Problem 1 (40%): Show that for the state equation

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

with \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) to be completely controllable it is necessary and sufficient that the only matrix \( X \in C^{n \times n} \) (\( C \) denotes the field of the complex numbers), satisfying

\[ XA = AX \quad XB = 0 \]

is \( X = 0 \).

Problem 2 (30%): Let \( A, F \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^{n} \) and \( G := A - F \). Show that the system

\[ \dot{x}(t) = e^{Ft}Ae^{-Ft}x(t) + e^{Ft}bu(t) \]

is completely controllable if and only if

\[ \det \begin{bmatrix} b & Gb & \cdots & G^{n-1}b \end{bmatrix} \neq 0. \]

Problem 3 (30%): Let \( A(t) \in \mathbb{R}^{n \times n} \) be continuous in \( t \). Show that the equation

\[ \dot{x}(t) = A(t)x(t) \]

is exponentially stable, if there exist constants \( c \geq 0 \) and \( a > 0 \) such that for all \( t_0 \) and \( t > t_0 \)

\[ \int_{t_0}^{t} \lambda_m(\tau) \, d\tau \leq -a(t - t_0) + c, \]

where \( \lambda_m(t) \) denotes the largest eigenvalue of \( A(t) + A^T(t) \).

By definition \( \dot{x}(t) = A(t)x(t) \) is called exponentially stable if there exist constants \( M > 0 \) and \( \gamma > 0 \) such that for every \( t_0 \in \mathbb{R}, \|x(t)\| \leq M\|x(t_0)\|e^{-\gamma(t-t_0)} \), for all \( t > t_0 \).
Problem 1 (40%): Let $A \in \mathbb{R}^{n \times n}$. It is proposed to integrate the differential equation $\dot{x} = Ax$, $x(0) = x_0$ by

(a) the Euler method: $\xi_{k+1} = (I + hA)\xi_k$, $\xi_0 = x_0$

(b) the backward Euler method: $\eta_{k+1} = (I - hA)^{-1}\eta_k$, $\eta_0 = x_0$

(Note that if $h > 0$ is very small such that $\|hA\| \ll 1$, the two methods give the same result $O(h)$.)

Prove that:

1. $\forall A \in \mathbb{R}^{n \times n}$ with eigenvalues in the open left-half complex plane, $\exists h_0 > 0$ such that $\forall h > h_0$, algorithm (a) diverges.

2. $\forall A \in \mathbb{R}^{n \times n}$ with eigenvalues in the open left-half complex plane and $\forall h > 0$, algorithm (b) yields a sequence which converges (of course, if $h$ is too large the answer is not accurate).

Problem 2 (30%): Recall that a linear, time-invariant system of the form

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ is stabilizable by linear state feedback if there exists a matrix $K \in \mathbb{R}^{m \times n}$ such that the control $u(t) = Kx(t)$ makes the closed loop system asymptotically stable to the origin.

Suppose that (2.1) is not stabilizable by linear state feedback. Is it possible that for some nonlinear function $f : \mathbb{R}^n \to \mathbb{R}^m$ the control assignment $u(t) = f(x(t))$ makes the closed loop system $\dot{x}(t) = Ax(t) + Bf(x(t))$ asymptotically stable to the origin? Explain fully.

Problem 3 (30%): Recall that

$$\dot{x}(t) = A(t)x(t), \quad x \in \mathbb{R}^n, \quad t \geq 0$$

is called uniformly stable if there is a constant $\gamma \in \mathbb{R}$ such that for all $t \geq \tau \geq 0$ the state transition matrix $\Phi(t, \tau)$ of (3.1) satisfies

$$\|\Phi(t, \tau)\| < \gamma.$$
Suppose now that (3.1) is uniformly stable and that $B(t) \in \mathbb{R}^{n \times n}$ is such that the integral

$$
\int_0^\infty B(\sigma) \, d\sigma
$$

exists and is finite. Prove that

$$
\dot{x}(t) = [A(t) + B(t)]x(t), \quad t \geq 0
$$

is uniformly stable.
Problem 1 (40%): Consider the differential equation
\[ \dot{x}(t) = A(t)x(t), \quad x \in \mathbb{R}^n, \quad t \geq 0 \] (1.1)
and suppose that the matrix \( A \) satisfies
\[ \int_0^\infty \|A(t)\| \, dt < \infty. \]
Is it true that if \( x(t) \) is any solution of (1.1), then the limit
\[ \lim_{t \to \infty} x(t) \]
e exists? Provide a proof or a counterexample.

Problem 2 (30%): Let \( V \) be a finite dimensional inner-product space over \( \mathbb{C} \) (the field of complex numbers), of dimension \( n \). We denote by \( \langle \cdot, \cdot \rangle \) the inner product on \( V \) and by \( \| \cdot \| \) the induced norm. A linear operator \( T : V \to V \) is called unitary if it satisfies \( T^*T = TT^* = I \), where \( T^* \) denotes the adjoint of \( T \). Prove that the following statements are equivalent.

(a) \( T \) is unitary.

(b) \( \|Tx\| = \|x\| \), for all \( x \in V \).

(c) If \( \{v_1, \ldots, v_n\} \) is an orthonormal basis of \( V \) then \( \{Tv_1, \ldots, Tv_n\} \) is also an orthonormal basis of \( V \).

Problem 3 (30%): Consider the control system
\[ \dot{x}(t) = g(t)\left[ Ax(t) + Bu(t) \right], \quad x \in \mathbb{R}^n, \]
where \( g(t) \) is a continuous and bounded real valued function satisfying
\[ 0 < \alpha \leq g(t) \leq \beta < \infty. \]
Show that if
\[ \text{rank} \left[ B \mid AB \mid \cdots \mid A^{n-1}B \right] = n \]
then given any \( T > 0 \) and any \( x_0, x_1 \in \mathbb{R}^n \), there exists a control \( u(\cdot) \) which steers \( x_0 \) at \( t = 0 \) to \( x_1 \) at \( t = T \).

Hint: Consider an appropriate change of time scales.
Problem 1 (30%): If $F$ and $Ce^{At}B$ are square $n \times n$ matrices, show that the weighting pattern

$$G(t, \sigma) = e^{-Ft}Ce^{A(t-\sigma)}Be^{F\sigma}$$

has a constant parameter realization if and only if

$$FCA^kB = CA^jBF, \quad j = 0, 1, 2, \ldots$$

Problem 2 (35%): Consider the differential equation

$$\ddot{x}(t) + a_1(t)\dot{x}(t) + a_2(t)x(t) = 0, \tag{1}$$

where $a_1(t)$ and $a_2(t)$ are real periodic functions of period $T > 0$. Note that (1) can be expressed in matrix form as

$$\frac{d}{dt} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_2(t) & -a_1(t) \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}. \tag{2}$$

Let $\Phi(t, 0)$ be the state transition matrix of (2) and $\Phi(t, 0) = Z(t)e^{Rt}$ its Floquet decomposition. Note that if $s$ is an eigenvalue of $\Phi(T, 0)$ then it satisfies

$$s^2 - bs + c = 0,$$

where

$$b := \text{trace}[\Phi(T, 0)] \quad c := \text{det}[\Phi(T, 0)].$$

Given $b$ and $c$ can you determine the stability properties of (2)? How?

Now suppose that $a_1(t) = 0$ so that (1) reduces to the special Hill’s equation. Show that

(a) If $|b| < 2$ then the equation is uniformly stable.

(b) If $b = 2$ the equation has at least one periodic solution of period $T$ and if $b = -2$ then there is at least one periodic solution of period $2T$. 

Problem 3 (35%): Find the control $u$ and the final time $T$ such that the scalar system

$$\dot{x}(t) = u(t)$$

is driven from $x(0) = 0$ to $x(T) = 1$ while minimizing (over $u$ and $T$) the quantity

$$J = \int_0^T u^2(t) \, dt + T.$$
Problem 1 (40%): Consider the controllability grammian

\[ W(t, t_1) := \int_t^{t_1} e^{A(t-\tau)}BB^*e^{A^*(t-\tau)}d\tau. \]

of the time invariant system

\[ \dot{x}(t) = Ax(t) + Bu(t), \] \hspace{1cm} (1)

(a) Prove that its derivative \( \frac{d}{dt} W(t, t_1) \) is a negative semi-definite matrix.

(b) Let \( t_1 > 0 \) be arbitrary and consider the feedback control

\[ u(t) := -B^*[W(0, t_1)]^{-1}x(t). \]

Under the assumption that the system in (1) is completely controllable, show that the closed loop system

\[ \dot{x}(t) = \left( A - BB^*[W(0, t_1)]^{-1} \right)x(t) \] \hspace{1cm} (2)

is asymptotically stable and that \( V(x) := \langle x, [W(0, t_1)]^{-1}x \rangle \) is a suitable Lyapunov function for (2).

Problem 2 (30%): In the following questions \( A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n \) and \( c \in \mathbb{R}^{1 \times n} \).

(a) If \( \{A, b\} \) is given and not controllable, is it always possible to choose \( c \) so that \( \{c, A\} \) is observable? A full explanation or a counterexample will suffice.

(b) If \( \{A, b\} \) is given and is controllable, can we always choose \( c \) so that \( \{c, A\} \) is observable?

Problem 3 (30%): Consider the differential equation

\[ \dot{x}(t) = A(t)x(t), \quad x \in \mathbb{R}^n, \quad t \geq 0 \] \hspace{1cm} (1)

and suppose that the matrix \( A \) satisfies

\[ \int_0^\infty \|A(t)\| dt < \infty. \]

Is it true that if \( x(t) \) is any solution of (1), then the limit

\[ \lim_{t \to \infty} x(t) \]

exists? Provide a proof or a counterexample.
In the problems that follow, all random variables are defined in a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and $\mathcal{E}$ denotes the expectation operator in this probability space.

**Problem 1.** For each positive integer $n$, let $X_n$ be a zero mean, unit variance Gaussian real random variable, and assume that this resulting set $\{X_n\}$ of random variables are mutually independent. Show that there exists no random variable $X$ such that $X_n$ converges in probability to $X$ as $n \to \infty$.

**Problem 2.** Let $X$ be a real random variable such that $\mathcal{E}|X| < \infty$ and $Y$, $Z$ real random variables such that the $\sigma$-algebra generated by $X$ and $Y$ is independent of the $\sigma$-algebra generated by $Z$. Prove that

$$\mathcal{E}[X \mid Y, Z] = \mathcal{E}[X \mid Y] \quad \text{a.e.}$$

**Problem 3.** Let $\{X_n\}$ be a sequence of real random variables and suppose that, for each $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathcal{P}\left\{ \sup_{m>n} \left| \sum_{k=n+1}^{m} X_k \right| > \varepsilon \right\} = 0.$$ 

Prove that $\sum_{k=1}^{n} X_k$ converges a.e. (equivalently: with probability 1) as $n \to \infty$. 
In the problems that follow, all random variables are defined in a probability space \((\Omega, \mathcal{F}, \mathcal{P})\), and \(\mathcal{E}\) denotes the expectation operator in this probability space.

**Problem 1 (30%)**: Let \(X\) be a real random variable which, for all nonnegative integers \(n\), satisfies

\[
x^n \mathbb{P}\{|X| \geq x\} \longrightarrow 0, \quad \text{as } x \to \infty. \tag{1}
\]

Show that all moments of \(X\) are finite, or, in other words, that \(\mathcal{E}|X|^n < \infty\), for all \(n \geq 0\).

Is the converse true? Prove or exhibit a counterexample.

**Problem 2 (30%)**: Let \(X\) be an integrable real random variable, \(Y\) a bounded real random variable and \(A\) a sub-\(\sigma\)-field of \(\mathcal{F}\). Show that

\[
\mathcal{E}\left[Y \mathcal{E}[X | A]\right] = \mathcal{E}\left[X \mathcal{E}[Y | A]\right].
\]

**Problem 3 (40%)**: Let \(\{Y_n\}, n = 0, 1, \ldots,\) be a sequence of independent, identically distributed, real random variables and let \(\mathcal{F}_n\) denote the \(\sigma\)-field generated by \(\{Y_0, Y_1, \ldots, Y_{n-1}\}\). Let \(\tau_1, \tau_2\) be integer valued random variables such that

\[
0 < \tau_1 < \tau_2 < \infty \quad \text{a.e.}
\]

In addition, suppose that the set \(\{\tau_i = n\} \in \mathcal{F}_n\), for all \(n = 1, 2, \ldots, i = 1, 2\).

Let \(Y_{\tau_i}\) denote the random variable whose value at \(\omega \in \Omega\) is \(Y_{\tau_i}(\omega)\).

Show that \(Y_{\tau_1}\) and \(Y_{\tau_2}\) are independent and identically distributed.

**Hint**: Start with

\[
P\{Y_{\tau_1} < x_1, Y_{\tau_2} < x_2\} = \sum_{0 < n_1 < n_2 < \infty} P\{\tau_1 = n_1, Y_{n_1} < x_1, \tau_2 = n_2, Y_{n_2} < x_2\}
\]
Problem 1. Recall that the set $\mathbb{Q}$ of rational numbers is countable and let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rationals. Define a function $f : \mathbb{R} \rightarrow \mathbb{R}$ via $f(x) = 0$ if $x$ is not rational and $f(x) = \frac{1}{n}$ if $x = r_n$. At what real numbers $x$ is $f$ continuous?

Problem 2. Let $\{f_n\}$ be a sequence of continuous real-valued functions defined on the real line, which converges uniformly to a function $f$ on a set $K \subset \mathbb{R}$. Prove that

$$\lim_{n \to \infty} f_n(x_n) = f(x),$$

for every sequence of points $x_n \in K$ such that $x_n \to x$, and $x \in K$.

Problem 3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that

$$\int_0^{\infty} f^2(x) \, dx < \infty.$$

Suppose, in addition, that $f$ is continuously differentiable and for some constant $M \in \mathbb{R}$ its derivative $f'$ satisfies

$$|f'(x)| < M, \quad \forall x \in \mathbb{R}.$$

Prove that

$$\lim_{x \to \infty} f(x) = 0.$$
Problem 1. Suppose \( \{ f_n \} \) is a sequence of monotonic functions on \([a, b]\), and \( \{ f_n \} \) converges pointwise to a continuous function \( f \) on \([a, b]\). Prove that the convergence is uniform on \([a, b]\).

Problem 2. Call a mapping \( f : X \to Y \) open if \( f(V) \) is an open set in \( Y \) whenever \( V \) is an open set in \( X \). Prove that every continuous open mapping of \( \mathbb{R} \) into \( \mathbb{R} \) is monotonic (\( \mathbb{R} \) stands for the real line).

Problem 3. Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function and \( f' \) denote its derivative. Suppose \( f' \) is continuous on \([a, b]\) and \( \varepsilon > 0 \). Prove that there exists \( \delta > 0 \) such that
\[
\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon
\]
whenever \( |t - x| < \delta \), \( a \leq x \leq b \), \( a \leq t \leq b \).