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Continuous selections of trajectories of hybrid systems ☆

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Abstract

We present results on existence of continuous selections of trajectories of hybrid systems evolving according to Lipschitz nonlinear inclusions. First, we utilize the Skorohod metric to define both a family of pseudo-metrics and a metric on the set of trajectories accepted by the hybrid automaton. We show that under a non-Zeno condition the latter metric is complete. Second, the existence of continuous selections with respect to the family of pseudo-metrics is proved under a relatively mild assumption of transversality.

Keywords: Hybrid systems; Differential inclusions; Skorohod metric

1. Introduction

Little work has appeared in the literature on hybrid systems studying the qualitative behavior of their trajectories. This is a difficult task because hybrid systems represent a very rich class of dynamical systems.

In this paper we study properties of the set of trajectories of a hybrid system evolving according to Lipschitz nonlinear inclusions. Specifically, we investigate the existence of continuous selections of trajectories with respect to initial conditions. In order to study continuity in a setting where trajectories can change discontinuously due to resets of the hybrid system, we adopt the Skorohod metric for the continuous part of the hybrid trajectories and augment it using the discrete metric on the sequence of locations visited by the automaton. In this manner, we introduce a metric topology on the set of trajectories accepted by the hybrid automaton and present conditions under which this metric space is complete. Our completeness result asserts that the limit of a Cauchy sequence of hybrid trajectories is itself a trajectory accepted by the automaton and thus it may be viewed as regularity property of solutions. Zeno trajectories are excluded by imposing a non-overlap condition on the guards and resets. Second, the existence of continuous selections with respect to a family of pseudo-metrics based on the Skorohod metric is proved under a relatively mild assumption of transversality.

An early paper by Witsenhausen [10] considers a model for switching between vector fields. The model eliminates non-determinacy by assuming that transitions are taken at the first time the enabling condition is reached, enabling conditions are non-overlapping (also a non-Zeno condition), and the reset map is the identity. The present work extends this model as

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we permit non-determinacy in several features of our model: (1) the dynamics follow a differential inclusion, (2) multiple enabling conditions (thus, multiple edges) can be reached from a state, (3) a transition can be taken at any time while an enabling condition is active, or not at all, and (4) the reset map is non-deterministic. A paper by Tavernini [9] considers a hybrid system with differential equations in each location. The paper obtains a result on continuity with respect to initial conditions based on a transversality hypothesis at the boundary of the enabling regions. Our result on continuity with respect to initial conditions is closely related, though we make use of a result by Cellina and Ornelas [4] on continuous selections of Lipschitz inclusions and a more general transversality condition suitable for inclusions. Also, the paper by Gupta et al. [7] introduces a metric for finite trajectories of timed automata. Finally, while this paper was still under review results on continuity of solutions of hybrid automata appeared in [8]. The systems studied in [8] are hybrid automata with complete vector fields at each location. Also, a key assumption is that the automata are deterministic, i.e., there is a unique hybrid trajectory for each initial condition, that the guards are components of the boundary of the domain of each location and that this boundary is C^1 and the vector field is transversal to it. Moreover, the topology used in [8] to compare hybrid trajectories is weaker than our metric topology and hence a parameterization of trajectories which is continuous in the topology utilized in [8] is not necessarily continuous in ours.

The paper is organized as follows. Section 2 contains some preliminary background on hybrid automata. Section 3 defines a suitable topology for trajectories of hybrid systems using the Skorohod metric. The study of continuity with respect to initial conditions for hybrid systems with Lipschitz inclusions, constituting the main contribution of the paper, is undertaken in Section 4.

2. Preliminaries

2.1. Notation

We denote by $|\cdot|$ the Euclidean norm and by $d(x,B)$ the distance from a point x to a set B defined by $d(x,B) = \inf_{y\in B}|x-y|$. $B(x,r)$ denotes the open ball centered at x of radius r, and A° the interior of a set A. The Hausdorff distance between two sets d_{H} is $d_{\mathrm{H}}(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}$. For an interval $I=[t_0,t_1]$, let $\mathscr{C}(I)$ and $\mathscr{C}_{\mathrm{ac}}(I)$ denote the spaces of continuous and absolutely continuous functions $f: I \to \mathbb{R}^n$, endowed with the sup norm $\|f\|_\infty$ and the norm $\|f\|_{\mathrm{ac}} = |f(t_0)| + \int_I |\dot{f}(s)|\,\mathrm{d}s$, respectively. Finally, $\mathscr{F}(\mathbb{R}^n)$ denotes the space of differential inclusions on $\mathbb{R}^n$ and $\mathscr{D}(I,\mathbb{R}^n)$ the space of all functions $f: I \to \mathbb{R}^n$ that are left continuous, $\lim_{t\uparrow a} f(t) = f(a)$, and have limits from the right.

2.2. Hybrid automata

A hybrid automaton is a tuple $H=(Q,\Sigma,D,E,G,R)$ consisting of the following components:

State space: $Q = L\times\mathbb{R}^n$ is the state space where L is a finite set of control locations.

Events: Σ is a finite observation alphabet.

Differential inclusions: $D: L \to \mathscr{F}(\mathbb{R}^n)$ is a function assigning a differential inclusion to each location. We use the notation $D(l) = F_l$.

Control switches: $E \subset L\times\Sigma\times L$ is a set of control switches. Each element $e=(l,\sigma,l')\in E$ is a directed edge between a source location l and a target location l' with observation σ.

Guard conditions: $G \subset 2^{\mathbb{R}^n}$ is the set of guard conditions on the continuous states. We use the notation $G(e) = g_e \subseteq \mathbb{R}^n$.

Reset conditions: R is the set of reset conditions. We use the notation $R(e) = r_e$, where $r_e: \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a set-valued map.

2.2.1. Semantics

A state is a pair $(l,x)\in Q$. $\Sigma(l)$ denotes the set of events possible at $l\in L$ and $E(l)$ denotes the set of edges possible at $l\in L$. For $\sigma\in\Sigma$, a *σ-step* is a binary relation $\to^\sigma \subset Q\times Q$ and we write $(l,x)\to^\sigma(l',x')$ iff (1) $e=(l,\sigma,l')\in E(l)$, (2) $x\in g_e$, and (3) $x'\in r_e(x)$. A σ-step need not be taken even if $x\in g_e$. Let $\varphi_t^l(x)$ be a trajectory of F_l starting from x and evolving for time t. For $t\in\mathbb{R}^+$, a *t-step* is a binary relation $\to^t \subset Q\times Q$ and we write $(l,x)\to^t(l',x')$ iff (1) $l=l'$, (2) at $t=0$, $x'=x$, and (3) for $t>0$, $x'=\varphi_t^l(x)$, where $\dot{\varphi}_t^l(x)\in F_l(\varphi_t^l(x))$. A *trajectory* π of H is a finite or infinite sequence $\pi: q_0\to^{\theta_0} q_1\to^{\theta_1} q_2\to^{\theta_2}\cdots$

where $q_i \in Q$ and $\theta_i \in \Sigma \cup \mathbb{R}^+$. A trajectory is *accepted* by H if each $q_i \to^{\theta_i} q_{i+1}$ is a t-step or σ-step of H, and we denote the space of all such trajectories by $\mathscr{H}$. A *step* of a trajectory refers to a t-step followed by a σ-step. Associated with the kth step of a trajectory is the data $I^0 = [0, t^1]$ or $I^k = (t^k, t^{k+1}]$, for $k \geqslant 1$, the time interval of the step, $\tau^k = t^{k+1} - t^k$, its duration, $e^k = (l^k, \sigma^k, l^{k+1})$, the edge, and $q^k(t) = (l^k, x^k(t))$, the state, where l^k is fixed over I^k and $x^k(t)$ satisfies $\dot{x}^k(t) \in F_{l^k}(x^k(t))$. Thus, the step can be represented as

$$(l^k, x^k(t^k+)) \xrightarrow{\tau^k} (l^k, x^k(t^{k+1})) \xrightarrow{\sigma^k} (l^{k+1}, x^{k+1}(t^{k+1}+)), \tag{1}$$

satisfying $x^k(t^{k+1}) \in g_{e^k}$ and $x^{k+1}(t^{k+1}+) \in r_{e^k}(x^k(t^{k+1}))$. We do not exclude the possibility $\tau^k = 0$, in which case the step is only a σ-step. A *run* of H is the projection to the discrete part of a trajectory in $\mathscr{H}$; namely, a finite or infinite sequence $l^0, l^1, l^2, \ldots$ of admissible locations. We also refer to $x(t) := \{x^k(t)\colon t \in I^k, k = 0, 1, \ldots\}$ as the *continuous part* of the trajectory.

Trajectories in $\mathscr{H}$ might exhibit finite escape time for the continuous state, or admit an infinite number of σ-steps in a bounded time interval (i.e., *Zeno* trajectories). Therefore, we define the *regular trajectory language* $\Pi \subset \mathscr{H}$ as those trajectories whose continuous part belongs to $\mathscr{D}(\mathbb{R}_+, \mathbb{R}^n)$ and has a finite number of discontinuities in any bounded interval of time. Dealing with a subset of all trajectories might pose difficulties in the analysis especially when trying to characterize convergence or continuity. For example, if we employ a topology suitable for functions in $\mathscr{D}(\mathbb{R}_+, \mathbb{R}^n)$ it is unclear whether Π is open in $\mathscr{H}$, or whether a converging sequence of elements of Π is always non-Zeno. We introduce a suitable definition below in order to surpass some of these difficulties.

Assumption 1. For each e, e' in E, g_e is a closed set, r_e has closed values and $d(r_e(g_e), g_{e'}) > 0$.

If Assumption 1 holds, and under mild conditions on the inclusion, i.e., each F_l has bounded values and is upper semicontinuous, then any trajectory of H, whose continuous part belongs to $\mathscr{D}(\mathbb{R}_+, \mathbb{R}^n)$, satisfies the non-Zeno condition a priori.

3. Topologies for hybrid systems

We introduce suitable topologies for Π, using the Skorohod metric. The Skorohod metric, denoted $d_s(\cdot,\cdot)$, was originally used in the study of stochastic processes with right (or left)-continuous sample paths, such as Poisson processes [2]. Given two functions $f \in \mathscr{D}(I_f, \mathbb{R}^n)$ and $g \in \mathscr{D}(I_g, \mathbb{R}^n)$, $d_s(f, g)$ is the infimum of $\varepsilon > 0$ for which there exists a strictly increasing, continuous, surjective function $\kappa : I_f \to I_g$ such that

(a) $\sup_{t \in I_f} |\kappa(t) - t| \leqslant \varepsilon$ and

(b) $\sup_{t \in I_f} |f(t) - g(\kappa(t))| \leqslant \varepsilon.$

3.1. The pseudo-metric space (Π, d^m)

We define a topology on Π via a family of pseudo-metrics that combine the Skorohod metric on the continuous parts of a pair of trajectories with the distance between the corresponding runs in the Cantor topology.

Let $\pi, \tilde{\pi} \in \Pi$ with $\pi = \{l^k, x^k(\cdot), t^k\}$ and $x(\cdot)$ the continuous part of π, where $x^k : (t^k, t^{k+1}] \to \mathbb{R}^n$. We adopt the analogous notation for $\tilde{\pi}$. Let $x^{(m)}, \tilde{x}^{(m)}$, $m \geqslant 1$, denote the restriction of $x, \tilde{x}$ on $[0, t^m]$ and $[0, \tilde{t}^m]$, respectively. We define the pseudo-metric $d^m(\cdot,\cdot)$ by

$$d^m(\pi, \tilde{\pi}) = d_s(x^{(m)}, \tilde{x}^{(m)}) + \sum_{k=0}^{m-1} \frac{1}{2^k} \mathbb{I}(l^k \neq \tilde{l}^k),$$

where $\mathbb{I}(\cdot)$ is the indicator function. Thus, $(\Pi, \{d^m\})$ is a topology on Π induced by the family of pseudometrics. Also, for fixed $m > 0$, (Π, d^m) denotes the pseudo-metric topology on the m-step trajectories of Π.

3.2. The metric space (Π, d^∞)

We define a metric topology that utilizes the Skorohod metric for functions in $\mathscr{D}(\mathbb{R}_+, \mathbb{R}^n)$ (see [5]). This approach has the advantage that properties of this metric are readily available, though its definition is somewhat more cumbersome. Let Λ be the collection of strictly increasing, Lipschitz continuous functions $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lambda(0) = 0$ and

$\lim_{t\to\infty} \lambda(t) = \infty$ such that

$$\gamma(\lambda) := \sup_{s>t\geqslant 0} \left| \log \frac{\lambda(s)-\lambda(t)}{s-t} \right| < \infty.$$

This function estimates how much $\lambda(t)$ increases relative to t. Note that when $\gamma(\lambda)$ is large, then the maximum or minimum rate of change of λ is different from one. Also, when $\gamma(\lambda)=0$, then $\lambda = t$. For $f,g \in \mathscr{D}(\mathbb{R}_+,\mathbb{R}^n)$, $\lambda\in\Lambda$ and $u\in\mathbb{R}_+$ define

$$\hat{d}_s(f,g,\lambda,u) := \sup_{t\geqslant 0} \min\{1, |f(t\wedge u) - g(\lambda(t)\wedge u)|\},$$

where $a\wedge b = \min\{a,b\}$. The *Skorohod metric* $d_s^\infty(\cdot,\cdot)$ is defined by

$$d_s^\infty(f,g) = \inf_{\lambda\in\Lambda} \left[\max\left\{ \gamma(\lambda), \int_0^\infty e^{-u} \hat{d}_s(f,g,\lambda,u)\,\mathrm{d}u \right\} \right].$$

Let $\pi,\tilde{\pi}\in\Pi$ be as in Section 3.1. We define the hybrid metric d^∞ by

$$d^\infty(\pi,\tilde{\pi}) = d_s^\infty(x,\tilde{x}) + \sum_{k=0}^{\infty} \frac{1}{2^k}\, \mathbb{I}(l^k \neq l'^k). \tag{2}$$

It is well known ([5, Theorem 5.6, p. 121]) that $(\mathscr{D}(\mathbb{R}_+,\mathbb{R}^n), d_s^\infty)$ is a complete metric space. The main result of this section is that the metric space (Π, d^∞) is also a complete metric space.

Theorem 2. *Suppose that H satisfies Assumption* 1, *that r_e has closed values and is upper semicontinuous, for all $e\in E$, and that at each location l, F_l has non-empty, compact, convex values and is upper semicontinuous. Then (Π,d^∞) is a complete metric space.*

Proof. Let $\{\pi_j, j\in\mathbb{N}\} \subset (\Pi,d^\infty)$, with $\pi_j = \{l_j^k, x_j^k(\cdot), t_j^k\}$, be a Cauchy sequence, where $x_j^k : (t_j^k, t_j^{k+1}] \to \mathbb{R}^n$ is a solution of $\dot{x}_j^k \in F_{l_j^k}(x_j^k)$. Let $x_j(\cdot)$ denote the continuous part of π_j. By (2), $\{x_j\}_{j\in\mathbb{N}}$ is Cauchy in $(\mathscr{D}(\mathbb{R}_+,\mathbb{R}^n), d_s^\infty)$ and thus converges to some $x\in\mathscr{D}(\mathbb{R}_+,\mathbb{R}^n)$. We must show that x is the continuous part of a trajectory $\pi\in\Pi$ and $d^\infty(\pi_j,\pi)\to 0$, as $j\to\infty$. By Proposition 5.2 in [5, p. 118], $\lim_{j\to\infty} d_s^\infty(x_j,x) = 0$ if and only if there exists $\{\lambda_j\}\subset\Lambda$ such that $\lim_{j\to\infty}\gamma(\lambda_j) = 0$ and

$$\lim_{j\to\infty} \sup_{0\leqslant t\leqslant T} |x_j(t) - x(\lambda_j(t))| = 0 \quad \text{for all } T>0. \tag{3}$$

Note also that $\lim_{j\to\infty}\gamma(\lambda_j)=0$ implies that

$$\lim_{j\to\infty} \sup_{0\leqslant t\leqslant T} |\lambda_j(t) - t| = 0 \quad \text{for all } T>0. \tag{4}$$

Since the inclusion F_l has compact values and is upper semicontinuous, it follows that all solutions that lie in a bounded domain are equicontinuous [6, Lemma 2, p. 78]. Using this fact along with (3)–(4) and Assumption 1, one can show that $x(t)$ has at most a finite number of discontinuities in each bounded time interval. Moreover, if $\{t^k\}_{k\geqslant 1}$ are the discontinuity points of x, then $t_j^k \to t^k$ as $j\to\infty$. Since $\{\pi_j\}$ is Cauchy, for each $k\in\mathbb{N}$ there exists $j_0 = j_0(k)\in\mathbb{N}$ such that $d^\infty(\pi_{j_0},\pi_j) < 1/2^k$, $\forall j>j_0$. This implies $l_j^k = l_{j_0}^k$, $\forall j > j_0$. Set $l^k = l_{j_0}^k$ and $e^k = (l^k,\sigma^k,l^{k+1})$. Let x^k denote the restriction of x on $(t^k,t^{k+1}]$. The equicontinuity of $x_j(t)$ on bounded domains along with (3)–(4) implies that $x_j^k(t)\to x^k(t)$, as $j\to\infty$, uniformly on compact subsets of (t^k,t^{k+1}). Hence, by Corollary 1 in [6, p. 77], x^k is a solution of the inclusion $\dot{x}^k\in F_{l^k}(x^k)$ on (t^k,t^{k+1}). By left-continuity we have $x_j^k(t^{k+1}) \to x^k(t^{k+1})$ which implies, since g_{e^k} is closed, that $x^k(t^{k+1})\in g_{e^k}$. The existence of right limits along with equicontinuity yields $x_j^k(t^k+) \to x^k(t^k+)$, and since the graph of r_{e^k} is closed, it follows that $x^k(t^k+)\in r_{e^k}(x^{k-1}(t^k))$. Therefore, π_j converges to $\pi := \{l^k, x^k(\cdot), t^k\}$ in (Π,d^∞). □

4. Continuity with respect to initial conditions

Continuity with respect to initial conditions for hybrid systems with Lipschitz differential inclusions is established under a *transversality* condition, stated in Definition 6. Let π_0 be a trajectory starting from $q_0\in Q$. We show that if π_0 satisfies the transversality condition, and under mild assumptions on the automaton stated in Assumption 3, there exists a continuous selection of trajectories from (Π,d^m) in a neighborhood of q_0. Implicit in the definition of transversality is that the steps of the trajectory have non-zero duration, i.e., pure σ-steps are excluded. The reason for this is the following. Consider a parameterized family of trajectories $\pi_\alpha=\{l_\alpha^k,x_\alpha^k(\cdot),t_\alpha^k\}$, with $\alpha\in[0,1]$, and suppose that for some $k\in\mathbb{N}$, $\tau_0^k=t_0^{k+1}-t_0^k=0$, but $\tau_\alpha^k>0$ for $\alpha>0$. Also suppose that $|x_\alpha^{k-1}(t_\alpha^k)-x_\alpha^k(t_\alpha^k+)|$ and $|x_\alpha^k(t_\alpha^{k+1})-x_\alpha^{k+1}(t_\alpha^{k+1}+)|$ are bounded below by a positive constant as $\alpha\to 0$. Then, π_α cannot converge to π_0 in (Π,d^m) (note that in the weaker topology utilized in [8] convergence under these circumstances is possible). In other words if π_0 has a step of zero duration at one location, this in general forces any continuous selection of trajectories, in the vicinity of π_0, to also have a corresponding step of zero duration. Satisfying this would involve conditions on the composition of successive reset maps, which we prefer to avoid.

Consider the problem

$$\dot x\in F(x),\quad x(0)=\xi, \tag{5}$$

on a time interval $[0,T]$, where ξ ranges over a compact $X\subset\mathbb{R}^n$ with diameter δ. In addition, we assume the following.

Assumption 3. The set-valued map F satisfies:

(a) The values of F are closed, non-empty subsets of $\mathbb{R}^n$.
(b) There exists $K>0$ such that $d_{\mathrm H}(F(x),F(x'))\leqslant K|x-x'|$, for all $x,x'\in\mathbb{R}^n$.

Under Assumption 3, an absolutely continuous solution to (5) exists for each $\xi\in X$ [6]. Let $\xi_0\in X$ and $x(\cdot)$ be a solution of (5) such that $x(0)=\xi_0$. It is shown in [4] that there exists a selection $\varphi_t(\xi)$ from the set of solutions of (5) which is continuous in $\xi\in X$ and such that $\varphi_t(\xi_0)=x(t)$. Such a selection is found by constructing a sequence of approximate trajectories, $\{\varphi_t^j(\xi)\}_{j=0}^\infty$ which are shown to form a Cauchy sequence in the normed space $\mathscr{C}_{\mathrm{ac}}([0,T])$. In particular, this sequence can be chosen to satisfy

$$\|\varphi^j(\xi)-\varphi^{j-1}(\xi)\|_{\mathrm{ac}}\leqslant\delta\left(\frac{(KT)^j}{j!}+\frac{\mathrm{e}^{2KT}}{2^{j+1}}\right).$$

Thus,

$$\|\varphi^j(\xi)-\varphi^0(\xi)\|_{\mathrm{ac}}\leqslant\delta(\mathrm{e}^{KT}+\mathrm{e}^{2KT}),\quad\forall j\in\mathbb{N},$$

where

$$\varphi_t^0(\xi)=\xi+\int_0^t\dot\varphi_s(\xi_0)\,\mathrm{d}s$$

is the initial guess of the approximate trajectories. Hence, we obtain the estimate

$$\|\varphi^j(\xi)-\varphi(\xi_0)\|_{\mathrm{ac}}\leqslant\delta(\mathrm{e}^{KT}+\mathrm{e}^{2KT}+1)\leqslant 3\delta\mathrm{e}^{2KT},\quad\forall j\in\mathbb{N}. \tag{6}$$

Assumption 4. The automaton H satisfies the following:

(a) The inclusion $\dot x\in F_l(x)$ at each location l satisfies Assumption 3.
(b) For each $e\in E$, g_e is either a closed, n-dimensional topological manifold with boundary, or an embedded $(n-1)$-dimensional C^1 submanifold.
(c) r_e is a lower semicontinuous reset map from $\mathbb{R}^n$ to the closed, convex subsets of $\mathbb{R}^n$.

Remark 5. Assumption 4(c) makes possible the use of Michael's selection theorem [1].

The following definition is essential for our main result. See Fig. 1.

Definition 6. Let $e=(l,\sigma,l')$ and $x(t)$, be a solution of $\dot x\in F_l(x)$ defined for $t\in[t_0,t_1]$, with $t_0<t_1$ and such that $x(t_1)\in g_e$. We say that $x(\cdot)$ is transversal to g_e at $x(t_1)$ if it fulfills the following requirements:

(1) If g_e is an $(n-1)$-dimensional submanifold we require that the solution $x(t)$ of $\dot x\in F_l(x)$ can be suitably extended on some interval $(t_1,s_1]$, $s_1>t_1$ in a manner that for some open neighborhood V of $x(t_1)$ and local coordinates $u=(u_1,\dots,u_n)$ centered at $x(t_1)$ and mapping V homeomorphicaly onto some open neighborhood of $\mathbb{R}^n$, and satisfying $u_n(V\cap g_e)=0$,

$$\dot x(t)\cdot\nabla u_n(v)\geqslant 1,\ \ \forall v\in V,\ \text{ a.e. on } \{t\colon x(t)\in V\}.$$

(2) If g_e is a topological n-manifold with boundary we require that the solution $x(t)$ of $\dot x\in F_l(x)$ can either be continued on some interval $(t_1,s_1]$, $s_1>t_1$ in a manner that

$$x(t)\in g_e{}^\circ,\quad\forall t\in(t_1,s_1], \tag{7a}$$

or there exists $s_0\in[t_0,t_1)$ such that

$$x(t)\in g_e{}^\circ,\quad\forall t\in[s_0,t_1). \tag{7b}$$

Note that if $x(t_1)$ is an interior point of g_e then (7b) is trivially satisfied.

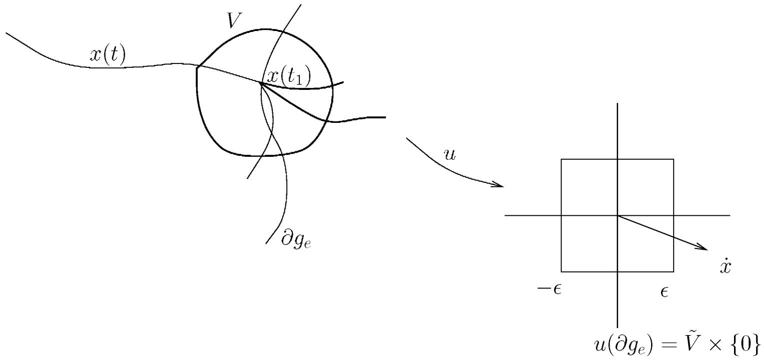

Fig. 1. Transversal trajectory of a hybrid system with differential inclusions.

We say that $\pi = \{l^k, x^k(\cdot), t^k\}$, whose steps are denoted as in (1), is a *transversal trajectory* if $x^k(t)$ is transversal to g_{e^k} at $x^k(t^{k+1})$ for all k.

Remark 7. If the enabling region is n-dimensional and has a differentiable boundary, a simple condition suffices for the solution to be continued in the interior of the region. Using the notation of Definition 6 and denoting by $T_{x(t_1)}g_e$ the tangent space to g_e at $x(t_1)$ and by $\boldsymbol{n}_{x(t_1)}$ the unit normal to $T_{x(t_1)}g_e$ in the direction of g_e^o, we require that $F(x(t_1))$ contains a vector η such that $\langle \eta, \boldsymbol{n}_{x(t_1)} \rangle > 0$, where $\langle \cdot, \cdot \rangle > 0$ is the standard inner-product in $\mathbb{R}^n$.

The transversality assumption allows for the following construction.

Lemma 8. *Let $\dot{x} \in F_l(x)$ be a Lipschitz inclusion satisfying Assumption* 3, *and let $x(t)$, $t \in [t_0, t_1]$, be a solution that is transversal to g_e, $e = (l, \sigma, l')$, at $x(t_1)$. Then there exist $s_1 \geqslant t_1$, an open neighborhood W of $x(t_0)$, and a continuous selection $\varphi : W \to \mathscr{C}_{\rm ac}([t_0, s_1])$ of solutions of $\dot{\varphi} \in F_l(\varphi)$ satisfying*:

(a) $\varphi_t(x(t_0)) = x(t)$, $t \in [t_0, t_1]$.

(b) *there exists a continuous* $\tilde{\tau} : W \to [t_0, s_1]$, satisfying $\tilde{\tau}(x(t_0)) = t_1$, such that $\varphi_{\tilde{\tau}(\xi)}(\xi) \in g_e, \forall \xi \in W$.

(c) *if g_e is $(n-1)$-dimensional, there exists $s_0 \in (t_0, t_1)$ such that, with u denoting the coordinates in Definition* 6,

$$\dot{\varphi}_t(\xi) \cdot \nabla u_n(\varphi_t(\xi)) \geqslant \tfrac{1}{2}, \quad \text{a.e. on} [s_0, s_1], \quad \forall \xi \in W.$$

Proof. We first consider the case when g_e is $(n-1)$-dimensional. By the transversality assumption there exists an open neighborhood V of $x(t_1)$ and coordinates $u : V \to \mathbb{R}^n$ such that x can be continued to $(t_1, s_1]$ for some $s_1 > t_1$ and

$$\dot{x}(t) \cdot \nabla u_n(v) \geqslant 1, \quad \text{a.e. on } \{t\colon x(t) \in V\}, \ \forall v \in V. \tag{8}$$

Select times $t_1' < t_1$, $t_1'' \in (t_1, s_1]$ and $\delta' > 0$ such that

$$B(x(t), \delta') \subset V, \quad \forall t \in [t_1', t_1''], \tag{9}$$

and if necessary shrink δ' even further so that

$$u_n < 0 \text{ on } B(x(t_1'), \delta') \text{ and } u_n > 0 \text{ on } B(x(t_1''), \delta'). \tag{10}$$

We use the construction in [4]. First, choose $\delta > 0$ to satisfy

$$3\delta \mathrm{e}^{2K(t_1'' - t_0)} \leqslant \delta', \tag{11a}$$

$$2\delta K \mathrm{e}^{2K(t_1'' - t_0)} \sup_{v \in V} |\nabla u_n(v)| \leqslant \tfrac{1}{2}. \tag{11b}$$

Let $\{\varphi_t^j(\xi)\}_{j=0}^{\infty}$ denote the sequence of approximate solutions in $\mathscr{C}_{\rm ac}([t_0, t_1''])$, with ξ in $B(x(t_0), \delta)$, converging to $\varphi_t(\xi)$ uniformly in $\mathscr{C}_{\rm ac}([t_0, t_1''])$. Let $\xi_0 := x(t_0)$. We claim that, for all $\xi \in B(\xi_0, \delta)$

$$\dot{\varphi}_t(\xi) \cdot \nabla u_n(\varphi_t(\xi)) \geqslant \tfrac{1}{2}, \quad \text{a.e. on } [t_1', t_1'']. \tag{12}$$

In order to prove this claim, using the construction in [4], we can derive the following property characterizing the sequence $\{\varphi_t^j\}_{j=0}^{\infty}$. Corresponding to each $j \geqslant 0$, and to each $\xi \in B(\xi_0, \delta)$, there exists a finite partition $\{I_i(\xi)\}_{i=1}^{n_j}$ of $[t_0, t_1'']$, and a finite subset of $B(\xi_0, \delta)$, denoted by $\Xi_j = \{\xi_i^\ell,\ 0 \leqslant \ell \leqslant$

$j-1,\ 1 \leqslant i \leqslant n_j\}$ (Ξ_j not depending on ξ) such that the following estimate holds a.e. on $I_i(\xi)$, for $i = 1, \dots, n_j$,

$$|\dot\varphi_t^\ell(\xi_i^\ell) - \dot\varphi_t^{\ell-1}(\xi_i^{\ell-1})| \leqslant \delta K \left[\frac{(K(t-t_0))^{\ell-1}}{(\ell-1)!} + \frac{e^{2K(t-t_0)}}{2^\ell} \right], \quad \ell \leqslant j-1,$$

$$|\dot\varphi_t^j(\xi) - \dot\varphi_t^{j-1}(\xi_i^{j-1})| \leqslant \delta K \left[\frac{(K(t-t_0))^{j-1}}{(j-1)!} + \frac{e^{2K(t-t_0)}}{2^j} \right]. \tag{13}$$

From (13), using a triangle inequality, we obtain

$$|\dot\varphi_t^j(\xi) - \dot\varphi_t^0(\xi_i^0)| \leqslant \delta K[e^{K(t-t_0)} + e^{2K(t-t_0)}] \leqslant 2\delta K e^{2K(t_1''-t_0)}, \tag{14}$$

a.e. on $I_i(\xi)$. By (6) and (11a),

$$|\varphi_t^j(\xi) - x(t)| \leqslant \delta', \quad \forall t \in [t_0, t_1''], \quad \forall \xi \in B(\xi_0, \delta). \tag{15}$$

Next, by (15) and (9), $\varphi_t^j(\xi) \in V$, for all $\xi \in B(\xi_0,\delta)$, $t \in [t_1', t_1'']$ and $j \in \mathbb{N}$. Hence, combining (8), (11b) and (14), and using the fact that $\dot\varphi_t^0(\xi_i^0) = \dot x(t)$, for all $i = 1, \dots, n_j$, a triangle inequality yields

$$\begin{aligned} &\dot\varphi_t^j(\xi) \cdot \nabla u_n(\varphi_t^j(\xi)) \\ &\quad \geqslant \dot x(t) \cdot \nabla u_n(\varphi_t^j(\xi)) \\ &\qquad -|\nabla u_n(\varphi_t^j(\xi))| \cdot |\dot\varphi_t^j(\xi) - \dot\varphi_t^0(\xi_i^0)| \\ &\quad \geqslant 1 - \tfrac{1}{2} = \tfrac{1}{2}, \quad \text{a.e. on } [t_1', t_1''], \quad \forall \xi \in B(\xi_0, \delta). \end{aligned}$$

Passing to the limit as $j \to \infty$, we establish (12). Parts (a) and (c) of the Lemma follow if we select $W = B(\xi_0, \delta)$. In order to establish part (b), first note that by (10) and (15)

$$u_n(\varphi_{t_1'}(\xi)) < 0 \quad \text{and} \quad u_n(\varphi_{t_1''}(\xi)) > 0, \quad \forall \xi \in W. \tag{16}$$

Hence (12) and (16) imply that for each $\xi \in W$, there exists a unique $\tilde\tau(\xi) \in (t_1', t_1'')$ satisfying $u_n(\varphi_{\tilde\tau(\xi)}(\xi)) = 0$, or equivalently, $\varphi_{\tilde\tau(\xi)}(\xi) \in g_e$. To prove continuity of $\tilde\tau(\cdot)$, we argue by contradiction. Suppose $\{\xi^{(k)}\} \subset W$ is a sequence converging to $\xi^* \in W$, as $k \to \infty$, but $\tilde\tau(\xi^{(k)}) \nrightarrow \tilde\tau(\xi^*)$. Then along some subsequence also denoted by $\{\xi^{(k)}\}$, $\tilde\tau(\xi^{(k)}) \to \tilde\tau^*$ for some $\tilde\tau^* \neq \tilde\tau(\xi^*)$. It follows that $\varphi_{\tilde\tau(\xi^{(k)})}(\xi^{(k)}) \to \varphi_{\tilde\tau^*}(\xi^*)$, and hence, $u_n(\varphi_{\tilde\tau(\xi^{(k)})}(\xi^{(k)})) \to u_n(\varphi_{\tilde\tau^*}(\xi^*))$. But $u_n(\varphi_{\tilde\tau(\xi^{(k)})}(\xi^{(k)})) = 0$ implying $u_n(\varphi_{\tilde\tau^*}(\xi^*)) = 0$ which contradicts the uniqueness of $\tilde\tau(\xi^*)$. This proves part (b).

We now turn to the case where g_e is a topological n-manifold with boundary. Suppose that (7a) applies. Let $\{\varphi_t(\cdot), t \in [t_0, s_1]\}$ be a continuous selection of solutions of $\dot\varphi \in F_l(\varphi)$, defined on some open neighborhood U of $\xi_0 = x(t_0)$. Using the continuity of the selection, we pick an open neighborhood $W \subset U$ of ξ_0 such that $\varphi_{s_1}(\xi) \in \mathring{g}_e$, for all $\xi \in W$. Let $\mu : W \to [t_0, s_1]$ be defined by

$$\mu(\xi) := \inf\{s \in [t_0, s_1] : \varphi_t(\xi) \in g_e{}^\circ, \ \forall t \in (s, s_1]\}.$$

The multivalued map $\mathscr{T}$ defined on W by $\mathscr{T}(\xi) := [\mu(\xi), s_1]$ has closed, convex values and satisfies $\mathscr{T}(\xi_0) := [t_1, s_1]$. It is also lower semicontinuous. To establish this fact we argue by contradiction. If not, then for some $\xi' \in W$, an open neighborhood $\mathscr{N}$ of $\mathscr{T}(\xi')$, and a sequence $\{\xi^{(k)}\} \subset W$ converging to ξ', we must have $\mathscr{T}(\xi^{(k)}) \cap \mathscr{N} = \emptyset$, for all $k \in \mathbb{N}$. Select $s' \in \mathscr{T}(\xi') \cap \mathscr{N}$, $s' \neq \mu(\xi')$. Then $\varphi_{s'}(\xi') \in \mathring{g}_e$. Since $s' \notin \mathscr{T}(\xi^{(k)})$, for each $k \in \mathbb{N}$, there exists $s^{(k)} \in (s', s_1)$ such that $\varphi_{s^{(k)}}(\xi^{(k)}) \in \partial g_e$. Let s^* be a subsequential limit of $\{s^{(k)}\}$. Passing to the limit as $k \to \infty$ along this subsequence, we conclude that $\varphi_{s^*}(\xi') \in \partial g_e$. However, $s^* \geqslant s'$ and therefore $s^* > \mu(\xi')$, yielding a contradiction. Applying Michael's selection theorem we obtain a continuous selection $\tilde\tau : W \to [t_0, s_1]$, passing through t_1 at ξ_0. By construction $\varphi_{\tilde\tau(\xi)}(\xi) \in g_e$, for all $\xi \in W$. In the event that (7b) applies, the proof is analogous. □

Theorem 9. *Suppose H satisfies Assumption* 4 *and let π_0 be a transversal trajectory of H with initial state $q_0 = (l^0, \xi_0) \in Q$. For each $m > 0$, there exists a neighborhood (l^0, U) of q_0, with $U \subset \mathbb{R}^n$ open, and $\Psi(t, \xi)$, a selection of trajectories of H, such that $\Psi(t, \xi_0) = \pi_0(t)$ and $\xi \mapsto \Psi(\cdot, \xi)$ is continuous in (Π, d^m).*

Proof. Suppose that π_0 has an m step run $l^0, \dots, l^{m-1}$, each step represented by (1), and visits the enabling conditions $g^0, \dots, g^{m-1}$, with $r^0, \dots, r^{m-1}$ denoting the corresponding reset maps. Observe that in order for $\xi \mapsto \Psi(\cdot, \xi)$ to be continuous in (Π, d^m), the m-step run of $t \mapsto \Psi(t, \xi)$ must be independent of ξ and therefore must equal $l^0, \dots, l^{m-1}$.

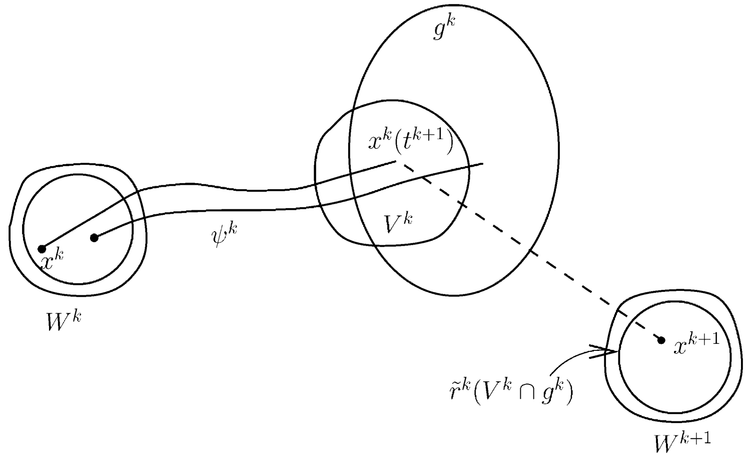

Fig. 2. Theorem 9: Continuity with respect to initial conditions.

First consider the reset of the kth step (see Fig. 2). Since r^k is locally selectionable, by Michael's Selection Theorem, there exists a continuous selection $\tilde{r}^k$ of r^k, satisfying

$$\tilde{r}^k(x^k(t^{k+1})) = x^{k+1}(t^{k+1}+). \tag{17}$$

Therefore, given an open neighborhood W^{k+1} of $x^{k+1}(t^{k+1}+)$, there exists an open subset V^k containing $x^k(t^{k+1})$ such that $\tilde{r}^k(V^k \cap g^k) \subset W^{k+1}$. If $x^k(t^{k+1}) \in g^k$, then by Lemma 8, for each open set V^k containing $x^k(t^{k+1})$, there exists an open neighborhood W^k of $x^k(t^k+)$, a time $\tilde{t}^{k+1} \geqslant t^{k+1}$, a continuous selection $\psi^k : W^k \to \mathscr{C}_{\mathrm{ac}}([0, \tilde{t}^{k+1} - t^k])$ of solutions of $\dot{\psi}^k = F_{l^k}(\psi^k)$, and a continuous map $\tilde{\tau}^k : W^k \to [0, \tilde{t}^{k+1} - t^k]$ such that

$$\begin{aligned} &\psi^k_t(x^k(t^k+)) = x^k(t+t^k), \quad t \in (0, t^{k+1} - t^k], \\ &\tilde{\tau}^k(x^k(t^k+)) = t^{k+1} - t^k, \\ &\psi^k_{\tilde{\tau}^k(w)}(w) \in V^k \cap g^k, \quad \forall w \in W^k. \end{aligned} \tag{18}$$

An iteration of the above arguments yields, for each $k = 0, \ldots, m-1$, open sets W^k and V^k along with a continuous selections ψ^k and continuous maps $\tilde{r}^k$ and $\tilde{\tau}^k$, as defined above, such that (17) and (18) hold.

Define $\tilde{\psi}^k : W^k \to V^k \cap g^k$ by $\tilde{\psi}^k(w) := \psi^k_{\tilde{\tau}^k(w)}(w)$. From the continuity of $w \mapsto \psi^k_t(w)$ and $w \mapsto \tilde{\tau}^k(w)$, the absolute continuity of $t \mapsto \psi^k_t(w)$, and the triangle inequality

$$\begin{aligned} |\tilde{\psi}^k(w) - \tilde{\psi}^k(w')| \leqslant{}& |\psi^k_{\tilde{\tau}^k(w)}(w) - \psi^k_{\tilde{\tau}^k(w')}(w)| \\ &+ |\psi^k_{\tilde{\tau}^k(w')}(w) - \psi^k_{\tilde{\tau}^k(w')}(w')|, \end{aligned}$$

we conclude that $\tilde{\psi}^k$ is continuous on W^k. Let $U = W^0$ and define for $\xi \in U$

$$\begin{aligned} &\beta^k(\xi) = \tilde{r}^{k-1} \circ \tilde{\psi}^{k-1} \circ \cdots \circ \tilde{r}^0 \circ \tilde{\psi}^0(\xi), \\ &\quad k = 1, \ldots, m, \ \beta^0(\xi) = \xi, \\ &t^k(\xi) = \sum_{\ell=0}^{k-1} \tilde{\tau}^\ell \circ \beta^\ell(\xi), \\ &I^0(\xi) = [0, t^1(\xi)], \ I^k(\xi) = (t^k(\xi), t^{k+1}(\xi)], \\ &\Psi(t,\xi) = \{(l^k, \psi^k_{t - t^k(\xi)} \circ \beta^k(\xi)), \ t \in I^k(\xi)\}_{k=0}^{m-1}. \end{aligned}$$

It follows that each $t^k(\cdot)$ is continuous on U. To show continuity of $\Psi(\cdot,\cdot)$ in (Π, d^m), we fix $\xi' \in U$ and define

$$\lambda(t) = \frac{t - t^k(\xi')}{t^{k+1}(\xi') - t^k(\xi')}, \ t \in I^k(\xi'), \ k = 0, \ldots, m-1.$$

Also, for each $\xi \in U$ we construct the functions κ_ξ and s_ξ with domain $[0, t^m(\xi')]$ by defining on each $I^k(\xi')$,

$k = 0, \dots, m-1$,

$$\kappa_\xi(t) = \lambda(t)t^{k+1}(\xi) + (1-\lambda(t))t^k(\xi),$$
$$s_\xi(t) = \lambda(t)\min\{t^{k+1}(\xi), t^{k+1}(\xi')\} + (1-\lambda(t))\max\{t^k(\xi), t^k(\xi')\}.$$

Note that $\kappa_\xi : [0, t^m(\xi')] \to [0, t^m(\xi)]$ is strictly increasing, continuous, and surjective. Also, for ξ sufficiently close to ξ',

$$\min\{t^{k+1}(\xi), t^{k+1}(\xi')\} > \max\{t^k(\xi), t^k(\xi')\}$$

and hence

$$s_\xi(t) \in \mathscr{I}(\xi, \xi') := \bigcup_{k=0}^{m-1} (I^k(\xi') \cap I^k(\xi)). \tag{19}$$

In addition, from the continuity of $t^k(\cdot)$ on U it follows that

$$|t - \kappa_\xi(t)| \underset{\xi \to \xi'}{\longrightarrow} 0$$

and

$$|t - s_\xi(t)| \underset{\xi \to \xi'}{\longrightarrow} 0 \quad \text{uniformly on } [0, t^m(\xi')]. \tag{20}$$

In order to avoid introducing new notation we identify Ψ with the continuous part of the trajectories. While it is certainly not the case that $\Psi(t, \cdot)$, is continuous on U, for arbitrary t, it holds that

$$\sup_{t \in \mathscr{I}(\xi, \xi')} |\Psi(t, \xi') - \Psi(t, \xi)| \underset{\xi \to \xi'}{\longrightarrow} 0, \tag{21}$$

where $\mathscr{I}(\xi, \xi')$ is defined in (19). Therefore, if we form the triangle inequality

$$|\Psi(t, \xi') - \Psi(\kappa(t), \xi)| \leqslant |\Psi(t, \xi') - \Psi(s_\xi(t), \xi')| + |\Psi(s_\xi(t), \xi') - \Psi(s_\xi(t), \xi)| + |\Psi(s_\xi(t), \xi) - \Psi(\kappa_\xi(t), \xi)|$$

and let $\xi \to \xi'$, the first and third term on the right-hand side converge by (20) and the uniform absolute continuity of $t \to \Psi(t, \cdot)$, on each $I^k(\cdot)$, while the middle term converges by (19) and (21), and the convergence is uniform in $t \in [0, t^m(\xi')]$. Therefore,

$$d^m(\Psi(t, \xi), \Psi(t, \xi')) \underset{\xi \to \xi'}{\longrightarrow} 0,$$

and the proof is complete. □

5. Conclusions

We have introduced some useful analytical tools and have demonstrated the existence of continuous selections of trajectories of hybrid automata with Lipschitz differential inclusions with respect to initial conditions. We believe that the basic properties of hybrid trajectories presented here will be useful in establishing new connections between observation equivalences for hybrid automata, including those that are bisimulations, and qualitative features of trajectories starting from equivalent points. This type of investigation was started in [3].

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