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On the controllability of a class of nonlinear stochastic systems ☆

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Abstract

We study the controllability properties of the class of stochastic differential systems characterized by a linear controlled diffusion perturbed by a smooth, bounded, uniformly Lipschitz nonlinearity. We obtain conditions that guarantee the weak and strong controllability of the system. Also, given any open set in the state space we construct a control, depending only on the Lipschitz constant and the infinity-norm of the nonlinear perturbation, such that the hitting time of the set has a finite expectation with respect to all initial conditions.

Keywords: Stochastic controllability; Diffusions; Recurrence

1. Introduction

In this paper, we investigate the weak and strong controllability of the class of nonlinear stochastic systems characterized by the Itô equation

$$\mathrm{d}X(t) = AX(t)\,\mathrm{d}t + Bu(t)\,\mathrm{d}t + C\,\mathrm{d}W_t + f(X(t))\,\mathrm{d}t, \quad X(0) = X_0, \tag{1.1}$$

where A, B and C are $n \times n$, $n \times l$ and $n \times m$ constant matrices, respectively, and the function f is smooth, bounded and uniformly Lipschitz continuous on $\mathbb{R}^n$. The m-dimensional Brownian motion $\{W_t, \mathscr{F}_t\}_{t \geqslant 0}$ and the initial distribution X_0 are defined on a complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$ and are independent, while $\{u(t)\}_{t \geqslant 0}$ is a *feedback control*, i.e., an $\mathbb{R}^l$-valued process which is progressively measurable with respect to the filtration $\mathscr{F}_t^0 = \sigma\{X(s),\ s \leqslant t\}$.

Definition 1.1. System (1.1) is said to be *weakly controllable* if, for any initial state $x_0 \in \mathbb{R}^n$ and any nonempty open set $V \subset \mathbb{R}^n$, there exists a feedback control $\{u(t)\}_{t \geqslant 0}$ such that the corresponding solution $X(\cdot)$ of (1.1) satisfies

$$P^{x_0}(X(t) \in V, \text{ for some } t > 0) > 0. \tag{1.2}$$

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System (1.1) is said to be *strongly controllable* if a feedback control can be found such that (1.2) holds and the hitting time

$$\tau_V = \inf\{t > 0,\ X(t) \in V\}$$

satisfies $E^{x_0}[\tau_V] < \infty$.

Consider the linear system

$$\mathrm{d}X(t) = AX(t)\,\mathrm{d}t + Bu(t)\,\mathrm{d}t + C\,\mathrm{d}W_t, \tag{1.3}$$

which is obtained from (1.1) by letting $f = 0$. Zabczyk has obtained the following necessary and sufficient conditions for the weak and strong controllability of linear system (1.3).

Theorem 1.1 (Zabczyk [9]). (i) *Linear system* (1.3) *is weakly controllable iff*

$$\operatorname{rank}[B,\ AB,\ \ldots,\ A^{n-1}B,\ C,\ AC, \ldots, A^{n-1}C] = n. \tag{1.4}$$

(ii) *Linear system* (1.3) *is strongly controllable iff it is weakly controllable and the matrix A is stable.*

Our objective in this paper is to study the controllability properties of the nonlinearly perturbed system (1.1). Using a Lyapunov function approach, Sunahara et al. [7,8], obtained conditions for the stochastic controllability of nonlinear systems. However, those conditions are often difficult to verify. We show that, when the nonlinearity f is a smooth, bounded, uniformly Lipschitz continuous function, the conditions obtained in Theorem 1.1 for linear systems are sufficient for the controllability of (1.1). A result which is utilized in our analysis concerning the controllability of the corresponding deterministic system is presented in Section 2. The main results of the paper are in Section 3.

2. Controllability of the corresponding deterministic system

Consider the nonlinear deterministic system

$$\dot{x} = Ax + Bu + f(x), \tag{2.1}$$

corresponding to stochastic system (1.1). As usual, system (2.1) is said to be controllable if for every $x_0, x' \in \mathbb{R}^n$ and $t' > 0$, there exist a control $\{u(t),\ 0 \leqslant t \leqslant t'\}$ such that $x(0) = x_0$ and $x(t') = x'$.

The following lemma is sufficient for our purpose; however, more general results are known (see [2]).

Lemma 2.1. *Suppose that* (2.1) *satisfies*

(1) $\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n$.

(2) *The function f is Lipschitz continuous and bounded on $\mathbb{R}^n$.*

Then system (2.1) *is controllable.*

3. Main results

In this section, we obtain sufficient conditions for the weak and strong controllability of stochastic system (1.1).

Theorem 3.1. *Suppose that* (1.4) *holds. Then, the nonlinear stochastic system* (1.1) *is weakly controllable.*

Proof. Consider the deterministic system

$$\dot{x}(t) = Ax(t) + Bu(t) + Cw(t) + f(x(t)), \quad x(0) = x_0, \tag{3.1}$$

where both $u(t) \in \mathbb{R}^l$ and $w(t) \in \mathbb{R}^m$ are control parameters. By Lemma 2.1, given any $t' > 0$ and $x' \in \mathbb{R}^n$, there exist controls $\tilde{u}(\cdot)$ and $\tilde{w}(\cdot)$ which steer $(x_0, 0)$ to (x', t'). Now fix the control $\tilde{u}(\cdot)$ in (3.1), and let V be an open neighborhood of x'. The accessibility set from x_0 at time t' of (3.1), with control parameter $w(\cdot)$, clearly contains the point x'. It follows from the Stroock–Varadhan support theorem [4] that $P^{x_0}(X(t') \in V) > 0$. $\square$

Next we state the strong controllability result.

Theorem 3.2. *Suppose that* (1.4) *holds and that the pair* (A, B) *is stabilizable. Then* (1.1) *is strongly controllable.*

The proof of Theorem 3.2 is presented in a series of separate theorems and lemmas which comprise the rest of the paper. The following theorem resolves the case when $B = 0$.

Theorem 3.3. *Suppose that* $B = 0$, A *is a stable matrix and* (A, C) *is a controllable pair. Then, the diffusion* $\{X(t)\}_{t\geqslant 0}$ *in* (1.1) *is positive recurrent, i.e., for any nonempty, open set* V *and* $x_0 \in \mathbb{R}^n$, $E^{x_0}[\tau_V] < \infty$.

Proof. Let $Q \in \mathbb{R}^{n\times n}$ be the solution to the Lyapunov equation

$$A^*Q + QA = -I.$$

If $v(x) := x^*Qx$ and L denotes the infinitesimal generator of (1.1) with $B = 0$, it holds

$$\lim_{\|x\|\to\infty} Lv(x) = -\infty.$$

By Theorem 3.1, the system satisfies property (1.2). Therefore, it suffices to show that $X(t)$ has an invariant probability measure (see [1,3,6]). Define the *occupation measures* $\{\rho_t,\ t \in \mathbb{R}_+\}$ of $X(\cdot)$ as follows:

$$\int g\,\mathrm{d}\rho_t = -\frac{1}{t}E\left[\int_0^t g(X(s))\,\mathrm{d}s\right] \quad \text{for } g \in C_b(\mathbb{R}^n).$$

The desired result would follow if we show that $\{\rho_t,\ t \in \mathbb{R}_+\}$ is tight. For $M > 0$, choose a compact set $K_M \subset \mathbb{R}^n$ such that

$$\sup_{x\in K_M^c} Lv(x) \leqslant -M.$$

Let $D = \sup_{x\in\mathbb{R}^n}\{Lv(x)\}$. By Itô's formula, assuming the second moment of $X(0)$ is finite,

$$0 \leqslant E[v(X(t))] = E[v(X(0))] + E\left[\int_0^t I_{\{X(s)\in K_M\}} Lv(X(s))\,\mathrm{d}s\right] + E\left[\int_0^t I_{\{X(s)\in K_M^c\}} Lv(X(s))\,\mathrm{d}s\right]$$

$$\leqslant E[v(X(0))] + Dt\rho_t(K_M) - Mt\rho_t(K_M^c).$$

Hence,

$$\rho_t(K_M^c) \leqslant \frac{D}{M+D} + \frac{E[v(X(0))]}{(M+D)t},$$

and the tightness of $\{\rho_t,\ t \in \mathbb{R}_+\}$ follows. $\square$

We now continue with some preliminary analysis needed for the proof of Theorem 3.2. Decomposing (1.1) with respect to the invariant subspace $\mathscr{R}[B, AB, \ldots, A^{n-1}B]$, where $\mathscr{R}$ denotes the range of a matrix, yields

$$\mathrm{d}X_1(t) = A_{11}X_1(t)\,\mathrm{d}t + A_{12}X_2(t)\,\mathrm{d}t + B_1u(t)\,\mathrm{d}t + F_1\,\mathrm{d}W_t + f_1(X_1(t), X_2(t))\,\mathrm{d}t,$$

$$\mathrm{d}X_2(t) = A_{22}X_2(t)\,\mathrm{d}t + C_2\,\mathrm{d}W_t + f_2(X_1(t), X_2(t))\,\mathrm{d}t,$$

where $X_1(t) \in \mathbb{R}^{n-d}$, $X_2(t) \in \mathbb{R}^d$ and (A_{11}, B_1), (A_{22}, C_2) are controllable pairs. Furthermore, we can select feedback of the form $u(t) = KX_1(t)$ such that $A'_{11} := A_{11} + KB_1$ is a stable matrix with spectrum disjoint from that of A_{22} [5]. Thus, if N is the solution of $NA_{22} - A'_{11}N = -A_{12}$, the transformation

$$\begin{pmatrix} I & N \\ 0 & I \end{pmatrix}\begin{pmatrix} A'_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} I & -N \\ 0 & I \end{pmatrix} = \begin{pmatrix} A'_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$$

results in a block-diagonal structure for A. Therefore, without loss of generality, throughout the rest of the paper we consider the stochastic system

$$\begin{aligned} \mathrm{d}X_1(t) &= A_1X_1(t)\,\mathrm{d}t + Bu(t)\,\mathrm{d}t + C_1\,\mathrm{d}W_t + f_1(X_1(t),X_2(t))\,\mathrm{d}t,\\ \mathrm{d}X_2(t) &= A_2X_2(t)\,\mathrm{d}t + C_2\,\mathrm{d}W_t + f_2(X_1(t),X_2(t))\,\mathrm{d}t, \end{aligned} \tag{3.2}$$

with $X(t)=(X_1(t),X_2(t))\in\mathbb{R}^{n-d}\times\mathbb{R}^d$. We denote the Lipschitz constant of $f=(f_1,f_2)$ by $\mathscr{L}_f$, while $\|\cdot\|$ denotes the standard Euclidean norm and $\mathscr{B}(x,r)$ the corresponding open ball of radius r, centered at the point x.

Let

$$\xi_i(t,\tau) := \int_\tau^t \mathrm{e}^{A_i(t-s)}C_i\,\mathrm{d}W_s, \quad 0\leqslant\tau<t, \quad i=1,2. \tag{3.3}$$

Since A_1, A_2 are stable, there exist constants $M_0>0$, $r_0>0$ and $c_0>0$ such that, for $i=1,2$,

$$\|\mathrm{e}^{A_it}\|\leqslant M_0\mathrm{e}^{-r_0t} \quad \text{for } t\geqslant 0, \tag{3.4a}$$

$$E[\|\xi_i(t,\tau)\|^2]\leqslant c_0|t-\tau|, \quad 0\leqslant\tau<t. \tag{3.4b}$$

As usual, $\{\theta_t\}_{t\geqslant 0}$ denotes the standard time-shift on the path space of the process $X(t)$.

In the proof of Theorem 3.2, the control is constructed explicitly and is in the class defined below.

Definition 3.1. We say that an $\mathbb{R}^l$-valued process $\{u(t)\}_{t\geqslant 0}$ is in the class $\mathscr{U}_s$ if there is a constant $T>0$, and a bounded measurable function $g:\mathbb{R}^{n-d}\times\mathbb{R}\to\mathbb{R}^l$ such that $u(t)=g(X_1(kT),t)$, for all $t\in[kT,(k+1)T)$, and all $k\in\mathbb{N}_0$.

Clearly, if the control $\{u(t)\}_{t\geqslant 0}$ is in the class $\mathscr{U}_s$, then (3.2) has a $\mathscr{P}$-a.s. unique strong solution on each interval $[kT,(k+1)T)$, $k\in\mathbb{N}_0$.

Lemma 3.4. *For any given $x'=(x_1',x_2')\in\mathbb{R}^{n-d}\times\mathbb{R}^d$, $\varepsilon>0$ and $T_0>0$, define the event*

$$\mathscr{H}(\varepsilon,T_0) := \left\{\sup_{0\leqslant t\leqslant T_0}\left\|\int_0^t \mathrm{e}^{A_2(t-s)}[f_2(X_1(s),X_2(s)) - f_2(x_1',X_2(s))]\,\mathrm{d}s\right\| < \frac{\varepsilon}{4}\exp\left(-\frac{M_0\mathscr{L}_f}{r_0}\right)\right\}.$$

Also, define the hitting time

$$\tau_2 := \inf\{t>0,\ X_2(t)\in\mathscr{B}(x_2',\varepsilon/2)\} \tag{3.5}$$

and let $K\subset\mathbb{R}^d$ be a compact set. Then there exist a constant $m=m(K,\varepsilon,x_2')$ such that for any $\mathscr{F}_t$-optional time v satisfying *$\mathscr{P}(v<\infty)=1$ and $\mathscr{P}(X_2(v)\in K)=1$, and for any control $\{u(t)\}_{t\geqslant 0}\in\mathscr{U}_s$, and all $T_0>0$,*

$$\mathscr{P}(\tau_2\circ\theta_v - v < T_0\,|\,\mathscr{F}_v) \geqslant \mathscr{P}(\theta_v^{-1}\mathscr{H}(\varepsilon,T_0)\,|\,\mathscr{F}_v) - \frac{m}{T_0}, \quad \mathscr{P}\text{-a.e.}$$

Proof. Consider the diffusion

$$\mathrm{d}Z(t) = A_2Z(t)\,\mathrm{d}t + C_2\,\mathrm{d}W_t + f_2(x_1',Z(t))\,\mathrm{d}t, \quad t\geqslant 0, \tag{3.6}$$

defined on the same probability space as (3.2) and let

$$s_2 := \inf\{t>0,\ Z(t)\in\mathscr{B}(x_2',\varepsilon/4)\}.$$

Since, by Theorem 3.3, $\{Z(t)\}_{t\geqslant 0}$ is positive recurrent,

$$m := \sup_{z\in K}\{E^z[s_2]\} \tag{3.7}$$

is a finite constant. Let $\{\tilde{X}(t)\}_{t\geqslant 0}$ be the diffusion governed by

$$\begin{aligned} \mathrm{d}\tilde{X}_1(t) &= A_1\tilde{X}_1(t)\,\mathrm{d}t + Bu(t)\,\mathrm{d}t + C_1\,\mathrm{d}W_t + f_1(\tilde{X}_1(t),\tilde{X}_2(t))\,\mathrm{d}t,\\ \mathrm{d}\tilde{X}_2(t) &= A_2\tilde{X}_2(t)\,\mathrm{d}t + C_2\,\mathrm{d}W_t + [I_{\{t\leqslant v\}}f_2(\tilde{X}_1(t),\tilde{X}_2(t)) + I_{\{t>v\}}f_2(x_1',\tilde{X}_2(t))]\,\mathrm{d}t, \end{aligned} \tag{3.8}$$

defined on $(\Omega, \mathscr{F}, \mathscr{P})$, and with the same initial distribution as $X(0)$. Note that the conditional law of $\tilde{X}_2(v+t)$ given $\mathscr{F}_v$ is the same as the law of $Z(t)$ (with initial distribution $Z(0)=X_2(v)$). Letting

$$\hat{X}_2(t):=X_2(t)-\tilde{X}_2(t)$$

and using (3.2) and (3.8), we form the triangle inequality

$$\|\hat{X}_2(v+t)\| \leqslant \left\| \int_0^t \mathrm{e}^{A_2(t-s)}[f_2(X_1(v+s),X_2(v+s))-f_2(x_1',X_2(v+s))]\,\mathrm{d}s \right\| + \int_0^t \mathscr{L}_f \|\mathrm{e}^{A_2(t-s)}\|\,\|\hat{X}_2(v+s)\|\,\mathrm{d}s, \quad \mathscr{P}\text{-a.e.}$$

By applying the Gronwall Lemma and using (3.4a), we obtain

$$\|\hat{X}_2(v+t)I_{\theta_v^{-1}\mathscr{H}(\varepsilon,T_0)}\| < \varepsilon/4, \quad \mathscr{P}\text{-a.e.} \quad \forall t\in[0,T_0].$$

It follows that

$$\{\tau_2\circ\theta_v - v \geqslant T_0\} \cap \theta_v^{-1}\mathscr{H}(\varepsilon,T_0) \cap \{\tilde{X}_2(v+t)\in\mathscr{B}(x_2',\varepsilon/4) \text{ for some } t<T_0\}$$

is a $\mathscr{P}$-null set. Hence,

$$\begin{aligned}\mathscr{P}(\tau_2\circ\theta_v - v < T_0 \mid \mathscr{F}_v) &\geqslant 1-\mathscr{P}(\theta_v^{-1}\mathscr{H}^{\mathrm{c}}(\varepsilon,T_0)\,|\,\mathscr{F}_v) - \mathscr{P}(s_2\circ\theta_v - v\geqslant T_0\,|\,\mathscr{F}_v)\\ &\geqslant \mathscr{P}(\theta_v^{-1}\mathscr{H}(\varepsilon,T_0)\,|\,\mathscr{F}_v) - \frac{m}{T_0}, \quad \mathscr{P}\text{-a.e.} \qquad \square\end{aligned}$$

Corollary 3.5. *Let*

$$\bar{R}:=\frac{2M_0}{r_0}\|f\|_\infty \exp\left(\frac{M_0\mathscr{L}_f}{r_0}\right).$$

Suppose $R_1>R_0>\bar{R}$, *and define* $v_1=0$, *and recursively, for* $k=1,2,\ldots,$

$$\begin{aligned} v_{2k} &= \inf\{t>v_{2k-1},\ X_2(t)\in\mathscr{B}(0,R_0)\},\\ v_{2k+1} &= \inf\{t>v_{2k},\ X_2(t)\notin\overline{\mathscr{B}(0,R_1)}\}. \end{aligned} \tag{3.9}$$

Then for all $x\in\mathbb{R}^n$ *and all controls in the class* $\mathscr{U}_s$

$$E^x[v_k]<\infty \quad \forall k\in\mathbb{N}.$$

Moreover, there exists a constant $\gamma=\gamma(R_0,R_1)$ *such that*

$$E[v_{2k+2}-v_{2k}\,|\,\mathscr{F}_{v_{2k}}]<\gamma, \quad \mathscr{P}\text{-a.e.}, \quad \forall k\in\mathbb{N}.$$

Proof. Consider the diffusion in (3.6), defined on the same probability space, with initial value $Z(0)=X_2(0)$, and let $R_0':=R_0-\bar{R}$, $R_1':=R_1+\bar{R}$. Define $v_1'=0$, and recursively, for $k=1,2,\ldots,$

$$\begin{aligned} v_{2k}' &= \inf\{t>v_{2k-1}',\ Z(t)\in\mathscr{B}(0,R_0')\},\\ v_{2k+1}' &= \inf\{t>v_{2k}',\ Z(t)\notin\overline{\mathscr{B}(0,R_1')}\}. \end{aligned}$$

Subtracting (3.6) from (3.2) and applying the Gronwall Lemma as in the proof of Lemma 3.4, we deduce that $\hat{Z}(t):=X_2(t)-Z(t)$ satisfies

$$\|\hat{Z}(t)\|\leqslant\bar{R}, \quad \mathscr{P}\text{-a.e.} \quad \forall t\geqslant 0.$$

Consequently, we have $v_k\leqslant v_k'$, $\mathscr{P}$-a.e., for all $k\in\mathbb{N}$, and the results follow from the positive recurrence and strong Markov property of $Z(t)$. $\square$

Lemma 3.6. *For any given* $\varepsilon > 0$, $T_0 > 0$ *and* $\alpha > 0$, *there exists a control* $\{u_0(t)\}_{t\geqslant 0}$ *in the class* $\mathscr{U}_s$ *such that* $\mathscr{P}(\theta_{t'}^{-1}\mathscr{H}(\varepsilon, T_0) \mid \mathscr{F}_{t'}) > 1-\alpha$, $\mathscr{P}$*-a.e., for all* $t' \geqslant 0$.

Proof. For $N \in \mathbb{N}$, let

$$\delta(N) := \left(\frac{4c_0T_0^2}{N}\right)^{1/3}. \tag{3.10}$$

Note that if N is large enough, the following inequalities hold:

$$\delta(N) \leqslant \max\left\{\frac{\alpha\varepsilon\exp(-M_0\mathscr{L}_f/r_0)}{4M_0(3\|f\|_\infty + T_0\mathscr{L}_f)}, \sqrt{\frac{c_0T_0}{M_0\|f\|_\infty}}, T_0\right\}, \tag{3.11}$$

$$(\mathrm{e}^{\|A_1\|T_0/N} - 1)\|x_1'\| \leqslant \frac{\delta(N)}{4}. \tag{3.12}$$

Fix such an N and let

$$t_\ell := \frac{\ell T_0}{N}, \quad \ell = 0, 1, \ldots\ . \tag{3.13}$$

The feedback control $\{u_0(t)\}_{t\geqslant 0} \in \mathscr{U}_s$ is defined by

$$u_0(t) := \begin{cases} \varphi(t; \delta(N)/N, t_\ell, X_1(t_\ell), x_1') & \text{for } t \in [t_\ell, t_\ell + \delta(N)/N), \quad \ell = 0, 1, \ldots \\ 0 & \text{otherwise,} \end{cases} \tag{3.14}$$

where

$$\begin{aligned} &\varphi(t; \delta', s, x_1, x_1' := B^*\mathrm{e}^{A_1^*(s+\delta'-t)}W_c^{-1}(\delta')(x_1' - \mathrm{e}^{A_1\delta'}x_1), \\ &W_c(\delta') := \int_0^{\delta'} \mathrm{e}^{A_1 s}BB^*\mathrm{e}^{A_1^* s}\,\mathrm{d}s. \end{aligned} \tag{3.15}$$

We continue with the derivation of the estimate in the lemma. Define

$$J_\ell := \left[t_\ell + \frac{\delta(N)}{N}, t_{\ell+1}\right), \quad J := \bigcup_{\ell=0}^{\infty} J_\ell,$$

$$\mathscr{D} := \{t \in [t', t'+T_0]\colon \|X_1(t) - x_1'\| > \delta(N)\},$$

$$\tilde{\mathscr{I}} := \int_{t'}^{t'+T_0} \|f_2(X_1(s), X_2(s)) - f_2(x_1', X_2(s))\|\,\mathrm{d}s.$$

By splitting the integral $\tilde{\mathscr{I}}$ on $\mathscr{D}\cap J$, $\mathscr{D}\cap J^{\mathrm{c}}$ and $[t', t'+T_0]\setminus\mathscr{D}$, and with μ denoting the Lebesgue measure on the real line, we obtain

$$\tilde{\mathscr{I}} \leqslant 2\|f\|_\infty\mu(\mathscr{D}\cap J) + 2\|f\|_\infty\delta(N) + T_0\mathscr{L}_f\delta(N), \quad \mathscr{P}\text{-a.e.} \tag{3.16}$$

If $t \in \mathscr{D}\cap J_\ell$, an easy calculation yields

$$\begin{aligned} \|X_1(t) - x_1'\| &\leqslant \|(\mathrm{e}^{A_1(t-t_\ell-\delta(N)/N)} - I)x_1'\| + \|\xi_1(t, t_\ell)\| \\ &\quad + \int_{t_\ell}^{t} \|\mathrm{e}^{A_1(t-s)}\|\,\|f_1(X_1(s), X_2(s))\|\,\mathrm{d}s \\ &\leqslant (\mathrm{e}^{\|A_1\|T_0/N} - 1)\|x_1'\| + \|\xi_1(t, t_\ell)\| + \frac{T_0}{N}M_0\|f\|_\infty, \quad \mathscr{P}\text{-a.e.,} \end{aligned} \tag{3.17}$$

where ξ_1 is defined in (3.3). By (3.11), $\delta^2(N) \leqslant c_0T_0/M_0\|f\|_\infty$, which combined with (3.10) yields

$$\frac{T_0}{N}M_0\|f\|_\infty \leqslant \frac{\delta(N)}{4}. \tag{3.18}$$

By (3.12), (3.17) and (3.18),

$$\|X_1(t) - x_1'\| \leqslant \|\xi_1(t, t_\ell)\| + \frac{\delta(N)}{2}, \quad \mathscr{P}\text{-a.e.} \quad \forall t \in \mathscr{D} \cap J_\ell. \tag{3.19}$$

Using the estimate in (3.19), the Chebyshev inequality, and (3.4b), we obtain

$$\begin{aligned}\mathscr{P}(\|X_1(t) - x_1'\| > \delta(N) \,|\, \mathscr{F}_{t'}) &\leqslant \mathscr{P}(\|\xi_1(t, t_\ell)\| > \frac{\delta(N)}{2} \,|\, \mathscr{F}_{t'}) \\ &\leqslant \frac{4c_0(t - t_\ell)}{\delta^2(N)}, \quad \mathscr{P}\text{-a.e.} \quad \forall t \in \mathscr{D} \cap J_\ell. \end{aligned} \tag{3.20}$$

Integrating (3.20), yields

$$\begin{aligned} E[\mu(\mathscr{D} \cap J) \,|\, \mathscr{F}_{t'}] &= \int_{\mathscr{D}\cap J} \mathscr{P}(\|X_1(t) - x_1'\| > \delta(N) \,|\, \mathscr{F}_{t'})\, \mathrm{d}t \\ &= \sum_{\ell=0}^{\infty} \int_{\mathscr{D}\cap J_\ell} \mathscr{P}(\|X_1(t) - x_1'\| > \delta(N) \,|\, \mathscr{F}_{t'})\, \mathrm{d}t \\ &\leqslant \frac{2c_0 T_0^2}{N\delta^2(N)} = \frac{\delta(N)}{2}, \quad \mathscr{P}\text{-a.e.} \end{aligned} \tag{3.21}$$

By (3.16), using (3.21) and (3.11),

$$\begin{aligned} E[\tilde{\mathscr{I}} \,|\, \mathscr{F}_{t'}] &\leqslant 2\|f\|_\infty E[\mu(\mathscr{D} \cap J) \,|\, \mathscr{F}_{t'}] + 2\|f\|_\infty \delta(N) + T_0 \mathscr{L}_f \delta(N) \\ &\leqslant (3\|f\|_\infty + T_0 \mathscr{L}_f)\delta(N) \\ &\leqslant \frac{\alpha\varepsilon}{4M_0} \exp\left(-\frac{M_0 \mathscr{L}_f}{r_0}\right), \quad \mathscr{P}\text{-a.e.} \end{aligned} \tag{3.22}$$

Finally, the estimate

$$\begin{aligned} \mathscr{P}(\theta_{t'}^{-1} \mathscr{H}^{\mathrm{c}}(\varepsilon, T_0) \,|\, \mathscr{F}_{t'}) &\leqslant \mathscr{P}\left(M_0 \tilde{\mathscr{I}} \geqslant \frac{\varepsilon}{4} \exp\left(-\frac{M_0 \mathscr{L}_f}{r_0}\right) \middle|\, \mathscr{F}_{t'}\right) \\ &\leqslant \frac{4M_0}{\varepsilon} \exp\left(\frac{M_0 \mathscr{L}_f}{r_0}\right) E[\tilde{\mathscr{I}} \,|\, \mathscr{F}_{t'}], \quad \mathscr{P}\text{-a.e.,} \end{aligned}$$

together with (3.22), yields the desired result. □

Using the right-continuity of the process $X(t)$, a standard technique allows Lemma 3.6 to be strengthened as follows:

Corollary 3.7. *Let v be a $\mathscr{P}$-a.e. finite, $\mathscr{F}_t$-optional time, and $\{u_0(t)\}_{t\geqslant 0}$ the control constructed in Lemma* 3.6. *Then it holds that $\mathscr{P}(\theta_v^{-1}\mathscr{H}(\varepsilon, T_0) \,|\, \mathscr{F}_v) > 1 - \alpha$, $\mathscr{P}$-a.e.*

We conclude with the proof of Theorem 3.2.

Proof of Theorem 3.2. Let V be a given open neighborhood of $x' = (x_1', x_2') \in \mathbb{R}^{n-d} \times \mathbb{R}^d$. Choose $\varepsilon > 0$ such that $\mathscr{B}(x_1', \varepsilon) \times \mathscr{B}(x_2', 2\varepsilon) \subset V$. Next select R_0 and R_1 large enough so that the hypothesis of Corollary 3.5 is satisfied and, in addition, $\mathscr{B}(x_2', \varepsilon) \subset \mathscr{B}(0, R_0)$ and

$$R_1 > 4\sqrt{c_0 T_0} + M_0\left(R_0 + \frac{\|f\|_\infty}{r_0}\right). \tag{3.23}$$

Set $K = \mathscr{B}(0, R_0)$ and let m be the corresponding value of $m(K, \varepsilon, x_2')$ defined in Lemma 3.4. Also, let $T_0 = 32\, m$ and $\alpha = \frac{1}{32}$. Finally, choose a positive integer N such that (3.11) and (3.12) hold and, in addition, the following inequalities are satisfied:

$$N > 32\frac{c_0 T_0}{\varepsilon^2}, \tag{3.24a}$$

$$(\mathrm{e}^{2\|A_2\|T_0/N} - 1)\left(\|x_2'\| + \frac{\varepsilon}{2}\right) + \frac{2T_0}{N}M_0\|f\|_\infty < \frac{\varepsilon}{2}. \tag{3.24b}$$

We apply the control $\{u_0(t)\}_{t\geqslant 0}$ defined in (3.13)–(3.15), with the parameters T_0 and N as specified above.

Define the hitting times τ_1 and τ_2' by

$$\tau_1 := \inf\{t > 0,\ X_1(t) \in \mathscr{B}(x_1', \varepsilon)\},$$

$$\tau_2' := \inf\{t > \tau_2,\ X_2(t) \notin \overline{\mathscr{B}(x_2', 3\varepsilon/2)}\},$$

where τ_2 is defined in (3.5). Recall the definition of $\{v_k\}_{k\in\mathbb{N}}$ in (3.9). With τ_V denoting the hitting time of the set V, observe that

$$\begin{aligned}\{\tau_V \circ \theta_{v_{2k}} < v_{2k+1}\} &\supset \{\tau_2 \circ \theta_{v_{2k}} < v_{2k+1}\} \cap \{\tau_2 \circ \theta_{v_{2k}} \leqslant \tau_1 \circ \theta_{v_{2k}} < \tau_2' \circ \theta_{v_{2k}}\}\\ &\supset \{\tau_2 \circ \theta_{v_{2k}} - v_{2k} < T_0\} \cap \{v_{2k+1} - v_{2k} > T_0\}\\ &\cap \left\{\tau_2' \circ \theta_{v_{2k}} - \tau_2 \circ \theta_{v_{2k}} \geqslant \frac{T_0 + \delta(N)}{N}\right\}\\ &\cap \left\{\tau_2 \circ \theta_{v_{2k}} \leqslant \tau_1 \circ \theta_{v_{2k}} < \tau_2 \circ \theta_{v_{2k}} + \frac{T_0 + \delta(N)}{N}\right\}.\end{aligned} \tag{3.25}$$

Next, we bound the probabilities of the complements of the sets on the right-hand side of (3.25), for $k \in \mathbb{N}$. By Lemma 3.4 and Corollary 3.7,

$$\begin{aligned}\mathscr{P}(\tau_2 \circ \theta_{v_{2k}} - v_{2k} \geqslant T_0 \mid \mathscr{F}_{v_{2k}}) &\leqslant 1 - \mathscr{P}(\theta_{v_{2k}}^{-1}\mathscr{H}(\varepsilon, T_0) \mid \mathscr{F}_{v_{2k}}) + \frac{m}{T_0}\\ &< 1 - (1-\alpha) + \frac{m}{T_0} = \frac{1}{16}, \quad \mathscr{P}\text{-a.e.}\end{aligned} \tag{3.26}$$

By (3.2) and (3.4a), for $t \geqslant 0$,

$$\begin{aligned}\|X_2(v_{2k} + t)\| &\leqslant \mathrm{e}^{A_2 t} X_2(v_{2k})\| + \|\xi_2(v_{2k} + t, v_{2k})\|\\ &+ \int_{v_{2k}}^{v_{2k}+t} \|\mathrm{e}^{A_2(v_{2k}+t-s)}\|\,\|f_2(X_1(s), X_2(s))\|\,\mathrm{d}s\\ &\leqslant M_0 R_0 + \|\xi_2(v_{2k} + t, v_{2k})\| + \frac{M_0}{r_0}\|f\|_\infty, \quad \mathscr{P}\text{-a.e.}\end{aligned} \tag{3.27}$$

From (3.27), using the Submartingale inequality, and the bounds in (3.4b) and (3.23), we obtain

$$\begin{aligned}\mathscr{P}(v_{2k+1} - v_{2k} \leqslant T_0 \mid \mathscr{F}_{v_{2k}}) &\leqslant \mathscr{P}\left(\sup_{0\leqslant t\leqslant 2T_0} \|X_2(v_{2k} + t)\| > R_1 \mid \mathscr{F}_{v_{2k}}\right)\\ &\leqslant \frac{1}{16}, \quad \mathscr{P}\text{-a.e.}\end{aligned} \tag{3.28}$$

By a similar calculation, using the inequalities (3.24a)–(3.24b), we obtain

$$\mathscr{P}\left(\tau_2' \circ \theta_{v_{2k}} - \tau_2 \circ \theta_{v_{2k}} < \frac{T_0 + \delta(N)}{N} \,\middle|\, \mathscr{F}_{v_{2k}}\right) \leqslant \frac{1}{4}, \quad \mathscr{P}\text{-a.e.} \tag{3.29}$$

Last, consider the event

$$\mathscr{A} := \left\{\tau_2 \leqslant \tau_1 < \tau_2 + \frac{T_0 + \delta(N)}{N}\right\}.$$

For a $\mathscr{P}$-a.e. finite, $\mathscr{F}_t$-optional time τ, define $\lambda(\tau):=\lceil N/T_0\tau\rceil T_0/N$. Observe that $\lambda(\tau)$ takes values, $\mathscr{P}$-a.e., in the set $\{t_\ell\colon \ell\in\mathbb{N}_0\}$, defined in (3.13). Thus, we obtain the estimate

$$\begin{aligned}\left\|X_1\left(\lambda(\tau)+\frac{\delta(N)}{N}\right)-x_1'\right\| &\leqslant \left\|\xi_1\left(\lambda(\tau)+\frac{\delta(N)}{N},\lambda(\tau)\right)\right\| \\ &\quad+\int_{\lambda(\tau)}^{\lambda(\tau)+\delta(N)/N}\|\mathrm{e}^{A_1(\lambda(\tau)+\delta(N)/N-s)}\|\,\|f_1(X_1(s),X_2(s))\|\,\mathrm{d}s \\ &\leqslant \left\|\xi_1\left(\lambda(\tau)+\frac{\delta(N)}{N},\lambda(\tau)\right)\right\|+\frac{\delta(N)}{N}M_0\|f\|_\infty \\ &< \left\|\xi_1\left(\lambda(\tau)+\frac{\delta(N)}{N},\lambda(\tau)\right)\right\|+\frac{\varepsilon}{2},\end{aligned} \tag{3.30}$$

where the last inequality follows from (3.24b). Also, since $\tau\leqslant\lambda(\tau)<\tau+T_0/N$, utilizing (3.30) and (3.24a), we obtain

$$\begin{aligned}\mathscr{P}(\theta_{v_{2k}}^{-1}\mathscr{A}^{\mathrm{c}}\,|\,\mathscr{F}_{v_{2k}}) &\leqslant \mathscr{P}\left(\left\|X_1\left(\lambda(\tau_2\circ\theta_{v_{2k}})+\frac{\delta(N)}{N}\right)-x_1'\right\|>\varepsilon\,\middle|\,\mathscr{F}_{v_{2k}}\right) \\ &\leqslant \mathscr{P}\left(\left\|\xi_1\left(\lambda(\tau_2\circ\theta_{v_{2k}})+\frac{\delta(N)}{N},\lambda(\tau_2\circ\theta_{v_{2k}})\right)\right\|>\frac{\varepsilon}{2}\,\middle|\,\mathscr{F}_{v_{2k}}\right) \\ &\leqslant \frac{4c_0\delta(N)}{N\varepsilon^2}<\frac{\delta(N)}{8T_0}\leqslant\frac{1}{8},\quad \mathscr{P}\text{-a.e.}\end{aligned} \tag{3.31}$$

Combining (3.25) with the estimates in (3.26), (3.28), (3.29) and (3.31), we obtain

$$\mathscr{P}(\tau_V\circ\theta_{v_{2k}}\geqslant v_{2k+1}\,|\,\mathscr{F}_{v_{2k}})<\frac{1}{2},\quad \mathscr{P}\text{-a.e.},\quad \forall k\geqslant 1. \tag{3.32}$$

It follows from (3.32) that

$$P^x(\tau_V\geqslant v_{2k+1})<\left(\frac{1}{2}\right)^k,\quad \forall k\in\mathbb{N},\quad \forall x\in\mathbb{R}^n. \tag{3.33}$$

Since from Corollary 3.5, $E[v_{2k+2}-v_{2k}\,|\,\mathscr{F}_{v_{2k}}]<\gamma\ \forall k\in\mathbb{N}$, and also $E^x[v_2]<\infty$, for all $x\in\mathbb{R}^n$, it is fairly standard to show (see [9]), using (3.33), that $E^x[\tau_V]<\infty$, for all $x\in\mathbb{R}^n$. □

4. Concluding remarks

In addition to weak and strong controllability, Zabczyk [9] provides a characterization of *controllability*, a property defined by the requirement that the probability in (1.2) be 1. In the case of a nonlinearly perturbed system, it seems difficult to find sufficient conditions for controllability without further knowledge of the structure of f. The Lyapunov function criterion in [1] could serve as a starting point. However, results based on this criterion would have to use the explicit form of the perturbation f, since recurrence may change under small perturbations.

The boundedness assumption on f cannot be relaxed in general. Note that even if f is linear the results do not hold without assuming stability. Finally, the control we constructed does not depend on the explicit form of the nonlinear perturbation, and it results in a finite expectation for τ_V for all initial states.

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