EE382m: Homework 6
Automata Theory and Mathematical Logic

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Due October 30 (Monday)

1. This problem is to designed increase your familiarity with describing strings using automata.
   Design deterministic finite state automata that accept the following languages:

   (a) alphabet : $\{a, b, c\}$, every occurence of a a is eventually followed by a b and then a c.

   (b) alphabet : $\{a, b\}$, has both ab and ba as substrings.

   (c) alphabet : $\{0, 1\}$, the substrings 0000 and 1111 never occur

   (d) alphabet : $\{\text{Thinking, Eating, Reading}\}^n$, given string $s = \langle s_1, s_2, \cdots, s_m \rangle$, it is in the language iff it is never true that for any $k, 1 \leq k \leq m$,

   $$\exists i \left(s(k)_i \quad = \quad \text{Eating} \; \wedge \; s(k)_{(i+1) \bmod n} = \text{Eating} \right)$$

   15 marks

2. This problem illustrates the power and the complexity of nondeterminism

   (a) Prove that NFA are exponentially more succint than DFA in the sense that there is a sequence of regular languages $\mathcal{L}_1, \mathcal{L}_2, \ldots$ with the property that

   - for each $\mathcal{L}_n$ there exists an NFA recognizing $\mathcal{L}_n$ with $c \cdot n$ states for some constant c.
   - the minimum state DFA for $\mathcal{L}_n$ requires at least d^n states, for some $d > 1$.

   Hint: consider the language T_k over $\{0, 1\}$ where $x \in T_k$ iff the k-th last letter in T_k is a 1.

   20 marks

3. Express the set of possible output traces for the design in Figure 1 as the language recognized as a DFA (assume that the netlist can power up in any state).

   15 marks

4. *(Challenging)*
   Let L be a language. Define $\frac{1}{2}L$ to be

   $$\{x \mid \text{for some } y \text{ such that} \, \text{length}(x) = \text{length}(y), \; x \cdot y \in L\}$$

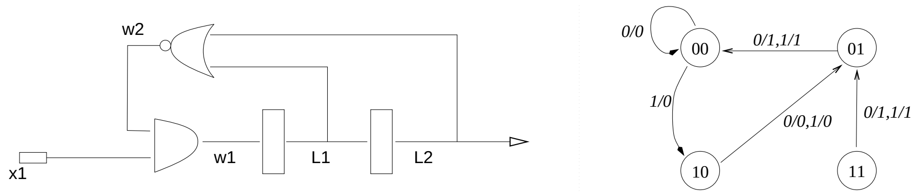

Figure 1: A netlist and its associated STG.

That is, $\frac{1}{2}L$ is the first halves of strings in L. Proce that if L is regular, then so is $\frac{1}{2}L$.

20 marks

5. There are three suspects for a murder: Adams, Brown, and Clark.

   - Adams says "I didn't do it. The victim was a friend of Brown's. But Clark hated him."
   - Brown states "I didn't do it. I didn't even know him. Besides, I was out of town that week."
   - Clark say "I didn't do it. I saw both Adams and Brown downtown with the victim that day; one of them must have done it."

   Assume that the two innocent men are telling the truth, but that the guilty man may not be. Who did it?

   (Hint: create atomic propositions corresponding to the individual facts stated by the men.)

15 marks

6. Recall that the *formulas* of propositional logic are sequences over the set

$$\Sigma_{\text{PL}} = \{(,), \neg, \wedge, \vee, \rightarrow, \leftrightarrow, A_0, A_1, A_2, \ldots\}$$

   The *well formed formulas* (wff's) of propositional logic were defined as follows:

   - For every A_i, the sequence $\langle A_i \rangle$ is a wff,
   - If f is a wff, then so is $\langle (\rangle \frown f \frown \langle) \rangle$,
   - If f and g are wff's, then so are $\langle (\rangle \frown f \frown \langle \wedge \rangle \frown g \frown \langle) \rangle$, $\langle (\rangle \frown f \frown \langle \vee \rangle \frown g \frown \langle) \rangle$ $\langle (\rangle \frown f \frown \langle \leftrightarrow g \frown \rangle \frown \langle) \rangle$ $\langle (\rangle \frown f \frown \langle \rightarrow g \frown \rangle \frown \langle) \rangle$
   - No formulas, other than those constrained by the above, are wff's.

   (Here $p \frown q$ denotes the concatenation of the two sequences p and q.)

   Let P, Q, R, S, T be arbitrary wff's. Prove the following:

   (a) If $P = \langle (\rangle \frown Q \frown \langle \vee \rangle \frown R \frown \langle) \rangle$, then $p \neq \langle (\rangle \frown S \frown \langle \wedge \rangle \frown T \frown \langle) \rangle$

   (b) If $P = \langle (\rangle \frown Q \frown \langle \wedge \rangle \frown R \frown \langle) \rangle$ and $P = \langle (\rangle \frown S \frown \langle \wedge \rangle \frown T \frown \langle) \rangle$ then $Q = S$ and $R = T$.

15 marks

7. Show that no one of the following sentences in first order logic is logically implied by the other two. (This is done by giving a structure in which the sentence in question is false, while the other two are true.)

   - $\forall x \forall y \forall z \big(Pxy \to (Pyz \to Pxz)\big)$
   - $\forall x \forall y \big(Pxy \to (Pyx \to (x = y))\big)$
   - $\forall x \exists y Pxy \ \to \ \exists y \forall x Pxy$

15 marks

8. The first order logic of *pure equality* is defined over the following alphabet:

$$\Sigma_{\mathtt{PE}} = \{(,), \neg, \to, =, \forall, x_0, x_1, \ldots\}$$

   Thus there are no parameter symbols.

   A sentence σ is said to be *existential* if it is of the form $\exists x_{i_1} \exists x_{i_2} \ldots \exists x_{i_k} \phi$ where ϕ is a formula with no quantifier symbols.

   Examples: $\exists x_1 \exists x_2 (x_1 \neq x_2)$ and $\exists x_3 \exists x_4 \exists x_2 (((x_3 = x_2) \wedge (x_4 = x_3)) \to (\neg(x_2 = x_4)))$ are existential sentences; $\exists x_1 (x_1 = x_2 \to x_1 = x_1)$ is not (since x_2 occurs free in it).

   Let σ be an existential sentence in this language, with n variable symbols occurring in it. Prove the following:

   Lemma: The sentence σ has a model if and only if it has a model where the universe has at most n elements.

   Additionally, show that the bound is "tight", i.e., there are σ on n variables so that no less than n elements are required in the universe.

   As an example, the sentence $\exists x_1 \exists x_2 (x_1 \neq x_2)$ has a model of size 2 (simply take any two element set as the universe, and σ is true in it).

15 marks