qualitative and quantitative results. The insightful understanding of these dynamics will help prevent ill-behaviors due to discretization in digital controllers design. Furthermore, it will help make use of some discretization behaviors for engineering applications such as oscillations generation. Of particular interest are the dynamical behaviors of the trajectories within some specified boundaries. There may be different domains of attractions within the boundaries for different symbolic sequences. Experience reveals that there might be rich bifurcating and even chaotic dynamics within such bounded regions. Further research is undertaken to investigate this kind of interesting phenomena from the chaos and fractal points of view.

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Linearization of Discrete-Time Systems via Restricted Dynamic Feedback

Hong-Gi Lee, Ari Arapostathis, and Steven I. Marcus

Abstract—**We extend the results from a previous paper of ours to multiple-input systems, and utilize these to obtain necessary and sufficient conditions for the linearization of discrete-time nonlinear systems via restricted dynamic feedback. We observe that for discrete-time nonlinear systems, the bound on the number of delays (or integrators) needed to synthesize the linearizing dynamic feedback differs from the continuous-time analogue.**

Index Terms—**Dynamic-feedback linearization, nonlinear discrete-time control systems, restricted dynamic feedback.**

I. Introduction

Linearization is a widely used tool for the control of nonlinear systems, because well-developed linear system theory techniques can be applied to the nonlinear plant, once this is linearized. The transformations employed for linearization usually involve a state coordinate change and feedback. Linearization via static feedback has been thoroughly studied and an abundance of results exist in the literature applicable to both continuous-time [8], [11], [15], [24] and discrete-time nonlinear systems [2], [3], [9], [13], [16], [18]. More recently, the use of dynamic feedback has been investigated, in the hope of augmenting the class of linearizable systems. However, despite the significant effort already invested in studying linearization via dynamic state feedback [1], [4]–[7], [10], [14], [17], [20]–[23], finding verifiable necessary and sufficient conditions to characterize the class of such linearizable systems is still an open problem. Restricted dynamic feedback refers to a compensator in the feedback loop that consists only of pure integrators. Lee *et al.* [17] obtained necessary and sufficient conditions for a continuous-time system to be linearizable via restricted dynamic feedback by establishing a bound on the maximum number of integrators needed for the input channels. In this note, we extend the study in [17] to discrete-time systems. The method relies on the multiple-input version of results in [16].

Consider a smooth nonlinear discrete-time system

$$x(t+1) = f(x(t), u(t)), \qquad f(0,0) = 0 \tag{1}$$

with state $x \in \Sigma \simeq \mathbb{R}^n$ and input $u \in \mathcal{U} \simeq \mathbb{R}^m$.

Definition 1: System (1) is linearizable by a state coordinate change, if there exists a smooth diffeomorphism $T : \Sigma \to \Sigma$ which transforms (1) to a reachable linear system, in the variable $\zeta = T(x)$

$$\zeta(t+1) = A\zeta(t) + Bu(t), \qquad \zeta \in \Sigma.$$

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Definition 2: System (1) is static-feedback linearizable, if there exists a smooth map $\gamma : \Sigma \times \mathcal{U} \to \mathcal{U}$ such that the feedback $u = \gamma(x, v)$ results in a closed-loop system

$$x(t+1) = f(x(t), \gamma(x(t), v(t))), \qquad x \in \Sigma, v \in \mathcal{U}$$

which is linearizable by a state coordinate change.

Dynamic state feedback amounts to the use of a controller with dynamics

$$z(t+1) = g(x(t), z(t), v(t)), \qquad z \in \Sigma_c \simeq \mathbb{R}^s,\ v \in \mathcal{U} \tag{2}$$

and a smooth map $h : \Sigma \times \Sigma_c \times \mathcal{U} \to \mathcal{U}$, which when combined with (1) yield the closed-loop system with extended state–space $\Sigma \times \Sigma_c$

$$\begin{bmatrix} x(t+1) \\ z(t+1) \end{bmatrix} = \begin{bmatrix} f(x(t), h(x(t), z(t), v(t))) \\ g(x(t), z(t), v(t)) \end{bmatrix}. \tag{3}$$

Definition 3: System (1) is dynamic-feedback linearizable, if there exists a smooth dynamic feedback (2) which yields a closed-loop system (3) that is linearizable by a state coordinate change.

Definition 4: System (1) is said to be linearizable via restricted dynamic feedback of index $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{Z}_+^m$, if there exists a dynamic compensator of the form

$$z_k^i(t+1) = \begin{cases} z_{k+1}^i(t), & \text{if } d_i \geq 1, \qquad 1 \leq k \leq d_i - 1 \\ v_i(t), & \text{if } d_i \geq 1, \qquad k = d_i \end{cases} \tag{4a}$$

and a smooth feedback $h = (h_1, \ldots, h_m)$, defined by

$$u_i = h_i(z, v) = \begin{cases} z_1^i, & \text{if } d_i \geq 1 \\ v_i, & \text{if } d_i = 0 \end{cases} \tag{4b}$$

such that the resulting closed-loop system is static-feedback linearizable.

In this note, we obtain necessary and sufficient conditions for (1) to be linearizable via restricted dynamic feedback as defined in Definition 4. First, we extend the results in [16] to multiple-input systems and then we follow the approach in [17]. The results we obtain are very similar to the continuous-time case. However, the bound obtained on the number of necessary delays is smaller than the corresponding one on the number of integrators.

II. Preliminaries and Definitions

In this section, we introduce some basic definitions and then extend the results of [16] to multiple-input systems. We refer the reader to [12], [19], and other papers in the references for basic results in nonlinear systems and differential geometry used in this note.

We view $\mathcal{B} := \Sigma \times \mathcal{U}$, $\pi : \mathcal{B} \to \Sigma$ as a vector bundle over Σ. With $\mathcal{B}_x \simeq \mathcal{U}$ denoting the fiber over $x \in \Sigma$, we define, for each nonnegative integer k, the *kth product bundle* $\mathcal{B}^k$ by

$$\mathcal{B}^k = \bigcup_{x \in \Sigma} \underbrace{\mathcal{B}_x \times \cdots \times \mathcal{B}_x}_{\text{k times}}.$$

Thus, $\mathcal{B}^k$ is a smooth vector bundle over Σ, and it may also be viewed as a vector bundle over $\mathcal{B}^{k-1}$, with $\pi : \mathcal{B}^k \to \mathcal{B}^{k-1}$ denoting the projection. Also, $\mathcal{B}^0 \simeq \Sigma$ denotes the zero-section $\Sigma \times \{0\}$. The response of a discrete-time system to a finite input sequence can be conveniently represented, albeit reversed in time, if one extends the definition of the system map $f : \mathcal{B} \to \Sigma$ to a map $f : \mathcal{B}^k \to \mathcal{B}^{k-1}$, for $k > 0$, by

$$f(x, u^1, \ldots, u^k) := (f(x, u^k), u^1, \ldots, u^{k-1})$$

where

$$u^j := \begin{bmatrix} u_1^j & \ldots & u_m^j \end{bmatrix}^T \in \mathcal{U}, \qquad 1 \leq j \leq k.$$

Also, for $k = 0$, f is interpreted as a map $f : \mathcal{B}^0 \to \mathcal{B}^0$, i.e., $f(x, 0) = (f(x, 0), 0)$. Then, the kth composition f^k is well-defined as a map from $\mathcal{B}^k$ to Σ, and more generally as a map from $\mathcal{B}^\ell$ to $\mathcal{B}^{(\ell-k)^+}$, where $(\cdot)^+$ denotes the positive part of $(\cdot)$. In particular, when the domain of f^k is selected as $\mathcal{B}$, then $f^k : \mathcal{B} \to \mathcal{B}^0 \simeq \Sigma$ is identified as the k-step impulse response of the system and we denote it as $\hat{f}^k$. In other words

$$\hat{f}^1(x, u) := f(x, u)$$
$$\hat{f}^\ell(x, u) := f(\hat{f}^{\ell-1}(x, u), 0), \qquad \ell \geq 2.$$

Identifying the fiber of $\mathcal{B}^k$ over $x \in \Sigma$ with the k-fold product $\mathcal{U}^k \simeq \mathcal{U} \times \cdots \times \mathcal{U}$, we often use the convenient notation $f_x^k(\mathbf{u}^k) = f^k(x, \mathbf{u}^k)$, with $\mathbf{u}^k = (u^1, \ldots, u^k)$ representing the generic element in $\mathcal{U}^k$.

Definition 5: For each $i \in \{1, \ldots, m\}$, let κ_i be the smallest nonnegative integer such that

$$\frac{\partial \hat{f}^{\kappa_i + 1}}{\partial u_i}(0) \in \operatorname{span}\left\{ \frac{\partial \hat{f}^{k+1}}{\partial u_j}(0) \,\middle|\, k \geq 0, j < m(\kappa_i - k) + i \right\}.$$

The Kronecker indexes of (1) are defined as the collection $\{\kappa_i\}$ and are represented by the multiple-index $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_m) \in \mathbb{Z}_+^m$.

It is well known that if $\sum_{i=1}^m \kappa_i = n$, then (1) is reachable around the origin.

For a nonnegative multiple-index $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_m)$ of length m, we set $\ell_{\max} := \max\{\ell_1, \ldots, \ell_m\}$ and $|\boldsymbol{\ell}| := \sum_i \ell_i$. We also define, for $k \geq 0$

$$\mathcal{U}(\boldsymbol{\ell}, k) = \operatorname{span}\{u_i^j \in \mathcal{U}^k \mid 1 \leq i \leq m, j > \ell_i\}$$
$$\mathcal{U}^\perp(\boldsymbol{\ell}, k) = \operatorname{span}\{u_i^j \in \mathcal{U}^k \mid 1 \leq i \leq m, j \leq \ell_i\} \tag{5}$$

and denote by $\pi_{\boldsymbol{\ell}}$ the projection of $\mathcal{U}^k = \mathcal{U}(\boldsymbol{\ell}, k) \times \mathcal{U}^\perp(\boldsymbol{\ell}, k)$ onto the first factor.

The three theorems and the remark that follow are a straightforward extension of the results in [16]. Thus, we omit the proofs. Let $\mathcal{F}$ denote the map $f_0^{\kappa_{\max}+1} : \mathcal{U}^{\kappa_{\max}+1} \to \Sigma$, and Ψ stand for the restriction of $f_0^{\kappa_{\max}}$ on $\mathcal{U}^\perp(\boldsymbol{\kappa}, \kappa_{\max})$.

Theorem 1: System (1) is linearizable by state coordinate change if and only if

i) $\sum_{i=1}^m \kappa_i = n$;
ii) $\mathcal{F}_*(\partial/\partial u_i^\ell)$ is a well-defined vector field, for each $i = 1, \ldots, m$, and $\ell = 1, \ldots, \kappa_i + 1$.

Furthermore, $\zeta = \Psi^{-1}(x)$ is a linearizing state coordinate transformation.

Theorem 2: System (1) is static-feedback linearizable if and only if

i) $\sum_{i=1}^m \kappa_i = n$;
ii) $\mathcal{F}_*(\Delta_i), i = 1, \ldots, \kappa_{\max} - 1$, are well-defined distributions, where

$$\Delta_i = \operatorname{span}\left\{ \frac{\partial}{\partial u_j^\ell} \,\middle|\, 1 \leq j \leq m, 1 \leq \ell \leq i \right\}. \tag{6}$$

We also state a useful variant of Theorem 2.

Theorem 3: System (1) is static-feedback linearizable if and only if there exist smooth functions $\psi_i : \Sigma \to \mathbb{R}, \psi_i(0) = 0$, defined for $i \in J_+ := \{i \mid \kappa_i > 0\}$, such that

i) $\sum_{i=1}^m \kappa_i = n$;
ii) $(\partial \psi_i \circ \hat{f}^\ell)/(\partial u) = 0$, for $i \in J_+$, and $\ell = 1, \ldots, \kappa_i - 1$;
iii) $\operatorname{rank}\{((\partial \psi_i \circ \hat{f}^{\kappa_i})/(\partial u))(0) \mid i \in J_+\} = |J_+|$, where $|J_+|$ denotes the cardinality of J_+.

Remark 1: Hypotheses ii) of Theorem 1 and Theorem 2 can be replaced by ii') and ii"), respectively [16], [25].

ii') $[(\partial/\partial u_j^\ell), \ker(\mathcal{F}_*)] \subset \ker(\mathcal{F}_*), 1 \leq j \leq m, \ell \leq \kappa_j + 1$.
ii") $[\Delta_i, \ker(\mathcal{F}_*)] \subset \Delta_i + \ker(\mathcal{F}_*), 1 \leq i \leq \kappa_{\max} - 1$.

III. Main Results

In this section, we obtain necessary and sufficient conditions for the discrete-time nonlinear system (1) to be linearizable via restricted dynamic feedback. Even though our approach is analogous to the one taken for the continuous-time case [17], the proof for the discrete-time case turns out to be somewhat simpler.

The closed-loop system of (1) with the compensator (4) is of the form

$$\begin{bmatrix} x(t+1) \\ z(t+1) \end{bmatrix} = \begin{bmatrix} f(x(t), h(z(t), v(t))) \\ g(z(t), v(t)) \end{bmatrix} = F(x(t), z(t), v(t))$$

with $x \in \Sigma \simeq \mathbb{R}^n$, $z \in \mathbb{R}^{|\mathbf{d}|}$ and $v \in \mathcal{U} \simeq \mathbb{R}^m$.

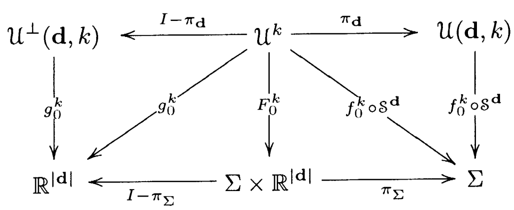
Consider the map F_0^k with domain $\mathcal{B}_0^k \simeq \mathcal{U}^k$. Recall the definitions in (5), and observe that $\ker g_0^k = \mathcal{U}(\mathbf{d}, k)$. Therefore, if we decompose $\mathcal{U}^k = \mathcal{U}(\mathbf{d}, k) \times \mathcal{U}^{\perp}(\mathbf{d}, k)$, it follows that the restriction $g_0^k : \mathcal{U}^{\perp}(\mathbf{d}, k) \to \mathbb{R}^{|\mathbf{d}|}$ is a linear isomorphism, provided $k \geq d_{\max}$. Next, define the map $\mathcal{S}^{\mathbf{d}} : \mathcal{U}^k \to \mathcal{U}^k$ by

$$(\mathcal{S}^{\mathbf{d}}(\mathbf{u}^k))_i^j = \begin{cases} u_i^{j+d_i}, & \text{if } j + d_i \leq k \\ 0, & \text{otherwise} \end{cases}.$$

It follows that $\ker \mathcal{S}^{\mathbf{d}} = \mathcal{U}^{\perp}(\mathbf{d}, k)$ and, hence, the restriction of $\mathcal{S}^{\mathbf{d}}$ on $\mathcal{U}(\mathbf{d}, k)$ is an isomorphism onto its range. In addition, if π_Σ denotes the projection $\Sigma \times \mathbb{R}^{|\mathbf{d}|} \to \Sigma$ on the first factor, i.e., $\pi_\Sigma(x, z) = x$, we obtain $\pi_\Sigma \circ F_0^k = f_0^k \circ \mathcal{S}^{\mathbf{d}} \circ \pi_{\mathbf{d}}$. Therefore, we have the decomposition

$$F_0^k = \left(f_0^k \circ \mathcal{S}^{\mathbf{d}}, g_0^k\right) : \mathcal{U}(\mathbf{d}, k) \times \mathcal{U}^{\perp}(\mathbf{d}, k) \to \Sigma \times \mathbb{R}^{|\mathbf{d}|}$$

which is also depicted in the following commutative diagram:



Definition 6: Given $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{Z}_+^m$, the *relative Kronecker indexes* $\tilde{\boldsymbol{\kappa}}(\mathbf{d}) = (\tilde{\kappa}_1(\mathbf{d}), \ldots, \tilde{\kappa}_m(\mathbf{d}))$ of (1) are defined as follows. For each $i \in \{1, \ldots, m\}$, $\tilde{\kappa}_i(\mathbf{d})$ is the smallest nonnegative integer such that

$$\frac{\partial \hat{f}^{\tilde{\kappa}_i(\mathbf{d})+1}}{\partial u_i}(0) \in \operatorname{span}\left\{\frac{\partial \hat{f}^{k+1}}{\partial u_j}(0)\right\}_{\substack{k \geq 0 \\ j < m(\tilde{\kappa}_i(\mathbf{d}) + d_i - d_j - k) + i}}.$$

Observe that $|\tilde{\boldsymbol{\kappa}}(\mathbf{d})| = |\boldsymbol{\kappa}|$. From the previous discussion, we obtain the following corollary to Theorem 2.

Corollary 1: System (1) is linearizable via restricted dynamic feedback if and only if there exists $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{Z}_+^m$, such that, with

$$\bar{\kappa}_{\mathbf{d}} := \max_{1 \leq i \leq m} \{\tilde{\kappa}_i(\mathbf{d}) + d_i\}$$
$$\tilde{\mathcal{F}}^{\mathbf{d}} := f_0^{\bar{\kappa}_{\mathbf{d}}+1} \circ \mathcal{S}^{\mathbf{d}} : \mathcal{U}(\mathbf{d}, \bar{\kappa}_{\mathbf{d}} + 1) \to \Sigma$$

and $\{\Delta_i\}$ as defined in (6)

i) $\sum_{i=1}^m \kappa_i = n$;
ii) $\tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_i)$, is a well-defined distribution, for $i = 1, \ldots, \bar{\kappa}_{\mathbf{d}} - 1$.

Lemma 1: Suppose (1) is linearizable via restricted dynamic feedback of index $\mathbf{d} = (d_1, \ldots, d_m)$, and for some $\alpha \geq 0$

$$\tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_\alpha) = \tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_{\alpha+1}) \tag{7}$$

with $\Delta_0 := 0$. Then, it is also linearizable via restricted dynamic feedback of index $\mathbf{d}' = (d_1', \ldots, d_m')$, where

$$d_i' = \begin{cases} d_i, & \text{if } d_i \leq \alpha \\ d_i - 1, & \text{otherwise.} \end{cases}$$

Proof: Let

$$J_\alpha := \{i \mid d_i \leq \alpha\} \quad J_\alpha^c := \{i \mid d_i > \alpha\}. \tag{8}$$

We define, for each $i = \alpha + 2, \ldots, \bar{\kappa}_{\mathbf{d}}$

$$\tilde{\Delta}_i = \operatorname{span}\left\{\frac{\partial}{\partial u_j^\ell} \,\middle|\, j \in J_\alpha^c, \qquad d_i < \ell \leq i\right\}.$$

Assumption (7) of the Lemma implies

$$\tilde{\mathcal{F}}_*^{\mathbf{d}}\left(\frac{\partial}{\partial u_j^\ell}\right) \in \tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_\alpha), \qquad \ell \geq \alpha + 1, j \in J_\alpha. \tag{9}$$

We may assume that $J_a^c \neq \oslash$, otherwise the conclusion of the Lemma is trivially true. By (9)

$$\tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_i) = \tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_\alpha + \tilde{\Delta}_i), \qquad i = \alpha + 2, \ldots, \bar{\kappa}_{\mathbf{d}}. \tag{10}$$

By Corollary 1, $\tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_i)$ is a well-defined distribution, for each $i = 1, \ldots, \bar{\kappa}_{\mathbf{d}} - 1$, and $\dim(\tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_{\bar{\kappa}_{\mathbf{d}}})) = n$. By (9), this assertion is also true for the map $\hat{\mathcal{F}}^{\mathbf{d}}$, which denotes the restriction of $\tilde{\mathcal{F}}^{\mathbf{d}}$ on the subspace

$$\hat{\mathcal{U}}_\alpha^{\bar{\kappa}_{\mathbf{d}}+1} := \mathcal{U}^{\bar{\kappa}_{\mathbf{d}}+1} \bigcap \left\{u_j^{\bar{\kappa}_{\mathbf{d}}+1} = 0 \,\middle|\, j \in J_\alpha\right\}.$$

Observe that the maximum in the definition of $\bar{\kappa}_{\mathbf{d}}$ is attained on J_α^c. From (9), we deduce that $\tilde{\boldsymbol{\kappa}}(\mathbf{d}') = \tilde{\boldsymbol{\kappa}}(\mathbf{d})$. Hence, $\bar{\kappa}_{\mathbf{d}'} = \bar{\kappa}_{\mathbf{d}} - 1$. Define the map

$$\varphi : \mathcal{U}(\mathbf{d}, \bar{\kappa}_{\mathbf{d}} + 1) \cap \hat{\mathcal{U}}_\alpha^{\bar{\kappa}_{\mathbf{d}}+1} \to \mathcal{U}(\mathbf{d}', \bar{\kappa}_{\mathbf{d}'} + 1)$$

by

$$(\varphi(\mathbf{u}))_i^j := \begin{cases} u_i^{j+1}, & \text{if } i \in J_a^c, \qquad j \geq d_i \\ u_i^j, & \text{otherwise.} \end{cases} \tag{11}$$

We obtain

$$\hat{\mathcal{F}}^{\mathbf{d}} = f_0^{\bar{\kappa}_{\mathbf{d}'}+1} \circ \mathcal{S}^{\mathbf{d}'} \circ \varphi = \tilde{\mathcal{F}}^{\mathbf{d}'} \circ \varphi. \tag{12}$$

Combining (10)–(12), we conclude that

$$\tilde{\mathcal{F}}_*^{\mathbf{d}'}(\Delta_i) = \begin{cases} \tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_i), & \text{if } i \leq \alpha \\ \tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_{i+1}), & \text{if } \alpha < i \leq \bar{\kappa}_{\mathbf{d}'} \end{cases}$$

and the proof follows from Corollary 1. ■

Lemma 2: If (1) is linearizable via restricted dynamic feedback of index $\mathbf{d} = (d_1, \ldots, d_m)$, and for some $\alpha \in \{1, \ldots, \bar{\kappa}_{\mathbf{d}}\}, \dim(\tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_\alpha)) = n$, then it is also linearizable via restricted dynamic feedback of index $\mathbf{d}' = (d_1', \ldots, d_m')$, defined by

$$d_i' = \min\{d_i, (\alpha - 1)\}, \qquad i = 1, \ldots, m.$$

Proof: Let $F : \Sigma \times \Sigma_c \times U \to \Sigma \times \Sigma_c$ denote the closed-loop system map, with a compensator of index $\mathbf{d}$. To simplify the notation, let $\boldsymbol{\kappa}$ denote the Kronecker indexes of the closed-loop system, i.e., $\kappa_i = \tilde{\kappa}_i(\mathbf{d}) + d_i, i = 1, \ldots, m$. Recall the definition of J_α in (8), and suppose that $J_{\alpha-1}^c \neq \oslash$. By Theorem 3, there exist smooth functions $\psi_i : \Sigma \times \Sigma_c \to \mathbb{R}$ such that properties ii)–iii) hold. Observe that we may choose $\psi_i, i \in J_{\alpha-1} \cap J_+$, so that these are independent of the coordinates $\{z^j \mid j \in J_{\alpha-1}^c\}$. Indeed, we may select a collection $\{\psi_i \mid i \in J_{\alpha-1} \cap J_+\}$ such that each $d\psi_i$ is orthogonal to the involutive distribution $(F_0^{\bar{\kappa}_{\mathbf{d}}+1})_*(\Delta_{\kappa_i - 1} + \Delta')$, where

$$\Delta' = \operatorname{span}\left\{\frac{\partial}{\partial u_j^\ell} \,\middle|\, j \in J_{\alpha-1}^c, 1 \leq \ell \leq d_j\right\}$$

and at the same time

$$\operatorname{rank}\left\{ \frac{\partial \psi_i \circ \hat{F}^{\kappa_i}}{\partial u}(0) \,\middle|\, i \in J_{\alpha-1} \cap J_+ \right\} = |J_{\alpha-1} \cap J_+|. \quad (13)$$

Also, $\{\psi_j | j \in J_{\alpha-1}^c\}$ may be selected in such a manner that each $d\psi_j$ is orthogonal to $(F_0^{\bar{\kappa}_{\mathbf{d}}+1})_*(\Delta_{\kappa_j-1} + \Delta'')$, with

$$\Delta'' = \operatorname{span}\left\{ \frac{\partial}{\partial u_i^\ell} \,\middle|\, i \in J_{\alpha-1} \cap J_+, 1 \le \ell \le \kappa_i \right\}$$

and they satisfy the rank condition analogous to (13). We use the decomposition $u = (\breve{u}, \breve{u}^c)$ where $\breve{u} = \{u_i \,|\, i \in J_{\alpha-1} \cap J_+\}$ and $\breve{u}^c$ consists of the remaining input coordinates. By construction

$$\frac{\partial \psi_i \circ \hat{F}^{\kappa_i}}{\partial \breve{u}^c}(0) = 0 \qquad \forall i \in J_{\alpha-1} \cap J_+. \quad (14)$$

By (13) and (14)

$$\operatorname{rank}\left\{ \frac{\partial \psi_i \circ \hat{F}^{\kappa_i}}{\partial \breve{u}}(0) \,\middle|\, i \in J_{\alpha-1} \cap J_+ \right\} = |J_{\alpha-1} \cap J_+|. \quad (15)$$

Equation (14), together with property ii) of Theorem 3, yields

$$\operatorname{rank}\left\{ \frac{\partial \psi_j \circ \hat{F}^{\kappa_j}}{\partial \breve{u}^c}(0) \,\middle|\, j \in J_{\alpha-1}^c \right\} = |J_{\alpha-1}^c|. \quad (16)$$

We claim that if we modify the collection $\{\psi_j \mid j \in J_{\alpha-1}^c\}$ by selecting $\psi_j(x,z) := z_1^j$, ii) and iii) of Theorem 3 still hold. Indeed, for $j \in J_{\alpha-1}^c$, since $\kappa_j = d_j$, and

$$\psi_j \circ \hat{F}^\ell(x,z,u) = \begin{cases} z_{\ell+1}^j, & \text{if } 1 \le \ell \le d_j - 1 \\ u_j, & \text{if } \ell = d_j \end{cases} \quad (17)$$

property ii) follows. Also, (17) implies (16) and

$$\frac{\partial \psi_j \circ \hat{F}^{\kappa_j}}{\partial \breve{u}}(0) = 0 \qquad \forall j \in J_{\alpha-1}^c. \quad (18)$$

Property iii) follows from (14)–(16).

Consider now the compensator with index $\mathbf{d}'$ and denote by F' and $\boldsymbol{\kappa}'$ be the corresponding system map and Kronecker indexes of the closed-loop system, respectively. Since $\kappa_i' = \kappa_i$, for $i \in J_{\alpha-1}$, and $\kappa_j' = \kappa_j - 1$, for $j \in J_{\alpha-1}^c$, property i) of Theorem 3 holds. Let the collection $\{\psi_i\}$ be as selected beforehand. Note that these are well defined in the new state–space. Property ii) of Theorem 3 easily follows. Also, (15), (16), and (18) hold for F' and $\boldsymbol{\kappa}'$, which together imply iii). ■

Using Lemma 1, we obtain the following.

Lemma 3: Suppose (1) is linearizable via restricted dynamic feedback of index $\mathbf{d} = (d_1, \ldots, d_m)$ and $d_i \ge 1$, for $1 \le i \le m$. Then, it is also linearizable via restricted dynamic feedback of index $\mathbf{d}' = (d_1', \ldots, d_m')$, where $d_i' = d_i - 1$, for $1 \le i \le m$.

Proof: Suppose that $d_i \ge 1$, for all $i \in \{1, \ldots, m\}$. Then, we have

$$\tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_0) = \tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_1) = 0.$$

The rest follows by Lemma 1. ■

Lemma 3 implies that if (1) is linearizable via restricted dynamic feedback, then the linearizing compensator (4) can be chosen so as to satisfy $d_{\min} := \min\{d_1, \ldots, d_m\} = 0$. This of course means that a single-input discrete-time nonlinear system is linearizable via restricted dynamic feedback only if it is static-feedback linearizable.

Theorem 4: If (1) is linearizable via restricted dynamic feedback, then a compensator of index $\mathbf{d}$ can be chosen, satisfying $d_{\min} = 0$, and $d_{\max} \le n - 1$, yielding the estimate

$$|\mathbf{d}| \le (m-1)(n-1). \quad (19)$$

Proof: By Lemma 1, we may choose the index $\mathbf{d}$ so that

$$\dim(\tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_i)) < \dim(\tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_{i+1})), \qquad 0 \le i \le \bar{\kappa}_{\mathbf{d}} - 1.$$

Thus, $\bar{\kappa}_{\mathbf{d}} \le n$. Since $\dim(\tilde{\mathcal{F}}_*^{\mathbf{d}}(\Delta_{\bar{\kappa}_{\mathbf{d}}})) = n$, applying Lemma 2, we can choose $\mathbf{d}$ such that $d_{\max} \le \bar{\kappa}_{\mathbf{d}} - 1$. Therefore, $d_{\max} \le n - 1$. Also, by Lemma 3, $\mathbf{d}$ can be chosen, so as to satisfy $d_{\min} = 0$. Hence, the estimate in (19) follows. ■

IV. EXAMPLES

By the results of the previous section, the validity of the conditions for linearization via restricted dynamic feedback needs to be verified only over a finite set of indexes. Therefore, we have obtained a set of decidable necessary and sufficient conditions. The bounds for the compensator index in Theorem 4 are sharp, as can be seen by the following example.

Example 1: Consider the system

$$\begin{aligned} x_1(t+1) &= x_2(t) + \sum_{i=2}^m x_1(t)u_i(t) \\ x_\ell(t+1) &= x_{\ell+1}(t) \quad \text{for } \ell = 2, \ldots, n-1 \\ x_n(t+1) &= \sum_{i=1}^m u_i(t). \end{aligned}$$

This system is linearizable via restricted dynamic feedback of index $d_1 = 0$ and $d_i = n - 1$, for $i = 2, \ldots, m$. However, it cannot be linearized if the index of the compensator is below the bound established in Theorem 4. An anonymous referee pointed out the following interesting observation concerning this example. Consider, for simplicity, the case $n = 3$ and $m = 2$, i.e.,

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \begin{bmatrix} x_2(t) + x_1(t)u_2(t) \\ x_3(t) \\ u_1(t) + u_2(t) \end{bmatrix}.$$

When x_1 is bounded away from 0, the static feedback

$$u_1 = v_2 - \frac{v_1 - x_2}{x_1} \quad u_2 = \frac{v_1 - x_2}{x_1} \quad (20)$$

yields the linear system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \begin{bmatrix} v_1(t) \\ x_3(t) \\ v_2(t) \end{bmatrix}.$$

Using restricted dynamic feedback in the form of the compensator $z_1(t+1) = z_2(t), z_2(t+1) = v_2(t)$, avoids the singularity at $x_1 = 0$ that arises in the static feedback in (20).

Example 2: Consider the system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \begin{bmatrix} x_2(t) + x_1(t)u_2(t) \\ u_1(t) \\ u_2(t) \end{bmatrix}. \quad (21)$$

The system in (21) is linearizable via restricted dynamic feedback of index $(d_1, d_2) = (0, 1)$. This feedback yields the closed-loop system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ z_1(t+1) \end{bmatrix} = \begin{bmatrix} x_2(t) + x_1(t)z_1(t) \\ v_1(t) \\ z_1(t) \\ v_2(t) \end{bmatrix}. \quad (22)$$

If we define new state variables $\zeta_1 = x_1, \zeta_2 = x_2 + x_1 z_1, \zeta_3 = x_3$, and $\zeta_4 = z_1$, (22) transforms to

$$\begin{bmatrix} \zeta_1(t+1) \\ \zeta_2(t+1) \\ \zeta_3(t+1) \\ \zeta_4(t+1) \end{bmatrix} = \begin{bmatrix} \zeta_2(t) \\ v_1(t) + (x_2(t) + x_1(t)z_1(t))v_2(t) \\ \zeta_4(t) \\ v_2(t) \end{bmatrix} \tag{23}$$

and, in turn, (23) can be linearized via the static state feedback

$$v = \gamma(x, z, v') = \begin{bmatrix} v_1' - (x_2 + x_1 z_1)v_2' \\ v_2' \end{bmatrix}.$$

In the following example, we present a system which is not linearizable via restricted dynamic feedback, but is dynamic-feedback linearizable.

Example 3: Consider the system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \begin{bmatrix} x_2(t) + x_1(t)(u_1(t) + u_2(t)) \\ u_1(t) \\ u_1(t) + u_2(t) \end{bmatrix}. \tag{24}$$

The application of Theorem 4, shows that (24) is not linearizable via restricted dynamic feedback. However, if we let $u_1 = v_1$ and $u_2 = v_2 - v_1$, then (24) transforms to (21). Therefore, (24) is dynamic-feedback linearizable.

V. Conclusion

We have formulated the problem of linearization via restricted dynamic feedback for discrete-time nonlinear systems in analogy to the continuous-time version [17]. We have shown that if a discrete-time nonlinear system is linearizable via restricted dynamic feedback, it is also linearizable without using a delay for at least one of the inputs. This means that the class of single-input systems linearizable by dynamic feedback is no larger than the class linearizable by static feedback, a fact which which also holds for continuous-time systems [17], [22]. We have also obtained sharp upper bounds on the number of delays necessary for the input channels. This bound is $n-1$, for each channel, whereas the analogous bound for the number of integrators used in the continuous-time case is $2n-3$ [17]. Our results yield verifiable necessary and sufficient conditions for linearization of discrete-time nonlinear systems via restricted dynamic feedback. However, the problem of linearization via general dynamic feedback is still wide open.

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