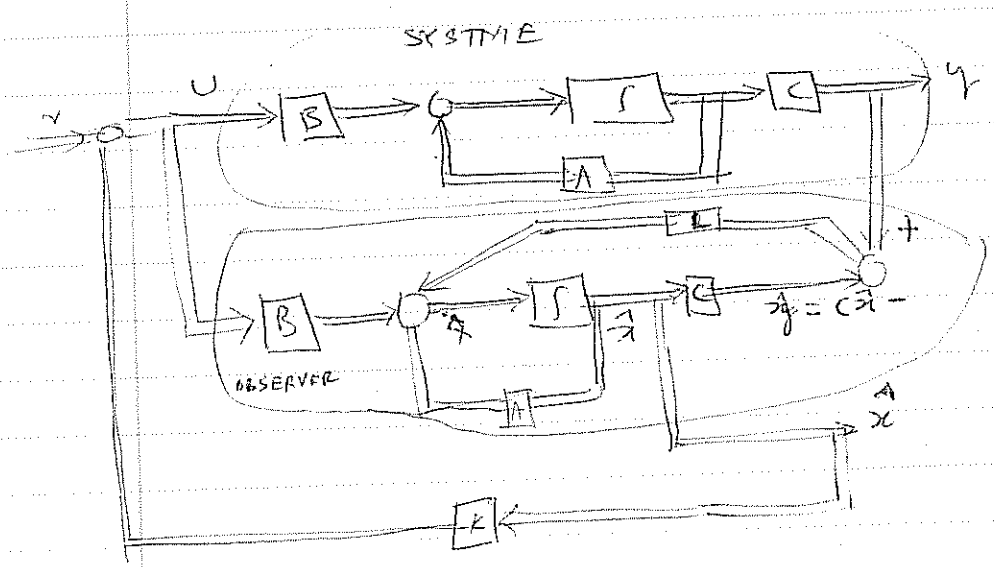
3. Full Order Observer - MIMO

Find L

1) Find K (MIMO)

s.t. $(A^T - C^T K)$ desired eigenvalues

(2) $L = K^T$; careful of sign!



use estimated states $\hat{x}(t)$ for state FB to move eigenvalues (overall sys response)

Q1: $\hat{x}(t) \to x(t)$

Q2: overall sys. response: does it go as e.v. of $A - BK$?

Look at ...: $\dot{x} = Ax - BK\hat{x} + Bv$

$\dot{\hat{x}} = A\hat{x} + LC(x - \hat{x}) + B(-K\hat{x} + v)$

$\dot{\hat{x}} = LCx + (A - LC - BK)\hat{x} + Bv$.

Estimation error: $\hat{x} - x$ $\quad \dot{e} = (A - LC)e$

Eqn of error again is negative

without

∴ ~~B s. FB box~~ s. FB doesn't affect obs. design

[Get Xeroxed Notes]

~~3) $\dot{x} = (A - BK)x$ ... BV~~

3) $\dot{x} = (A - BK)x - BK\epsilon + BV$.

$\epsilon(t) \to 0$

$\approx \dot{x} = (A - BK)x + BV$

Transfer fn of overall system
$V \mapsto y$

$\dot{x} = (A-BK)x - BK\epsilon + BV$
$\dot{\epsilon} = (A - LC)\epsilon$
$y = Cx$ $\qquad y = [C \;\; 0]\begin{bmatrix} x \\ \epsilon \end{bmatrix}$

~~$H(s) = [C\,0]$~~

$$A = \begin{bmatrix} A-BK & -BK \\ 0 & A-LC \end{bmatrix}$$

$$\therefore H(s) = [C \;\; 0]\begin{bmatrix} sI-(A-BK) & BK \\ 0 & sI-(A-LC) \end{bmatrix}^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix} = C[sI-(A-BK)]$$

$$= [C \;\; 0]\begin{bmatrix} [sI-(A-BK)]^{-1} & \times \\ 0 & \times \end{bmatrix}\begin{bmatrix} B \\ 0 \end{bmatrix}$$ same as w/ st. FB only!

char. poly $d(s) = \underbrace{\det[sI-(A-BK)]}\underbrace{\det[sI-(A-LC)]}_{\text{observer}}$

uncontrollable from the i/p.
(cf. eqn $\dot{\epsilon} = (A-LC)\epsilon$)

can do observer & FB designs indep. (for linear system) same thing in stochastic case

only requirement on C so for mas
$[A, C]$ c.o.

sp. case: $C = I$ $\quad y = x$; observer complexity vanishes

n^{th} order $\rightarrow$ 0^{th} order

sp. case C sq. nonsingular
this too is zeroth order

in general $\quad \begin{array}{cc} & n \\ m & \boxed{C} \end{array} \quad m < n$

$\boxed{y} = \boxed{C} \; \boxed{x}$: Do have some knowledge of x

∴ scope for designing a reduced order observer

17.2 avg /30
max 29 whatsmie!!
min 8
SD = 4.9

Friday
01 Dec '89

$\underset{\sim}{\ell}$ST ~~tec~~ discussion

Dr Wu will ask Questions from discussions sections (Hill examine notes of Ben)

$$x_{k+1} = Ax_k \qquad \longrightarrow \qquad x_k = A^k x_0$$
$$y_k = Cx_k \qquad\qquad\qquad y_k = CA^k x_0$$

$$y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} CA^0 x_0 \\ CA^1 x_0 \\ \vdots \\ CA^{n-1} x_0 \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{R} x_0$$

$$R = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^*$$

$$x_0 = U_1 \Sigma^{-1} V_1^* y$$

show $R\bar{x}_0 = y$

Multiplication: $R\bar{x}_0 = [V_1 \mid V_2] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} U_1 \Sigma^{-1} V_1^* y$

$$= [V_1 \mid V_2] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \Sigma^{-1} V_1^* y$$

$$= [V_1 \mid V_2] \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \Sigma^{-1} V_1^* y$$

$$= [V_1 \mid V_2] \begin{bmatrix} I \\ 0 \end{bmatrix} V_1^* y$$

$$R\bar{x}_0 = \boxed{V_1 V_1^*}^{\neq I} y \qquad \text{we know } V_1 V_1^* + V_2 V_2^* = I$$

$$Iy = [V_1 \mid V_2] \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} y$$

$$= V_1 V_1^* y + V_2 V_2^* y$$

$$y \in R(R) \Rightarrow y \perp V_2$$

$$\boxed{y = V_1 V_1^* y}$$

(doing an SVD)

Q3 $M(0,t) = \int_0^t \Phi(t,\tau) B \cdot B^* \Phi^*(t,\tau)\, d\tau = [U_1 \mid U_2] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix}$

Let $u(\tau) = B^* \Phi^*(t,\tau) U_1 \Sigma^{-1} U_1^* z$

$x(t) = z$

Assgn $x_{k+1} = Ax_k + bu_k$

$Q = [b \mid Ab \mid \cdots A^{n-1}b]$ has full rank

$b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$

$\Rightarrow Q^{-1}$ exists

$$x_n = A^n x_0 + \sum_{j=0}^{n-1} A^j b u_{n-j+1} = A^n x_0 + QU \qquad U = \begin{bmatrix} u_{n-1} \\ \vdots \\ u_0 \end{bmatrix}$$

$x_n = 0$

$0 = A^n x_0 + QU$

$-A^n x_0 = QU$

$-Q^{-1}A^n x_0 = U$

U is the unique i/p which steers $x \to 0$ at time n_0.

$Q^{-1} = \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} \quad \to u_i = -v_{n-i}^T A^n x_0$

(i) Assume $\exists \{u_0, \ldots u_k\}$ $(k<n-1)$ which steers $x_0 \to 0$ at time $k+1$.

Then the sequence $\{u_0 \cdots u_k, 0 \ldots 0\}$ steers $x_0 \to 0$ at time n_0

$x_{k+1} = Ax_k + bu_k$

$\hookrightarrow$ But we know from above its unique. ∎

$(PAP^{-1})(PAP^{-1})$

$= PAP^{-1}(P^{-1})^T A^T P^T$ | Singular values of similar matrices are different.

$[A, b, c]$ c.o. & c.c.

Many realizations which give same t.f.

① is this the "optimal" realization for the given $h(s)$ (transfer fn)?

- What does optimal mean?

(i) Don't want very large & very small numbers

(ii) balanced realization spec. we don't want ——— nearly singular

$Q \to$ nearly singular $\Rightarrow$ need large gains to control

$R \to$ " " $\Rightarrow$ Hard to observe states

A

eigenvalues are defined by t.f. so realization makes no difference

could have A singular but still have c.c.

× small perturbations may drastically change the system!

$$A = \begin{bmatrix} -1 & 20 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1/\sqrt{20} & \sqrt{20} \end{bmatrix} = \begin{bmatrix} 0.22 & 4.47 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 19 \\ 1 & -1 \end{bmatrix} \qquad R = \begin{bmatrix} 0.22 & 4.47 \\ -0.22 & 0 \end{bmatrix}$$

$\sigma_1 \cong 19$

$\sigma_2 \cong 1$

$\sigma_1 = 4.47$

$\sigma_2 = 0.22$

condition # $= \sigma_1/\sigma_2 = 18$

$\sigma_1/\sigma_2 = 20$

big cond # $\Rightarrow$ close to being singular

Transform to Jordan form Realization:

$$A_J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \quad B_J = \begin{bmatrix} .22 \\ 4.47 \end{bmatrix} \quad C_J = [1 \ \ 1]$$

$Q_J = \sigma_1/\sigma_2 \sim 2.5$ $\quad R_J$ $\sigma_1/\sigma_2 \sim 2.6$ $\quad Q_J = \begin{bmatrix} .22 & 4.\,? \\ 4.47 & -4.4? \end{bmatrix}$

Done lots better $\quad R_J = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ (?)

cont. canonical form:

$$A_c = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \qquad b_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad c_c = [9.17 \quad 4.69]$$

Q_c

$\sigma_1/\sigma_2 \approx 5.8$ $\qquad$ $R_c \doteq \sigma_1/\sigma_2 \sim 6.2$

obs. canonical form:

Everything same except transposed (Duals)

Q has 6.2

R has 5.8

MATRIXX in Sim

$$W_o = \begin{bmatrix} .025 & .75 \\ .75 & 25 \end{bmatrix}$$

solns of $AW_c + W_c A^* + BB^* = 0$

$A^* W_o + W_o A + C^* C = 0$

$$W_c = \begin{bmatrix} 110.5 & 5.5 \\ 5.5 & 0.5 \end{bmatrix} \qquad R = \begin{bmatrix} 0.054 & 0.15 \\ 0.15 & 5.0 \end{bmatrix}$$

$\int_0^t \|y(t)\|^2 dt$ o/p energy (maximize?)

$\int_{-\infty}^0 \|u(t)\|^2 dt$ i/p energy (want to min)

$$T = \begin{bmatrix} 0.07 & 2.26 \\ -0.07 & 0.93 \end{bmatrix} \qquad A_b = \begin{bmatrix} -0.56 & 0.44 \\ -0.44 & -1.44 \end{bmatrix} \qquad b_b = \begin{bmatrix} 2.32 \\ 0.86 \end{bmatrix}$$

$$C_b = [2.32 \quad -0.86]$$

Q_b: $\sigma_1/\sigma_2 = 2.3$ $\qquad$ R_b: $r_1/r_2 = 2.3$

Balanced: antisymm,

idea balance of i/p & balance of o/p.

i/p affects state 1 say $\Rightarrow$ affects o/p same.

can get closest approx to system by dropping the state corresp. to lowest singular value

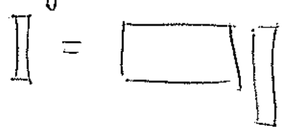
Tuesday Dec 5/99

KST-lecture

5. Reduced Order Observer

1) derivation

$$y = Cx$$



$C^{m \times n}$ $(m < n)$

Assume C has full rank!

coordinate Transformation

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} C \\ C_1 \end{bmatrix} x \qquad \begin{bmatrix} C \\ C_1 \end{bmatrix} \text{ nonsingular}$$

$$\bar{x}_1 = y$$

system: $\dot{x} = Ax + Bu$
$\quad y = Cx$

new coordinates $\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u$

$$y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

observer: $\bar{x}_1 = y$ available (no observer needed)
$\bar{x}_2 = ?$

open loop observer
$$\dot{\hat{x}}_2 = \bar{A}_{22}\hat{x}_2 + \bar{A}_{21} y + \bar{B}_2 u$$

asymptotic observer
$$\dot{\hat{x}}_2 = \bar{A}_{22}\hat{x}_2 + \bar{A}_{21} y + \bar{B}_2 u + (\text{fdbk correction term})$$

$(\bar{x}_2 - \hat{x}_2)$ $\quad$ $\bar{x}_2$ not accessible to us

Compromise
look for l.c. form of $\bar{x}_2$

$$\dot{\bar{x}}_1 = \underbrace{A_{11}\bar{x}_1 + \bar{A}_{12}\bar{x}_2}\, + \bar{B}_1 u$$

$$= A_{11} y + \bar{B}_1 u + \bar{A}_{12}\bar{x}_2$$

$$y_2 := \underbrace{\dot{y} - \bar{A}_{11} y - \bar{B}_1 u}_{\text{computable}} = \bar{A}_{12}\bar{x}_2$$

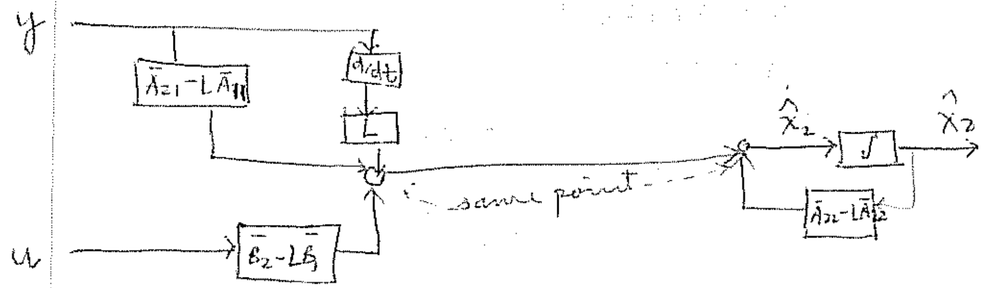
($\bar{A}_{12}$ typically would be rectangular)

$L\bar{A}_{12}(\bar{x}_2 - \hat{x}_2)$
↑ to keep correct order

$\therefore$ use $L(y_2 - \bar{A}_{12}\hat{x}_2)$ as fdbck

$$\dot{\hat{x}}_2 = \bar{A}_{22}\hat{x}_2 + \bar{A}_{21} y + \bar{B}_2 u + L(y_2 - \bar{A}_{12}\hat{x}_2)$$

$$= (\bar{A}_{22} - L\bar{A}_{12})\hat{x}_2 + (\bar{A}_{21} - L\bar{A}_{11}) y + L\dot{y} + (\bar{B}_2 - L\bar{B}_1) u$$



Q. Does it work?
Look @ the error and show that the error goes to zero.

estimation error $e = \bar{x}_2 - \hat{x}_2$

system $\dot{\bar{x}}_2 = \bar{A}_{22}\bar{x}_2 + \bar{A}_{21} y + \bar{B}_2 u$

observer $\dot{\hat{x}}_2 = \bar{A}_{22}\hat{x}_2 + \bar{A}_{21} y + \bar{B}_2 u + L\bar{A}_{12} e$

$$\dot{e} = \bar{A}_{22} e - L\bar{A}_{12} e = (\bar{A}_{22} - L\bar{A}_{12}) e$$

encouraging
- error ind of state.
- error ind of input.

$e(t) \to 0$ if e-values of $(\bar{A}_{22} - L\bar{A}_{12})$ can be placed in the open LHP.

Question becomes can we move eigenvalues of $(\bar{A}_{22} - L\bar{A}_{12})$?

Looks familiar! Observability type!

System $\bar{c}$ equ $(\bar{A}_{22}, \bar{A}_{12})$ c.o.?

Original c.o. $\left\{ \begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array} \right\}$

$\therefore$ new coord $\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \bar{A} \ldots$ c.o.

$$y = [\underbrace{I \quad 0}_{\bar{C}}] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

by defn $u[0,t]$, $y[0,t]$ det uniquely $\left\{ \begin{array}{l} \bar{x}_1(0) \\ \bar{x}_2(0) \end{array} \right. \Rightarrow \begin{array}{l} \bar{x}_1(0) \\ \bar{x}_2(0) \end{array}$

can be determined as $\begin{pmatrix} \bar{x}_1(0) \\ \bar{x}_2(0) \end{pmatrix} = \begin{pmatrix} C \\ C_1 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$

rewrite

Good stuff

$$\dot{\bar{x}}_2 = \bar{A}_{22}\bar{x}_2 + \overbrace{(\bar{A}_{21} y + \bar{B}_2 u)}^{u_2}$$

$$\dot{\bar{x}}_1 - \bar{A}_{11} y - \bar{B}_1 u = y_2 = \bar{A}_{12}\bar{x}_2$$

o/p matrix

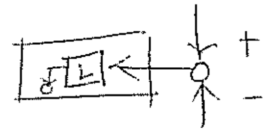
$u_2[0,t]$, $y_2[0,t]$ det uniquely $\bar{x}_2(0)$

Basically, rearranging eqns doesn't change observability.

Hence, can infact move eigenvalues for $(\bar{A}_{22}, \bar{A}_{12})$ c.o.

Answer: Yes, it works.

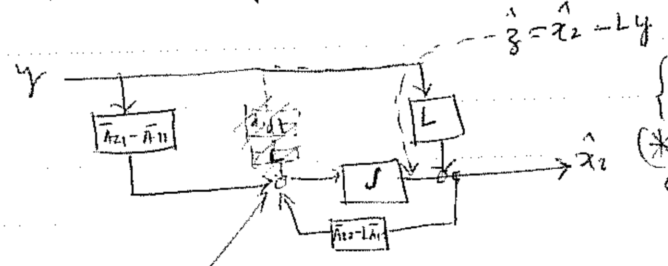
L: F.B. of the error!
doesn't affect observability.
($\because$ its inside the observer)



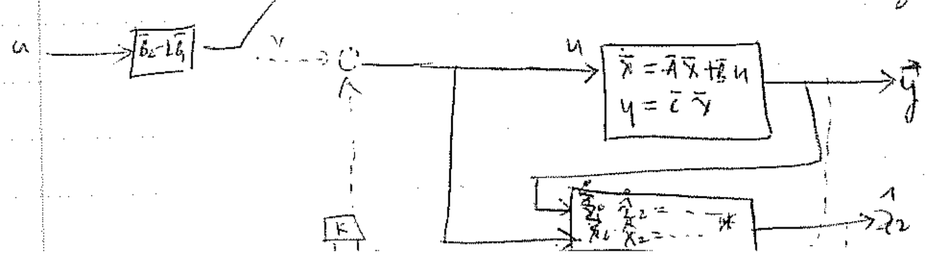
Poor design: $\because$ differentiator

Can we eliminate the differentiator?

Move after the integrator!



$$(*)\begin{cases} \dot{\hat{z}} = (\bar{A}_{22} - L\bar{A}_{11})(\hat{z} + Ly) + (\bar{A}_{21} - L\bar{A}_{11})y + (\bar{B}_2 - L\bar{B}_1)u \\ \hat{x}_2 = \hat{z} + Ly \end{cases}$$



Starting from here, can show

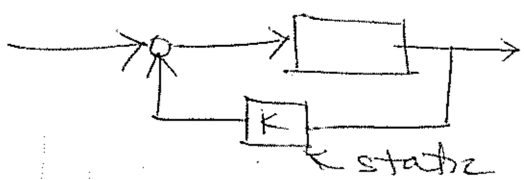
system $\begin{cases} \dot{\bar{X}} = \bar{A}\bar{X} + \bar{B}u \\ \dot{e} = (\bar{A}_{22} - L\bar{A}_{12})e \end{cases}$

whole system
$$d(s) = \left[\det(sI - A)\det(sI - (\bar{A}_{22} - L\bar{A}_{12}))\right]$$

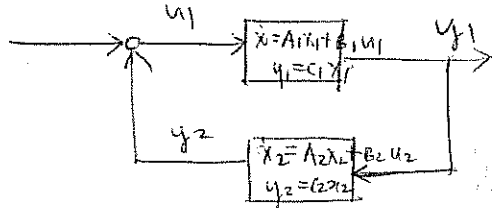
eigenvalues of system ↑ ↑ e.values of observer

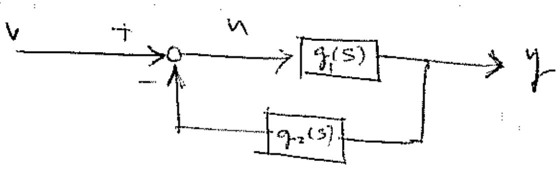
VII MULTIVARIABLE NYQUIST

1 Motivation: one impt application of state F.B. is to stablize the system.



more generally use dynamics in K

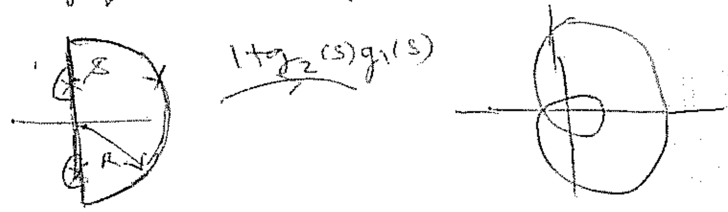




Nyquist: graphical method
gives insight

$$y(s) = g_1(s)u(s)$$
$$u(s) = v(s) - g_2(s)y(s)$$
$$= v(s) - g_2(s)g_1(s)u(s)$$
$$y(s) = \boxed{\frac{g_1(s)}{1 + g_2(s)g_1(s)}}\, v(s)$$

Nyquist diagram



$P_{of} :=$ # of $P(g_2(s)\,g_1(s))$ in $\bar{C}_+$ [closed RHP (includes $j\omega$ axis)]

...that $1+g_1(s)g_2(s)$ and $g_1(s)$ cannot ever have common zeros ...

The closed loop system $= g_1(s) \big/ 1+g_2(s)g_1(s)$ is exp stable

$\Leftrightarrow$ (i) Nyquist diagram of $1+g_2(s)g_1(s)$ doesn't go through the origin
(ii) Does encircle origin P_{of} times in CCW dirn.

The proof is based on the principle of the argument.

C : closed curve on s-plane (simple curve)
$F(s)$: rational fn

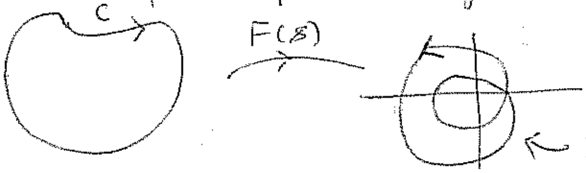
as s traverses C in the CW dirn, the trajectory of $F(s)$ encircles the origin $(p-z)$ times in the CCW dirn.

p: # of poles of $F(s)$ enclosed in C
z: # of zeroes of $F(s)$ enclosed in C.

Thursday
Dec 7, 1989

1ST-lecture

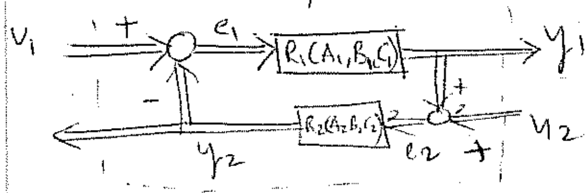
Principle of the argument



$\leftarrow$ # CCW encirclement of origin
$=$ # p $-$ # z

p: poles of $F(s)$ enclosed in C
z: zeroes of $F(s)$

3.5 System Representation



R_1: $\dot{x}_1 = A_1 x_1 + B_1 e_1$
$y_1 = C_1 x_1$

R_2: $\dot{x}_2 = A_2 x_2 + B_2 e_2$
$y_2 = C_2 x_2$

connexion $\quad e_1 = u_1 - y_2$
$\qquad e_2 = u_2 + y_1$

combined system: i/p are u_1 & u_2 o/p y_1, y_2
to formulate eqns eliminate e_1, e_2

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}}^{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$R_c(A,B,C)$: Representation of closed loop system

<u>Stability</u>: e-values of A
$\quad$ zeros $\rightarrow$ det $(sI - A)$

$$(sI - A) = \begin{bmatrix} sI - A_1 & B_1 C_2 \\ -B_2 C_1 & sI - A_2 \end{bmatrix}$$

want to relate these eigenvalues
with those of original systems

Perform Gaussian elimination (L-U decomposition)

$$sI - A = \begin{bmatrix} sI-A_1 & 0 \\ 0 & sI-A_2 \end{bmatrix} \begin{bmatrix} I & (sI-A_1)^{-1}B_1C_2 \\ -(sI-A_2)^{-1}B_2C_1 & I \end{bmatrix}$$

$$= \begin{bmatrix} sI-A_1 & 0 \\ 0 & sI-A_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ -(sI-A_2)^{-1}B_2C_1 & I \end{bmatrix} \begin{bmatrix} I & (sI-A_1)^{-1}B_1C_2 \\ 0 & I+(sI-A_2)^{-1}B_2C_1(sI-A_1)^{-1}B_1C_2 \end{bmatrix}$$

$$\det(sI-A) = \det(sI-A_1).\det(sI-A_2).\det\{I+(sI-A_2)^{-1}B_2C_1(sI-A_1)^{-1}B_1C_2\}$$

switch

$*$ $C_1(sI-A_1)^{-1}B_1$: T.F matrix $G_1(s)$

becomes

$G_2(s)$ commute?

In general, $\det(I+MN)$ → could it be $\det\{I+NM\}$

$$\begin{bmatrix} I & -N \\ M & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ M & I \end{bmatrix} \begin{bmatrix} I & -N \\ 0 & I+MN \end{bmatrix}$$

Want TF

$$\Delta = \det\begin{bmatrix} I & -N \\ 0 & I \end{bmatrix} \det\begin{bmatrix} I+NM & 0 \\ M & I \end{bmatrix} = \det(I+NM)$$

$$\det\begin{bmatrix} I & -N \\ M & I \end{bmatrix} = \det\{I+MN\}$$

$$\Delta = \det(I+MN) \quad \begin{bmatrix} I & -N \\ M & I \end{bmatrix} \equiv \begin{bmatrix} I & -N \\ 0 & I \end{bmatrix} \begin{bmatrix} I+NM & 0 \\ M & I \end{bmatrix}$$

; Gauss Elim is essentially matrix factorization

$$\det[\quad] = \det\{I+NM\}$$

elements are rational fns

$$\det(sI-A) = \det(sI-A_1)\det(sI-A_2)\det(I+G_2(s)G_1(s))$$

poly   poly   poly   rational fn.

Transfer functions of original subsystems

In other words, some of the eigenvalues will get cancelled out ($\because$ poles of rational fn have to cancel out)

Under what conditions is the closed loop ~~system~~ stable?

Turns out that the closed loop system
$R_c(A,B,C)$ is exp stable
$\Leftrightarrow$ The Nyquist diagram of $\det\{I+G_2(s)G_1(s)\}$
(i) Doesnt go thru the origin
(ii) does encircle the origin
p_{o+} times in CCW dirn.
$p_{o+} := \#$ of $\overline{z}\,\{\det(sI-A_1)\det(sI-A_2)\}$
in closed RHP.

Proof

$$\underbrace{\det(sI-A)}_{d_A(s)} = \underbrace{\det(sI-A_1)\times\det(sI-A_2)\times\det\{I+G_2(s)G_1(s)\}}_{P_R(s)}$$

$R_c(A,B,C)$ exp stable

Sketchy!

$\Leftrightarrow$ $d_A(s)$ no zeroes in $\overline{C}_+$

$\Leftrightarrow$ $P_R(s)$ no zeroes in $\overline{C}_+$

algebraic requirement on both sides only polys

(i) All poles of $\det(I+G_2G_1)$ cancelled by $\det(sI-A_1)\det(sI-A_2)$

so is not there could have more poles cancelled!

(ii) Maximum # of poles of $\det[I+G_2G_1]$ in $\overline{C}_+$
$= p_{o+}$ (# of zeroes of $\det(sI-A_1)\cdot\det(sI-A_2)$ in $\overline{C}_+$ counting multiplicities)

because $\det(I+G_2(s)G_1(s)) = \dfrac{\det(sI-A)}{\det(sI-A_1)\det(sI-A_2)}$ $\Rightarrow$ # poles of $\det(I+G_2G_1)$ = # zeroes of $\det(sI-A_1)\det(sI-A_2)$

$\therefore$ $R_c(A,B,C)$ exp stable

$\Leftrightarrow^*$ (i) all $\overline{C}_+$-zeroes of $\det(sI-A_1)\det(sI-A_2)$ must be cancelled by the $\overline{C}_+$ poles of $\det[I+G_2G_1]$

(ii) $\det[I+G_2G_1]$ has no zeros in $\overline{C}_+$
(else these would not be cancelled)

want to show (*) ⇒ Nyquist

*(i) ⇒ # of $\bar{C}_+$ poles of $\det[I+G_2G_1] = p_{0+}$

*(ii) ⇒ # of $\bar{C}_+$ zeros of $\det[I+G_2G_1] = 0$

(*)(ii) ⇒ Nyquist

most of Nyquist (2)

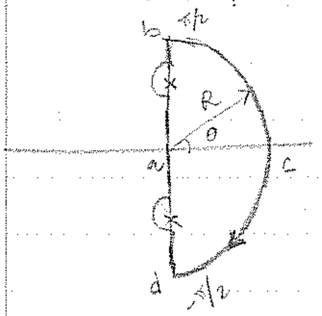
principle of the argument

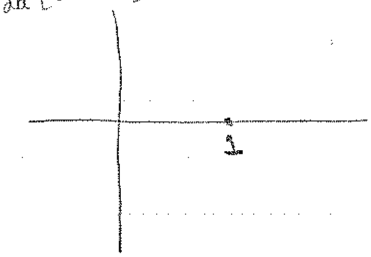
⇒ # c.c.w. encirclement $= p_{0+}$ [Nyquist (ii) ✓]

Nyquist (ii)

Consider Nyquist (i)

$\det[I+G_2(s)G_1(s)] = \det\left[I + \frac{C_2B_2}{s}\frac{C_1B_1}{s} + \cdots \to 0\right] \approx 1$





s: $b \to c \to d$ $s = Re^{j\theta}$ $\theta \sim \frac{\pi}{2} \to -\frac{\pi}{2}$

$R \to \infty$ $|s| \to \infty$ $1/s$ small

$G_1(s) = C_1[sI - A_1]^{-1}B_1$ if $s > \max|\lambda(A)|$

$= C_1B_1/s + C_1A_1B_1/s^2 + \cdots$

$G_2(s) = C_2B_2/s + C_2A_2B_2/s^2 + \cdots$

C_2 $C_1 \to 0$ since $s = Re^{j\theta} \to \infty$ as $R \to \infty$

$\det[I + G_2(s)G_1(s)] = \det\left[I + \frac{C_2B_2C_1B_1}{s^2} + \cdots\right]$

$\cong 1$

∴ the portion $b \to c \to d$ is close to 1. (won't pass thru origin)

s: $d \to a \to b$ on $j\omega$ axis

∴ $s = j\omega$ ω: $-R$ to $+R$ (later $R \to \infty$)

$*$ (ii) $\Rightarrow \det[I + G_2(j\omega)G_1(j\omega)] \neq 0$

$\therefore$ NYQUIST (i) is satisfied!

$\therefore$ NYQUIST conditions are SUFFICIENT.

he got these mixed up

To show Nyquist conditions are necessary

Nyquist diagram $\det[I+G_2(s)G_1(s)]$ encircles the origin p_{0+} times in CCW dirn.

principle of the argument,

$$p_{0+} = \begin{pmatrix} \#\text{ of poles} \\ \text{of } \det[I+G_2(s)G_1(s)] \\ \text{inside } \bar{C}_+ \end{pmatrix} - \begin{pmatrix} \#\text{ of zeros} \\ \text{of } \det[I+G_2(s)G_1(s)] \text{ inside } \bar{C}_+ \end{pmatrix}$$

$0 \leq \cdot \leq p_{0+}$ $\qquad\qquad$ $0 \leq \cdot \leq ?$

from (ii) cancellation by O.L.

but only comb which allows this integer eqn to be satisfied is

$\#$ of poles $= p_{0+}$ $\qquad\qquad$ $\#$ of zeros $= 0$

concl: $\#$ of RH side poles of this det must be p_{0+}

$\#$ of RH side zeros " " " 0

$\therefore$ Nyquist is necessary

① not c.c. / c.o. ← still looking at all modes (π of total)
(not need c.c./c.o.) Very STRONG.

② not. cc/co of component systems
(but have to know # of RH poles/zeroes of cc/co)

If origin is passed thru, then how do you know it encircles.

Friday 1/12/89

LST - discussion

$$f_1(x_1,x_2,x_3) = \dot{x}_1 = x_1(x_3 - x_2) + 3(x_2^2 - x_2x_3 + x_2) + 4x_3 - 5$$
$$f_2(x_1,x_2,x_3) = \dot{x}_2 = 6x_1 - 5x_2 - 4x_3 + 51$$
$$f_3(x_1,x_2,x_3) = \dot{x}_3 = -2x_1^2 + x_1x_2 + x_2^2 - x_2x_3 - 5x_2 + 3x_3 + 13 + u$$
$$y = \tfrac{1}{2}(x_1 - x_2 - x_3)$$

$x = \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix}$ ← an $\underset{\text{e.p.}}{\rightleftarrows}$ pt of the system.

Q: Is it a stable equilibrium point?
Q: If not how do you design observer to approx sys?
Q: How do you design controller

linearize: take Jacobian

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \\ \delta\dot{x}_3 \end{bmatrix} = \overbrace{\begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 \\ \partial f_2/\partial x_1 & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}^{A}_{3\times 3} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix} + \overbrace{\begin{bmatrix} \partial f_1/\partial u \\ \partial f_2/\partial u \\ \partial f_3/\partial u \end{bmatrix}}^{b} u$$

$$\Big|_{@\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix}}$$

$$A = \begin{bmatrix} 5 & -4 & -4 \\ 6 & -5 & -4 \\ 1 & -2 & -2 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\delta y = \overset{\longleftarrow c \longrightarrow}{\begin{bmatrix} 1/2 & -1/2 & -1/2 \end{bmatrix}} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix}$$

$$X_1 = 1 + \delta x_1$$
$$X_2 = 5 + \delta x_2$$
$$X_3 = 8 + \delta x_3$$

$$y = -6 + \delta y$$

$$\det(sI - A) = s^3 - 2s^2 - 5s - 6$$
$$= (s-2)(s+1)(s+3)$$

eigenvalue in RHP
∴ system unstable.

if eigenvector for that mode is in nullspace of Q then could still control that mode (eg cost Q in LTI)

Form R & Q matrices : Full Rank
$\Rightarrow$ c.c. & c.o.

<u>Observer</u>:
change coord so in obs. can. form

$$z = T^{-1}\delta x \;;\; (\delta x = Tz)$$
$$\dot z = TAT^{-1}z + Tbu \qquad = A_o z + b_o u$$
$$\delta y = cTz \qquad = c_o z$$

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & -1 \end{bmatrix} \qquad A_o = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 0 & 5 \\ 0 & 1 & -2 \end{bmatrix}$$

could write by looking at ch. poly right off

$$b_o = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad c_o = [\,0 \quad 0 \quad 1\,]$$

$$T^{-1}A_c = AT^{-1} \;;\; b_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = Tb$$

$$T^{-1}b_c = b$$

$$T^{-1}b_c = \begin{bmatrix} | & | & | \\ t_1 & t_2 & t_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = b \Rightarrow t_3 = b$$

Look into this

and $\begin{bmatrix} | & | & | \\ t_1 & t_2 & b \\ | & | & | \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A} \end{bmatrix} \begin{bmatrix} t_1 & t_2 & b \end{bmatrix}$

$$t_2 - \alpha_1 b = Ab$$
$$t_2 = (A + \alpha_1 I) b$$

canonicalobservable form (take transpose to get c.c)

$$\dot{z} = A_o z + b_o u$$

$$\dot{\bar{z}} = A_o \bar{z} + b_o u + \underbrace{L_o c (z - \bar{z})}_{L_o(\delta y - \delta \bar{y})}$$

$$\dot{\bar{z}} = (A_o - L_o C_o)\bar{z} + (bu + L_o C_o z)$$

"input"

go "rapidly" to ~~states~~ of system
eg $\lambda = -3$ then want -6 (so faster damping)

$$\begin{array}{c} A_o \\ \begin{bmatrix} 0 & 0 & 6 \\ 1 & 0 & 5 \\ 0 & 1 & -2 \end{bmatrix} \end{array} - \begin{array}{c} L_o \\ \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \end{array} \begin{array}{c} C_o \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{array}{cc} \begin{bmatrix} 0 & 0 & 6 - l_1 \\ 1 & 0 & 5 - l_2 \\ 0 & 1 & -2 - l_3 \end{bmatrix} & \begin{array}{c} -\alpha_3 \\ -\alpha_2 \\ -\alpha_1 \end{array} \end{array}$$

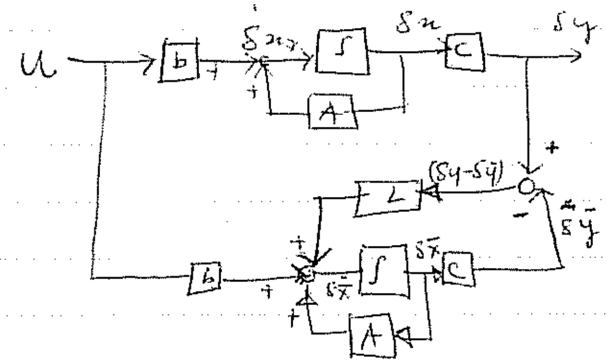
$\alpha_1' = 18$
$\alpha_2' = 108$
$\alpha_3' = 216$

$$\therefore L_o = \begin{bmatrix} 222 \\ 113 \\ 16 \end{bmatrix}$$

$$\dot{\bar{z}} = (A_o - L_o C_o)\bar{z} + b_o u + L_o C_o y$$
$$= A_o \bar{z} + b_o u + L_o C_o (z - \bar{z})$$
$$\dot{\bar{z}} = A_o \bar{z} + b_o u + L_o (\delta y - \delta \bar{y})$$

$$T^{-1} \dot{\bar{\delta x}} = A_o T^{-1} \bar{x} + b_o u + L_o (\delta y - \delta \bar{y})$$
$$\dot{\bar{\delta x}} = \underbrace{T A_o T^{-1}}_{A} \bar{x} + \underbrace{T b_o}_{b} u + \underbrace{T L_o}_{L} (\delta y - \delta \bar{y})$$

$\dot{\delta x} = A\delta\bar{x} + bu + L(\delta y - \delta\bar{y})$



<u>controller</u> Transform to c.c. form, by "observation" choose f.b. matrix K_c, and transform m back to original coords.

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{bmatrix} \qquad b_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad c = [\cdot \;\; - \;\; -]$$

not imp

move unstable pole on ~~stable~~ side of $j\omega$ axis.

$s = 2$ $\qquad$ $s = -1$

char. polynomial we want $(s+1)^2(s+3)$

$s^3 + 5s^2 + 7s + 3$

$$A_c - b_c K_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -7 & -5 \end{bmatrix} \qquad K_c = \begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \\ {[9} & 12 & 3] \end{array}$$

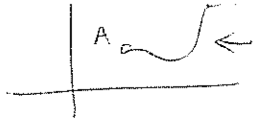
$\dot{\delta x} = A\delta x - bK\delta\bar{x}$

$\dot{\delta\bar{x}} = A\delta\bar{x} - bK\delta\bar{x} + L(\delta y - \delta\bar{y})$

$\delta y = c\,\delta x$

$\delta\bar{y} = c\,\delta\bar{x}$

behaviour at eq pt of linearized system should be same as on original system at eq pt.



← can we find any trajectory

$$\dot{\delta x} = \begin{bmatrix} 5 & -4 & -4 \\ 6 & -5 & -4 \\ 1 & -2 & -2 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

suppose we want trajectory to be course.

then $\delta x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \dot{\delta x} = \begin{bmatrix} -0.3 \\ -0.3 \\ -0.3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix}$

Previous End Sem -

1. $\dot{x} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u \qquad x(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(a) For $u(t) \equiv 0$, what is the space in $\mathbb{R}^3$ in which $x(t)$ lies?

(b) What conditions on b_1, b_2, b_3 will guarantee that $\exists\, u(t)$ $t \in [0,1]$ s.t. ~~x(t)~~ $x(1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

2. $\ddot{x} + d\dot{x} + kx = 0$

(a) find the eq. pts

(b) for what values of k & d is the eq. pt asymptotically stable?

(c) (b) but stable i.s.L.

(d) $\ddot{x} + d\dot{x} + kx = u$ $\quad$ K & d bibs stable

Suppose for some $\lambda_k \in \sigma(A)$ $\exists\, \eta$ st

$\begin{bmatrix} \lambda_k I - A \\ \hline C \end{bmatrix} \eta = \theta$ $\quad$ Show set of all observable states is orthogonal to η.

$\dot{x} = Ax + Bu$

consider sampled data system with 0-order hold ($u(t)$ const. over $t = [k, k+h]$)

$$A^T P A - P = -Q$$

$$A A^T P A - A P = -A Q$$

$$x_d(k+1) = A_d x_d(k) + B_d u(k)$$

$$A_d = \exp(A h)$$

$$B_d = \int_0^h e^{At}\,dt\, B$$

$$PA - (A^T)^{-1} A^{-1} A P = \cancel{(A^T)^{-1} Q} = -(A^T)^{-1} Q$$

$$PA = (A^T)^{-1} P - (A^T)^{-1} Q = (A^T)^{-1}(P - Q)$$

$$A (A^T)^{-1} = P^{-1}(P - Q)$$

show that if $\{A, B\}$ not c.c., then $\{A_d, B_d\}$ not c.c

Suppose char poly $= d(s) = (s+1)^4$

min poly $= \psi(s) = (s+1)^2$

Does ZSR of $\dot{x} = Ax + bu$, $y = Cx$ have a term $a t^4 e^{-t}$

$u(t) =$ unit step fn.

Let $C : \mathbb{R}^n \to \mathbb{R}^m$

given $y \in \mathbb{R}^m$, consider

$$Cx = y$$

(i) suppose C is <u>onto</u>

find min norm soln

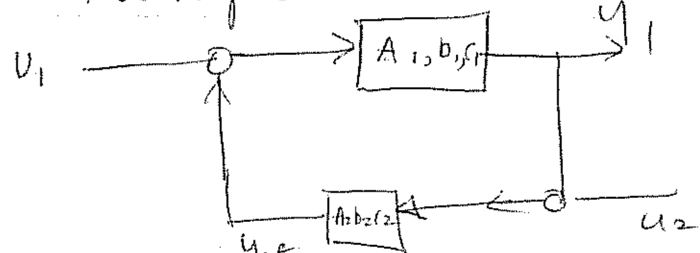
(ii) suppose C is 1-1

find best "approx" soln

Suppose $[A, B, C, 0]$ is a minimal-system

define & characterize zeros of the representation

Given the system



derive A, B, C matrices for total system

9/88

Sample 221A prelims

Richard and I tried to make the following tests resemble actual full prelims as given by each of the usual 221 examiners. Most of the questions come straight from old prelims. The solutions are at best sketchy, and in some cases may be wrong, but should at least point you in the right direction.

Kris Pister
Richard Murray

Sample prelim 1 [Sastry]

(1) Consider the linear operator $L: \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ given by

$$L(X) = BXA - X$$

a. What are the eigenvalues of L in terms of the eigenvalues of A & B?

b. When is the map L bijective?

c. What are the eigenvalues of the linear map $M: \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by

$$M(Y) = AY + YB$$

(2) You are given a T-periodic linear system $\dot{x} = A(t)x + B(t)u$ with a periodic input $u(t)$ of period T.

a. Find a T-periodic trajectory of the system

b. when do soln's of the differential equation converge towards the periodic solution.

(3) You are given a transfer function $G(s) \in \mathbb{R}^{n_o \times n_i}(s)$. Describe how you would build a minimal realization of $G(s)$.

[Hint: Start w/ a partial fraction expansion of $G(s)$

$$G(s) = \sum_{i=1}^{p} \frac{R_i}{(s-\lambda_i)}$$
]

Sample prelim 2 [Varaiya]

① True or false

a. $AB = BA$, $f(\cdot)$ analytic $\Rightarrow$ $f(A)e^B = e^B f(A)$ ✓

to expand by Taylor then use commutativity

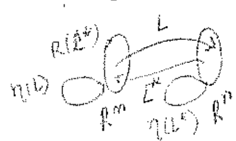
b. $\mathcal{R}(L) = \mathcal{R}(L^* L)$ ✗

c. $\mathcal{R}(A) \perp \mathcal{N}(A^*)$. Prove or disprove ✓

d. $L: \mathbb{R}^m \to \mathbb{R}^n$ is a surjective map. The least norm sol'n of $Lx = y$ is given by

$$x = L^T (L L^T)^{-1} y$$

$n < m$



① onto $\Rightarrow$ given y $\exists x$ $\Rightarrow$ $\dim(R(L)) = n$

② $x = L^* \xi$ $\Rightarrow$ $\dim R(LL^*) = n$

$LL^* \xi = y$ $\therefore$ invertible $n \times n \cdot m \times n$

$\xi = (LL^*)^{-1} y$

③ $x = L^*(LL^*)^{-1} \xi$

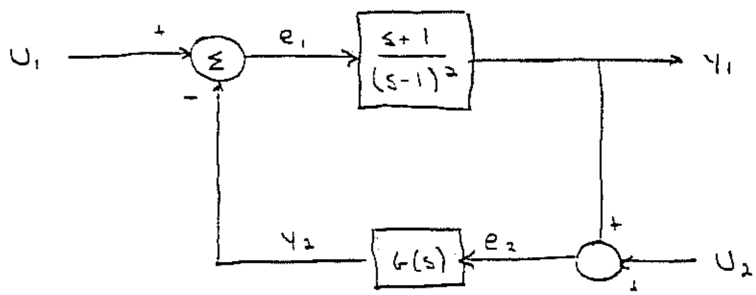
② Consider the system

$$\frac{\partial^3 y}{\partial t^3} + y = \dot{u} + u$$

a. Write down the controllable canonical form of the system

b. Is it observable? Why?

③ Consider the block diagram



a) Is the system open loop stable?

b) what is $G(s)$ so that the characteristic polynomial is $(s+2)^2$

④ Explain the design of a reduced-order observer.

Give bounds on the dimension of an observer given all information about a plant (i.e. # inputs, # outputs, # states (controllable, observable, etc))

Sample prelim 3 [Polak]

① a. Define complete controllability (be precise)

b. Describe all the tests you know of for determining if a system is completely controllable. Describe the conditions under which each test may be used.

② a. $\dot{x} = Ax + Bu$

$y = Cx$

Define a minimal representation of this system

b. How would you extract a minimal representation? make co/cc by a transformation?

c. Effect of state feedback on controllability and observability (proof or counterexample)

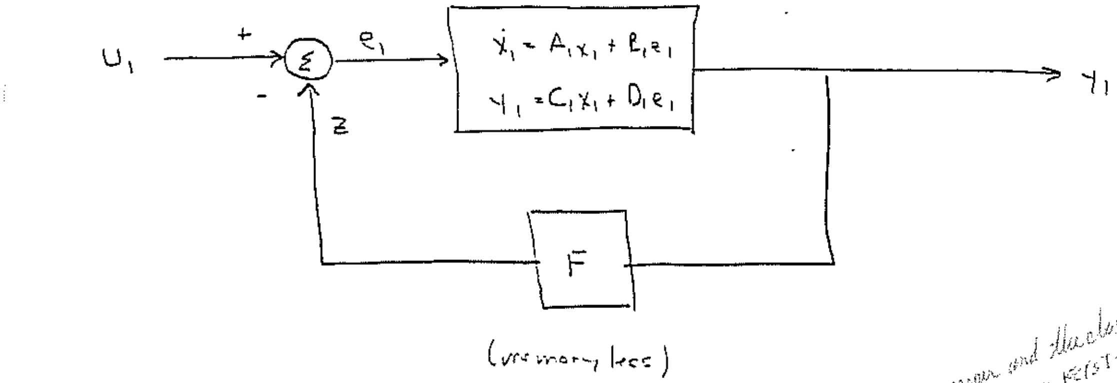
d. Effect of output feedback on controllability & observability (proof or counter example)

③ $\dot{x} = Ax + Bu$

$$A = \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{bmatrix} \qquad B = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Find a feedback of the form $u = v + Kx$ (K diagonal) so that the eigenvalues of the closed loop system have low sensitivity to perturbations in the eigenvalues of A.

Sample prelim 4 [Wu]



① When is the overall system well-posed? *if every subsystem is proper and the closed loop system is proper and well-def. i.e. $H_{yu}(s) = c(sI-A)^{-1}B \cdot (I + F c(sI-A)^{-1}B)^{-1}$ (assuming $D_1 = 0$)*

Suppose it is well-posed:

a. A_1 has an eigenvalue in $\mathbb{C}_+$. Will it appear in the overall system?

b. Can we use F to eliminate such an eigenvalue

c. A_1 has eigenvalue 1 of multiplicity 2. Characterize the behavior of this mode

② Replace F with
$$\dot{x}_2 = A_2 x_2 + B_2 y_1$$
$$z = C_2 x_2 + D_2 y_1$$

a. The overall system has an eigenvalue at λ_1. Presuming no other knowledge of the transfer function, how would you determine whether or not this mode is observed in the transfer function? *use an input at that frequency and see if $y(s) = 0$.*

b. What conditions imply the overall system transfer function exists and is proper? Are these conditions necessary?

c. Give the conditions for exponential stability of the overall system transfer function (in terms of $G_1(s)$ & $G_2(s)$) *eigenvalues of closed loop system in $\mathbb{C}_-$*

$$\begin{cases} G_1(s) = C_1(sI - A_1)^{-1} B_1 + D_1 \\ G_2(s) = C_2(sI - A_2)^{-1} B_2 + D_2 \end{cases}$$

d. How would you check for exponential stability

[Desoer] Sample Prelim 5

① $\dot{x} = A(t)x + B(t)u(t)$

$A(t) = A(t+T) \quad \forall t$

a) show $\Phi(t,\tau)$ satisfies $\Phi(t+T, \tau+T) = \Phi(t,\tau)$

b) what conditions are necessary for a T periodic solution if:

i) $B(t) = 0$

ii) $B(t+T)u(t+T) = B(t)u(t) \quad \forall t$

c) assume $B(t) \equiv 0$, and the eigenvalues of $\Phi(T,0)$ lie inside the unit disk, except 1 at -1.

$x(t)$ periodic?

② P, Q pos. def.

$A^T P A - P = -Q$

what can you say about the eigenvalues of A?

(consider discrete time)

[illegible] eigenvalue $= |(\lambda_i \lambda_j - 1)| < 0$

$\Rightarrow -1 < \lambda_i \lambda_j < 1$

special case when

$|\lambda_i^2| < 1$

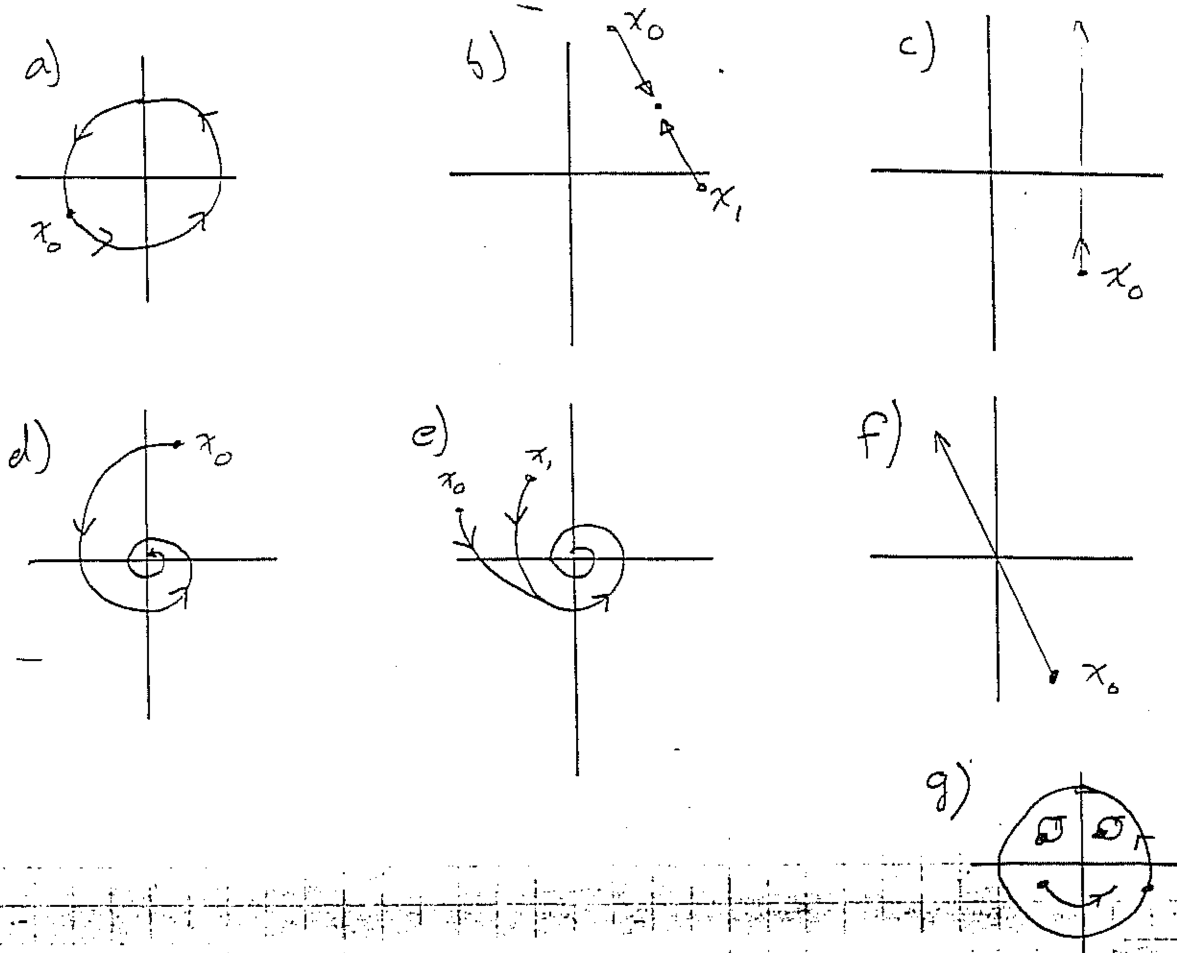
$\Rightarrow -1 < \lambda_i < 1$

Sample Prelim 6

① [wu]

$$\dot{x}(t) = Ax(t) \qquad A \in \mathbb{R}^{2\times 2}$$

which of the following state space plots is possible?
why, why not, who cares?



[varaiya]

② a) give a controllable canonical realization of

$$\frac{d^3}{dt^3}y(t) + y(t) = \frac{d}{dt}u(t) + u(t)$$

is it observable?

b) ocf of same. controllable?

c) How do controllability and observability pertain to transfer functions?

STUDY GUIDE FOR THE DOCTORAL PRELIMINARY EXAMINATION

Published by the Electrical Engineering Graduate Students Association

University of California at Berkeley

GENERAL INFORMATION

This document is a collection of practice questions for most of the courses in which students can take the doctoral preliminary examination. It is by no means complete but we hope it will help students in preparing for the exam.

Be aware that the prelim system at Berkeley has the unfortunate feature that professors test based on the material they covered (or would like to have covered) when they taught the course, and this content can vary widely from professor to professor. To prepare properly for the prelims, you really must cover not just the topics listed in the course description but whatever has been taught in the course plus whatever might have been taught had there been more time. In other words, it is the ill-defined closure of the subject for which you are responsible.

The questions in this study guide are not meant to be comprehensive. They should however give you an idea of what type of questions to expect from the examiners. Notice that there is a wide disparity between the easiest questions and the most difficult ones. This is another feature of prelims at Berkeley. Different professors have different styles. Most of these questions are questions from previous prelim exams, most of them from the quarter version of the courses; however, every effort has been made to include questions from last spring's exam, the first one on the semester version of the courses.

A given examination may involve a number questions, so speed of response is important. Hints are often given freely. Take advantage of them! If you get stuck, it is often better to ask for a hint than to flounder, wasting precious time. If you don't understand a question or a hint, don't pretend! You won't fool anyone, and a simple question might clear things up very quickly.

Finally, remember that this document in no way reflects official departmental policy. It is prepared by graduate students in an attempt to make practice questions *uniformly available* to all students taking prelims.

The EEGSA would like to thank all the students who helped in the preparation of this document.

Spring 1986

EECS 221A - LINEAR SYSTEM THEORY

Problem 1

$R = (A, B, C, D)$ is a binary system, i.e. every value appearing in this system is either 0 or 1. Is the following system observable? Is it completely reachable from the origin? Is it completely controllable to the origin?

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Problem 2

a) Is the system $\dot{x} = Ax + Bxu$ linear?

b) If u is the unit step function i.e. $\dot{x} = Ax + Bx, t > 0$ can we write $x(t) = e^{At} e^{Bt} x_0$?

Problem 3

Consider the response of the system in question 1,

$$x(k+1) = A(k)x \qquad x(0) = x_0$$
$$y(k) = C(k)x$$

Define the map $\alpha : x_0 \longmapsto y(.)$ (the function $y(.)$). Find the adjoint of α.

Problem 4

Show that $\dot{x} = A(t)x + B(t)u$ has a unique solution.

Problem 5

Show that: (A,b) is c.c. <==> There exists k such that $\sigma(A - bk)$ can be assigned to an arbitrary spectrum ("constellation") Σ.

3

4

1. Let an operator Λ be defined by:

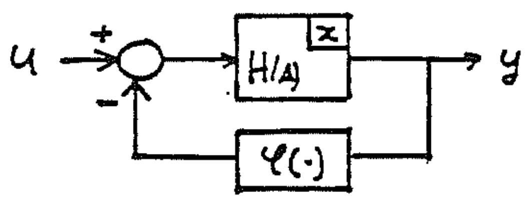
$$\Lambda u = y(t) = \int_0^{\infty} \lambda(\tau) u(t-\tau) d\tau$$

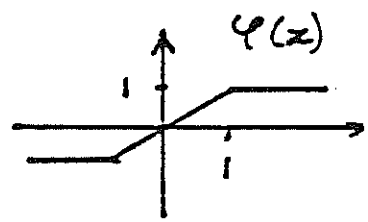
When is Λ BIBO stable? What if λ is a matrix?

2. We are given (A,b). For what polynomials $p(s)$ can we find a row vector such that $p(s) = \det(sI - A - bf)$? If (A,b) is controllable, how would you find such an f?

3. $\dot{x} = A(t)x$, $A(t+T) = A(t)$ and $A(t)$ is continuous.
(a) What can we say about $\Phi(t,\tau)$?
(b) When is this system stable? (in terms of Φ)
(c) When does it have a periodic solution?

4. The realization H(s) is minimal. We apply non-linear feedback:





Is the closed loop system controllable? Observable?

5. Consider the following implicit algorithm to integrate $\dot{x} = f(x)$:

$$\frac{1}{h}(x_{n+1} - x_n) = \frac{1}{2}(f(x_n) + f(x_{n+1}))$$
$$x_n = \hat{x}(nh)$$

Professor B uses this algorithm to simulate $\dot{x} = Ax$; one simulation run is exponentially unstable, so he says $\dot{x} = Ax$ is unstable. Professor A, however, says "A could be stable - you just used a stepsize h which is too big." Who's right? Discuss.

B is right
$\left|\frac{1+h\lambda}{1-h\lambda}\right| < 1$
$|1-h\lambda| > |1+h\lambda|$
$h\lambda < 0$
$\Rightarrow$ true if $\lambda < 0$ for any $h > 0$
$\Rightarrow$ false if $\lambda > 0$ for any $h > 0$

6. C(s) is 1-input, 3-output, P is 3-input, 1 output and both are stable. An old timer control engineer who is not familiar with the subtleties of MIMO control systems sketches the Nyquist plot of PC: Is the system actually stable? Discuss.

a

EE221A Fall 85. Prof. P. P. Varaiya.

Question 1 : True or false

a. If $A \in \mathbb{C}^{n\times n}$. then $\det(A) = 0$.

b. If $R(A) = R(A^TA)$?

c. If $AB = BA$, then $e^{A+B} = e^{A} \cdot e^{B}$.

d. $R(A) \perp N(A^*)$. prove it.

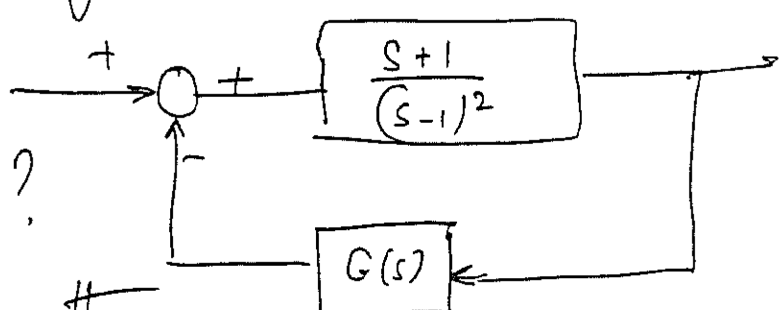
Question 2. System.

$$\frac{\partial^3 y}{\partial t^3} + y = \dot{u} + u.$$

(a) Give the controllable canonical form of the system.

(b). Is it observable.? why?

Question 3. Given.



(a) Is it open-loop stable?

(b). What is $G(s)$ so that the characteristic polynomial is $(s+2)^2$

Question 4.

Explain design of reduced-order (minimal) compensator.

Given $\dot{x} = A\dot{x}$, $A \in \mathbb{R}$, $x \in \mathbb{R}^n$ $\sigma(A) \in \mathbb{C}^-$, $x(0) = x_0$,

discretize the system using the following formulae

$$\text{BE)} \quad x_{k+1} = x_k + h\,\dot{x}_{k+1}$$

$$\text{FE)} \quad x_{k+1} = x_k + h\,\dot{x}_k$$

$$\text{TR)} \quad x_{k+1} = x_k + \frac{1}{2}h\,(\dot{x}_{k+1} + \dot{x}_k)$$

where $h \triangleq (t_{k+1} - t_k)$.

discuss the stability of the discretized system. as a function of h and $\sigma(A)$.

A.S.U.

Compute

$$\frac{d}{dh}(I - hD)^{-1}$$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix, $D \in \mathbb{R}^{n\times n}$; $h \in \mathbb{R}^1$.

Compute the sum of the series

$$I + A + A^2 + \dots$$

A.S.V.

//

(iii) Apply state feedback $u(t) = F\,x(t)$. the matrix F satisfies $Fx_0 = u_0$ (where $x_0 \in \mathbb{R}^n$ is an initial condition such that when an input $u(t) = u_0 e^{\lambda t}$, $t \geq 0$ is applied results in a zero output (this can be done if λ is a zero ...)) .
Is the feedback system still c.o. ?

221 A / Desoer

1. Consider the linear system $\dot{x} = A(t)\,x$ such that

$$A(t) = A(t+T) \qquad \forall t \in \mathbb{R} \qquad (\text{for some } T > 0)$$

- Under what conditions does the above system accept at least one periodic solution ?
- When is the system stable ?

2. Consider now $\dot{x} = A(t)\,x + u(t)$ with

$$A(t) = A(t+T)\,, \quad u(t) = u(t+T) \qquad \forall t \in \mathbb{R} \qquad (\text{for some } T > 0)$$

- The same questions.

221 A / Polak

1. • Given a feedback system as in the next figure. Write down the A, B, C matrices of the total system. /3

Problem 1

SASTRY

You are told the linear system

$$\dot{x} = A(t)x + B(t)u$$

$$y = C(t)x$$

is completely observable on $[t_0, t_1]$. Given $y(t)$ on the interval $[t_0, t_1]$ find $x(t_0)$.

15

14

Reduced Order Observers

221
Sastry

You are given the linear system

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n, \; u \in \mathbb{R}^{n_i}$$

$$y = Cx \qquad y \in \mathbb{R}^{n_o}$$

Further you are told that C has the form

$$\begin{array}{cc} & \overset{n}{\longleftrightarrow} \\ n_o \updownarrow & \left[\begin{array}{c:c} I_{n_o} & 0 \end{array} \right] \end{array}$$

Write down a reduced order ($n - n_o$) observer for this system.

II.

Consider

$$\dot{x} = Ax + Bu$$

You are told rank $[sI - A \,\vdots\, B] = n$ for all $s \in \mathbb{C}_+$ the closed right half plane $\iff$ The linear system is stabilisable

Is this true or false? If true prove, if false give counterexample

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} sI - A & | & B \end{bmatrix} = \begin{bmatrix} s-1 & 0 & | & 1 \\ 0 & s+1 & | & 0 \end{bmatrix}$$

$\therefore$ full rank $\forall$ all $s \in \mathbb{C}_+$

$$A + Bf^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [f_1 \; f_2] = \begin{bmatrix} 1+f_1 & f_2 \\ 0 & -1 \end{bmatrix}$$

but $\lambda = 1$ is eigenvalue $\Rightarrow$ unstable, and $[B \,\vdots\, AB] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow$ not C.

but does it mean we cant get to the point at 1?

19

16

[illegible])

You are given a T-periodic linear system

$$\dot{x} = A(t)x + B(t)u \qquad (1)$$

with periodic input $u(t)$ of period T. Find a T-period trajectory of the system. Discuss when all solutions of (1) converge towards the periodic solution.

2. Define the matrix $P(t) \in \mathbb{R}^{n \times n}$ for given $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n_o \times n}$ by

$$P(t) = \int_0^t e^{A^T(t-\tau)} C^T C\, e^{A(t-\tau)}\, d\tau .$$

$\dot P(t) = e^{A^T(t-t)} C^T C e^{A(t-t)} - 0 + A^T P(t) + P(t) A(t) = C^T C + A^T P(t) + P(t) A(t)$

$P(0) = 0_n$

(a) Find the differential equation that P satisfies?

(b) Find the limit as $t \to \infty$ of $P(t)$ given that $\sigma(A) \subset \mathbb{C}_-$.

$\to 0$ by $e^{A^T t} \int_0^t e^{-A^T \tau} C^T C e^{-A\tau} d\tau\, e^{At}$

(c) Give conditions so that $P(\infty)$ is positive definite. [10 points]

14

3. You are given a transfer function $G(s) \in \mathbb{R}^{n_o \times n_i}(s)$. How would you proceed to build a minimal realisation of it

HINT:- You may wish to start with a partial fraction expansion of $G(s)$ ($\displaystyle\sum_{i=1}^{p} \frac{R_i}{(s-\lambda_i)}$, for starters for e.g.).

[10 points]

PROBLEM 2

Consider the system:

$$\text{S:} \begin{cases} \dot{x} = Ax + Bu \\ y_1 = Cx + Du \end{cases}$$

where A is an $n \times n$ matrix.

(a) Denote the characteristic polynomial of this system by $\chi(s)$. Show that **S** is asymptotically stable if and only if there exists a polynomial $d(s)$, with real coefficients, of the same degree as $\chi(s)$, such that all of its zeros are in the *open left half complex plane*, $\overset{o}{C}_-$, and

$$\mathrm{Re}[\chi(j\omega)/d(j\omega)] > 0, \quad \forall\ \omega \in (-\infty, \infty)$$

(b) Suppose that there exist constants $T, \beta \in (0, \infty)$, and $\alpha \in (0,1)$ such that for all $x_0 \in \mathbb{R}^n$,

$$\|e^{TA}x_0\| \leq \alpha\|x_0\|$$

and

$$\|e^{tA}x_0\| \leq \beta\|x_0\|, \quad \forall\ t \in [0, T]$$

Show that the system **S** is asymptotically stable.

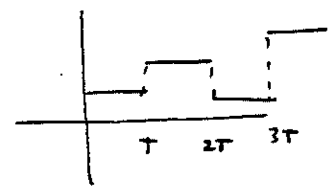
221A

I. Consider the LTI system described by

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t)$$

Suppose one uses piecewise constant control inputs:

$$u(t) = u_k \quad , \quad kT \le t < (k+1)T$$



Then there is a representation of the discrete time system in the form:

$$x_{k+1} = A_d\,x_k + B_d\,u_k$$
$$y_k = C_d\,x_k$$

where $x_k := x(kT)$, $y_k = y(kT)$.

Find A_d, B_d, C_d in terms of A, B, C, T.

II Consider

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $x \in R^n$, $u \in R^1$. Suppose

$$u(t) = u_0 \cos \omega_0 t \quad ; \quad u_0, \omega_0 \text{ are constant.}$$

Suppose

$$\sigma(A) \subset \mathring{\mathbb{C}}_-$$

Show that the state trajectory $t \to x(t)$ tends to an ellipse, centered at 0, and lying in a two-dimensional subspace of R^n.

III A rotating wheel is governed by

$$\ddot{\theta} + 0.01\,\dot{\theta} + \theta = 0$$

An instrument measures θ accurately. Design an observer to measure $\dot{\theta}$ with a time constant of 1.

E. Polak PRELIMS S221A Fall 1989

PROBLEM 1

(a) Consider the polynomial ratio

$$r(s) = \frac{n(s)}{d(s)} ,$$

where $d(s) = \prod_{i=1}^{N} (s - \lambda_i)$, with the λ_i known. Establish a frequency domain test for determining if all the zeros of $n(s)$ are in $\overset{o}{\mathbb{C}}_-$. What would be an advantageous selection of the polynomial $d(s)$?

(b) Consider the system

$$\text{S:} \quad \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) \end{cases} ,$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$ and the matrices $A(t)$, $B(t)$, $C(t)$ are continuous.

Define define what is meant by this system being (i) b.i.b.o. stable, and (ii) asymptotically stable.

(c) Consider the interconnected system:

$$\text{S}_1\text{:} \quad \begin{cases} \dot{x}_1 = A_1x_1 + B_1u_1 \\ y_1 = C_1x_1 \end{cases}$$

$$\text{S}_2\text{:} \quad \begin{cases} \dot{x}_2 = A_2x_2 + B_2u_2 \\ y_2 = C_2x_2 \end{cases}$$

$$\begin{cases} u_1 = v_1 - y_2 \\ u_2 = v_2 + y_1 \end{cases} .$$

(i) Obtain two formulae for the characteristic polynomial of the above interconnected system.

(ii) State and prove the multivariable Nyquist stability criterion, as it applies to the above interconnected system, and discuss its value in determining the asymptotic stability of interconnected systems.

27

PROBLEM 2

(a) Consider the system:

$$S: \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

where A is an $n \times n$ matrix.

Denote the characteristic polynomial of this system by $\chi(s)$. Show that S is asymptotically stable if and only if there exists a polynomial $d(s)$, with real coefficients, of the same degree as $\chi(s)$, such that all of its zeros are in the *open left half complex plane*, $\overset{o}{C}_-$, and

$$\mathrm{Re}[\chi(j\omega)/d(j\omega)] > 0, \quad \forall\, \omega \in (-\infty, \infty)$$

(b) How would you modify the above test so as to be able to determine if the zeros of $\chi(s)$ are in the sector $C \triangleq \{ s = \alpha + j\omega \in \mathbb{C} \mid \alpha = -2|\omega|,\ \omega \in (-\infty, \infty) \}$.

3. Given A, B a completely controllable pair and $\phi: \mathbb{R}^n \to \mathbb{R}^m$ ~~of~~ a Lipschitz function; can you say something about the controllability of

$$\dot{x} = Ax + B\phi(x) + Bu$$

(5 points)

2 Given a matrix

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

How many (minimum no) outputs do you need to get a completely observable system?

(5 points)

30

Prelim 221A Feb 14, 1989 Sanders

① What is Jordan form for matrix $A \in \mathbb{R}^{n \times n}$?

a) How does one obtain Jordan Form ?
(distinct λ_i)

b) Non-distinct eigenvalues ?

c) Given system

$$\dot{x} = Jx + bu$$

How does one test controllability ?
– example

d) True or False : Any two LTI systems with identical char. polynomial have similar A-matrices

e) T or F : Any two LTI controllable, single-input systems with same char. poly have similar A-matrices

3

PROBLEM 1

Consider a controllable system

$$\dot{x} = Ax + bu .$$

your friend from CMU says that he can use state feedback $u = f^T x + v$ to place all the closed loop eigenvalues at $s = 0$ and thereby get

$$\dot{x} = bv$$

TRUE or FALSE ?

221A PRELIMS SPRING '89

PROBLEM 2.

Consider a minimal, SISO linear system

$$\dot{x} = Ax + bu \qquad u \in \mathbb{R}$$

$$y = cx + du \qquad y \in \mathbb{R}$$

The transfer function $\hat{h}(s) = c(sI-A)^{-1}b + d$.

Consider the following feedback laws:

① $\dot{x} = Ax + b(f^T x + v)$ STATE FEEDBACK

$y = cx + d(f^T x + v)$

② $\dot{x} = Ax + bu + \underline{l}\,(cx + du)$ OUTPUT INJECTION

$y = cx + du$

Discuss what happens to controllability, observability ~~and the~~ in each case

What are the 'zeros' of the linear systems (1), (2) (before cancellation).

PROBLEM 3.

3. ~~What~~ You are told that

$$z_1 = c_1 x$$
$$z_2 = c_2 x \qquad x \in \mathbb{R}^n, \; z_i \in \mathbb{R}$$
$$\vdots$$
$$z_n = c_n x.$$

and that c_i's are linearly independent rows and they are surjective ~~and surjective~~

What is the linear least squares estimate of x given z_1? (Call it $\hat{x}_1$)
How about the linear least squares estimate of x given z_1, z_2? (Call it $\hat{x}_2$). Can you relate $\hat{x}_2$ to $\hat{x}_1$.

Do you see the recursion?

EECS 221a HW#1 Due 9/8/89

1. Define addition and multiplication on $\{0, 1\}$ to form a field.

2. Show that the set of all polynomials in s of degree k or less with real coefficients is a vector space over $\mathbb{R}$. Find a basis. What is the dimension of the vector space.

3. Consider the derivative operator on the set of polynomials of degree 3 or less. Is it a linear transformatión? If so find its matrix representatión with respect to the bases of your choice.

4. Consider the set of sequences $\{f_k\}_{k=0}^{\infty} := \{f_0, f_1, f_2, \dots\}$ satisfying $f_k = f_{k-1} + f_{k-2}$ where f_0 and f_1 are arbitrary real numbers. Is this a subspace in the vector space of all sequences of real numbers?

5. The unit sphere is defined by $\|x\| = 1$. Draw the unit spheres of the normed vector space $(\mathbb{R}^2, \mathbb{R}, \|\cdot\|)$ with the norm defined by (i) $\|(x_1, x_2)\|_\infty = \max(|x_1|, |x_2|)$, (ii) $\|(x_1, x_2)\|_1 = |x_1| + |x_2|$, and (iii) $\|(x_1, x_2)\|_2 = \sqrt{x_1^2 + x_2^2}$