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UNIFORM RECURRENCE PROPERTIES OF CONTROLLED DIFFUSIONS AND APPLICATIONS TO OPTIMAL CONTROL*

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Abstract. In this paper we address an open problem which was stated in [A. Arapostathis et al., *SIAM J. Control Optim.*, 31 (1993), pp. 282–344] in the context of discrete-time controlled Markov chains with a compact action space. It asked whether the associated invariant probability distributions are necessarily tight if all stationary Markov policies are stable, in other words if the corresponding chains are positive recurrent. We answer this question affirmatively for controlled nondegenerate diffusions modeled by Itô stochastic differential equations. We apply the results to the ergodic control problem in its average formulation to obtain fairly general characterizations of optimality without resorting to blanket Lyapunov stability assumptions.

Key words. controlled diffusions, Markov processes, uniform stability, optimal control

AMS subject classifications. Primary, 93E15, 93E20; Secondary, 60J25, 60J60, 90C40

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1. Introduction.

This paper is concerned with controlled diffusion processes $X = \{X_t,\ t \ge 0\}$ taking values in the d-dimensional Euclidean space $\mathbb{R}^d$ and governed by the Itô stochastic differential equation

$$\mathrm{d}X_t = b(X_t, U_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\,. \tag{1.1}$$

All random processes in (1.1) live in a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Here, W is a d-dimensional standard Wiener process independent of the initial condition X_0. The control process U takes values in a compact, metrizable set $\mathbb{U}$, and $U_t(\omega)$ is jointly measurable in $(t, \omega) \in [0, \infty) \times \Omega$. In addition, it is *nonanticipative*: For $s < t$, $W_t - W_s$ is independent of

$$\mathfrak{F}_s \triangleq \text{the completion of } \sigma\{X_0, U_r, W_r,\ r \le s\} \text{ relative to } (\mathfrak{F}, \mathbb{P})\,.$$

Such a process U is called an *admissible control*, and we let $\mathfrak{U}$ denote the set of all admissible controls. We adopt the *relaxed control framework* (see section 3.2), and we assume that the diffusion is nondegenerate; i.e., σ is nonsingular. Standard assumptions on the drift b and the diffusion matrix σ to guarantee existence and uniqueness of solutions to (1.1) are discussed in section 3. Recall that a control is called *stationary Markov* if $U_t = v(X_t)$ for a measurable map $v : \mathbb{R}^d \mapsto \mathbb{U}$. Let $\mathfrak{U}_{\mathrm{SM}}$ denote the set of stationary Markov controls. Under $v \in \mathfrak{U}_{\mathrm{SM}}$, the process X is strong Markov, and we denote its transition function by $P^v(t, x, \cdot)$. We let $\mathbb{P}^v_x$ denote the probability measure and $\mathbb{E}^v_x$ the expectation operator on the canonical space of the process under the control $v \in \mathfrak{U}_{\mathrm{SM}}$, conditioned on the process X starting from $x \in \mathbb{R}^d$ at $t = 0$.

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The term *domain* in $\mathbb{R}^d$ refers to a nonempty open subset of the Euclidean space $\mathbb{R}^d$. We denote by $\tau(A)$ the *first exit time* of the process $\{X_t\}$ from the set $A \subset \mathbb{R}^d$, defined by

$$\tau(A) \triangleq \inf \{t > 0 : X_t \notin A\}.$$

Consider (1.1) under a stationary Markov control $v \in \mathfrak{U}_{\mathrm{SM}}$. The controlled process is called *recurrent relative to a domain* D, or D*-recurrent*, if $\mathbb{P}^v_x(\tau(D^c) < \infty) = 1$ for all $x \in D^c$ (D^c denotes the complement of D). Otherwise, it is called *transient* (relative to D). A D-recurrent process is called *positive* if $\mathbb{E}^v_x[\tau(D^c)] < \infty$ for all $x \in D^c$; otherwise it is called *null*. We refer to $\tau(D^c)$ as the *recurrence time to* D, or the *first hitting time* of D.

A controlled process is called *(positive) recurrent* if it is (positive) D-recurrent for all bounded domains $D \subset \mathbb{R}^d$. It is well known that for a nondegenerate diffusion the recurrence properties are independent of the particular domain. Thus a nondegenerate diffusion is either recurrent or transient, and if it is recurrent, then it is either positive or null, relative to all bounded domains [19]. A control $v \in \mathfrak{U}_{\mathrm{SM}}$ is called *stable* if the associated diffusion is positive recurrent. We let $\mathfrak{U}_{\mathrm{SSM}} \subset \mathfrak{U}_{\mathrm{SM}}$ denote the set of all stable stationary Markov controls.

The first relatively surprising result is that if all stationary Markov controls are stable, i.e., $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$, then $\mathbb{E}^v_x[\tau(D^c)]$ is bounded uniformly in $v \in \mathfrak{U}_{\mathrm{SSM}}$, for every fixed domain D and $x \in \bar{D}^c$ (Corollary 5.2). We call this property *uniform positive recurrence*. It might appear that uniform positive recurrence is due to the compactness of $\mathbb{U}$ and that it can be derived by a simple compactness argument. However, this is not the case. A sequence $\{v_n\} \subset \mathfrak{U}_{\mathrm{SSM}}$ may satisfy $\mathbb{E}^{v_n}_x\bigl[\tau(D^c)\bigr] \to \infty$, while at the same time $v_n \to v^* \in \mathfrak{U}_{\mathrm{SSM}}$, as $n \to \infty$, and therefore at the limit $\mathbb{E}^{v^*}_x\bigl[\tau(D^c)\bigr] < \infty$. Here, convergence of v_n is in the topology of Markov controls described in section 3.3. For example, with $d = 1$, let $\sigma(x) = \sqrt{2}$, $b(x,u) = u$, and

$$v_n(x) = \begin{cases} -\operatorname{sign}(x) & \text{if } |x| \le n, \\ -n^{-1}\mathrm{e}^{-n}\operatorname{sign}(x) & \text{if } |x| > n. \end{cases}$$

Then $v_n(x) \to v^*(x) = -\operatorname{sign}(x)$, as $n \to \infty$, and the corresponding diffusions, including the limiting one with drift $b(x) = -\operatorname{sign}(x)$, are all positive recurrent, even though the mean recurrence times of any bounded interval grow unbounded as $n \to \infty$. Note that the corresponding invariant probability distributions μ_n satisfy

$$\mu_n\left([-n,n]^c\right) \approx \frac{n}{1+n}.$$

Uniform positive recurrence relies on the fact that Markov controls can be spatially concatenated. If G is an open set and v' and v'' in $\mathfrak{U}_{\mathrm{SM}}$, then the control defined by

$$(v, G, v')(x) \triangleq \begin{cases} v(x) & \text{if } x \in G, \\ v'(x) & \text{if } x \in G^c \end{cases} \tag{1.2}$$

is clearly a stationary Markov control. If G and G' are bounded domains in $\mathbb{R}^d$, we use the notation $G \Subset G'$ to indicate that $\bar{G} \subset G'$. We say that a subset $\mathcal{U} \subset \mathfrak{U}_{\mathrm{SM}}$ is *closed under concatenations* if there exists a collection of bounded domains with $\mathcal{C}^2$ boundaries which is ordered by $\Subset$, is a cover of $\mathbb{R}^d$, and satisfies $(v, G, v') \in \mathcal{U}$, whenever $v, v' \in \mathcal{U}$. Theorem 5.1 asserts that the diffusion is uniformly positive recurrent over any $\mathcal{U} \subset \mathfrak{U}_{\mathrm{SSM}}$ which is closed in $\mathfrak{U}_{\mathrm{SSM}}$ (in the topology of Markov controls), and is also closed under concatenations.

It is well known that under a stable Markov control $v \in \mathfrak{U}_{\mathrm{SSM}}$ the diffusion has a (unique) invariant probability measure, which we denote by μ_v. In other words, μ_v satisfies

$$\int_{\mathbb{R}^d} \mu_v(\mathrm{d}x) P^v(t, x, A) = \mu_v(A) \quad \forall t \geq 0\,,$$

and all Borel sets $A \subset \mathbb{R}^d$. In [11] the concept of uniform stability was introduced: $\mathfrak{U}_{\mathrm{SSM}}$ is called *uniformly stable* if the associated invariant probability measures $\mathcal{I} \triangleq \{\mu_v : v \in \mathfrak{U}_{\mathrm{SSM}}\}$ are tight. In general, uniform positive recurrence does not imply tightness of the corresponding invariant probability measures, as the following example shows. Consider a one-dimensional controlled diffusion with

$$\sigma(x) = \sqrt{2} \quad \text{and} \quad b(x, u) = (1 + |x|)u\,, \quad u \in [-1, 1]\,.$$

Define a sequence of controls by

$$v_n(x) = \begin{cases} -\frac{\mathrm{sign}(x)}{1+|x|} & \text{if } |x| \leq n \text{ or } |x| \geq n + \sqrt{n}, \\ \frac{x\,\mathrm{sign}(x)}{1+|x|} & \text{if } n < |x| < n + \sqrt{n}\,. \end{cases}$$

Then $\{v_n\} \subset \mathfrak{U}_{\mathrm{SSM}}$, and it can be easily verified that $\sup_n \mathbb{E}_x^{v_n}[\tau(D^c)] < \infty$ for any bounded domain D. Also v_n converges, as $n \to \infty$, to $v_\infty(x) = -\frac{\mathrm{sign}(x)}{1+|x|}$, which is a stable control. Therefore the controlled diffusion is uniformly positive recurrent under $\{v_n\,,\ 1 \leq n \leq \infty\} \subset \mathfrak{U}_{\mathrm{SSM}}$. However, $\mu_{v_n}([-n, n]^c) \geq 1/2$, so the family $\{\mu_{v_n}\}$ is not tight.

An open problem stated in the framework of discrete-time, controlled Markov chains in [1, Remark 5.10, p. 314] is whether $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$ implies that $\mathcal{I}$ is necessarily tight. This is settled in the affirmative in Theorem 8.3. The importance of the result can be appreciated in the context of ergodic control problems. Suppose that g is a bounded, continuous, nonnegative functional defined on $\mathbb{R}^d$. If $v \in \mathfrak{U}_{\mathrm{SSM}}$, Birkhoff's ergodic theorem asserts that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}_x^v[g(X_t)]\,\mathrm{d}t = \int_{\mathbb{R}^d} g(x)\,\mu_v(\mathrm{d}x)\,, \tag{1.3}$$

and of course, (1.3) also holds a.s., without the expectation operator, and for any measurable g which is integrable with respect to μ_v. Thus when minimizing (1.3) over $v \in \mathfrak{U}_{\mathrm{SSM}}$ in the stable case, i.e., under the assumption that $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$, tightness of $\mathcal{I}$, and therefore also compactness, since $\mathcal{I}$ is closed, guarantees the existence of an optimal stationary Markov control. When treating the problem in the stable case, a blanket Lyapunov stability assumption is usually imposed to guarantee tightness of $\mathcal{I}$ [9, 12, 14]. Theorem 8.3 dispenses with the need for Lyapunov stability conditions. Moreover, a converse Lyapunov theorem is asserted. For $f \in \mathcal{C}^2(\mathbb{R}^d)$, where $\mathcal{C}^2(\mathbb{R}^d)$ denotes the space of twice continuously differentiable real-valued functions on $\mathbb{R}^d$, define the operator $L : \mathcal{C}^2(\mathbb{R}^d) \mapsto \mathcal{C}(\mathbb{R}^d \times \mathbb{U})$ by

$$Lf(x, u) = \sum_{i,j} a^{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_i b^i(x, u) \frac{\partial f}{\partial x_i}(x)\,, \quad u \in \mathbb{U}\,, \tag{1.4}$$

where $a \triangleq \frac{1}{2}\upsigma\,\upsigma^{\mathsf{T}}$. Then, provided $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$, it follows from Theorem 5.6 that there exist nonnegative functions $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$ and $h : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\max_{u\in\mathbb{U}} L\mathcal{V}(x,u) \le -h(x) \quad \forall x \in \mathbb{R}^d\,,$$

and

$$\lim_{|x|\to\infty} h(x) \to \infty\,.$$

The proof of uniform stability is made possible by some sharp equicontinuity estimates for the resolvents of the process, which are obtained in Theorem 6.2 and are important in their own right. Moreover, Corollary 6.3 asserts that as long as the running cost is integrable with respect to the invariant probability measure of some stable stationary Markov control, then the α-discounted value functions are equicontinuous. One approach to the ergodic control problem in the stable case is to express the running cost functional as the difference of two near-monotone functions and then utilize the results obtained from the study of the near-monotone case [9, 14]. The results obtained in section 6 facilitate a general treatment of ergodic control in the stable case, without the need of blanket Lyapunov stability hypotheses. A by-product of the analysis of the ergodic control problem is the uniform stability property stated in Theorem 8.3. This leads to a fairly general existence result in Theorem 8.5, which can be viewed as the analogue of the well-known result for the linear-quadratic-Gaussian problem, which states that when the system is stabilizable there always exists a stationary Markov optimal control. Theorem 8.5 asserts, without assuming that all stationary controls are stable, that provided the α-discounted optimal controls have a limit point in $\mathfrak{U}_{\mathrm{SSM}}$ (as $\alpha \to 0$) which results in a finite ergodic cost, then there exists a solution to the ergodic Hamilton–Jacobi–Bellman (HJB) equation, and a control $v \in \mathfrak{U}_{\mathrm{SSM}}$ with finite ergodic cost is optimal if and only if it is a measurable selector from the minimizer in the HJB.

Most of the notation used is summarized in section 2 for quick reference. In section 3 we review the model of controlled diffusions. Section 4 is devoted to invariant probability measures and their properties. Uniform positive recurrence is proved in section 5, and equivalent characterizations of uniform stability are provided. Section 6 is dedicated to continuity estimates of the α-discounted value function. The ergodic control problem, along with the proof of uniform stability, occupies sections 7 and 8. Concluding remarks are in section 9. A summary of results on elliptic partial differential equations (PDEs) used in this paper occupies Appendix A. Some proofs are in Appendix B.

2. Notation. The standard Euclidean norm in $\mathbb{R}^d$ is denoted by $|\cdot|$, and $\langle\cdot\,,\cdot\rangle$ stands for the inner product. The set of nonnegative real numbers is denoted by $\mathbb{R}_+$, $\mathbb{N}$ stands for the set of natural numbers, and $\mathbb{I}$ denotes the indicator function. As introduced in section 1, $\uptau(A)$ denotes the first exit time from the set $A \subset \mathbb{R}^d$. The closure and the boundary of a set $A \subset \mathbb{R}^d$ are denoted by $\bar{A}$ and ∂A, respectively. Also $|A|$ denotes the Lebesgue measure of A. The open ball of radius R in $\mathbb{R}^d$, centered at the origin, is denoted by B_R, and we let $\uptau_R \triangleq \uptau(B_R)$ and $\breve{\uptau}_R \triangleq \uptau(B_R^c)$.

The Borel σ-field of a topological space E is denoted by $\mathfrak{B}(E)$. Metric spaces are in general viewed as equipped with their Borel σ-field, and therefore the notation $\mathfrak{P}(E)$ for the set of probability measures on $\mathfrak{B}(E)$ of a metric space E is unambiguous.

The space $\mathfrak{P}(E)$ is always viewed as endowed with the topology of weak convergence of probability measures (the Prohorov topology).

We introduce the following notation for spaces of real-valued functions on a domain $D \subset \mathbb{R}^d$. The space $\mathcal{L}^p(D)$, $p \in [1, \infty)$, stands for the Banach space of (equivalence classes) of measurable functions f satisfying $\int_D |f(x)|^p \, \mathrm{d}x < \infty$, and $\mathcal{L}^\infty(D)$ is the Banach space of functions that are essentially bounded in D. The space $\mathcal{C}^k(D)$ ($\mathcal{C}^\infty(D)$) refers to the class of all functions whose partial derivatives up to order k (of any order) exist and are continuous, $\mathcal{C}^k_c(D)$ is the space of functions in $\mathcal{C}^k(D)$ with compact support, and $\mathcal{C}^k_b(\mathbb{R}^d)$ is the subspace of $\mathcal{C}^k(\mathbb{R}^d)$ consisting of those functions whose derivatives up to order k are bounded. Also, the space $\mathcal{C}^{k,r}(D)$ is the class of all functions whose partial derivatives up to order k are Hölder continuous of order r. Therefore $\mathcal{C}^{0,1}(D)$ is precisely the space of Lipschitz continuous functions on D.

The standard Sobolev space of functions on D, whose generalized derivatives up to order k are in $\mathcal{L}^p(D)$, equipped with its natural norm, is denoted by $\mathcal{W}^{k,p}(D)$, $k \geq 0$, $p \geq 1$. The closure of $\mathcal{C}^\infty_c(D)$ in $\mathcal{W}^{k,p}(D)$ is denoted by $\mathcal{W}^{k,p}_0(D)$. It is well known that if B is an open ball, then $\mathcal{W}^{k,p}_0(B)$ consists of all functions in $\mathcal{W}^{k,p}(B)$ which, when extended by zero outside B, belong to $\mathcal{W}^{k,p}(\mathbb{R}^d)$.

In general if $\mathcal{X}$ is a space of real-valued functions on D, $\mathcal{X}_{\mathrm{loc}}$ consists of all functions f such that $f\varphi \in \mathcal{X}$ for every $\varphi \in \mathcal{C}^\infty_c(D)$. In this manner we obtain the spaces $\mathcal{L}^p_{\mathrm{loc}}(D)$ and $\mathcal{W}^{2,p}_{\mathrm{loc}}(D)$.

Let $h \in \mathcal{C}(\mathbb{R}^d)$ be a positive function. We denote by $\mathcal{O}(h)$ the set of functions $f \in \mathcal{C}(\mathbb{R}^d)$ having the property

$$\limsup_{|x| \to \infty} \frac{|f(x)|}{h(x)} < \infty \tag{2.1}$$

and by $\mathfrak{o}(h)$ the subset of $\mathcal{O}(h)$ over which the limit in (2.1) is zero.

We adopt the notation $\partial_i \triangleq \frac{\partial}{\partial x_i}$ and $\partial_{ij} \triangleq \frac{\partial^2}{\partial x_i \partial x_j}$. We often use the standard summation rule that repeated subscripts and superscripts are summed from 1 through d. For example,

$$a^{ij}\partial_{ij}\varphi + b^i \partial_i \varphi \triangleq \sum_{i,j=1}^{d} a^{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i \frac{\partial \varphi}{\partial x_i} \,.$$

3. Controlled diffusions. In integral form, (1.1) is written as

$$X_t = X_0 + \int_0^t b(X_s, U_s)\, \mathrm{d}s + \int_0^t \sigma(X_s)\, \mathrm{d}W_s \,. \tag{3.1}$$

The second term on the right-hand side of (3.1) is an Itô stochastic integral. We say that a process $X = \{X_t(\omega)\}$ is a solution of (1.1) if it is $\mathfrak{F}_t$-adapted, continuous in t, defined for all $\omega \in \Omega$ and $t \in [0, \infty)$, and satisfies (3.1) for all $t \in [0, \infty)$ at once a.s. We impose the following conditions on the drift and diffusion matrix of (1.1).

Local Lipschitz continuity. The functions

$$b = \left[b^1, \dots, b^d\right]^{\mathsf{T}} : \mathbb{R}^d \times \mathbb{U} \mapsto \mathbb{R}^d \quad \text{and} \quad \sigma = \left[\sigma^{ij}\right] : \mathbb{R}^d \mapsto \mathbb{R}^{d \times d}$$

are locally Lipschitz in x with a Lipschitz constant K_R depending on $R > 0$. In other words, for all $x, y \in B_R$ and $u \in \mathbb{U}$,

$$|b(x,u) - b(y,u)| + \|\sigma(x) - \sigma(y)\| \leq K_R |x - y| \,, \tag{3.2}$$

where $\|\sigma\|^2 \triangleq \operatorname{trace}\left(\sigma\sigma^{\mathsf{T}}\right)$. In addition, b is continuous in (x, u).

Growth condition. b and σ satisfy a global "linear growth condition" of the form

$$|b(x,u)|^2 + \|\sigma(x)\|^2 \le K_1\bigl(1+|x|^2\bigr) \quad \forall (x,u) \in \mathbb{R}^d \times \mathbb{U}\,. \tag{3.3}$$

The linear growth assumption (3.3) guarantees that trajectories do not suffer an explosion in finite time. This assumption is quite standard but may be restrictive for some applications. As far as the results of this paper are concerned it may be replaced by the weaker condition

$$2\langle x, b(x,u)\rangle + \|\sigma(x)\|^2 \le K_1\bigl(1+|x|^2\bigr) \quad \forall (x,u) \in \mathbb{R}^d \times \mathbb{U}\,. \tag{3.4}$$

Nondegeneracy. For each $R > 0$, there exists a positive constant κ_R such that

$$\sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \ge \kappa_R|\xi|^2 \quad \forall x \in B_R\,, \tag{3.5}$$

for all $\xi = (\xi_1,\dots,\xi_d) \in \mathbb{R}^d$.

Remark 3.1. Let $(\Omega,\mathfrak{F},\mathbb{P})$ be a complete probability space, and let $\{\mathfrak{F}_t\}$ be a filtration on $(\Omega,\mathfrak{F})$ such that each $\mathfrak{F}_t$ is complete relative to $\mathfrak{F}$. Recall that a d-dimensional Wiener process $(W_t,\mathfrak{F}_t)$, or $(\mathfrak{F}_t)$-Wiener process, is an $\mathfrak{F}_t$-adapted Wiener process such that $W_t - W_s$ and $\mathfrak{F}_s$ are independent for all $t > s \ge 0$. An equivalent definition of the model for the controlled diffusion in (1.1) starts with a d-dimensional Wiener process $(W_t,\mathfrak{F}_t)$ and requires that the control process U be $\mathfrak{F}_t$-adapted. Note then that U is necessarily nonanticipative.

We summarize here some standard results from [15, 21].

THEOREM 3.2. *Let W, $U \in \mathfrak{U}$, and X_0 be given on a complete probability space $(\Omega,\mathfrak{F},\mathbb{P})$, and let X be a solution of* (1.1). *Under* (3.4),

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t|^2\right] \le \bigl(1+\mathbb{E}\,|X_0|^2\bigr)\,\mathrm{e}^{4K_1T}\,.$$

With $\tau_n \triangleq \inf\{t>0 : |X_t| > n\}$, applying Chebyshev's inequality we obtain

$$\begin{aligned}\mathbb{P}(\tau_n \le t) &= \mathbb{P}\left(\sup_{s\le t}|X_s| \ge n\right)\\ &\le \frac{\bigl(1+\mathbb{E}\,|X_0|^2\bigr)\,\mathrm{e}^{4K_1t}}{n^2} \xrightarrow[n\to\infty]{} 0\,,\end{aligned} \tag{3.6}$$

from which it follows that $\tau_n \uparrow \infty$, as $n\to\infty$, $\mathbb{P}$-a.s. If in addition (3.2) *and* (3.5) *hold, then there exists a pathwise unique solution to* (1.1) *in* $(\Omega,\mathfrak{F},\mathbb{P})$.

Of fundamental importance in the study of functionals of X is Itô's formula. For $f \in \mathcal{C}^2(\mathbb{R}^d)$ and with L as defined in (1.4),

$$f(X_t) = f(X_0) + \int_0^t Lf(X_s,U_s)\,\mathrm{d}s + M_t \quad \text{a.s.}, \tag{3.7}$$

where

$$M_t \triangleq \int_0^t \bigl\langle \nabla f(X_s), \sigma(X_s)\,\mathrm{d}W_s\bigr\rangle$$

is a local martingale. Krylov's extension of the Itô formula [20, p. 122] extends (3.7) to functions f in the Sobolev space $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$.

With $u \in \mathbb{U}$ treated as a parameter, (1.4) also gives rise to a family of operators $L^u : \mathcal{C}^2(\mathbb{R}^d) \mapsto \mathcal{C}(\mathbb{R}^d)$, defined by $L^u f(x) = Lf(x,u)$. We refer to L^u as the *controlled extended generator* of the diffusion.

3.1. Markov controls. An admissible control U is called *Markov* if it takes the form $U_t = v_t(X_t)$ for a measurable map $v : \mathbb{R}^d \times [0,\infty) \mapsto \mathbb{U}$. It is evident that U cannot be specified a priori. Instead, one has to make sense of (1.1) with U_t replaced by $v_t(X_t)$. In Theorem 3.2, X_0, W, and U are prescribed on a probability space and a solution X is constructed on the same space. This is the strong formulation. Correspondingly, the equation

$$X_t = x_0 + \int_0^t b\bigl(X_s, v_s(X_s)\bigr)\,\mathrm{d}s + \int_0^t \upsigma(X_s)\,\mathrm{d}W_s \tag{3.8}$$

is said to have a *strong solution* if, given a Wiener process $(W_t, \mathfrak{F}_t)$ on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, there exists a process X on $(\Omega, \mathfrak{F}, \mathbb{P})$, with $X_0 = x_0 \in \mathbb{R}^d$, which is continuous, $\mathfrak{F}_t$-adapted, and satisfies (3.8) for all t at once a.s. A strong solution is called *unique* if any two such solutions X and X' agree $\mathbb{P}$-a.s. when viewed as elements of $\mathcal{C}\bigl([0,\infty), \mathbb{R}^d\bigr)$.

Let $\bigl\{\mathfrak{F}_t^W\bigr\}$ be the filtration generated by W. It is evident that if X_t is $\mathfrak{F}_t^W$-adapted, then such a solution X is a strong solution. We say that (3.8) has a *weak solution* if we can find processes X and W on some probability space $(\Omega', \mathfrak{F}', \mathbb{P}')$ such that $X_0 = x_0$, W is a standard Wiener process, and (3.8) holds with $W_t - W_s$ independent of $\bigl\{X_{s'}, W_{s'}\,,\ s' \le s\bigr\}$ for all $s \le t$. The weak solution is *unique* if any two weak solutions X and X', possibly defined on different probability spaces, agree in law when viewed as $\mathcal{C}\bigl([0,\infty), \mathbb{R}^d\bigr)$-valued random variables.

It is well known that under (3.2), (3.4), and (3.5), for any Markov control v_t, (3.8) has a unique weak solution [16]. Weak solutions are also guaranteed for *feedback controls*, which are defined as admissible controls that are progressively measurable with respect to the natural filtration $\bigl\{\mathfrak{F}_t^X\bigr\}$ of X. We do not elaborate further on feedback controls, as we do not need these results in this paper. The analysis in this paper is based on weak solutions. Nevertheless, we mention parenthetically that the results in [25, 26], based on the method in [28], assert that under the assumptions (3.2), (3.3), and (3.5), for any Markov control v_t, (3.8) has a pathwise strong solution which is a Feller (and therefore strong Markov) process.

It follows from the work of [6, 24] that under $v \in \mathfrak{U}_{\mathrm{SM}}$, the transition probabilities of X have densities which are locally Hölder continuous. Thus L^v is the generator of a strongly continuous semigroup on $\mathcal{C}_b(\mathbb{R}^d)$, which is strong Feller.

As in the case of stationary Markov controls, we let $\mathbb{P}_x^U$ denote the probability measure on the canonical space of the process X starting at $X_0 = x$, under the control $U \in \mathfrak{U}$. The associated expectation operator is denoted by $\mathbb{E}_x^U$.

3.2. Relaxed controls. We describe the *relaxed control* framework, originally introduced for deterministic control in [27]. This entails the following: The space $\mathbb{U}$ is replaced by $\mathfrak{P}(\mathbb{U})$, where $\mathfrak{P}(\mathbb{U})$ denotes the space of probability measures on $\mathbb{U}$ endowed with the Prohorov topology, and b^i, $1 \le i \le d$, is replaced by

$$\bar{b}^i(x,v) \triangleq \int_{\mathbb{U}} b^i(x,u)v(du)\,, \qquad x \in \mathbb{R}^d\,, \quad v \in \mathfrak{P}(\mathbb{U}), \quad 1 \le i \le d\,.$$

Note that $\bar{b}$ inherits the same continuity, linear growth and Lipschitz (in its first argument) properties from b. The space $\mathfrak{P}(\mathbb{U})$, in addition to being compact, is convex when viewed as a subset of the space of finite signed measures on $\mathbb{U}$. One may view $\mathbb{U}$ as the "original" control space and view the passage from $\mathbb{U}$ to $\mathfrak{P}(\mathbb{U})$ as a "relaxation" of the problem that allows $\mathfrak{P}(\mathbb{U})$-valued controls that are analogous to randomized controls in the discrete-time setup. Note that a $\mathbb{U}$-valued control trajectory $\tilde{U}$ can be identified with the $\mathfrak{P}(\mathbb{U})$-valued trajectory $U_t = \delta_{\tilde{U}_t}$, where δ_q denotes the Dirac measure at q. Henceforth, "control" means relaxed control, with Dirac measure-valued controls (which correspond to original $\mathbb{U}$-valued controls) being referred to as *precise* controls. The class of stationary Markov controls is still denoted by $\mathfrak{U}_{\mathrm{SM}}$, and $\mathfrak{U}_{\mathrm{SD}} \subset \mathfrak{U}_{\mathrm{SM}}$ is the subset corresponding to precise controls.

DEFINITION 3.3. *To facilitate the passage to relaxed controls we introduce the following notation. In general, for a measurable function* $h\colon \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}^k$, $k \in \mathbb{N}$, *we denote by* $\bar{h}\colon \mathbb{R}^d \times \mathfrak{P}(\mathbb{U}) \to \mathbb{R}^k$ *its extension to relaxed controls defined by*

$$\bar{h}(x,\nu) \triangleq \int_{\mathbb{U}} h(x,u)\nu(\mathrm{d}u)\,, \quad \nu \in \mathfrak{P}(\mathbb{U})\,. \tag{3.9}$$

Since a relaxed stationary Markov control $v \in \mathfrak{U}_{\mathrm{SM}}$ *is a Borel measurable kernel on* $\mathfrak{P}(\mathbb{U}) \times \mathbb{R}^d$, *we adopt the notation* $v(x) = v(\mathrm{d}u \mid x)$. *For any fixed* $v \in \mathfrak{U}_{\mathrm{SM}}$ *and* h *as above,* $x \mapsto \bar{h}\bigl(x, v(x)\bigr)$ *is a Borel measurable function, and in the interest of notational economy, treating* v *as a parameter, we define* $h_v\colon \mathbb{R}^d \to \mathbb{R}^k$ *by*

$$h_v(x) \triangleq \bar{h}\bigl(x, v(x)\bigr) = \int_{\mathbb{U}} h(x,u)\, v(du \mid x)\,. \tag{3.10}$$

Also for $v \in \mathfrak{U}_{\mathrm{SM}}$,

$$L^v \triangleq a^{ij}\partial_{ij} + b^i_v\partial_i$$

denotes the extended generator of the diffusion governed by v.

3.3. The topology of Markov controls. We endow $\mathfrak{U}_{\mathrm{SM}}$ with the topology that renders it a compact metric space. We refer to it as "the" topology since, as is well known, the topology of a compact Hausdorff space has a certain rigidity and cannot be weakened or strengthened without losing the Hausdorff property or compactness, respectively [22, p. 60]. This can be accomplished by viewing $\mathfrak{U}_{\mathrm{SM}}$ as a subset of the unit ball of $\mathcal{L}^\infty\bigl(\mathbb{R}^d, \mathfrak{M}_s(\mathbb{U})\bigr)$ under its weak*-topology, where $\mathfrak{M}_s(\mathbb{U})$ denotes the set of signed Borel measures on $\mathbb{U}$ under the weak*-topology. The space $\mathcal{L}^\infty\bigl(\mathbb{R}^d, \mathfrak{M}_s(\mathbb{U})\bigr)$ is the dual of $\mathcal{L}^1\bigl(\mathbb{R}^d, \mathcal{C}(\mathbb{U})\bigr)$, and by the Banach–Alaoglu theorem the unit ball is weak*-compact. Since the space of probability measures is closed in $\mathfrak{M}_s(\mathbb{U})$, it follows that $\mathfrak{U}_{\mathrm{SM}}$ is weak*-closed in $\mathcal{L}^\infty\bigl(\mathbb{R}^d, \mathfrak{M}_s(\mathbb{U})\bigr)$, and since it is a subset of the unit ball of the latter, it is weak*-compact. Moreover, $\mathcal{L}^1\bigl(\mathbb{R}^d, \mathcal{C}(\mathbb{U})\bigr)$ is separable, which implies that the weak*-topology of $\mathcal{L}^\infty\bigl(\mathbb{R}^d, \mathfrak{M}_s(\mathbb{U})\bigr)$ is metrizable. We have the following criterion for convergence in $\mathfrak{U}_{\mathrm{SM}}$ [10].

LEMMA 3.4. *For* $v_n \to v$ *in* $\mathfrak{U}_{\mathrm{SM}}$ *it is necessary and sufficient that*

$$\int_{\mathbb{R}^d} g(x)\bigl(h_{v_n}(x) - h_v(x)\bigr)\,\mathrm{d}x \xrightarrow[n\to\infty]{} 0$$

for all $g \in \mathcal{L}^1(\mathbb{R}^d)$ *and* $h \in \mathcal{C}_b(\mathbb{R}^d \times \mathbb{U})$, *where* h_v *is as defined in* (3.10).

Throughout this paper, convergence and, in general, any topological properties of $\mathfrak{U}_{\mathrm{SM}}$, are with respect to the compact metrizable topology introduced above. We make frequent use of the following convergence result.

LEMMA 3.5. *Let* $\{v_n\} \subset \mathfrak{U}_{\mathrm{SM}}$ *be a sequence that converges to* $v \in \mathfrak{U}_{\mathrm{SM}}$ *in the topology of Markov controls, and let* $\{\varphi_n\} \subset \mathcal{W}^{2,p}(D)$, $p > d$, *be a sequence of solutions of* $L^{v_n}\varphi_n = h_n$, $n \in \mathbb{N}$, *on a bounded* $\mathcal{C}^2$ *domain* $D \subset \mathbb{R}^d$. *Suppose that for some* *constant M, $\|\varphi_n\|_{\mathcal{W}^{2,p}(D)} \le M$ for all $n \in \mathbb{N}$, and that h_n converges weakly in $\mathcal{L}^p(D)$, for $p > 1$, to some function h. Then any weak limit φ of $\{\varphi_n\}$ in $\mathcal{W}^{2,p}(D)$, as $n \to \infty$, satisfies $L^v\varphi = h$ in D.*

Proof. We have

$$L^v\varphi - h = a^{ij}\partial_{ij}(\varphi - \varphi_n) + b^i_{v_n}\partial_i(\varphi - \varphi_n) + (b^i_v - b^i_{v_n})\partial_i\varphi - (h - h_n)\,. \tag{3.11}$$

Since $p > d$, by the compactness of the embedding $\mathcal{W}^{2,p}(D) \hookrightarrow \mathcal{C}^{1,r}(\bar{D})$, $r < 1 - \frac{d}{p}$ (see Theorem A.11), we can select a subsequence such that $\varphi_{n_k} \to \varphi$ in $\mathcal{C}^{1,r}(\bar{D})$. Thus $b^i_{v_n}\partial_i(\varphi - \varphi_n)$ converges to 0 in $\mathcal{L}^\infty(D)$. By Lemma 3.4, and since D is bounded, $(b^i_v - b^i_{v_n})\partial_i\varphi$ converges weakly to 0 in $\mathcal{L}^p(D)$ for any $p > 1$. The remaining two terms in (3.11) converge weakly to 0 in $\mathcal{L}^p(D)$ by hypothesis. Since the left-hand side of (3.11) is independent of $n \in \mathbb{N}$, it solves $L^v\varphi - h = 0$. ∎

4. Invariant probability measures. We start the presentation with some useful bounds of mean recurrence times. For uncontrolled diffusions, these are well known. The next lemma extends them to the controlled case. This is made possible by Harnack's inequality for L^v-harmonic functions [17, Corollary 9.25, p. 250], or by its extension to a class of L^v-superharmonic functions (Theorem A.9). The proof is fairly standard and can be found in Appendix B.

LEMMA 4.1. *Let D_1 and D_2 be two open balls in $\mathbb{R}^d$, satisfying $D_1 \Subset D_2$. Then*

$$0 < \inf_{\substack{x\in\bar{D}_1\\ v\in\mathfrak{U}_{\mathrm{SM}}}} \mathbb{E}^v_x\bigl[\tau(D_2)\bigr] \le \sup_{\substack{x\in\bar{D}_1\\ v\in\mathfrak{U}_{\mathrm{SM}}}} \mathbb{E}^v_x\bigl[\tau(D_2)\bigr] < \infty\,, \tag{4.1a}$$

$$\inf_{\substack{x\in\partial D_2\\ v\in\mathfrak{U}_{\mathrm{SSM}}}} \mathbb{E}^v_x\bigl[\tau(D_1^c)\bigr] > 0\,, \tag{4.1b}$$

$$\sup_{x\in\partial D_2} \mathbb{E}^v_x\bigl[\tau(D_1^c)\bigr] < \infty \quad \forall v \in \mathfrak{U}_{\mathrm{SSM}}\,, \tag{4.1c}$$

$$\inf_{v\in\mathfrak{U}_{\mathrm{SM}}}\ \inf_{x\in\Gamma} \mathbb{P}^v_x\bigl(\tau(D_2) > \tau(D_1^c)\bigr) > 0 \tag{4.1d}$$

for all compact sets $\Gamma \subset D_2 \setminus \bar{D}_1$.

The following construction due to Has'minskiĭ which characterizes the invariant probability measure of the diffusion via an embedded Markov chain is standard [19, Theorem 4.1, p. 119]. What we have added here is the continuous dependence of the invariant probability distribution of the embedded Markov chain on $v \in \mathfrak{U}_{\mathrm{SSM}}$. The proof is in Appendix B.

THEOREM 4.2. *Let D_1 and D_2 be as in Lemma 4.1. Let $\hat{\tau}_0 = 0$, and for $k = 0, 1, \dots$ define inductively an increasing sequence of stopping times by*

$$\hat{\tau}_{2k+1} = \inf\,\{t > \hat{\tau}_{2k} : X_t \in D_2^c\}\,,$$
$$\hat{\tau}_{2k+2} = \inf\,\{t > \hat{\tau}_{2k+1} : X_t \in D_1\}\,.$$

(i) *The process $\tilde{X}_n \triangleq X_{\hat{\tau}_{2n}}$, $n \ge 1$, is a ∂D_1-valued ergodic Markov chain, under any $v \in \mathfrak{U}_{\mathrm{SSM}}$. Moreover there exists a constant $\delta \in (0,1)$, which does not depend on v, such that if $\tilde{P}_v$ and $\tilde{\mu}_v$ denote the transition kernel and the stationary distribution of $\tilde{X}$ under $v \in \mathfrak{U}_{\mathrm{SSM}}$, respectively, then for all $x \in \partial D_1$,*

$$\begin{aligned} &\bigl\|\tilde{P}^{(n)}_v(x,\cdot) - \tilde{\mu}_v(\cdot)\bigr\|_{\mathrm{TV}} \le \delta^n \quad \forall n \in \mathbb{N}\,,\\ &\delta\tilde{P}_v(x,\cdot) \le \tilde{\mu}_v(\cdot)\,. \end{aligned} \tag{4.2}$$

(ii) *The map* $v \mapsto \tilde{\mu}_v$ *from* $\mathfrak{U}_{\mathrm{SSM}}$ *to* $\mathfrak{P}(\partial D_1)$ *is continuous in the topology of Markov controls.*

(iii) *Define* $\mu_v \in \mathfrak{P}(\mathbb{R}^d)$ *by*

$$\int_{\mathbb{R}^d} f\,\mathrm{d}\mu_v = \frac{\int_{\partial D_1} \mathbb{E}_x^v\Big[\int_0^{\hat{\tau}_2} f(X_t)\,\mathrm{d}t\Big]\,\tilde{\mu}_v(\mathrm{d}x)}{\int_{\partial D_1} \mathbb{E}_x^v\left[\hat{\tau}_2\right]\,\tilde{\mu}_v(\mathrm{d}x)}\,, \quad f \in \mathcal{C}_b(\mathbb{R}^d)\,.$$

Then μ_v *is the unique invariant probability measure of* X*, under* $v \in \mathfrak{U}_{\mathrm{SSM}}$*.*

Let $v \in \mathfrak{U}_{\mathrm{SSM}}$. A Borel probability measure ν on $\mathbb{R}^d$ is called *infinitesimally invariant* if

$$\int_{\mathbb{R}^d} L^v f(x)\,\nu(\mathrm{d}x) = 0 \quad \forall f \in \mathcal{C}_c^2(\mathbb{R}^d)\,. \tag{4.3}$$

The invariant probability measure of the Markov semigroup generated by L^v is infinitesimally invariant, and for the model considered the converse is also true. We state this without proof as a theorem. For recent work on these issues see [6, 7, 8].

THEOREM 4.3. *A Borel probability measure* ν *on* $\mathbb{R}^d$ *is an invariant measure for the process associated with* L^v*,* $v \in \mathfrak{U}_{\mathrm{SSM}}$*, if and only if* (4.3) *holds. Moreover, if* ν *satisfies* (4.3)*, then it has a density* $\varphi \in \mathcal{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ *with respect to the Lebesgue measure which is a generalized solution to the adjoint equation given by*

$$\big(L^v\big)^*\varphi(x) = \sum_{i=1}^d \frac{\partial}{\partial x_i}\left(\sum_{j=1}^d a^{ij}(x)\frac{\partial \varphi}{\partial x_j}(x) + \hat{b}_v^i(x)\varphi(x)\right) = 0\,, \tag{4.4}$$

where

$$\hat{b}_v^i = \sum_{j=1}^d \frac{\partial a^{ij}}{\partial x_j} - b_v^i\,.$$

4.1. Ergodic occupation measures. Let $c : \mathbb{R}^d \times \mathbb{U} \mapsto \mathbb{R}_+$ be a continuous function, serving as the running cost.

The *ergodic control problem* in its average formulation seeks to minimize over all admissible $U \in \mathfrak{U}$ the functional

$$F(U) \triangleq \limsup_{t\to\infty}\ \frac{1}{t}\int_0^t \mathbb{E}^U\left[\bar{c}(X_s, U_s)\right]\mathrm{d}s\,.$$

We say that $U^* \in \mathfrak{U}$ is *average-cost optimal* if $F(U^*) = \inf_{U\in\mathfrak{U}}\ F(U)$, and that it is *average-cost optimal in* $\mathcal{U}$, for some collection $\mathcal{U} \subset \mathfrak{U}$, if $F(U^*)$ attains the value of its infimum over $\mathcal{U}$.

By Birkhoff's ergodic theorem, if $v \in \mathfrak{U}_{\mathrm{SSM}}$, then provided c_v is integrable with respect to μ_v,

$$\lim_{T\to\infty}\ \frac{1}{T}\int_0^T c_v(X_t)\,\mathrm{d}t = \int_{\mathbb{R}^d}\int_{\mathbb{U}} c(x,u)v(\mathrm{d}u \mid x)\mu_v(\mathrm{d}x) \quad \text{a.s.} \tag{4.5}$$

This motivates the following definition. We define the *ergodic occupation measure* $\pi_v \in \mathfrak{P}(\mathbb{R}^d \times \mathbb{U})$, corresponding to $v \in \mathfrak{U}_{\mathrm{SSM}}$, by

$$\pi_v(\mathrm{d}x, \mathrm{d}u) \triangleq \mu_v(\mathrm{d}x)v(\mathrm{d}u \mid x)\,.$$

We denote the set of all ergodic occupation measures by $\mathcal{M}$. By (4.5), the ergodic control problem over $\mathfrak{U}_{\mathrm{SSM}}$ is equivalent to a linear optimization problem over $\mathcal{M}$. It is well known that the set of ergodic occupation measures $\mathcal{M}$ is closed and convex, and its extreme points belong to the class of stable precise controls denoted as $\mathfrak{U}_{\mathrm{SSD}}$ [9].

Let $\varphi[\mu]$ denote the density of $\mu \in \mathcal{I}$, and for $\mathcal{K} \subset \mathcal{I}$, let

$$\Phi(\mathcal{K}) \triangleq \{\varphi[\mu] : \mu \in \mathcal{K}\}\,.$$

If $\mathcal{K}$ is tight, then Harnack's inequality for (4.4) [17, Theorem 8.20, p. 199] implies that there exist $R_0 > 0$ and a constant $C_H = C_H(R)$ such that for every $R > R_0$, with $|B_R|$ denoting the volume of $B_R \subset \mathbb{R}^d$,

$$\frac{1}{2C_H|B_R|} \le \inf_{B_R}\, \varphi \le \sup_{B_R}\, \varphi \le \frac{C_H}{|B_R|} \quad \forall \varphi \in \Phi(\mathcal{K})\,. \tag{4.6}$$

Moreover, the Hölder estimates for solutions of (4.4) [17, Theorem 8.24, p. 202] imply that there exists a constant $C_1 = C_1(R, \mathcal{K}) > 0$, and $a_1 > 0$, such that

$$|\varphi(x) - \varphi(y)| \le C_1|x - y|^{a_1} \quad \forall x, y \in B_R\,, \quad \forall \varphi \in \Phi(\mathcal{K})\,. \tag{4.7}$$

Invariant probability measures enjoy the following continuity properties with respect to $v \in \mathfrak{U}_{\mathrm{SSM}}$.

LEMMA 4.4. *For a subset $\mathcal{U} \subset \mathfrak{U}_{\mathrm{SSM}}$ let $\mathcal{I}_{\mathcal{U}}$ and $\mathcal{M}_{\mathcal{U}}$ denote the set of associated invariant measures and ergodic occupation measures, respectively. Suppose $\mathcal{I}_{\mathcal{U}}$ is tight. Then*

(i) *the map $v \mapsto \mu_v$ from $\bar{\mathcal{U}}$ to $\mathcal{I}_{\bar{\mathcal{U}}}$ is continuous under the total variation norm topology of $\mathcal{I}$.*

(ii) *the map $v \mapsto \pi_v$ from $\bar{\mathcal{U}}$ to $\mathcal{M}_{\bar{\mathcal{U}}}$ is continuous in $\mathfrak{P}(\mathbb{R}^d \times \mathbb{U})$.*

Proof. The proof is in Appendix B. ◻

5. Stability of controlled diffusions. Stability for controlled diffusions can be characterized with the aid of Lyapunov equations involving the operator L^u. We first review two sets of stochastic Lyapunov conditions. Recall that $f \in \mathcal{C}(\mathcal{X})$, where $\mathcal{X}$ is a topological space, is called *inf-compact* if the set $\{x \in \mathcal{X} : f(x) \le \lambda\}$ is compact (or empty) for every $\lambda \in \mathbb{R}$.

Consider the following Lyapunov conditions, each holding for some nonnegative, inf-compact function $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$:

1. For some bounded domain D

$$L^u\mathcal{V}(x) \le -1 \quad \forall x \in D^c, \quad \forall u \in \mathbb{U}\,. \tag{5.1}$$

2. There exist a nonnegative, inf-compact $h \in \mathcal{C}(\mathbb{R}^d)$ and a constant $k_0 \ge 0$ satisfying

$$L^u\mathcal{V}(x) \le k_0 - h(x)\,, \quad \forall x \in \mathbb{R}^d \quad \forall u \in \mathbb{U}\,. \tag{5.2}$$

The Lyapunov condition (5.1) is equivalent to the finiteness of the mean recurrence times to D, uniformly over all admissible controls. The main result in this section is that if all stationary controls are stable, then (5.1) holds (see Corollary 5.2 below). The stronger condition (5.2) is equivalent to the tightness of the invariant probability measures (Theorem 5.6). A central result in this paper is that (5.1) and (5.2) are in fact equivalent. This is shown in Theorem 8.3, and its proof is interleaved with the analysis of the ergodic control problem.

We next present a key result that establishes a uniform bound of a certain class of functionals of the controlled process over subsets $\mathcal{U} \subset \mathfrak{U}_{\mathrm{SSM}}$ that are closed under concatenations, as defined in section 1.

THEOREM 5.1. *Let $\mathcal{U}$ be a closed subset of $\mathfrak{U}_{\mathrm{SSM}}$ which is also closed under concatenations. Suppose that for some nonnegative function $h \in \mathcal{C}(\mathbb{R}^d \times \mathbb{U})$, some bounded domain D, and some $x \in \bar{D}^c$, we have (using the notation in* (3.9)*)*

$$\mathbb{E}^v_x\left[\int_0^{\uptau(D^c)} \bar{h}(X_t, U_t)\,\mathrm{d}t\right] < \infty \quad \forall v \in \mathcal{U}\,.$$

Then for any bounded domain $G \subset \mathbb{R}^d$ and any compact $\Gamma \subset \bar{G}^c$,

$$\sup_{v\in\mathcal{U}}\ \sup_{x\in\Gamma}\ \mathbb{E}^v_x\left[\int_0^{\uptau(G^c)} \bar{h}(X_t, U_t)\,\mathrm{d}t\right] < \infty\,. \tag{5.3}$$

Proof. We argue by contradiction. Define

$$\beta^v_x[\uptau] \triangleq \mathbb{E}^v_x\left[\int_0^{\uptau} \bar{h}(X_t, U_t)\,\mathrm{d}t\right].$$

If (5.3) does not hold, then there exist a sequence $\{v_n\} \subset \mathcal{U}$, a bounded domain $G \subset \mathbb{R}^d$, and a compact $\Gamma \subset \bar{G}^c$ such that $\sup_{x\in\Gamma}\ \beta^{v_n}_x[\uptau(G^c)] \to \infty$, as $n \to \infty$. Then, by Harnack's inequality, for all compact $\Gamma \subset \bar{G}^c$,

$$\inf_{x\in\Gamma}\ \beta^{v_n}_x[\uptau(G^c)] \to \infty\,. \tag{5.4}$$

One can show that (5.4) holds for any bounded domain $G \subset \mathbb{R}^d$ and compact $\Gamma \subset \bar{G}^c$ by following standard arguments as in [19, Lemma 3.1, p. 116]. This also follows directly from Lemma 5.3 below.

Fix a ball G_0 and let $\Gamma \subset \bar{G}^c_0$. Select $v_0 \in \mathcal{U}$ such that $\inf_{x\in\Gamma}\ \beta^{v_0}_x[\uptau(G^c_0)] > 2$. Since $\mathcal{U}$ is closed under concatenations, there exists a cover $\mathcal{G}$ of $\mathbb{R}^d$ consisting of bounded domains ordered by $\Subset$, with the property that for any $v', v'' \in \mathcal{U}$, their concatenation as defined in (1.2) is in $\mathcal{U}$. Let $G_1 \in \mathcal{G}$ such that $\Gamma \cup G_0 \Subset G_1$, satisfying

$$\beta^{v_0}_x[\uptau(G^c_0)] \le 2\beta^{v_0}_x[\uptau(G^c_0) \wedge \uptau(G_1)] \quad \forall x \in \Gamma\,.$$

This is always possible since, with $\uptau_R = \uptau(B_R)$, as defined earlier, we have

$$\beta^{v_0}_x[\uptau(G^c_0) \wedge \uptau_R)] \uparrow \beta^{v_0}_x[\uptau(G^c_0)] \quad \text{as } R \to \infty\,,$$

uniformly on Γ. Select any $\tilde{G}_1 \in \mathcal{G}$ satisfying $\tilde{G}_1 \Supset G_1$, and let

$$p_1 \triangleq \inf_{v\in\mathcal{U}}\ \inf_{x\in\Gamma}\ \mathbb{P}^v_x\bigl(\uptau(\tilde{G}_1) < \uptau(G^c_0)\bigr)\,. \tag{5.5}$$

By (4.1d), $p_1 > 0$. By (5.4), we select $v_1 \in \mathcal{U}$ such that

$$\inf_{x\in\partial\tilde{G}_1}\ \beta^{v_1}_x[\uptau(G^c_1)] > 8p_1^{-1}\,, \tag{5.6}$$

and let

$$\check{v}_1 = (v_0, G_1, v_1)\,.$$

It follows by (5.5) and (5.6) that

$$\inf_{x\in\Gamma}\ \beta_x^{\check{v}_1}\,[\uptau(G_0^c)] \ge \biggl(\inf_{x\in\Gamma}\ \mathbb{P}_x^{\check{v}_1}\bigl(\uptau(\tilde{G}_1) < \uptau(G_0^c)\bigr)\biggr)\ \biggl(\inf_{x\in\partial\tilde{G}_1}\ \beta_x^{v_1}\,[\uptau(G_1^c)]\biggr) \ge 8\,. \tag{5.7}$$

Therefore, there exists $G_2 \Supset G_1$ in $\mathcal{G}$ satisfying

$$\beta_x^{\check{v}_1}\,[\uptau(G_0^c)\wedge\uptau(G_2)] > 4\,.$$

We proceed inductively as follows. Suppose $\check{v}_{k-1}\in\mathcal{U}$ and $G_k\in\mathcal{G}$ are such that

$$\beta_x^{\check{v}_{k-1}}\,[\uptau(G_0^c)\wedge\uptau(G_k)] > 2^k\,.$$

First pick any $\tilde{G}_k\in\mathcal{G}$ such that $\tilde{G}_k\Supset G_k$, and then select $v_k\in\mathcal{U}$ satisfying

$$\inf_{x\in\partial\tilde{G}_k}\ \beta_x^{v_k}\bigl[\uptau(G_k^c)\bigr] > 2^{k+2}\Bigl(\inf_{v\in\mathcal{U}}\ \inf_{x\in\Gamma}\ \mathbb{P}_x^{v}\bigl(\uptau(\tilde{G}_k) < \uptau(G_0^c)\bigr)\Bigr)^{-1}\,.$$

This is always possible by (5.4). Proceed by defining the concatenated control

$$\check{v}_k = (\check{v}_{k-1}, G_k, v_k)\,.$$

It follows as in (5.7) that

$$\inf_{x\in\Gamma}\ \beta_x^{\check{v}_k}\,[\uptau(G_0^c)] > 2^{k+2}\,.$$

Subsequently choose $G_{k+1}\Supset\tilde{G}_k$, such that

$$\inf_{x\in\Gamma}\ \beta_x^{\check{v}_k}\,[\uptau(G_0^c)\wedge\uptau(G_{k+1})] > \frac{1}{2}\ \inf_{x\in\Gamma}\ \beta_x^{\check{v}_k}\,[\uptau(G_0^c)]\ ,$$

thus yielding

$$\beta_x^{\check{v}_k}\,[\uptau(G_0^c)\wedge\uptau(G_{k+1})] > 2^{k+1}\,. \tag{5.8}$$

By construction, each $\check{v}_k$ agrees with $\check{v}_{k-1}$ on G_k. It is also evident that the sequence $\{\check{v}_k\}$ converges to some control $v^*\in\mathcal{U}$, which agrees with $\check{v}_k$ on G_k, for each $k\ge 1$. Hence, by (5.8),

$$\inf_{x\in\Gamma}\ \beta_x^{v^*}\,[\uptau(G_0^c)\wedge\uptau(G_k)] > 2^k \quad \forall k\in\mathbb{N}\,.$$

Thus $\beta_x^{v^*}\,[\uptau(G_0^c)] = \infty$, contradicting the original hypothesis. ∎

When $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$, a direct application of Theorem 5.1 yields uniform positive recurrence. This is summarized as follows.

Corollary 5.2. *Suppose that all stationary Markov controls are stable, i.e., $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$. Then if D is a bounded domain with $\mathcal{C}^{2,1}$ boundary, there exists a function $\mathcal{V}\in\mathcal{C}^2(\mathbb{R}^d)$ which solves $\max_u L^u\mathcal{V} = -1$ on $\bar{D}^c$, with $\mathcal{V} = 0$ on ∂D. Moreover, for any $x\in\bar{D}^c$,*

$$\mathcal{V}(x) = \sup_{v\in\mathfrak{U}_{\mathrm{SSM}}}\ \mathbb{E}_x^{v}[\uptau(D^c)] = \sup_{U\in\mathfrak{U}}\ \mathbb{E}_x^{U}[\uptau(D^c)]\,. \tag{5.9}$$

Proof. Applying Theorem 5.1, with $h \equiv 1$, yields $\sup_{v\in\mathfrak{U}_{\mathrm{SSM}}} \mathbb{E}_x^v[\tau(D^c)] < \infty$. It is then straightforward to show, using Theorem A.15, that the Dirichlet problem

$$\max_{u\in\mathbb{U}} L^u\mathcal{V} = -1 \quad \text{in } \bar{D}^c, \qquad \mathcal{V} = 0 \quad \text{on } \partial D$$

has a unique solution $\mathcal{V} \in \mathcal{C}^2(\bar{D}^c)$, and that $\mathcal{V}(x) = \sup_{v\in\mathfrak{U}_{\mathrm{SSM}}} \mathbb{E}_x^v[\tau(D^c)]$ for all $x \in \bar{D}^c$. The second equality in (5.9) follows via a straightforward application of Itô's formula. $\square$

In the next lemma we extend a well-known result of Has'minskiĭ [18] to controlled diffusions. The proof is in Appendix B.

LEMMA 5.3. *Let $D \subset \mathbb{R}^d$ be a bounded domain and $G \subset \mathbb{R}^d$ a compact set. Define*

$$\xi_{D,G}^v(x) \triangleq \mathbb{E}_x^v\left[\int_0^{\tau(D^c)} \mathbb{I}_G(X_t)\,\mathrm{d}t\right].$$

Then

(i) $\sup_{v\in\mathfrak{U}_{\mathrm{SM}}} \sup_{x\in\bar{D}^c} \xi_{D,G}^v(x) < \infty$;

(ii) *if X is recurrent under $v \in \mathfrak{U}_{\mathrm{SM}}$, then $\xi_{D,G}^v$ is the unique bounded solution in $\mathcal{W}^{2,p}_{\mathrm{loc}}(D^c) \cap \mathcal{C}(\bar{D})$, $p > 1$, of the Dirichlet problem $L^v\xi = -\mathbb{I}_G$ in D^c and $\xi = 0$ on ∂D;*

(iii) *if $\mathcal{U} \subset \mathfrak{U}_{\mathrm{SM}}$ is a closed set of controls under which X is recurrent, the map $(v, x) \mapsto \xi_{D,G}^v(x)$ is continuous on $\mathcal{U} \times \bar{D}^c$.*

Now let $D_1 \Subset D_2$ be two fixed open balls in $\mathbb{R}^d$, and let $\hat{\tau}_2$ be as defined in Theorem 4.2. Let $h \in \mathcal{C}_b(\mathbb{R}^d \times \mathbb{U})$ be a nonnegative function and define

$$\Phi_R^v(x) \triangleq \mathbb{E}_x^v\left[\int_0^{\tau(D_1^c)} \mathbb{I}_{B_R^c}(X_t)\bar{h}(X_t, U_t)\,\mathrm{d}t\right], \quad x \in \partial D_2\,, \quad v \in \mathfrak{U}_{\mathrm{SSM}}\,. \tag{5.10}$$

Let $R_0 > 0$ such that $B_{R_0} \Supset D_2$. Then, provided $R > R_0$, $\Phi_R^v(x)$ satisfies $L^v\Phi_R^v = 0$ in $B_{R_0} \cap \bar{D}_1^c$, and by Harnack's inequality, there exists a constant C_H, independent of $v \in \mathfrak{U}_{\mathrm{SSM}}$, such that $\Phi_R^v(x) \le C_H\Phi_R^v(y)$, for all $x, y \in \partial D_2$ and $v \in \mathfrak{U}_{\mathrm{SM}}$. Harnack's inequality also holds for the function $x \mapsto \mathbb{E}_x^v[\hat{\tau}_2]$ on ∂D_1 (for this we apply Theorem A.9). Also, by Lemma 4.1, for some constant $C_0 > 0$,

$$\inf_{v\in\mathfrak{U}_{\mathrm{SSM}}} \inf_{x\in\partial D_2} \mathbb{E}_x^v[\tau(D_1^c)] \ge C_0 \sup_{v\in\mathfrak{U}_{\mathrm{SSM}}} \sup_{x\in\partial D_1} \mathbb{E}_x^v[\tau(D_2)]\,.$$

Consequently, using these estimates and applying Theorem 4.2(iii) with $f = h_v$, we obtain positive constants k_1 and k_2, which depend only on D_1, D_2, and R_0, such that for all $R > R_0$ and $x \in \partial D_2$,

$$k_1 \int_{B_R^c\times\mathbb{U}} h\,\mathrm{d}\pi_v \le \frac{\Phi_R^v(x)}{\inf_{x\in\partial D_2} \mathbb{E}_x^v[\tau(D_1^c)]} \le k_2 \int_{B_R^c\times\mathbb{U}} h\,\mathrm{d}\pi_v \quad \forall v \in \mathfrak{U}_{\mathrm{SSM}}\,. \tag{5.11}$$

Similarly, applying Theorem 4.2(iii) with $f = \mathbb{I}_{D_1}$, there exists a positive constant k_3, which depends only on D_1 and D_2, such that

$$\mu_v(D_1) \sup_{x\in\partial D_2} \mathbb{E}_x^v[\tau(D_1^c)] \le k_3 \sup_{x\in\partial D_1} \mathbb{E}_x^v[\tau(D_2)] \quad \forall v \in \mathfrak{U}_{\mathrm{SSM}}\,. \tag{5.12}$$

Recall the definition of $\mathcal{M}_{\mathcal{U}}$ in Lemma 4.4. We obtain the following useful variation of Theorem 5.1.

Corollary 5.4. *Let $\mathcal{U} \subset \mathfrak{U}_{\mathrm{SSM}}$ be closed in $\mathfrak{U}_{\mathrm{SSM}}$ and be also closed under concatenations. Suppose that a nonnegative $h \in \mathcal{C}(\mathbb{R}^d \times \mathbb{U})$ is integrable with respect to all $\pi \in \mathcal{M}_{\mathcal{U}}$. Then $\sup_{\pi\in\mathcal{M}_{\mathcal{U}}} \int h \,\mathrm{d}\pi < \infty$.*

Proof. By Corollary 5.2, $\sup_{v\in\mathcal{U}} \mathbb{E}_x^v[\uptau(D_1^c)] < \infty$. Then, since by hypothesis $\int h \mathrm{d}\pi_v < \infty$ for all $v \in \mathcal{U}$, (5.11) implies that $\Phi_R^v(x) < \infty$, for all $v \in \mathcal{U}$ and $x \in \partial D_2$. Therefore, applying Theorem 5.1, we obtain $\sup_{v\in\mathcal{U}} \Phi_R^v(x) < \infty$, and the result follows by (4.1b) and the left-hand side inequality of (5.11). ∎

Recall that a collection of stationary Markov controls $\mathcal{U} \in \mathfrak{U}_{\mathrm{SSM}}$ is called uniformly stable if the set $\mathcal{I}_{\mathcal{U}} = \{\mu_v\,,\ v \in \mathcal{U}\}$ is tight. Corollary 5.4 implies that if $\mathcal{U} \subset \mathfrak{U}_{\mathrm{SSM}}$ is closed in $\mathfrak{U}_{\mathrm{SSM}}$ and also closed under concatenations, and if some nonnegative, inf-compact function h is integrable with respect to every $\pi \in \mathcal{M}_{\mathcal{U}}$, then $\mathcal{U}$ is uniformly stable.

The next theorem provides some important equivalences of uniform stability. This is an augmented version of the results in [11]. We need the following definition.

Definition 5.5. *Let $\mathcal{C}$ denote the class of nonnegative functions $h \in \mathcal{C}(\mathbb{R}^d \times \mathbb{U})$ that are locally Lipschitz in their first argument, uniformly in $u \in \mathbb{U}$. More specifically, for some $C_h \colon \mathbb{R}_+ \to \mathbb{R}_+$, h satisfies*

$$\bigl|h(x,u) - h(y,u)\bigr| \le C_h(R)|x-y| \quad \forall x,y \in B_R\,,$$

for all $u \in \mathbb{U}$ and $R > 0$.

Theorem 5.6. *Let $\mathcal{U}$ be an arbitrary subset of $\mathfrak{U}_{\mathrm{SSM}}$. The following statements are equivalent (with $h \in \mathcal{C}$ an* inf*-compact function which is common to* (i)–(iv)*):*

(i) *For some open ball $D \subset \mathbb{R}^d$ and some $x \in \bar{D}^c$,*

$$\sup_{v\in\mathcal{U}} \mathbb{E}_x^v\biggl[\int_0^{\uptau(D^c)} \bar{h}(X_t,U_t)\,\mathrm{d}t\biggr] < \infty \quad \forall v \in \mathcal{U}\,.$$

(ii) *For all open balls $D \subset \mathbb{R}^d$ and compact sets $\Gamma \subset \mathbb{R}^d$,*

$$\sup_{v\in\mathcal{U}} \sup_{x\in\Gamma} \mathbb{E}_x^v\biggl[\int_0^{\uptau(D^c)} \bar{h}(X_t,U_t)\,\mathrm{d}t\biggr] < \infty\,.$$

(iii) *A uniform bound holds:*

$$\sup_{v\in\mathcal{U}} \int_{\mathbb{R}^d\times\mathbb{U}} h(x,u)\pi_v(\mathrm{d}x,\mathrm{d}u) < \infty\,. \tag{5.13}$$

(iv) *Provided $\mathcal{U} = \mathfrak{U}_{\mathrm{SM}}$, there exist a nonnegative,* inf*-compact function $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$ and a constant k_0 satisfying*

$$L^u\mathcal{V}(x) \le k_0 - h(x,u) \quad \forall u \in \mathbb{U}\,, \quad \forall x \in \mathbb{R}^d\,.$$

(v) *Provided $\mathcal{U} = \mathfrak{U}_{\mathrm{SM}}$, for any compact $K \subset \mathbb{R}^d$ and $t_0 > 0$, the* mean empirical measures

$$\bigl\{\bar{\nu}_{x,t}^U : x \in K\,,\ t \ge t_0\,,\ U \in \mathfrak{U}\bigr\}$$

defined by

$$\int_{\mathbb{R}^d\times\mathbb{U}} f\,\mathrm{d}\bar{\nu}_{x,t}^U = \frac{1}{t}\int_0^t \mathbb{E}_x^U\bigl[\bar{f}(X_s,U_s)\bigr]\,\mathrm{d}s\,, \quad t > 0\,, \tag{5.14}$$

for all $f \in \mathcal{C}_b(\mathbb{R}^d \times \mathbb{U})$, are tight.

(vi) $\mathfrak{I}_{\mathcal{U}}$ *is tight.*
(vii) $\mathcal{M}_{\mathcal{U}}$ *is tight.*
(viii) $\mathcal{M}_{\bar{\mathcal{U}}}$ *is compact.*
(ix) *For some open ball* $D \subset \mathbb{R}^d$ *and* $x \in \bar{D}^c$, *the family* $\{(\uptau(D^c), \mathbb{P}^v_x),\ v \in \mathcal{U}\}$ *is uniformly integrable, i.e.,*

$$\sup_{v \in \mathcal{U}} \mathbb{E}^v_x \left[\uptau(D^c)\, \mathbb{I}_{[t,\infty)}(\uptau(D^c))\right] \downarrow 0 \quad \text{as } t \uparrow \infty\,.$$

(x) *The family* $\{(\uptau(D^c), \mathbb{P}^v_x),\ v \in \mathcal{U},\ x \in \Gamma\}$ *is uniformly integrable for all open balls* $D \subset \mathbb{R}^d$ *and compact sets* $\Gamma \subset \mathbb{R}^d$.

Proof. It is clear that (ii) $\Rightarrow$ (i) and (x) $\Rightarrow$ (ix). Since $\mathbb{U}$ is compact, (vi) $\Leftrightarrow$ (vii). By Prohorov's theorem, (viii) $\Rightarrow$ (vii). With $D_1 \Subset D_2$ any two open balls in $\mathbb{R}^d$, we apply (5.11) and (5.12). Letting $D = D_1$, (i) $\Rightarrow$ (iii) follows by (5.11). It is evident that (iii) $\Rightarrow$ (vii). Therefore, since under (iii) $\mathfrak{I}_{\mathcal{U}}$ is tight, (4.6) implies $\inf_{v \in \mathcal{U}} \mu_v(D_1) > 0$. In turn, by (4.1a) and (5.12),

$$\sup_{v \in \mathcal{U}} \sup_{x \in \partial D_2} \mathbb{E}^v_x[\uptau(D_1^c)] < \infty\,. \tag{5.15}$$

Hence applying (5.11) and Lemma 5.3(i), we obtain (iii) $\Rightarrow$ (ii). We continue by proving (vi) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i), (ix) $\Rightarrow$ (vii), and (iv) $\Rightarrow$ (v) $\Rightarrow$ (viii) $\Rightarrow$ (x).

(vi) $\Rightarrow$ (iii): Let

$$\hat{h}_v(x) \triangleq \Bigl(\mu_v\bigl(B^c_{|x|}\bigr)\Bigr)^{-1/2}, \quad v \in \mathcal{U}\,, \quad x \in \mathbb{R}^d\,,$$

and define $h \triangleq \inf_{v \in \mathcal{U}} \hat{h}_v$. A simple calculation yields $\int_{\mathbb{R}^d} \hat{h}_v \,\mathrm{d}\mu_v = 2$. Next, we show that h is locally Lipschitz continuous. Let $R > 0$ and $x, x' \in B_R$. Then, with $g(x) \triangleq \mu_v\bigl(B^c_{|x|}\bigr)$,

$$|\hat{h}_v(x) - \hat{h}_v(x')| = \frac{|g(x) - g(x')|}{\sqrt{g(x)g(x')}\bigl(\sqrt{g(x)} + \sqrt{g(x')}\bigr)}\,. \tag{5.16}$$

By (4.6), the denominator of (5.16) is uniformly bounded away from zero on B_R, while the numerator has the upper bound $\bigl(\sup_{B_R} \varphi_v\bigr)\,\bigl||B_{|x|}| - |B_{|x'|}|\bigr|$, where φ_v is the density of μ_v. Therefore, by Lemma 4.4 and (5.16), $(x, v) \mapsto \hat{h}_v(x)$ is continuous in $\mathbb{R}^d \times \bar{\mathcal{U}}$ and locally Lipschitz in the first argument. Since $\bar{\mathcal{U}}$ is compact, local Lipschitz continuity of h follows. Thus (5.13) holds. Since $\mathfrak{I}_{\mathcal{U}}$ is tight, $\sup_{v \in \mathcal{U}} \mu_v\bigl(B^c_{|x|}\bigr) \to 0$, as $|x| \to \infty$, and thus $\lim_{|x| \to \infty} h(x) = \infty$.

(iii) $\Rightarrow$ (iv): By Theorem A.15, the Dirichlet problem

$$\begin{aligned} \max_{u \in \mathbb{U}} \left[L^u f_r(x) + h(x,u)\right] &= 0\,, \quad x \in B_r \setminus \bar{D}_1, \\ f_r\,\bigr|_{\partial D_1 \cap \partial B_r} &= 0 \end{aligned} \tag{5.17}$$

has a solution $f_r \in \mathcal{C}^{2,s}(\bar{B}_r \setminus D_1)$, $s \in (0,1)$. Let $v_r \in \mathfrak{U}_{\mathrm{SD}}$ be a measurable selector from the maximizer in (5.17). Then using (5.10) and (5.11), with $r > R > R_0$, and since, as shown earlier, under the hypothesis of (iii) equation (5.15) holds, we obtain

$$\begin{aligned} f_r(x) &= \mathbb{E}_x^{v_r}\biggl[\int_0^{\uptau(D_1^c)\wedge\uptau_r} \bar{h}(X_t,U_t)\,\mathrm{d}t\biggr] \\ &\le \Bigl(\sup_{B_R\times\mathbb{U}} h\Bigr)\,\xi^{v_r}_{D_1,B_R}(x) + \mathit{\Phi}_R^{v_r}(x) \\ &\le \Bigl(\sup_{B_R\times\mathbb{U}} h\Bigr)\,\xi^{v_r}_{D_1,B_R}(x) + k_2' \int_{\mathbb{R}^d} h\,\mathrm{d}\pi_{v_r}\,, \quad x\in\partial D_2\,, \end{aligned}$$

for some constant $k_2' > 0$ that depends only on D_1, D_2, and R_0. Therefore, by (iii) and Lemma 5.3(i), f_r is bounded above, and since it is monotone in r, it converges by Lemma A.16, as $r\to\infty$, to some $\mathcal{V}\in\mathcal{C}^2(D_1^c)$ satisfying

$$L^u\mathcal{V}(x) \le -h(x,u) \quad \forall u\in\mathbb{U}\,,\quad \forall x\in\bar{D}_1^c\,.$$

It remains to extend $\mathcal{V}$ to a smooth function. This can be accomplished, for instance, by selecting $D_4 \Supset D_3 \Supset D_1$, and with ψ any smooth function that equals zero on D_3 and $\psi = 1$ on D_4^c, to define $\tilde{\mathcal{V}} = \psi\mathcal{V}$ on D_1^c and $\tilde{\mathcal{V}} = 0$ on D_1. Then $L^u\tilde{\mathcal{V}} \le -h$ on D_4^c, for all $u\in\mathbb{U}$, and since $|L^u\tilde{\mathcal{V}}|$ is bounded in $\bar{D}_4$, uniformly in $u\in\mathbb{U}$, (iv) follows.

(iv) $\Rightarrow$ (i): Let D be an open ball such that $h(x,u)\ge 2k_0$ for all $x\in D^c$ and $u\in\mathbb{U}$. By Itô's formula, for any $R>0$ and $v\in\mathfrak{U}_{\mathrm{SM}}$,

$$\mathbb{E}_x^v\biggl[\int_0^{\uptau(D^c)\wedge\uptau_R} \bigl(\bar{h}(X_t,U_t)-k_0\bigr)\,\mathrm{d}t\biggr] \le \mathcal{V}(x) \quad \forall x\in\bar{D}^c. \tag{5.18}$$

Since $h\le 2(h-k_0)$ on D^c, the result follows by taking limits as $R\to\infty$ in (5.18).

(ix) $\Rightarrow$ (vii): By (5.10) and (5.11) with $h\equiv 1$, we obtain, for any $t_0\ge 0$ and $R>R_0$,

$$\begin{aligned} \pi_v(B_R^c\times\mathbb{U}) &\le k_1'\,\mathbb{E}_x^v\biggl[\int_0^{\uptau(D_1^c)} \mathbb{I}_{B_R^c}(X_t)\,\mathrm{d}t\biggr] \\ &\le k_1' t_0\,\mathbb{P}_x^v(\uptau_R\le t_0) + k_1'\,\mathbb{E}_x^v\bigl[\uptau(D_1^c)\mathbb{I}(\uptau(D_1^c)\ge t_0)\bigr]\,, \quad x\in\partial D_2\,, \end{aligned}$$

for some constant $k_1'>0$ that depends only on D_1, D_2, and R_0. By (ix), we can select t_0 large enough so that the second term on the right-hand side is as small as desired, uniformly in $v\in\mathcal{U}$ and $x\in\partial D_2$. By (3.6), for any fixed $t_0>0$,

$$\sup_{v\in\mathfrak{U}_{\mathrm{SM}}}\ \sup_{x\in\partial D_2}\ \mathbb{P}_x^v(\uptau_R\le t_0) \xrightarrow[R\to\infty]{} 0\,,$$

and (vii) follows.

(iv) $\Rightarrow$ (v): Applying Itô's formula, we have

$$\mathbb{E}_x^U\left[\mathcal{V}(X_{t\wedge\uptau_n})\right] - \mathcal{V}(x) = k_0\,\mathbb{E}_x^U[t\wedge\uptau_n] - \mathbb{E}_x^U\left[\int_0^{t\wedge\uptau_n}\bar{h}(X_s,U_s)\,\mathrm{d}s\right]. \tag{5.19}$$

Letting $n\to\infty$ in (5.19), using monotone convergence and rearranging terms, we obtain that for any ball $B_R\subset\mathbb{R}^d$,

$$\begin{aligned} \Bigl(\min_{B_R^c\times\mathbb{U}} h\Bigr)\int_0^t \mathbb{E}_x^U\left[\mathbb{I}_{B_R^c}(X_s)\right]\mathrm{d}s &\le \int_0^t \mathbb{E}_x^U\left[\bar{h}(X_s,U_s)\right]\mathrm{d}s \\ &\le k_0 t + \mathcal{V}(x)\,. \end{aligned} \tag{5.20}$$

By (5.20), for all $x \in \mathbb{R}^d$,

$$\frac{1}{t}\int_0^t \mathbb{E}_x^U \left[\mathbb{I}_{B_R^c}(X_s)\right] \mathrm{d}s \le \frac{k_0 t + \mathcal{V}(x)}{t\left(\min_{B_R^c \times \mathbb{U}} h\right)} \quad \forall U \in \mathfrak{U}\,,\ \forall t > 0\,,$$

and tightness of the mean empirical measures follows.

(v) ⇒ (viii): Since the mean empirical measures are tight, their closure is compact by Prohorov's theorem. Tightness also implies that every accumulation point of a sequence of mean empirical measures is an ergodic occupation measure [9, 23]. Also, if $v \in \mathfrak{U}_{\mathrm{SSM}}$, then $\bar{\nu}^v_{x,t}$ converges as $t \to \infty$ to π_v [19, Lemma 2.1, p. 72]. Therefore, tightness implies that the set of accumulation points of sequences of mean empirical measures is precisely the set of ergodic occupation measures $\mathcal{M}$, and hence, being closed, $\mathcal{M}$ is compact.

(viii) ⇒ (x): Let $D = D_1$, and without loss of generality, $\mathit{\Gamma} = \partial D_2$. Then (5.11) implies

$$\sup_{v\in\mathcal{U}}\ \sup_{x\in\partial D_2}\ \mathbb{E}_x^v\left[\int_0^{\tau(D_1^c)} \mathbb{I}_{B_R^c}(X_t)\,\mathrm{d}t\right] \xrightarrow[R\to\infty]{} 0\,. \tag{5.21}$$

Given any sequence $\{(v_n, x_n)\} \subset \mathcal{U} \times \partial D_2$ converging to some $(v, x) \in \bar{\mathcal{U}} \times \partial D_2$, Lemma 5.3(iii) asserts that, for all R such that $D_2 \Subset B_R$,

$$\mathbb{E}_{x_n}^{v_n}\left[\int_0^{\tau(D_1^c)} \mathbb{I}_{B_R}(X_t)\,\mathrm{d}t\right] \xrightarrow[n\to\infty]{} \mathbb{E}_x^v\left[\int_0^{\tau(D_1^c)} \mathbb{I}_{B_R}(X_t)\,\mathrm{d}t\right]. \tag{5.22}$$

Combining (5.21) and (5.22), we obtain $\mathbb{E}_{x_n}^{v_n}[\tau(D_1^c)] \to \mathbb{E}_x^v[\tau(D_1^c)]$ as $n \to \infty$, and (x) follows. □

6. Equicontinuity of the α-discounted value functions. In the analysis of the ergodic problem, we follow the vanishing discount approach. Let $\alpha > 0$ be a constant which we refer to as the *discount factor*. For any admissible control $U \in \mathfrak{U}$, we define the *α-discounted cost* by

$$J_\alpha^U(x) \triangleq \mathbb{E}_x^U\left[\int_0^\infty \mathrm{e}^{-\alpha t} c(X_t, U_t)\,\mathrm{d}t\right],$$

and we let

$$V_\alpha(x) \triangleq \inf_{U\in\mathfrak{U}} J_\alpha^U(x)\,. \tag{6.1}$$

The following theorem is standard [4, 9].

THEOREM 6.1. *Let $c \in \mathcal{C}$ (see Definition 5.5). Then V_α defined in (6.1) is the minimal nonnegative solution in $\mathcal{C}^2(\mathbb{R}^d) \cap \mathcal{C}_b(\mathbb{R}^d)$ of*

$$\min_{u\in\mathbb{U}}\left[L^u V_\alpha(x) + c(x,u)\right] = \alpha V_\alpha(x)\,. \tag{6.2}$$

Moreover, $v \in \mathfrak{U}_{\mathrm{SM}}$ is α-discounted optimal if and only if v a.e. realizes the pointwise minimum in (6.2), i.e., if and only if

$$\sum_{i=1}^d b_v^i(x)\frac{\partial V_\alpha}{\partial x_i}(x) + c_v(x) = \min_{u\in\mathbb{U}}\left[\sum_{i=1}^d b^i(x,u)\frac{\partial V_\alpha}{\partial x_i}(x) + c(x,u)\right] \quad a.e.\ x \in \mathbb{R}^d\,,$$

where b^i_v and c_v are as in Definition 3.3.

We next show that for a stable control $v \in \mathfrak{U}_{\mathrm{SSM}}$, the resolvents J^v_α are bounded in $\mathcal{W}^{2,p}(B_R)$, uniformly in $\alpha \in (0,1)$, for any $R > 0$. For $v \in \mathfrak{U}_{\mathrm{SSM}}$, and $\pi_v \in \mathcal{M}$ the corresponding ergodic occupation measure, we define

$$\varrho_v \triangleq \int_{\mathbb{R}^d \times \mathbb{U}} c(x,u)\pi_v(\mathrm{d}x,\mathrm{d}u)\,.$$

THEOREM 6.2. *There exists a positive constant $C_0 = C_0(R)$ depending only on the radius $R > 0$ such that, for all $v \in \mathfrak{U}_{\mathrm{SSM}}$ and $\alpha \in (0,1)$,*

$$\bigl\|J^v_\alpha - J^v_\alpha(0)\bigr\|_{\mathcal{W}^{2,p}(B_R)} \le \frac{C_0(R)}{\mu_v(B_{2R})}\left(\frac{\varrho_v}{\mu_v(B_{2R})} + \sup_{B_{4R}\times\mathbb{U}} c\right), \tag{6.3a}$$

$$\sup_{B_R}\, \alpha J^v_\alpha \le C_0(R)\left(\frac{\varrho_v}{\mu_v(B_R)} + \sup_{B_{2R}\times\mathbb{U}} c\right). \tag{6.3b}$$

Proof. Let $\hat{\uptau} = \inf\,\{t > \uptau_{2R} : X_t \in B_R\}$. For $x \in \partial B_R$, we have

$$\begin{aligned} J^v_\alpha(x) &= \mathbb{E}^v_x\left[\int_0^{\hat{\uptau}} \mathrm{e}^{-\alpha t} c_v(X_t)\,\mathrm{d}t + \mathrm{e}^{-\alpha\hat{\uptau}} J^v_\alpha(X_{\hat{\uptau}})\right] \\ &= \mathbb{E}^v_x\left[\int_0^{\hat{\uptau}} \mathrm{e}^{-\alpha t} c_v(X_t)\,\mathrm{d}t + J^v_\alpha(X_{\hat{\uptau}}) - (1-\mathrm{e}^{-\alpha\hat{\uptau}}) J^v_\alpha(X_{\hat{\uptau}})\right]. \end{aligned} \tag{6.4}$$

Let $\tilde{P}_x(A) = \mathbb{P}^v_x(X_{\hat{\uptau}} \in A)$. By Theorem 4.2, there exists $\delta \in (0,1)$ depending only on R such that

$$\bigl\|\tilde{P}_x - \tilde{P}_y\bigr\|_{\mathrm{TV}} \le 2\delta \quad \forall x,y \in \partial B_R\,.$$

Therefore,

$$\left|\mathbb{E}^v_x\left[J^v_\alpha(X_{\hat{\uptau}})\right] - \mathbb{E}^v_y\left[J^v_\alpha(X_{\hat{\uptau}})\right]\right| \le \delta \operatorname*{osc}_{\partial B_R}\, J^v_\alpha \quad \forall x,y \in \partial B_R\,. \tag{6.5}$$

Thus (6.4) and (6.5) yield

$$\begin{aligned} \operatorname*{osc}_{\partial B_R}\, J^v_\alpha \le \frac{1}{1-\delta} \sup_{x\in\partial B_R}\, &\mathbb{E}^v_x\left[\int_0^{\hat{\uptau}} \mathrm{e}^{-\alpha t} c_v(X_t)\,\mathrm{d}t\right] \\ &+ \frac{1}{1-\delta}\sup_{x\in\partial B_R}\, \mathbb{E}^v_x\left[(1-\mathrm{e}^{-\alpha\hat{\uptau}}) J^v_\alpha(X_{\hat{\uptau}})\right]. \end{aligned} \tag{6.6}$$

Next, we bound the terms on the right-hand side of (6.6). First,

$$\begin{aligned} \mathbb{E}^v_x\left[(1-\mathrm{e}^{-\alpha\hat{\uptau}}) J^v_\alpha(X_{\hat{\uptau}})\right] &\le \mathbb{E}^v_x\left[\alpha^{-1}(1-\mathrm{e}^{-\alpha\hat{\uptau}})\right] \sup_{x\in\partial B_R}\, \alpha J^v_\alpha(x) \\ &\le \left(\sup_{\partial B_R}\, \alpha J^v_\alpha\right) \mathbb{E}^v_x[\hat{\uptau}] \quad \forall x \in \partial B_R\,. \end{aligned} \tag{6.7}$$

Define

$$M(R) \triangleq \sup_{B_R\times\mathbb{U}} c\,, \quad R > 0\,.$$

The function

$$\varphi_\alpha = \frac{M(2R)}{\alpha} + J^v_\alpha$$

belongs to $\mathscr{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$ for all $p > 1$ and satisfies

$$L^v\varphi_\alpha(x) - \alpha\varphi_\alpha(x) = -c_v(x) - M(2R) \quad \forall x \in B_{2R}\,, \tag{6.8}$$

and thus

$$M(2R) \le \bigl|(L^v - \alpha)\varphi_\alpha(x)\bigr| \le 2M(2R) \quad \forall x \in B_{2R}\,. \tag{6.9}$$

By (6.9),

$$\bigl\|(L^v - \alpha)\varphi_\alpha\bigr\|_{\mathcal{L}^\infty(B_{2R})} \le 2|B_{2R}|^{-1}\bigl\|(L^v - \alpha)\varphi_\alpha\bigr\|_{\mathcal{L}^1(B_{2R})}\,. \tag{6.10}$$

Hence $\varphi_\alpha \in \mathfrak{K}(2, B_{2R})$ (see Definition A.7), and by Theorem A.9, there exists a constant $\tilde{C}_H > 0$ depending only on R such that

$$\varphi_\alpha(x) \le \tilde{C}_H\varphi_\alpha(y) \quad \forall x\,, y \in B_R \text{ and } \alpha \in (0,1)\,. \tag{6.11}$$

Integrating with respect to μ_v, and using Fubini's theorem, we have

$$\int_{\mathbb{R}^d} \alpha J^v_\alpha(x)\mu_v(\mathrm{d}x) = \varrho_v \quad \forall v \in \mathfrak{U}_{\mathrm{SSM}}\,. \tag{6.12}$$

By (6.12), $\inf_{B_R}\ \alpha J^v_\alpha \le \frac{\varrho_v}{\mu_v(B_R)}$. Thus (6.11) yields

$$\sup_{B_R}\ \alpha J^v_\alpha \le \tilde{C}_H\left(M(2R) + \frac{\varrho_v}{\mu_v(B_R)}\right)\,, \tag{6.13}$$

which establishes (6.3b). On the other hand, the function

$$\psi_\alpha(x) = \mathbb{E}^v_x\left[\int_0^{\hat{\uptau}} \mathrm{e}^{-\alpha t}\bigl(M(2R) + c_v(X_t)\bigr)\,\mathrm{d}t\right]$$

also satisfies (6.8)–(6.10) in B_{2R}, and therefore (6.11) holds for ψ_α. Thus

$$\begin{aligned}\sup_{x\in\partial B_R}\ \mathbb{E}^v_x\left[\int_0^{\hat{\uptau}} \mathrm{e}^{-\alpha t}c_v(X_t)\,\mathrm{d}t\right] &\le \tilde{C}_H \inf_{x\in\partial B_R}\ \mathbb{E}^v_x\left[\int_0^{\hat{\uptau}} \bigl(M(2R) + c_v(X_t)\bigr)\mathrm{d}t\right]\\ &\le \tilde{C}_H\bigl(M(2R) + \varrho_v\bigr) \sup_{x\in\partial B_R}\ \mathbb{E}^v_x[\hat{\uptau}]\,.\end{aligned} \tag{6.14}$$

By (6.6), (6.7), (6.13), and (6.14),

$$\operatorname*{osc}_{\partial B_R}\ J^v_\alpha \le \frac{1 + \tilde{C}_H}{1 - \delta}\left(M(2R) + \frac{\varrho_v}{\mu_v(B_R)}\right) \sup_{x\in\partial B_R}\ \mathbb{E}^v_x[\hat{\uptau}]\,. \tag{6.15}$$

Applying Theorem A.9 to the L^v-superharmonic function $x \mapsto \mathbb{E}^v_x[\hat{\uptau}]$, we have

$$\sup_{x\in\partial B_R}\ \mathbb{E}^v_x[\hat{\uptau}] \le \tilde{C}'_H \inf_{x\in\partial B_R}\ \mathbb{E}^v_x[\hat{\uptau}] \tag{6.16}$$

for some constant $\tilde{C}'_H = \tilde{C}'_H(R) > 0$. By (4.1a), (6.16), and the estimate

$$\inf_{x\in\partial B_R} \mathbb{E}^v_x[\hat{\tau}] \le \frac{1}{\mu_v(B_R)} \sup_{x\in\partial B_R} \mathbb{E}^v_x[\tau_{2R}]\,,$$

which is obtained from Theorem 4.2(iii), we have

$$\sup_{x\in\partial B_R} \mathbb{E}^v_x[\hat{\tau}] \le \frac{\tilde{C}_1}{\mu_v(B_R)}\,, \tag{6.17}$$

for some positive constant $\tilde{C}_1 = \tilde{C}_1(R)$. By Theorem A.3, there exists a constant $\tilde{C}'_1 > 0$, depending only on R, such that $\mathbb{E}^v_x[\tau_R] \le \tilde{C}'_1$ for all $x \in B_R$, and thus

$$\sup_{x\in B_R} \mathbb{E}^v_x\left[\int_0^{\tau_R} \mathrm{e}^{-\alpha t} c_v(X_t)\,\mathrm{d}t\right] \le \tilde{C}'_1 \sup_{B_R\times\mathbb{U}} c\,. \tag{6.18}$$

By (6.15), (6.17), and (6.18),

$$\begin{aligned}\operatorname*{osc}_{B_R} J^v_\alpha &\le \operatorname*{osc}_{\partial B_R} J^v_\alpha + \sup_{x\in B_R} \mathbb{E}^v_x\left[\int_0^{\tau_R} \mathrm{e}^{-\alpha t} c_v(X_t)\,\mathrm{d}t\right]\\ &\le \frac{\tilde{C}_2}{\mu_v(B_R)}\left(M(2R) + \frac{\varrho_v}{\mu_v(B_R)}\right)\end{aligned} \tag{6.19}$$

for some positive constant $\tilde{C}_2 = \tilde{C}_2(R)$. Let $\bar{\varphi}_\alpha \triangleq J^v_\alpha - J^v_\alpha(0)$. Then

$$L^v\bar{\varphi}_\alpha - \alpha\bar{\varphi}_\alpha = -c_v + \alpha J^v_\alpha(0) \quad \text{in } B_{2R}\,.$$

Applying Lemma A.5 to $\bar{\varphi}_\alpha$, relative to the operator $L^v - \alpha$, with $D = B_{2R}$ and $D' = B_R$, we obtain, for some positive constant $\tilde{C}_3 = \tilde{C}_3(R)$,

$$\begin{aligned}\bigl\|\bar{\varphi}_\alpha\bigr\|_{\mathcal{W}^{2,p}(B_R)} &\le \tilde{C}_3\left(\bigl\|\bar{\varphi}_\alpha\bigr\|_{\mathcal{L}^p(B_{2R})} + \bigl\|L^v\bar{\varphi}_\alpha - \alpha\bar{\varphi}_\alpha\bigr\|_{\mathcal{L}^p(B_{2R})}\right)\\ &\le \tilde{C}_3\bigl|B_{2R}\bigr|^{1/p}\left(\operatorname*{osc}_{B_{2R}} J^v_\alpha + M(2R) + \sup_{B_{2R}} \alpha J^v_\alpha\right),\end{aligned}$$

and the required bound follows from (6.13) and (6.19). ∎

The bounds in (6.3) along with Theorem 5.1 imply that if $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$, then as long as $\varrho_v < \infty$ for all $v \in \mathfrak{U}_{\mathrm{SM}}$, the functions $J^v_\alpha - J^v_\alpha(0)$ are bounded in $\mathcal{W}^{2,p}(B_R)$ on any ball B_R, uniformly in $\alpha \in (0,1)$ and $v \in \mathfrak{U}_{\mathrm{SSM}}$. The estimates in the corollary that follows imply that, provided $\varrho_v < \infty$, for some $v \in \mathfrak{U}_{\mathrm{SSM}}$, the α-discounted value functions $\{V_\alpha\}$ defined in (6.1) are bounded in $\mathcal{W}^{2,p}(B_R)$ on any ball B_R, uniformly in $\alpha \in (0,1)$.

COROLLARY 6.3. *There exists a constant $\tilde{C}_0(R) > 0$ depending only on the radius $R > 0$ such that, for all $\alpha \in (0,1)$ and all $v \in \mathfrak{U}_{\mathrm{SSM}}$,*

$$\bigl\|V_\alpha - V_\alpha(0)\bigr\|_{\mathcal{W}^{2,p}(B_R)} \le \frac{\tilde{C}_0(R)}{\mu_v(B_{2R})}\left(\frac{\varrho_v}{\mu_v(B_{2R})} + \sup_{B_{4R}\times\mathbb{U}} c\right), \tag{6.20a}$$

$$\sup_{B_R} \alpha V_\alpha \le \tilde{C}_0(R)\left(\frac{\varrho_v}{\mu_v(B_R)} + \sup_{B_{2R}\times\mathbb{U}} c\right). \tag{6.20b}$$

Proof. With $\hat{\uptau}$ as in the proof of Theorem 6.2, and $v_\alpha \in \mathfrak{U}_{\mathrm{SM}}$ an α-discounted optimal control, define the admissible control $U \in \mathfrak{U}$ by

$$U_t = \begin{cases} v & \text{if } t \le \hat{\uptau}, \\ v_\alpha & \text{otherwise.} \end{cases}$$

Since U is in general suboptimal for the α-discounted criterion, we have

$$V_\alpha(x) \le \mathbb{E}_x^v\biggl[\int_0^{\hat{\uptau}} \mathrm{e}^{-\alpha t} c_v(X_t)\,\mathrm{d}t + \mathrm{e}^{-\alpha\hat{\uptau}} V_\alpha(X_{\hat{\uptau}})\biggr]. \tag{6.21}$$

Invoking Theorem 4.2 as in the proof of Theorem 6.2, we obtain

$$\bigl|\mathbb{E}_x^v\left[V_\alpha(X_{\hat{\uptau}})\right] - \mathbb{E}_y^v\left[V_\alpha(X_{\hat{\uptau}})\right]\bigr| \le \delta \operatorname*{osc}_{\partial B_R} V_\alpha \quad \forall x, y \in \partial B_R\,, \tag{6.22}$$

and thus by (6.21) and (6.22)

$$\begin{aligned} \operatorname*{osc}_{\partial B_R} V_\alpha \le \frac{1}{1-\delta} \sup_{x\in\partial B_R} \mathbb{E}_x^v\biggl[\int_0^{\hat{\uptau}} \mathrm{e}^{-\alpha t} c_v(X_t)\,\mathrm{d}t\biggr] \\ + \frac{1}{1-\delta} \sup_{x\in\partial B_R} \mathbb{E}_x^v\Bigl[(1-\mathrm{e}^{-\alpha\hat{\uptau}}) V_\alpha(X_{\hat{\uptau}})\Bigr]\,. \end{aligned} \tag{6.23}$$

Since $V_\alpha \le J_\alpha^v$, (6.20b) follows from (6.13). Moreover, since the right-hand sides of (6.6) and (6.23) are equal, we can use (6.15) and (6.17) to obtain

$$\operatorname*{osc}_{\partial B_R} V_\alpha \le \frac{(1+\tilde{C}_H)\tilde{C}_1}{(1-\delta)\mu_v(B_R)} \left(M(2R) + \frac{\varrho_v}{\mu_v(B_R)}\right) \quad \forall v \in \mathfrak{U}_{\mathrm{SSM}}\,. \tag{6.24}$$

Using (6.24) and the bound

$$\operatorname*{osc}_{B_R} V_\alpha \le \operatorname*{osc}_{\partial B_R} V_\alpha + \sup_{x\in B_R} \mathbb{E}_x^{v_\alpha}\biggl[\int_0^{\uptau_R} \mathrm{e}^{-\alpha t} c_{v_\alpha}(X_t)\,\mathrm{d}t\biggr],$$

we proceed as in the proof of Theorem 6.2 to derive (6.20a). ∎

7. Analysis of the ergodic control problem. Throughout the rest of this paper we assume that $c \in \mathcal{C}$. We start the analysis with a useful lemma concerning control Lyapunov functions. We use the notation $\breve{\uptau}_r \triangleq \uptau(B_r^c)$ for $r > 0$. Also we extend the definition of $\mathfrak{o}$ to functions on $\mathcal{C}(\mathbb{R}^d \times \mathbb{U})$ as follows: For $h \in \mathcal{C}(\mathbb{R}^d \times \mathbb{U})$, with $h > 0$,

$$g \in \mathfrak{o}(h) \iff \limsup_{|x|\to\infty} \sup_{u\in\mathbb{U}} \frac{|g(x,u)|}{h(x,u)} = 0\,.$$

Lemma 7.1. *Suppose*

$$\sup_{v\in\mathfrak{U}_{\mathrm{SSM}}} \int_{B_R^c\times\mathbb{U}} (1 + c(x,u))\pi_v(\mathrm{d}x,\mathrm{d}u) \xrightarrow[R\to\infty]{} 0\,. \tag{7.1}$$

Then there exist a constant $k_0 \in \mathbb{R}$ and a pair of nonnegative, inf*-compact functions $(\mathcal{V}, h) \in \mathcal{C}^2(\mathbb{R}^d) \times \mathcal{C}$ with $1 + c \in \mathfrak{o}(h)$ such that*

$$L^u\mathcal{V}(x) \le k_0 - h(x,u) \quad \forall u \in \mathbb{U}\,, \quad \forall x \in \mathbb{R}^d\,. \tag{7.2}$$

Moreover,

(i) *for any* $r > 0$,

$$x \mapsto \mathbb{E}_x^v\biggl[\int_0^{\breve{\tau}_r} \bigl(1 + c_v(X_t)\bigr)\,\mathrm{d}t\biggr] \in \mathfrak{o}(\mathcal{V}) \quad \forall v \in \mathfrak{U}_{\mathrm{SSM}}\,. \tag{7.3}$$

(ii) *if* $\varphi \in \mathfrak{o}(\mathcal{V})$, *then for all* $x \in \mathbb{R}^d$, *and all* $v \in \mathfrak{U}_{\mathrm{SSM}}$,

$$\lim_{t\to\infty} \frac{1}{t}\,\mathbb{E}_x^v\bigl[\varphi(X_t)\bigr] = 0\,, \tag{7.4}$$

and for any $t \ge 0$,

$$\lim_{R\to\infty} \mathbb{E}_x^v\left[\varphi(X_{t\wedge\tau_R})\right] = \mathbb{E}_x^v\left[\varphi(X_t)\right]\,. \tag{7.5}$$

Conversely, if (7.2) *holds for a pair* $(\mathcal{V}, h) \in \mathcal{C}^2(\mathbb{R}^d) \times \mathcal{C}$ *of nonnegative,* inf-*compact functions, satisfying* $1 + c \in \mathfrak{o}(h)$, *then* (7.1) *and* (i)–(ii) *hold.*

Proof. Let

$$\check{c}_v(x) \triangleq 1 + \int_{\mathbb{U}} c(x,u)v(\mathrm{d}u \mid x)\,, \quad v \in \mathfrak{U}_{\mathrm{SSM}}\,.$$

Recall that if $\sum_n a_n$ is a convergent series of positive terms, and if $r_n \triangleq \sum_{k\ge n} a_k$ are its remainders, then $\sum_n r_n^{-\lambda} a_n$ converges for all $\lambda \in (0,1)$. Thus, if we define

$$\breve{g}(r) \triangleq \left(\sup_{v\in\mathfrak{U}_{\mathrm{SSM}}} \int_{B_r^c} \check{c}_v(x)\mu_v(\mathrm{d}x)\right)^{-1/2}, \quad r > 0\,,$$

it follows from (7.1) that

$$\int_{\mathbb{R}^d} \check{c}_v(x)\breve{g}^{\beta}(|x|)\mu_v(\mathrm{d}x) < \infty \quad \forall v \in \mathfrak{U}_{\mathrm{SSM}}, \quad \forall \beta \in [0,2)\,. \tag{7.6}$$

Let

$$h(x,u) \triangleq \bigl(1 + c(x,u)\bigr)\breve{g}(|x|)\,.$$

By (7.1), $\breve{g}$ is inf-compact, and it is straightforward to verify, using an estimate analogous to (5.16), that it is also locally Lipschitz. Adopting the notation in (3.10), we write $h_v(x) = \check{c}_v(x)\breve{g}(|x|)$. By Theorem 5.1 and (7.6),

$$\sup_{v\in\mathfrak{U}_{\mathrm{SSM}}} \int_{\mathbb{R}^d} h_v(x)\breve{g}^{\beta}(|x|)\mu_v(\mathrm{d}x) < \infty \quad \forall \beta \in [0,1)\,.$$

It then follows from Theorem 5.6 that there exists a nonnegative, inf-compact function $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$ which satisfies

$$L^u\mathcal{V}(x) \le k_0 - h(x,u) \qquad \forall x \in \mathbb{R}^d\,, \quad \forall u \in \mathbb{U}\,. \tag{7.7}$$

Next we prove (7.3). With $R > 0$ large enough so that $x \in B_R$, applying Itô's formula to (7.7) we obtain

$$\mathbb{E}_x^v\bigl[\mathcal{V}(X_{\breve{\tau}_r\wedge\tau_R})\bigr] - \mathcal{V}(x) \le \mathbb{E}_x^v\biggl[\int_0^{\breve{\tau}_r\wedge\tau_R} \bigl[k_0 - h_v(X_t)\bigr]\,\mathrm{d}t\biggr] \tag{7.8}$$

for all $v \in \mathfrak{U}_{\mathrm{SSM}}$. Therefore

$$\mathbb{E}_x^v\biggl[\int_0^{\breve{\uptau}_r\wedge\uptau_R} h_v(X_t)\,\mathrm{d}t\biggr] \le \mathcal{V}(x) + k_0\,\mathbb{E}_x^v[\breve{\uptau}_r\wedge\uptau_R]\,. \tag{7.9}$$

Taking limits as $R\to\infty$ in (7.9), and since $\mathbb{E}_x^v[\breve{\uptau}_r]\in\mathcal{O}(\mathcal{V})$, we obtain

$$\mathbb{E}_x^v\biggl[\int_0^{\breve{\uptau}_r} h_v(X_t)\,\mathrm{d}t\biggr] \in \mathcal{O}(\mathcal{V})\,. \tag{7.10}$$

For each $x\in B_r^c$, select the maximal radius $\rho(x)$ satisfying

$$\mathbb{E}_x^v\biggl[\int_0^{\breve{\uptau}_r} \mathbb{I}_{B_{\rho(x)}}(X_t)\check{c}_v(X_t)\,\mathrm{d}t\biggr] \le \frac{1}{2}\,\mathbb{E}_x^v\biggl[\int_0^{\breve{\uptau}_r} \check{c}_v(X_t)\,\mathrm{d}t\biggr]\,. \tag{7.11}$$

By (7.10) and (7.11),

$$\begin{aligned}
\mathbb{E}_x^v\biggl[\int_0^{\breve{\uptau}_r} \check{c}_v(X_t)\,\mathrm{d}t\biggr] &\le 2\,\mathbb{E}_x^v\biggl[\int_0^{\breve{\uptau}_r} \mathbb{I}_{B^c_{\rho(x)}}(X_t)\check{c}_v(X_t)\,\mathrm{d}t\biggr]\\
&\le \frac{2}{\check{g}(\rho(x))}\,\mathbb{E}_x^v\biggl[\int_0^{\breve{\uptau}_r} \mathbb{I}_{B^c_{\rho(x)}}(X_t)\check{c}_v(X_t)\check{g}(|X_t|)\,\mathrm{d}t\biggr]\\
&\le \frac{2}{\check{g}(\rho(x))}\,\mathbb{E}_x^v\biggl[\int_0^{\breve{\uptau}_r} h_v(X_t)\,\mathrm{d}t\biggr]\\
&\in \mathcal{O}\left(\frac{\mathcal{V}}{\check{g}\circ\rho}\right)\,.
\end{aligned} \tag{7.12}$$

Since for any fixed ball B_ρ the function

$$x\mapsto \mathbb{E}_x^v\biggl[\int_0^{\breve{\uptau}_r} \mathbb{I}_{B_\rho}(X_t)\check{c}_v(X_t)\,\mathrm{d}t\biggr]$$

is bounded on B_r^c by Lemma 5.3(i), whereas the function on the right-hand side of (7.11) grows unbounded as $|x|\to\infty$, it follows that $\liminf_{|x|\to\infty}\ \rho(x)\to\infty$. Therefore, (7.3) follows from (7.12).

We now turn to (7.4). Applying Itô's formula and Fatou's lemma, (7.7) yields

$$\mathbb{E}_x^v\bigl[\mathcal{V}(X_t)\bigr] \le k_0 t + \mathcal{V}(x) \quad \forall v\in\mathfrak{U}_{\mathrm{SSM}}\,. \tag{7.13}$$

If φ is $\mathfrak{o}(\mathcal{V})$, then there exists $\check{f}\colon\mathbb{R}_+\to\mathbb{R}_+$ satisfying $\check{f}(R)\to\infty$, as $R\to\infty$, and $\mathcal{V}(x)\ge|\varphi(x)|\check{f}(|x|)$. Define

$$R(t) \triangleq \inf\bigl\{|x| : |\varphi(x)|\ge\sqrt{t}\bigr\}\wedge t\,,\quad t\ge 0\,.$$

Then, by (7.13),

$$\begin{aligned}
\mathbb{E}_x^v\bigl|\varphi(X_t)\bigr| &\le \mathbb{E}_x^v\bigl[\bigl|\varphi(X_t)\bigr|\mathbb{I}_{B_{R(t)}}(X_t)\bigr] + \frac{\mathbb{E}_x^v\bigl[\mathcal{V}(X_t)\mathbb{I}_{B^c_{R(t)}}(X_t)\bigr]}{\check{f}(R(t))}\\
&\le \sqrt{t} + \frac{k_0 t + \mathcal{V}(x)}{\check{f}(R(t))}\,,
\end{aligned} \tag{7.14}$$

and dividing (7.14) by t, and taking limits as $t \to \infty$, (7.4) follows.

To prove (7.5), first write

$$\mathbb{E}_x^v\left[\varphi(X_{t\wedge\tau_R})\right] = \mathbb{E}_x^v\left[\varphi(X_t)\mathbb{I}\{t<\tau_R\}\right] + \mathbb{E}_x^v\left[\varphi(X_{\tau_R})\mathbb{I}\{t\geq\tau_R\}\right]. \tag{7.15}$$

By (7.13),

$$\mathbb{E}_x^v\left[\varphi(X_{\tau_R})\mathbb{I}\{t\geq\tau_R\}\right] \leq \left[k_0 t + \mathcal{V}(x)\right] \sup_{x\in\partial B_R} \frac{\varphi(x)}{\mathcal{V}(x)},$$

and since $\varphi \in \mathfrak{o}(\mathcal{V})$, this shows that the second term on the right-hand side of (7.15) vanishes as $R\to\infty$. Since $\left|\varphi(X_t)\right| \leq M\mathcal{V}(X_t)$, for some constant $M>0$, applying Fatou's lemma yields

$$\begin{aligned}\mathbb{E}_x^v\left[\varphi(X_t)\right] &\leq \liminf_{R\to\infty}\ \mathbb{E}_x^v\left[\varphi(X_t)\mathbb{I}\{t<\tau_R\}\right]\\ &\leq \limsup_{R\to\infty}\ \mathbb{E}_x^v\left[\varphi(X_t)\mathbb{I}\{t<\tau_R\}\right] \leq \mathbb{E}_x^v\left[\varphi(X_t)\right],\end{aligned}$$

thus obtaining (7.5).

The converse statement follows from Theorem 5.6. ◻

Remark 7.2. We observe that the estimates used in the proof of Lemma 7.1 are uniform in $v \in \mathfrak{U}_{\mathrm{SSM}}$. Therefore, the conclusions in (i) and (ii) can be strengthened to

$$x \mapsto \sup_{v\in\mathfrak{U}_{\mathrm{SSM}}} \mathbb{E}_x^v\left[\int_0^{\breve{\tau}_r}\big(1+c_v(X_t)\big)\,\mathrm{d}t\right] \in \mathfrak{o}(\mathcal{V}) \quad \forall r>0$$

and

$$\sup_{v\in\mathfrak{U}_{\mathrm{SSM}}} \frac{\mathbb{E}_x^v\big[\varphi(X_t)\big]}{t} \xrightarrow[t\to\infty]{} 0 \quad \forall x\in\mathbb{R}^d, \quad \forall\varphi\in\mathfrak{o}(\mathcal{V}),$$

respectively.

DEFINITION 7.3. *For $r>0$ and $x \in \bar{B}_r^c$, define*

$$\begin{aligned}\mathit{\Psi}^v(x;\varrho) &\triangleq \liminf_{r\downarrow 0}\ \mathbb{E}_x^v\left[\int_0^{\breve{\tau}_r}\big(c_v(X_t)-\varrho\big)\,\mathrm{d}t\right], \quad v\in\mathfrak{U}_{\mathrm{SSM}},\\ \mathit{\Psi}^*(x;\varrho) &\triangleq \liminf_{r\downarrow 0}\ \inf_{v\in\mathfrak{U}_{\mathrm{SSM}}}\ \mathbb{E}_x^v\left[\int_0^{\breve{\tau}_r}\big(c_v(X_t)-\varrho\big)\,\mathrm{d}t\right].\end{aligned}$$

Recall that $\varrho_v \triangleq \int_{\mathbb{R}^d} c_v(x)\mu_v(\mathrm{d}x)$, and define

$$\varrho^* \triangleq \inf_{v\in\mathfrak{U}_{\mathrm{SSM}}} \varrho_v\,.$$

We always assume that $\varrho^* < \infty$ or, in other words, that for some $\hat{v} \in \mathfrak{U}_{\mathrm{SSM}}$, $\varrho_{\hat{v}} < \infty$. In the next lemma we relax (7.1), and thus we cannot assume the existence of a control which is average-cost optimal in $\mathfrak{U}_{\mathrm{SSM}}$. Therefore, we have to argue via ε-optimality which is defined as follows. For $\varepsilon \geq 0$, $\pi_\varepsilon^* \in \mathcal{M}$ is called *ε-optimal* if it satisfies

$$\varrho^* \leq \int_{\mathbb{R}^d\times\mathbb{U}} c\,\mathrm{d}\pi_\varepsilon^* \leq \varrho^* + \varepsilon\,.$$

LEMMA 7.4. *Assume $\varrho^* < \infty$. The following hold:*

(i) *For each sequence* $\alpha_n \downarrow 0$ *there exist a further subsequence also denoted as* $\{\alpha_n\}$, $V \in \mathcal{C}^2(\mathbb{R}^d)$, *and* $\varrho \in \mathbb{R}$ *such that, as* $n \to \infty$, $\bar{V}_{\alpha_n} \triangleq V_{\alpha_n} - V_{\alpha_n}(0) \to V$ *uniformly on compact subsets of* $\mathbb{R}^d$, *and* $\alpha_n V_{\alpha_n}(0) \to \varrho$. *The pair* (V, ϱ) *satisfies*

$$\min_{u\in\mathbb{U}} \big[L^u V(x) + c(x,u)\big] = \varrho\,, \quad x \in \mathbb{R}^d. \tag{7.16}$$

Moreover,

$$V(x) \le \Psi^*(x;\varrho) \quad \text{and} \quad \varrho \le \varrho^*\,.$$

(ii) *If* $\hat{v} \in \mathfrak{U}_{\mathrm{SSM}}$ *and* $\varrho_{\hat{v}} < \infty$, *then there exist* $\hat{\varrho} \in \mathbb{R}^d$ *and* $\hat{V} \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, *for any* $p > 1$, *satisfying* $L^{\hat{v}}\hat{V} - c_{\hat{v}} = \hat{\varrho}$ *in* $\mathbb{R}^d$, *and such that, as* $\alpha \downarrow 0$, $\alpha J^{\hat{v}}_\alpha(0) \to \hat{\varrho}$ *and* $J^{\hat{v}}_\alpha - J^{\hat{v}}_\alpha(0) \to \hat{V}$ *uniformly on compact subsets of* $\mathbb{R}^d$. *Moreover,*

$$\hat{V}(x) = \Psi^{\hat{v}}(x;\hat{\varrho}) \quad \text{and} \quad \hat{\varrho} \le \varrho_{\hat{v}}\,.$$

Proof. By Theorem 6.2, $\alpha V_\alpha(0)$ is bounded, and $\bar{V}_\alpha = V_\alpha - V_\alpha(0)$ is bounded in $\mathcal{W}^{2,p}(B_R)$, $p > 1$, uniformly in α in a neighborhood of 0. Therefore, we start with (6.2), and applying Lemma A.16 we deduce that $\bar{V}_{\alpha_n}$ converges uniformly on any bounded domain along some subsequence $\alpha_n \downarrow 0$ to $V \in \mathcal{C}^2(\mathbb{R}^d)$ satisfying (7.16), with ϱ being the corresponding limit of $\alpha_n V_{\alpha_n}(0)$.

We first show $\varrho \le \varrho^*$. Let $v_\varepsilon \in \mathfrak{U}_{\mathrm{SSM}}$ be an ε-optimal control and select $R \ge 0$ large enough such that $\mu_{v_\varepsilon}\big(B_R\big) \ge 1 - \varepsilon$. Since $V_\alpha \le J^{v_\varepsilon}_\alpha$, by integrating with respect to μ_{v_ε} and using Fubini's theorem, we obtain

$$\Big(\inf_{B_R}\, V_\alpha\Big)\mu_{v_\varepsilon}\big(B_R\big) \le \int_{\mathbb{R}^d} V_\alpha(x)\mu_{v_\varepsilon}(\mathrm{d}x) \le \int_{\mathbb{R}^d} J^{v_\varepsilon}_\alpha(x)\mu_{v_\varepsilon}(\mathrm{d}x) \le \frac{\varrho^* + \varepsilon}{\alpha}\,.$$

Therefore,

$$\inf_{B_R}\, V_\alpha \le \frac{(\varrho^* + \varepsilon)}{\alpha(1-\varepsilon)}\,,$$

and since $V_\alpha(0) - \inf_{B_R}\, V_\alpha$ is bounded uniformly in $\alpha \in (0,1)$, we obtain

$$\varrho \le \limsup_{\alpha\downarrow 0}\, \alpha V_\alpha(0) \le \frac{(\varrho^* + \varepsilon)}{(1-\varepsilon)}\,.$$

Since ε was arbitrary, $\varrho \le \varrho^*$.

Let $v_\alpha \in \mathfrak{U}_{\mathrm{SM}}$ be an α-discounted optimal control. For $v \in \mathfrak{U}_{\mathrm{SSM}}$ and $r < R$, define the admissible control $U \in \mathfrak{U}$ by

$$U_t = \begin{cases} v & \text{if } t \le \breve{\tau}_r \wedge \tau_R, \\ v_\alpha & \text{otherwise.} \end{cases}$$

Since U is in general suboptimal for the α-discounted criterion, using the strong Markov property relative to the stopping time $\breve{\tau}_r \wedge \tau_R$, we have for $x \in B_R \setminus \bar{B}_r$,

$$\begin{aligned} V_\alpha(x) &\le \mathbb{E}^U_x\left[\int_0^\infty \mathrm{e}^{-\alpha t}\bar{c}(X_t, U_t)\,\mathrm{d}t\right] \\ &= \mathbb{E}^v_x\left[\int_0^{\breve{\tau}_r\wedge\tau_R} \mathrm{e}^{-\alpha t}c_v(X_t)\,\mathrm{d}t + \mathrm{e}^{-\alpha(\breve{\tau}_r\wedge\tau_R)}V_\alpha(X_{\breve{\tau}_r\wedge\tau_R})\right]. \end{aligned} \tag{7.17}$$

Since $v \in \mathfrak{U}_{\mathrm{SSM}}$, applying Fubini's theorem, $\int_{\mathbb{R}^d} \alpha J^v_\alpha(x)\, \mu_v(\mathrm{d}x) = \varrho_v < \infty$. Hence, J^v_α is a.e. finite, and since $V_\alpha \le J^v_\alpha$, by Theorem A.12 and Remark A.13, we have

$$\mathbb{E}^v_x\Big[\mathbb{I}\{\breve{\tau}_r \ge \tau_R\}\mathrm{e}^{-\alpha\tau_R} V_\alpha(X_{\tau_R})\Big] \le \mathbb{E}^v_x\big[\mathrm{e}^{-\alpha\tau_R} J^v_\alpha(X_{\tau_R})\big] \xrightarrow[R\to\infty]{} 0 \tag{7.18}$$

for all $v \in \mathfrak{U}_{\mathrm{SSM}}$. Decomposing the term $\mathrm{e}^{-\alpha(\breve{\tau}_r\wedge\tau_R)} V_\alpha(X_{\breve{\tau}_r\wedge\tau_R})$ in (7.17), then taking limits as $R \to \infty$, applying (7.18) and monotone convergence, and subtracting $V_\alpha(0)$ from both sides of the inequality, we obtain

$$\begin{aligned}\bar{V}_\alpha(x) &\le \mathbb{E}^v_x\left[\int_0^{\breve{\tau}_r} \mathrm{e}^{-\alpha t} c_v(X_t)\,\mathrm{d}t + \mathrm{e}^{-\alpha\breve{\tau}_r} V_\alpha(X_{\breve{\tau}_r}) - V_\alpha(0)\right] \\ &= \mathbb{E}^v_x\left[\int_0^{\breve{\tau}_r} \mathrm{e}^{-\alpha t}\big(c_v(X_t) - \varrho\big)\,\mathrm{d}t\right] + \mathbb{E}^v_x\Big[V_\alpha(X_{\breve{\tau}_r}) - V_\alpha(0)\Big] \\ &\qquad + \mathbb{E}^v_x\Big[\alpha^{-1}\big(1 - \mathrm{e}^{-\alpha\breve{\tau}_r}\big)\big[\varrho - \alpha V_\alpha(X_{\breve{\tau}_r})\big]\Big].\end{aligned} \tag{7.19}$$

Since $\mathbb{E}^v_x\left[\alpha^{-1}\big(1 - \mathrm{e}^{-\alpha\breve{\tau}_r}\big)\right] \le \mathbb{E}^v_x[\breve{\tau}_r]$ and, by Corollary 6.3, $\sup_{B_r} |\varrho - \alpha_n V_{\alpha_n}| \to 0$, letting $\alpha \to 0$ along the subsequence $\{\alpha_n\}$, (7.19) yields

$$V(x) \le \mathbb{E}^v_x\left[\int_0^{\breve{\tau}_r} \big(c_v(X_t) - \varrho\big)\,\mathrm{d}t + V(X_{\breve{\tau}_r})\right] \quad \forall v \in \mathfrak{U}_{\mathrm{SSM}}\,. \tag{7.20}$$

Since $V(0) = 0$,

$$\lim_{r\downarrow 0}\ \sup_{v\in\mathfrak{U}_{\mathrm{SSM}}}\ \mathbb{E}^v_x\big[V(X_{\breve{\tau}_r})\big] = 0\,.$$

Therefore,

$$V(x) \le \liminf_{r\downarrow 0}\ \inf_{v\in\mathfrak{U}_{\mathrm{SSM}}}\ \mathbb{E}^v_x\left[\int_0^{\breve{\tau}_r} \big(c_v(X_t) - \varrho\big)\,\mathrm{d}t\right].$$

This shows $V(x) \le \Psi^*(x; \varrho)$, and the proof of (i) is complete.

If $\varrho_{\hat{v}} < \infty$, then using the bounds in Theorem 6.2 along with Lemma A.16 and Remark A.17, it follows that $J^{\hat{v}}_{\alpha_n} - J^{\hat{v}}_{\alpha_n}(0)$ and $\alpha_n J^{\hat{v}}_{\alpha_n}(0)$ converge along some sequence $\alpha_n \to 0$ to $\hat{V}$ and $\hat{\varrho}$, respectively, satisfying $L^{\hat{v}}\hat{V} + c_{\hat{v}} = \hat{\varrho}$. Since (7.19) and (7.20) hold with equality if we replace V_α with $J^{\hat{v}}_\alpha$, ϱ with $\hat{\varrho}$, and v with $\hat{v}$, first letting $\alpha_n \to 0$ and then $r \to 0$, we obtain $\hat{V} = \Psi^{\hat{v}}(x; \hat{\varrho})$. The bound

$$\left(\inf_{B_R}\ \alpha J^{\hat{v}}_\alpha\right) \mu_{\hat{v}}(B_R) \le \int_{\mathbb{R}^d} \alpha J^{\hat{v}}_\alpha(x)\mu_{\hat{v}}(\mathrm{d}x) = \varrho_{\hat{v}} \quad \forall R > 0$$

yields

$$\alpha J^{\hat{v}}_\alpha(0)\mu_{\hat{v}}(B_R) \le \varrho_{\hat{v}} + \alpha\left(J^{\hat{v}}_\alpha(0) - \inf_{B_R}\ J^{\hat{v}}_\alpha\right)\mu_{\hat{v}}(B_R)\,. \tag{7.21}$$

Taking limits as $\alpha \to 0$ in (7.21) and using (6.3a), we obtain

$$\hat{\varrho}\,\mu_{\hat{v}}(B_R) \le \varrho_{\hat{v}} \quad \forall R > 0\,,$$

from which it follows that $\hat{\varrho} \le \varrho_{\hat{v}}$. This completes the proof of (ii). $\square$

We need the following definition.

DEFINITION 7.5. *Let $\mathfrak{V}$ be the class of nonnegative functions $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$ satisfying* (7.2) *for some nonnegative,* inf*-compact $h \in \mathfrak{C}$, with $1 + c \in \mathfrak{o}(h)$. We denote by $\mathfrak{o}(\mathfrak{V})$ the class of functions V satisfying $V \in \mathfrak{o}(\mathcal{V})$ for some $\mathcal{V} \in \mathfrak{V}$.*

The next theorem assumes (7.1). In other words, we assume that $1+c$ is uniformly integrable with respect to $\{\pi_v\,,\ v \in \mathfrak{U}_{\mathrm{SSM}}\}$. Note that if $c \in \mathcal{C}_b(\mathbb{R}^d \times \mathbb{U})$, Theorem 5.6 asserts that (7.1) is equivalent to uniform stability of $\mathfrak{U}_{\mathrm{SSM}}$, and thus (7.1) is automatically satisfied when $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$, and when the running cost is bounded by Theorem 8.3, which is stated later in section 8. The main reason for assuming (7.1) in Theorem 7.6 below is to assert that there exists a solution of the HJB equation in $\mathfrak{o}(\mathfrak{V})$. Then Theorem 7.7 which follows asserts that this solution is unique in $\mathfrak{o}(\mathfrak{V})$.

THEOREM 7.6. *Assume* (7.1) *holds. Then the HJB equation*

$$\min_{u\in\mathbb{U}} \bigl[L^u V(x) + c(x,u)\bigr] = \varrho\,, \quad x \in \mathbb{R}^d\,, \tag{7.22}$$

admits a solution with $\varrho \in \mathbb{R}$ and $V \in \mathcal{C}^2(\mathbb{R}^d) \cap \mathfrak{o}(\mathfrak{V})$, satisfying $V(0) = 0$. Moreover, $\varrho = \varrho^*$, and if $v^* \in \mathfrak{U}_{\mathrm{SM}}$ is a measurable selector from the minimizer in (7.22)*, i.e., if it satisfies*

$$\min_{u\in\mathbb{U}} \left[\sum_{i=1}^d b^i(x,u)\frac{\partial V}{\partial x_i}(x) + c(x,u)\right] = \sum_{i=1}^d b^i_{v^*}(x)\frac{\partial V}{\partial x_i}(x) + c_{v^*}(x) \quad a.e., \tag{7.23}$$

then

$$\varrho_{v^*} = \varrho^* = \inf_{U\in\mathfrak{U}}\ \limsup_{T\to\infty}\ \frac{1}{T}\,\mathbb{E}^U_x\left[\int_0^T c\,(X_t,U_t)\,\mathrm{d}t\right]. \tag{7.24}$$

Proof. The existence of a solution to (7.22) with $V \in \mathcal{C}^2(\mathbb{R}^d)$ and $\varrho \le \varrho^*$ is asserted by Lemma 7.4. By (7.3) and (7.20), $V \in \mathfrak{o}(\mathfrak{V})$. Suppose $v^* \in \mathfrak{U}_{\mathrm{SM}}$ satisfies (7.23). By Itô's formula,

$$\begin{aligned} \mathbb{E}^{v^*}_x\bigl[V(X_{t\wedge\tau_R})\bigr] - V(x) &= \mathbb{E}^{v^*}_x\left[\int_0^{t\wedge\tau_R} L^{v^*}V(X_s)\,\mathrm{d}s\right] \\ &= \mathbb{E}^{v^*}_x\left[\int_0^{t\wedge\tau_R}\bigl[\varrho - c_{v^*}(X_s)\bigr]\,\mathrm{d}s\right]. \end{aligned} \tag{7.25}$$

Taking limits as $R \to \infty$ in (7.25), by applying (7.5) to the left-hand side, and decomposing the right-hand side as

$$\varrho\,\mathbb{E}^{v^*}_x\,[t\wedge\tau_R] - \mathbb{E}^{v^*}_x\left[\int_0^{t\wedge\tau_R} c_{v^*}(X_s)\,\mathrm{d}s\right],$$

and employing monotone convergence, we obtain

$$\mathbb{E}^{v^*}_x\bigl[V(X_t)\bigr] - V(x) = \mathbb{E}^{v^*}_x\left[\int_0^t\bigl[\varrho - c_{v^*}(X_s)\bigr]\,\mathrm{d}s\right]. \tag{7.26}$$

Dividing (7.26) by t, and applying (7.4) as we let $t \to \infty$, we obtain $\varrho_{v^*} = \varrho$, which implies $\varrho^* \le \varrho$. Since $\varrho \le \varrho^*$, we have equality. One more application of Itô's formula to (7.22), relative to $U \in \mathfrak{U}$, yields (7.24). ∎

Concerning uniqueness of solutions to the HJB equation, the following applies.

THEOREM 7.7. *Let V^* denote the solution of* (7.22) *obtained via the vanishing discount limit in Theorem* 7.6*, and let v^* be a measurable selector from the minimizer* $\min_{u\in\mathbb{U}}\bigl[L^u V^*(x) + c(x,u)\bigr]$. *The following hold:*

(i) $V^*(x) = \Psi^*(x;\varrho^*)$.

(ii) $\hat{v} \in \mathfrak{U}_{\mathrm{SSM}}$ *is average-cost optimal in* $\mathfrak{U}_{\mathrm{SSM}}$*, i.e.,* $\varrho_{\hat{v}} = \varrho^*$*, if and only if it satisfies*

$$b^i_{\hat{v}}(x)\partial_i V^*(x) + c_{\hat{v}}(x) = \min_{u\in\mathbb{U}}\;\bigl[b^i(x,u)\partial_i V^*(x) + c(x,u)\bigr] \quad \text{a.e.}$$

(iii) *If a pair* $(\tilde{V},\tilde{\varrho}) \in \bigl(\mathcal{C}^2(\mathbb{R}^d)\cap\mathfrak{o}(\mathfrak{V})\bigr)\times\mathbb{R}$ *satisfies* (7.22) *and* $\tilde{V}(0)=0$*, then* $(\tilde{V},\tilde{\varrho}) = (V^*,\varrho^*)$.

Proof. By Lemma 7.4(i), since V^* is obtained as a limit of $\bar{V}_{\alpha_n}$ as $\alpha_n \to 0$, we have $V^* \le \Psi^*(x;\varrho)$, and by Theorem 7.6, $\varrho = \varrho^*$. Suppose $\hat{v} \in \mathfrak{U}_{\mathrm{SSM}}$ is optimal. By Lemma 7.4(ii), there exists $\hat{V} \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p>1$, satisfying $L^{\hat{v}}\hat{V} - c_{\hat{v}} = \hat{\varrho}$ in $\mathbb{R}^d$. Also, $\hat{V} = \Psi^{\hat{v}}(x;\hat{\varrho})$ and $\hat{\varrho} \le \varrho_{\hat{v}}$. Thus by the optimality of $\hat{v}$, $\hat{\varrho} \le \varrho^*$, and we obtain

$$L^{\hat{v}}(V^* - \hat{V}) \ge \varrho^* - \hat{\varrho} \ge 0 \tag{7.27}$$

and

$$\begin{aligned} V^*(x) - \hat{V}(x) &\le \Psi^*(x;\varrho^*) - \Psi^{\hat{v}}(x;\hat{\varrho}) \\ &\le \Psi^*(x;\varrho^*) - \Psi^{\hat{v}}(x;\varrho^*) \le 0\,. \end{aligned}$$

Since $V^*(0) = \hat{V}(0)$, the strong maximum principle (Theorem A.4) yields $V^* = \hat{V}$, and in turn by (7.27), $\hat{\varrho} = \varrho^*$. This completes the proof of (i)–(ii).

Now suppose $(\tilde{V},\tilde{\varrho}) \in \bigl(\mathcal{C}^2(\mathbb{R}^d)\cap\mathfrak{o}(\mathfrak{V})\bigr)\times\mathbb{R}$ is any solution of (7.22), and $\tilde{v} \in \mathfrak{U}_{\mathrm{SSM}}$ is an associated measurable selector from the minimizer. We apply Itô's formula and (7.5), since $\tilde{V} \in \mathfrak{o}(\mathcal{V})$, to obtain (7.26) with $\tilde{V}$, $\tilde{v}$, and $\tilde{\varrho}$ replacing V, v^*, and ϱ, respectively. Dividing by t, and applying (7.4) while taking limits as $t\to\infty$, we obtain $\varrho_{\tilde{v}} = \tilde{\varrho}$. Therefore $\varrho^* \le \tilde{\varrho}$. One more application of Itô's formula to (7.22) relative to the control v^* yields

$$\mathbb{E}^{v^*}_x\bigl[\tilde{V}(X_t)\bigr] - \tilde{V}(x) \ge \mathbb{E}^{v^*}_x\left[\int_0^t \bigl[\tilde{\varrho} - c_{v^*}(X_s)\bigr]\,\mathrm{d}s\right]. \tag{7.28}$$

Once more, dividing (7.28) by t, letting $t\to\infty$, and applying (7.4), we obtain $\tilde{\varrho} \le \varrho^*$. Thus, $\tilde{\varrho} = \varrho^*$. Next we show that $\tilde{V} \ge \Psi^*(x;\varrho^*)$. For $x\in\mathbb{R}^d$, choose $R>r>0$ such that $r<|x|<R$. Using (7.22) and Itô's formula,

$$\begin{aligned} \tilde{V}(x) = \mathbb{E}^{\tilde{v}}_x\biggl[\int_0^{\breve{\tau}_r\wedge\tau_R} \bigl(c_{\tilde{v}}(X_t) - \varrho^*\bigr)\,\mathrm{d}t + \mathbb{I}\{\breve{\tau}_r < \tau_R\}\tilde{V}(X_{\breve{\tau}_r}) \\ + \mathbb{I}\{\breve{\tau}_r \ge \tau_R\}\tilde{V}(X_{\tau_R})\biggr]. \end{aligned} \tag{7.29}$$

By (7.8),

$$\mathbb{E}^v_x\bigl[\mathcal{V}(X_{\tau_R})\,\mathbb{I}\{\tau_R \le \breve{\tau}_r\}\bigr] \le k_0\,\mathbb{E}^v_x[\breve{\tau}_r] + \mathcal{V}(x) \quad \forall v \in \mathfrak{U}_{\mathrm{SSM}}\,. \tag{7.30}$$

Since $\tilde{V} \in \mathfrak{o}(\mathcal{V})$, (7.30) implies that

$$\sup_{v\in\mathfrak{U}_{\mathrm{SSM}}}\;\mathbb{E}^v_x\Bigl[\tilde{V}(X_{\tau_R})\,\mathbb{I}\{\tau_R \le \breve{\tau}_r\}\Bigr] \xrightarrow[R\to\infty]{} 0\,.$$

Hence, letting $R \to \infty$ in (7.29), and using Fatou's lemma, we obtain

$$\begin{aligned}\tilde{V}(x) &\geq \mathbb{E}_x^{\tilde{v}}\biggl[\int_0^{\breve{\tau}_r} \bigl(c_{\tilde{v}}(X_t) - \varrho^*\bigr)\,\mathrm{d}t + \tilde{V}(X_{\breve{\tau}_r})\biggr] \\ &\geq \inf_{v\in\mathfrak{U}_{\mathrm{SSM}}} \mathbb{E}_x^{v}\biggl[\int_0^{\breve{\tau}_r} \bigl(c_v(X_t) - \varrho^*\bigr)\,\mathrm{d}t\biggr] + \inf_{B_r} \tilde{V}\,.\end{aligned}$$

Next, letting $r \to 0$ and using the fact that $\tilde{V}(0) = 0$ yields $\tilde{V} \geq \Psi^*(x;\varrho^*)$. It follows that $V^* - \tilde{V} \leq 0$ and $L^{\tilde{v}}(V^* - \tilde{V}) \geq 0$. Therefore, by the strong maximum principle, $\tilde{V} = V^*$. This completes the proof of (iii). ∎

8. Optimality under weakened hypotheses. In this section we relax the assumption in (7.1). Under the assumption $\mathfrak{U}_{\mathrm{SM}} = \mathfrak{U}_{\mathrm{SSM}}$, the existence of an average-cost optimal control in $\mathfrak{U}_{\mathrm{SSM}}$ is guaranteed by Theorem 8.1 and Remark 8.2 below. This is used subsequently to establish that $\mathfrak{U}_{\mathrm{SSM}}$ is uniformly stable. Therefore, $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$ implies that the mean empirical measures defined in Theorem 5.6 are tight, and this shows in retrospect that the optimality asserted in Theorem 8.1 is in fact over all admissible controls $\mathfrak{U}$.

THEOREM 8.1. *Suppose that $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$ and $\varrho_v < \infty$ for all $v \in \mathfrak{U}_{\mathrm{SSM}}$. Then the HJB equation in* (7.22) *admits a solution $V^* \in \mathcal{C}^2(\mathbb{R}^d)$ and $\varrho \in \mathbb{R}$ satisfying $V(0) = 0$. Moreover, $\varrho = \varrho^*$, and any $v \in \mathfrak{U}_{\mathrm{SSM}}$ is average-cost optimal in $\mathfrak{U}_{\mathrm{SSM}}$ if and only if it satisfies* (7.23).

Proof. By Lemma 7.4(i), we obtain a solution (V^*, ϱ) to (7.22), via the vanishing discount limit, satisfying $\varrho \leq \varrho^*$.

Let $v^* \in \mathfrak{U}_{\mathrm{SSM}}$ be a measurable selector from the minimizer in (7.22). We construct a stochastic Lyapunov function relative to v^*. Employing the technique in the proof of Lemma 7.1, we define

$$h_{v^*}(x) \triangleq \bigl(1 + c_{v^*}(x)\bigr)\left(\int_{B^c_{|x|}} \bigl(1 + c_{v^*}(y)\bigr)\mu_{v^*}(\mathrm{d}y)\right)^{-1/2} \tag{8.1}$$

and construct a nonnegative, inf-compact $\mathcal{V}^* \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, which satisfies, for some $k_0 \in \mathbb{R}$,

$$L^{v^*}\mathcal{V}^*(x) \leq k_0 - h_{v^*}(x) \quad \forall x \in \mathbb{R}^d\,. \tag{8.2}$$

It follows as in the proof of Lemma 7.1 that for any $r > 0$,

$$\mathbb{E}_x^{v^*}\biggl[\int_0^{\breve{\tau}_r} \bigl(1 + c_{v^*}(X_t)\bigr)\,\mathrm{d}t\biggr] \in \mathfrak{o}(\mathcal{V}^*)\,, \tag{8.3}$$

and for any $\varphi \in \mathfrak{o}(\mathcal{V}^*)$,

$$\lim_{t\to\infty} \frac{1}{t}\,\mathbb{E}_x^{v^*}\bigl[\varphi(X_t)\bigr] = 0 \tag{8.4}$$

and

$$\lim_{R\to\infty} \mathbb{E}_x^{v^*}\left[\varphi(X_{t\wedge\tau_R})\right] = \mathbb{E}_x^{v^*}\left[\varphi(X_t)\right]\,. \tag{8.5}$$

To show that $V^* \in \mathfrak{o}(\mathcal{V}^*)$, let $r < R$, and define the admissible control $U \in \mathfrak{U}$ by

$$U_t = \begin{cases} v^* & \text{if } t \leq \breve{\tau}_r \wedge \tau_R, \\ v_\alpha & \text{otherwise.}\end{cases}$$

Since U is in general suboptimal for the α-discounted criterion, using the strong Markov property as in (7.17), and taking limits as $R \to \infty$, we obtain

$$\begin{aligned}(8.6)\ V_\alpha(x) - V_\alpha(0) \le \mathbb{E}_x^{v^*}\biggl[\int_0^{\breve{\tau}_r} \mathrm{e}^{-\alpha t}\bigl(c_{v^*}(X_t) - \varrho\bigr)\,\mathrm{d}t\biggr] + \mathbb{E}_x^{v^*}\Bigl[V_\alpha(X_{\breve{\tau}_r}) - V_\alpha(0)\Bigr] \\ + \mathbb{E}_x^{v^*}\Bigl[\alpha^{-1}\bigl(1-\mathrm{e}^{-\alpha\breve{\tau}_r}\bigr)\bigl[\varrho - \alpha V_\alpha(X_{\breve{\tau}_r})\bigr]\Bigr]\,.\end{aligned}$$

By (8.3), the first term on the right-hand side of (8.6) is $\mathfrak{o}(\mathcal{V}^*)$, and the remaining two terms are bounded by Theorem 6.2. Hence $\bar{V}_\alpha \in \mathfrak{o}(\mathcal{V}^*)$ uniformly in α in some neighborhood of 0, and it follows that $V^* \in \mathfrak{o}(\mathcal{V}^*)$. Using Itô's formula as in (7.25) and applying (8.5), we obtain (7.26). Next, using (8.4) to take limits as $t \to \infty$, we obtain $\varrho_{v^*} = \varrho$, and therefore, $\varrho = \varrho^*$.

To prove the second assertion, suppose that some $\hat{v} \in \mathfrak{U}_{\mathrm{SSM}}$ is average-cost optimal in $\mathfrak{U}_{\mathrm{SSM}}$. By Lemma 7.4(ii), $\hat{v}$ satisfies $L^{\hat{v}}\hat{V} + c_{\hat{v}} = \hat{\varrho}$ for some $\hat{V} \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$ and $\hat{\varrho} \le \varrho^*$. Thus

$$(8.7)\qquad L^{\hat{v}}(V^* - \hat{V}) \ge \varrho^* - \hat{\varrho} \ge 0\,.$$

Also, by Lemma 7.4, $V^*(x) \le \Psi^*(x;\varrho^*)$ and $\hat{V} = \Psi^{\hat{v}}(x;\hat{\varrho})$. Hence $V^* - \hat{V} \le 0$, and since $V^*(0) = \hat{V}(0)$, the strong maximum principle yields $V^* = \hat{V}$, and in turn by (8.7), $\hat{\varrho} = \varrho^*$. Thus $L^{\hat{v}}V^* + c_{\hat{v}} = \varrho^*$. ◻

Remark 8.2. If we only assume that $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$, without requiring that $\varrho_v < \infty$ for all $v \in \mathfrak{U}_{\mathrm{SSM}}$, then it follows from the proof of Theorem 8.1 that any measurable selector $\hat{v}$ from the minimizer in (7.22) satisfying $\varrho_{\hat{v}} < \infty$ is average-cost optimal in $\mathfrak{U}_{\mathrm{SSM}}$. Moreover, one can show that any limit point v^* along some sequence $\alpha_n \downarrow 0$ of the family of α-discounted controls $\{v_\alpha\} \subset \mathfrak{U}_{\mathrm{SSM}}$ satisfies (7.23) and is average-cost optimal in $\mathfrak{U}_{\mathrm{SSM}}$. In order to prove this, we define the truncated running cost $c^M \triangleq \min\{c, M\}$, where $M > 0$ is a constant. Let $J^{v_\alpha}_{\alpha,M}$ denote the α-discounted cost relative to c^M under the control v_α. Applying Theorem 6.2 and Lemma 3.5 to take limits in

$$L^{v_\alpha}\bigl(J^{v_\alpha}_{\alpha,M} - J^{v_\alpha}_{\alpha,M}(0)\bigr) = \alpha J^{v_\alpha}_{\alpha,M} - c^M_{v_\alpha}\,,$$

along the sequence $\{\alpha_n\}$, it follows that v^* satisfies $L^{v^*}V_M + c^M_{v^*} = \varrho_M$, for some $V_M \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p > 1$, and $\varrho_M \in \mathbb{R}$. We construct a stochastic Lyapunov function $\mathcal{V}^*_M$ relative to $c^M_{v^*}$ as in (8.1)–(8.2) and follow the steps in the proof of Theorem 8.1 to show that $V_M \in \mathfrak{o}(\mathcal{V}^*_M)$ and $\varrho_M = \int c^M_{v^*}\,\mathrm{d}\mu_{v^*}$. Therefore,

$$\begin{aligned}\int c^M_{v^*}\,\mathrm{d}\mu_{v^*} = \varrho_M &= \lim_{n\to\infty}\ \alpha_n J^{v_{\alpha_n}}_{\alpha_n,M}(0) \\ &\le \lim_{n\to\infty}\ \alpha_n V_{\alpha_n}(0) = \varrho\,,\end{aligned}$$

and using monotone convergence to take the limit as $M \to \infty$, it follows that $\varrho_{v^*} \le \varrho$. Since $\varrho \le \varrho^*$, we have $\varrho_{v^*} = \varrho^*$, and hence v^* is optimal.

It is evident that if c is bounded, the assumption that $\varrho_v < \infty$ for all $v \in \mathfrak{U}_{\mathrm{SSM}}$ can be dropped from the statement of Theorem 8.1 as it is automatically satisfied. Let $\bar{\mathfrak{I}}$ be the closure of $\mathfrak{I}$ in $\mathfrak{P}(\bar{\mathbb{R}}^d)$. Theorem 8.1 shows that if $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$, then $\inf_{\mu\in\bar{\mathfrak{I}}} \int_{\bar{\mathbb{R}}^d} g\,\mathrm{d}\mu$ is attained in $\mathfrak{I}$ for all $g \in \mathcal{C}_b(\bar{\mathbb{R}}^d)$. We next prove that this implies that $\mathfrak{I}$ is tight, thus solving the open problem discussed in section 1.

THEOREM 8.3. *If* $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$, *then* $\mathfrak{I}$ *is tight.*

Proof. Consider the sequence $c_n(x) = \bigl(1 + \frac{\|x\|^2}{n}\bigr)^{-1}$. If $\mathcal{I}$ is not tight, then there exists $\varepsilon > 0$ such that

$$\varrho_n^* \triangleq \inf_{\mu\in\mathcal{I}} \int_{\mathbb{R}^d} c_n \,\mathrm{d}\mu < 1-\varepsilon \quad \forall n\in\mathbb{N}\,. \tag{8.8}$$

Let $V_\alpha^{(n)}$ be the α-discounted value function relative to c_n, and let $v_\alpha^{(n)} \in \mathfrak{U}_{\mathrm{SM}}$ denote a corresponding α-discounted optimal control. Since $\alpha V_\alpha^{(n)}(0) \to \varrho_n^*$, as $\alpha \downarrow 0$, we can select $\alpha_n \in (0,1)$ such that

$$\bigl|\alpha_n V_{\alpha_n}^{(n)}(0) - \varrho_n^*\bigr| \le \frac{1}{n}\,, \quad n\in\mathbb{N}\,. \tag{8.9}$$

It is evident that $\alpha_n \to 0$ as $n\to\infty$. Extract any subsequence of $n\in\mathbb{N}$ over which $v_{\alpha_n}^{(n)}$ converges to a limit $v\in\mathfrak{U}_{\mathrm{SSM}}$. By Corollary 6.3, $\bar{V}_{\alpha_n}^{(n)}$ is bounded in $\mathcal{W}^{2,p}(D)$ uniformly in $n\in\mathbb{N}$ for any bounded domain D. Hence, by Lemma 3.5, dropping perhaps to a further subsequence, which is also denoted by $\{n\}$, there exists $V\in\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p>1$, such that as $n\to\infty$, $\alpha V_\alpha^{(n)}(0)$ converges to a constant, $\bar{V}_{\alpha_n}^{(n)} \to V$, uniformly on compact subsets of $\mathbb{R}^d$, and

$$L^v V = -1 + \lim_{n\to\infty} \alpha_n V_{\alpha_n}^{(n)}(0)\,.$$

By (8.8) and (8.9), we obtain at the limit

$$L^v V \le -\varepsilon \quad \text{on } \mathbb{R}^d\,. \tag{8.10}$$

Since $v\in\mathfrak{U}_{\mathrm{SSM}}$, applying (8.1)–(8.2) (with $c\equiv 1$), we construct nonnegative, inf-compact functions $\mathcal{V}\in\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$ and $h\colon\mathbb{R}^d\to\mathbb{R}_+$, satisfying $L^v\mathcal{V}(x)\le k_0 - h(x)$, for some constant $k_0\in\mathbb{R}$, and such that $V\in\mathfrak{o}(\mathcal{V})$. As in (8.4),

$$\lim_{t\to\infty} \frac{1}{t}\,\mathbb{E}_x^v\bigl[V(X_t)\bigr] = 0\,. \tag{8.11}$$

By Itô's formula, which can be applied as in the derivation of (7.26) since V is $\mathfrak{o}(\mathcal{V})$, (8.10) yields

$$\mathbb{E}_x^v\bigl[V(X_t)\bigr] - V(x) \le -\varepsilon t\,. \tag{8.12}$$

Dividing (8.12) by t and letting $t\to\infty$, while applying (8.11), yields a contradiction. Therefore, $\mathcal{I}$ must be tight. ∎

Using Theorem 8.3 we can improve the results in Theorem 8.1.

COROLLARY 8.4. *Under the assumptions of Theorem* 8.1*, any measurable selector from the minimizer in the HJB equation* (7.22) *obtained via the vanishing discount limit is average-cost optimal.*

Proof. Since the hypothesis $\mathfrak{U}_{\mathrm{SSM}} = \mathfrak{U}_{\mathrm{SM}}$ implies that $\mathfrak{U}_{\mathrm{SSM}}$ is uniformly stable, by Theorem 5.6 the mean empirical measures $\bigl\{\bar{\nu}_{x,t}^U\bigr\}$ defined in (5.14) are tight. Consequently, since as noted in the proof of Theorem 5.6 the set of accumulation points of $\bar{\nu}_{x,t}^U$, as $t\to\infty$, equals $\mathcal{M}$, we have

$$\begin{aligned}\liminf_{t\to\infty}\ \frac{1}{t}\int_0^t \mathbb{E}_x^U\bigl[\bar{c}(X_s,U_s)\bigr]\mathrm{d}s &= \liminf_{t\to\infty}\ \int_{\mathbb{R}^d\times\mathbb{U}} c(z,u)\bar{\nu}_{x,t}^U(\mathrm{d}z,\mathrm{d}u)\\ &\ge \min_{v\in\mathfrak{U}_{\mathrm{SSM}}} \int_{\mathbb{R}^d\times\mathbb{U}} c\,\mathrm{d}\pi_v \quad \forall U\in\mathfrak{U}\,,\end{aligned}$$

and it follows that if $v \in \mathfrak{U}_{\mathrm{SM}}$ is average-cost optimal in $\mathfrak{U}_{\mathrm{SSM}}$, it is also average-cost optimal over all admissible controls. ☐

It follows from the proof of Corollary 8.4 that when $\mathfrak{U}_{\mathrm{SM}} = \mathfrak{U}_{\mathrm{SSM}}$ we obtain a stronger form of optimality, namely

$$\varrho^* \le \inf_{U \in \mathfrak{U}} \left(\liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}_x^U \left[\bar{c}(X_t, U_t) \right] \mathrm{d}t \right).$$

Relaxing the assumption $\mathfrak{U}_{\mathrm{SM}} = \mathfrak{U}_{\mathrm{SSM}}$, we obtain the following result.

THEOREM 8.5. *Suppose that the family of α-discounted optimal controls $\{v_\alpha\}$ has an accumulation point $\breve{v} \in \mathfrak{U}_{\mathrm{SSM}}$, as $\alpha \to 0$, and suppose that $\varrho_{\breve{v}} < \infty$. Then*

(i) *the HJB equation in* (7.22) *admits a solution $\breve{\varrho} \in \mathbb{R}$ and $\breve{V} \in \mathcal{C}^2(\mathbb{R}^d)$ satisfying $V(0) = 0$. Moreover, $\breve{\varrho} = \varrho^*$, and $\breve{v}$ is a measurable selector from the minimizer in*

$$\min_{u \in \mathbb{U}} \left[L^u \breve{V}(x) + c(x,u) \right]; \tag{8.13}$$

(ii) *any measurable selector $v^* \in \mathfrak{U}_{\mathrm{SSM}}$ from the minimizer in* (8.13)*, satisfying $\varrho_{v^*} < \infty$, is average-cost optimal;*

(iii) *$\hat{v} \in \mathfrak{U}_{\mathrm{SSM}}$ is average-cost optimal in $\mathfrak{U}_{\mathrm{SSM}}$ only if it satisfies*

$$b^i_{\hat{v}}(x) \partial_i \breve{V}(x) + c_{\hat{v}}(x) = \min_{u \in \mathbb{U}} \left[b^i(x,u) \partial_i \breve{V}(x) + c(x,u) \right] \quad \textit{a.e.}$$

Proof. Let $\breve{v} \in \mathfrak{U}_{\mathrm{SSM}}$ be the limit of α-discounted optimal controls $\{v_\alpha\}$ over some sequence as $\alpha \to 0$, and let $\breve{V}$ be the limit of $\bar{V}_\alpha$ and $\breve{\varrho}$ be the limit of $\alpha V_\alpha(0)$ over a common subsequence $\{\alpha_n\}$. By Lemma 7.4(i), $(\breve{V}, \breve{\varrho})$ is a solution of the HJB equation (7.22) and satisfies $\breve{\varrho} \le \varrho^*$ and $\breve{V}(x) \le \Psi^*(x; \breve{\varrho})$. Taking limits as $n \to \infty$ in

$$L^{v_{\alpha_n}} \bar{V}_{\alpha_n} = \alpha_n \bar{V}_{\alpha_n} + \alpha_n V_{\alpha_n}(0) - c_{v_{\alpha_n}},$$

and applying Lemma 3.5, it follows that $\breve{v}$ satisfies $L^{\breve{v}} \breve{V} + c_{\breve{v}} = \breve{\varrho}$, and therefore $\breve{v}$ is a measurable selector from the minimizer in the HJB. Since $\varrho_{\breve{v}} < \infty$, we can employ a stochastic Lyapunov function $\tilde{\mathcal{V}}$, defined relative to $\breve{v}$ as in (8.1)–(8.2), and follow the steps in the proof of Theorem 8.1 to obtain $\varrho_{\breve{v}} = \breve{\varrho}$, and thus $\varrho_{\breve{v}} = \varrho^*$, which also shows that $\breve{v}$ is average-cost optimal in $\mathfrak{U}_{\mathrm{SSM}}$. This completes the proof of (i).

Concerning (ii), if $v^* \in \mathfrak{U}_{\mathrm{SSM}}$ is a measurable selector from the minimizer in (8.13), and $\varrho_{v^*} < \infty$, the proof of Theorem 8.1 shows that v^* is average-cost optimal in $\mathfrak{U}_{\mathrm{SSM}}$. A standard application of a Tauberian theorem, which asserts that for all $U \in \mathfrak{U}$

$$\begin{aligned} \varrho^* = \limsup_{n \to \infty} \alpha_n V_{\alpha_n}(x) &\le \limsup_{n \to \infty} \alpha_n \int_0^\infty \mathrm{e}^{-\alpha_n t} \, \mathbb{E}_x^U \left[\bar{c}(X_t, U_t) \right] \mathrm{d}t \\ &\le \limsup_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}_x^U \left[\bar{c}(X_t, U_t) \right] \mathrm{d}t \,, \end{aligned}$$

shows that $v^* \in \mathfrak{U}_{\mathrm{SSM}}$ is in fact average-cost optimal in $\mathfrak{U}$.

Turning to (iii), suppose that some $\hat{v} \in \mathfrak{U}_{\mathrm{SSM}}$ satisfies $\varrho_{\hat{v}} = \varrho^*$. As in the last paragraph of the proof of Theorem 8.1, it follows that $J^{\hat{v}}_\alpha - J^{\hat{v}}_\alpha(0) \to \breve{V}$, as $\alpha \downarrow 0$, and that $\hat{v}$ is a measurable selector from the minimizer in (8.13). ☐

Remark 8.6. It follows from the proof of part (iii) of Theorem 8.5 that there is a unique $\breve{V} \in \mathcal{C}^2(\mathbb{R}^d)$ which is obtained as a limit of $\bar{V}_\alpha$ over any subsequence $\alpha_n \downarrow 0$, and satisfying $\lim_{n\to\infty} v_{\alpha_n} = \breve{v} \in \mathfrak{U}_{\mathrm{SSM}}$, with $\varrho_{\breve{v}} < \infty$.

We conclude this section by noting that the class of models with near-monotone running cost can be handled directly by Theorem 8.5. Recall that the running cost function c is called *near-monotone* if

$$\liminf_{|x|\to\infty} \inf_{u\in\mathbb{U}} c(x,u) > \varrho^* .$$

It is well known that if the running cost has the near-monotone property, then V is bounded below [9]. Thus the HJB takes the form of a stochastic Lyapunov equation, and this implies that any measurable selector v^* from the minimizer in the HJB is stable, and $\varrho_{v^*} < \infty$. Therefore, by Theorem 8.5, it is average-cost optimal.

9. Conclusion. In the context of elliptic PDEs the main result of this paper can be summarized as follows. The statement that v is a stable, stationary Markov control is equivalent to the existence of an inf-compact $\mathcal{V} \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p > 1$, such that $-L^v\mathcal{V}$ is inf-compact. On the other hand, uniform stability is equivalent to the existence of an inf-compact $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$ such that $-\max_{u\in\mathbb{U}} L^u\mathcal{V}$ is inf-compact.

We would like to point out that in the case of one-dimensional diffusions, there is a straightforward analytical proof for Theorem 8.3, which goes as follows. Let

$$\breve{b}(x) \triangleq \max_{u\in\mathbb{U}} \,[b(x,u)\,\mathrm{sign}(x)]\,. \tag{9.1}$$

Then assuming that all stationary Markov controls are stable, by solving a Dirichlet problem on $(-1,1)^c$, we can construct $\psi \in \mathcal{C}^2\left([-1,1]^c\right) \cap \mathcal{C}\left((-1,1)^c\right)$ such that $-(a\,\partial_x^2\psi + \breve{b}\,\partial_x\psi)$ is nonnegative and inf-compact on $(-1,1)^c$. It is straightforward to show that $\psi(x)$ is monotone, nondecreasing in $[1,\infty)$ (and nonincreasing in $(-\infty,1]$). Hence, $b(x,u)\,\partial_x\psi(x) \le \breve{b}(x)\,\partial_x\psi(x)$ a.e. on $[-1,1]^c$, which implies that $-\max_{u\in\mathbb{U}} L^u\psi$ is inf-compact on $(-1,1)^c$, and this is sufficient for uniform stability.

In closing we remark that there is a stronger property that holds for $d = 1$. Let $\bar{v} \in \mathfrak{U}_{\mathrm{SSM}}$ be a measurable selector from the maximizer in (9.1). An application of the comparison principle (for ordinary differential equations) to the Fokker–Planck equation (4.4) for the density φ_v of $\mu_v \in \mathfrak{I}$ yields

$$\frac{\varphi_v(x)}{\varphi_v(0)} \le \frac{\varphi_{\bar{v}}(x)}{\varphi_{\bar{v}}(0)} \quad \forall x \in \mathbb{R}\,, \quad \forall v \in \mathfrak{U}_{\mathrm{SSM}}\,. \tag{9.2}$$

The inequality in (9.2) can also be derived from the explicit solution for the density φ_v which takes a simple form when $d = 1$ [23]. On the other hand, since $\mathfrak{I}$ is tight, applying (4.6) for some fixed $R > 0$, we obtain $\varphi_v(0) \le 2C_H^2\varphi_{\bar{v}}(0)$ for all $v \in \mathfrak{U}_{\mathrm{SSM}}$, which combined with (9.2) shows that $\bar{\varphi} \triangleq \sup_{v\in\mathfrak{U}_{\mathrm{SSM}}} \varphi_v$ satisfies $\bar{\varphi} \le 2C_H^2\varphi_{\bar{v}}$, and hence belongs to $\mathcal{L}^1(\mathbb{R})$. Whether this is true or not for higher dimensions is an open problem.

Appendix A. Results from elliptic PDEs. The model in (1.1) gives rise to a class of elliptic operators, with $v \in \mathfrak{U}_{\mathrm{SM}}$ appearing as a parameter. To facilitate describing properties that are uniform over the class of operators we adopt the following parameterization.

DEFINITION A.1. *Let $\gamma : (0,\infty) \mapsto (0,\infty)$ be a positive function that plays the role of a parameter. Using the standard summation rule for repeated indices, we denote* *by* $\mathfrak{L}(\gamma)$ *the class of operators*

$$L = a^{ij}\partial_{ij} + b^i\partial_i - \lambda\,,$$

with $a^{ij} = a^{ji}$, $\lambda \ge 0$, *and whose coefficients* $\{a^{ij}, b^i, \lambda\}$ *are measurable and satisfy, on each ball* $B_R \subset \mathbb{R}^d$,

$$\sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \ge \gamma^{-1}(R)|\xi|^2 \quad \forall x \in B_R\,, \tag{A.1a}$$

for all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, *and*

$$\begin{gathered}\max_{i,j}\, |a^{ij}(x) - a^{ij}(y)| \le \gamma(R)|x-y| \quad \forall x, y \in B_R\,,\\ \sum_{i,j=1}^{d} \|a^{ij}\|_{\mathcal{L}^\infty(B_R)} + \sum_{i,j=1}^{d} \|b^i\|_{\mathcal{L}^\infty(B_R)} + \|\lambda\|_{\mathcal{L}^\infty(B_R)} \le \gamma(R)\,.\end{gathered} \tag{A.1b}$$

Also, we let $\mathfrak{L}_0(\gamma)$ *denote the class of operators in* $\mathfrak{L}(\gamma)$ *satisfying* $\lambda = 0$.

Remark A.2. Note that the linear growth condition is not imposed on the class $\mathfrak{L}$. Either of the assumptions in (3.3) or (3.4) guarantees that $\tau_n \uparrow \infty$ a.s., as $n \to \infty$, a property which we impose separately when needed.

Of fundamental importance to the study of elliptic equations is the following estimate due to Alexandroff, Bakelman, and Pucci (see Gilberg and Trudinger [17, Theorem 9.1, p. 220]).

THEOREM A.3. *Let* $D \subset \mathbb{R}^d$ *be a bounded domain. There exists a constant* C_a *depending only on* d, D, *and* γ *such that if* $\psi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(D) \cap \mathcal{C}(\bar{D})$ *satisfies* $L\psi \ge f$, *with* $L \in \mathfrak{L}(\gamma)$, *then*

$$\sup_D\, \psi \le \sup_{\partial D}\, \psi^+ + C_a\|f\|_{\mathcal{L}^d(D)}\,.$$

When $f \equiv 0$, Theorem A.3 yields generalizations of the classical weak and strong maximum principles [17, Theorems 9.5 and 9.6, p. 225]. We state the latter as follows.

THEOREM A.4. *If* $\varphi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(D)$ *and* $L \in \mathfrak{L}(\gamma)$ *satisfy* $L\varphi \ge 0$ *in a bounded domain* D, *with* $\lambda = 0$ ($\lambda > 0$), *then* φ *cannot attain a maximum (nonnegative maximum) in* D *unless it is a constant.*

We quote the well-known a priori estimate [13, Lemma 5.3, p. 48] as follows.

LEMMA A.5. *If* $\varphi \in \mathcal{W}^{2,p}_{\mathrm{loc}}(D) \cap \mathcal{L}^p(D)$, *with* $p \in (1,\infty)$, *then for any bounded subdomain* $D' \Subset D$, *we have*

$$\|\varphi\|_{\mathcal{W}^{2,p}(D')} \le C_0\Big(\|\varphi\|_{\mathcal{L}^p(D)} + \|L\varphi\|_{\mathcal{L}^p(D)}\Big) \quad \forall L \in \mathfrak{L}(\gamma)\,,$$

with the constant C_0 *depending only on* d, D, D', p, *and* γ.

We use the following result concerning solutions of the Dirichlet problem [17, Theorem 9.15 and Lemma 9.17, pp. 241–242].

THEOREM A.6. *Let* D *be a bounded* $\mathcal{C}^2$ *domain in* $\mathbb{R}^d$, *and let* $L \in \mathfrak{L}(\gamma)$, $\lambda \ge 0$, *and* $p \in (1,\infty)$. *For each* $f \in \mathcal{L}^p(D)$ *and* $g \in \mathcal{W}^{2,p}(D)$ *there exists a unique* $\varphi \in \mathcal{W}^{2,p}(D)$ *satisfying* $\varphi - g \in \mathcal{W}^{1,p}_0(D)$ *and* $L\varphi = -f$ *in* D. *Moreover, we have the estimate*

$$\|\varphi\|_{\mathcal{W}^{2,p}(D)} \le C_0'\Big(\|f\|_{\mathcal{L}^p(D)} + \|Lg\|_{\mathcal{L}^p(D)} + \|g\|_{\mathcal{W}^{2,p}(D)}\Big)$$

for some constant $C_0' = C_0'(d, p, D, \gamma)$.

A function $\varphi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(D)$ satisfying $L\varphi = 0$ ($L\varphi \le 0$) in a domain D is called L-harmonic (L-superharmonic). In this paper we employ some specialized results which pertain to a class of L-superharmonic functions. These are summarized as follows.

Definition A.7. *For* $\delta > 0$ *and* D *a bounded domain, let* $\mathfrak{K}(\delta, D) \subset \mathcal{L}^\infty(D)$ *denote the positive convex cone*

$$\mathfrak{K}(\delta, D) \triangleq \Bigl\{ f \in \mathcal{L}^\infty(D) : f \ge 0\,,\ \bigl\|f\bigr\|_{\mathcal{L}^\infty(D)} \le \delta |D|^{-1} \bigl\|f\bigr\|_{\mathcal{L}^1(D)} \Bigr\}\,.$$

We use the following theorem from [2].

Theorem A.8. *There exists a constant* $\tilde{C}_a = \tilde{C}_a(d, \gamma, R, \delta)$ *such that for every* $\varphi \in \mathcal{W}^{2,p}_{\mathrm{loc}}(B_R) \cap \mathcal{W}^{1,p}_0(B_R)$ *satisfying* $L\varphi = -f$ *in* B_R *and* $\varphi = 0$ *on* ∂B_R, *with* $f \in \mathfrak{K}(\delta, B_R)$ *and* $L \in \mathfrak{L}(\gamma)$,

$$\inf_{B_{R/2}}\ \varphi \ge \tilde{C}_a \bigl\|f\bigr\|_{\mathcal{L}^1(B_R)}\,.$$

Harnack's inequality plays a central role in the study of L-harmonic functions. For strong solutions we refer to [17, Corollary 9.25, p. 250] for this result. Harnack's inequality has been extended in [2, Corollary 2.2] to the class of superharmonic functions satisfying $-L\varphi \in \mathfrak{K}(\delta, D)$. This result is often used in this paper and is quoted as follows.

Theorem A.9. *Let* D *be a domain and* $K \subset D$ *a compact set. There exists a constant* $\tilde{C}_H = \tilde{C}_H(d, D, K, \gamma, \delta)$, *such that if* $\varphi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(D)$ *satisfies* $L\varphi = -f$ *and* $\varphi \ge 0$ *in* D, *with* $f \in \mathfrak{K}(\delta, D)$ *and* $L \in \mathfrak{L}(\gamma)$, *then*

$$\varphi(x) \le \tilde{C}_H \varphi(y) \quad \forall x, y \in K\,.$$

A.1. Embeddings. We summarize some useful embedding results used in this paper [13, Proposition 1.6, p. 211], [17, Theorem 7.22, p. 167]. We start with a definition.

Definition A.10. *Let* X *and* Y *be Banach spaces, and let* $X \subset Y$. *If, for some constant* C, *we have* $\|x\|_Y \le C\|x\|_X$ *for all* $x \in X$, *then we say that* X *is* continuously embedded *in* Y *and refer to* C *as the* embedding constant. *In such a case we write* $X \hookrightarrow Y$. *We say that the embedding is compact if bounded sets in* X *are precompact in* Y.

Theorem A.11. *Let* $D \subset \mathbb{R}^d$ *be a bounded* $\mathcal{C}^{0,1}$ *domain and* $k \in \mathbb{N}$. *Then*

(i) *for* $p > d$, $\mathcal{W}^{1,p}_0(D) \hookrightarrow \mathcal{C}(\bar{D})$ *is compact;*

(ii) *if* $kp < d$, *then* $\mathcal{W}^{k,p}(D) \hookrightarrow \mathcal{L}^q(D)$ *is compact for* $p \le q < \frac{pd}{d-kp}$ *and continuous for* $p \le q \le \frac{pd}{d-kp}$;

(iii) *if* $\ell p > d$ *and* $\ell \le k$, *then* $\mathcal{W}^{k,p}(D) \hookrightarrow \mathcal{C}^{k-\ell,r}(\bar{D})$ *is compact for* $r < \ell - \frac{d}{p}$ *and continuous for* $r \le \ell - \frac{d}{p}$ $(r \le 1)$.

In particular, $\mathcal{W}^{2,d}(D) \hookrightarrow \mathcal{C}^{0,r}(\bar{D})$ *is compact for* $r < 1$, *and* $\mathcal{W}^{2,p}(D) \hookrightarrow \mathcal{C}^{1,r}(\bar{D})$ *is compact for* $p > d$ *and* $r < 1 - \frac{d}{p}$.

A.2. The resolvent. We define the α-resolvent $\mathcal{R}_\alpha$ for $\alpha \in (0, \infty)$ by

$$\mathcal{R}_\alpha[f](x) \triangleq \mathbb{E}_x \left[\int_0^\infty \mathrm{e}^{-\alpha t} f(X_t)\, \mathrm{d}t \right], \quad f \in \mathcal{L}^\infty(\mathbb{R}^d)\,.$$

Note that $\mathcal{R}_\alpha[f]$ is also well defined if f is nonnegative and belongs to $\mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}^d)$.

Let $f \in \mathcal{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}^d)$, $f \geq 0$, and $\alpha \in (0,\infty)$. If $\mathcal{R}_\alpha[f] \in \mathcal{C}(\mathbb{R}^d)$, then it satisfies Poisson's equation in $\mathbb{R}^d$ that

$$L\psi - \alpha\psi = -f\,. \tag{A.2}$$

If $f \in \mathcal{L}^{\infty}(\mathbb{R}^d)$ and $\alpha \in (0,\infty)$, then $\mathcal{R}_\alpha[f]$ is the unique solution of Poisson's equation in $\mathbb{R}^d$ in the class $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathcal{L}^{\infty}(\mathbb{R}^d)$, $p \in (1,\infty)$. More generally, we have the following.

THEOREM A.12. *Suppose $f \in \mathcal{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}^d)$, $f \geq 0$, and $\mathcal{R}_\alpha[f](x_0) < \infty$ at some $x_0 \in \mathbb{R}^d$, $\alpha \in (0,\infty)$. Then $\mathcal{R}_\alpha[f] \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$ for all $p \in (1,\infty)$ and satisfies* (A.2) *in $\mathbb{R}^d$.*

Remark A.13. It follows from Theorem A.12 and the decomposition

$$\mathcal{R}_\alpha[f](x) = \mathbb{E}_x\left[\int_0^{\tau_R} \mathrm{e}^{-\alpha t} f(X_t)\,\mathrm{d}t\right] + \mathbb{E}_x\left[\mathrm{e}^{-\alpha\tau_R}\mathcal{R}_\alpha[f](X_{\tau_R})\right]$$

that if $f \geq 0$, $f \in \mathcal{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}^d)$, and $\mathcal{R}_\alpha[f]$ is finite at some point in $\mathbb{R}^d$, then

$$\mathbb{E}_x\left[\mathrm{e}^{-\alpha\tau_R}\mathcal{R}_\alpha[f](X_{\tau_R})\right] \xrightarrow[R\to\infty]{} 0\,.$$

We refer the reader to [3] for these and other results on resolvents.

A.3. Quasi-linear elliptic operators. HJB equations that are of interest to us involve *quasi-linear operators* of the form

$$\begin{aligned} S\psi(x) &\triangleq a^{ij}(x)\partial_{ij}\psi(x) + \inf_{u\in\mathbb{U}} \hat{b}(x,u,\psi)\,, \\ \hat{b}(x,u,\psi) &\triangleq b^i(x,u)\partial_i\psi(x) - \alpha\psi(x) + c(x,u)\,. \end{aligned} \tag{A.3}$$

We suitably parameterize families of quasi-linear operators of this form as follows.

DEFINITION A.14. *For a nondecreasing function $\gamma\colon (0,\infty) \to (0,\infty)$ we denote by $\mathfrak{Q}(\gamma)$ the class of operators of the form* (A.3)*, whose coefficients b^i and c belong to $\mathcal{C}(\mathbb{R}^d \times \mathbb{U})$, and satisfy* (A.1a)–(A.1b) *and*

$$\max_{u\in\mathbb{U}}\left\{\max_i \left|b^i(x,u) - b^i(y,u)\right| + \left|c(x,u) - c(y,u)\right|\right\} \leq \gamma(R)|x-y|$$
$$\sum_{i,j=1}^d |a^{ij}(x)| + \sum_{i=1}^d \max_{u\in\mathbb{U}} |b^i(x,u)| + \max_{u\in\mathbb{U}} |c(x,u)| \leq \gamma(R)$$

for all $x, y \in B_R$.

The Dirichlet problem for quasi-linear equations is more involved than the linear case. Here we investigate existence of solutions to the problem

$$S\psi(x) = 0 \quad \text{in } D\,, \qquad \psi = 0 \quad \text{on } \partial D \tag{A.4}$$

for a sufficiently smooth bounded domain D. We can follow the approach in [17, section 11.2], which utilizes the Leray–Schauder fixed point theorem, to obtain the following result.

THEOREM A.15. *Let D be a bounded $\mathcal{C}^{2,1}$ domain in $\mathbb{R}^d$. Then the Dirichlet problem in* (A.4) *has a solution in $\mathcal{C}^{2,r}(\bar{D})$, $r \in (0,1)$, for any $S \in \mathfrak{Q}(\gamma)$.*

We conclude with a useful convergence result.

LEMMA A.16. *Let D be a bounded $\mathcal{C}^2$ domain. Suppose $\{\psi_n\} \subset \mathcal{W}^{2,p}(D)$ and $\{h_n\} \subset \mathcal{L}^p(D)$, $p > 1$, are a pair of sequences of functions satisfying the following:*

(i) *$S\psi_n = h_n$ in D for all $n \in \mathbb{N}$ for some $S \in \mathfrak{Q}(\gamma)$.*
(ii) *For some constant M, $\|\psi_n\|_{\mathcal{W}^{2,p}(D)} \le M$ for all $n \in \mathbb{N}$.*
(iii) *h_n converges in $\mathcal{L}^p(D)$ to some function h.*

Then there exist $\psi \in \mathcal{W}^{2,p}(D)$ and a sequence $\{n_k\} \subset \mathbb{N}$ such that $\psi_{n_k} \to \psi$ in $\mathcal{W}^{1,p}(D)$, as $k \to \infty$, and

$$S\psi = h \quad \text{in } D\,. \tag{A.5}$$

If in addition $p > d$, then $\psi_{n_k} \to \psi$ in $\mathcal{C}^{1,r}(D)$ for any $r < 1-\frac{d}{p}$. Also, if $h \in \mathcal{C}^{0,\rho}(D)$, then $\psi \in \mathcal{C}^{2,\rho}(D)$.

Proof. By the weak compactness of $\bigl\{\varphi : \|\varphi\|_{\mathcal{W}^{2,p}(D)} \le M\bigr\}$ and the compactness of the imbedding $\mathcal{W}^{2,p}(D) \hookrightarrow \mathcal{W}^{1,p}(D)$, we can select $\psi \in \mathcal{W}^{2,p}(D)$ and $\{n_k\}$ such that $\psi_{n_k} \to \psi$, weakly in $\mathcal{W}^{2,p}(D)$ and strongly in $\mathcal{W}^{1,p}(D)$, as $k \to \infty$. The inequality

$$\Bigl|\inf_{u\in\mathbb{U}}\ \hat{b}(x,u,\psi) - \inf_{u\in\mathbb{U}}\ \hat{b}(x,u,\psi')\Bigr| \le \sup_{u\in\mathbb{U}}\Bigl|\hat{b}(x,u,\psi) - \hat{b}(x,u,\psi')\Bigr| \tag{A.6}$$

shows that $\inf_{u\in\mathbb{U}}\ \hat{b}(\,\cdot\,,u,\psi_{n_k})$ converges in $\mathcal{L}^p(D)$. Since, by weak convergence,

$$\int_D g(x)\partial_{ij}\psi_{n_k}(x)\,\mathrm{d}x \xrightarrow[k\to\infty]{} \int_D g(x)\partial_{ij}\psi(x)\,\mathrm{d}x$$

for all $g \in \mathcal{L}^{\frac{p}{p-1}}(D)$, and $h_n \to h$ in $\mathcal{L}^p(D)$, we obtain

$$\int_D g(x)\bigl(S\psi(x) - h(x)\bigr)\,\mathrm{d}x = \lim_{k\to\infty}\int_D g(x)\bigl(S\psi_{n_k}(x) - h_{n_k}(x)\bigr)\,\mathrm{d}x = 0$$

for all $g \in \mathcal{L}^{\frac{p}{p-1}}(D)$. Thus the pair (ψ, h) satisfies (A.5).

If $p > d$, the compactness of the embedding $\mathcal{W}^{2,p}(D) \hookrightarrow \mathcal{C}^{1,r}(\bar{D})$, $r < 1-\frac{d}{p}$, allows us to select the subsequence such that $\psi_{n_k} \to \psi$ in $\mathcal{C}^{1,r}(\bar{D})$. The inequality (A.6) shows that $\inf_{u\in\mathbb{U}}\ \hat{b}(\,\cdot\,,u,\psi_{n_k})$ converges uniformly on D, while the inequality

$$\Bigl|\inf_{u\in\mathbb{U}}\ \hat{b}(x,u,\psi) - \inf_{u\in\mathbb{U}}\ \hat{b}(y,u,\psi)\Bigr| \le \sup_{u\in\mathbb{U}}\Bigl|\hat{b}(x,u,\psi) - \hat{b}(y,u,\psi)\Bigr| \tag{A.7}$$

implies that the limit belongs to $\mathcal{C}^{0,r}(D)$.

If $h \in \mathcal{C}^{0,\rho}(D)$, then $\psi \in \mathcal{W}^{2,p}(D)$ for all $p > 1$. Using the continuity of the embedding $\mathcal{W}^{2,p}(D) \hookrightarrow \mathcal{C}^{1,r}(\bar{D})$ for $r \le 1-\frac{d}{p}$, and (A.7), we conclude that $\inf_{u\in\mathbb{U}}\hat{b}(\,\cdot\,,u,\psi) \in \mathcal{C}^{0,r}$ for all $r < 1$. Thus ψ satisfies $a^{ij}\partial_{ij}\psi \in \mathcal{C}^{0,\rho}(D)$, and it follows from elliptic regularity [17, Theorem 9.19, p. 243] that $\psi \in \mathcal{C}^{2,\rho}(D)$. ∎

Remark A.17. If we replace $S \in \mathfrak{Q}(\gamma)$ with $L \in \mathfrak{L}(\gamma)$ in Lemma A.16, all the assertions of the lemma other than the last sentence follow. The proof is identical.

Appendix B. Proofs.

Proof of Lemma 4.1. Let h be the unique solution in $\mathcal{W}^{2,p}(D_2) \cap \mathcal{W}^{1,p}_0(D_2)$, $p \ge 2$, of $L^v h = -1$ in D_2 and $h = 0$ on ∂D_2. By Itô's formula,

$$h(x) = \mathbb{E}^v_x\bigl[\uptau(D_2)\bigr] \quad \forall x \in D_2\,.$$

The positive lower bound in (4.1a) follows from Theorem A.8, while the finite upper bound results from the weak maximum principle of Alexandroff, Theorem A.3. In order to prove (4.1b), we select an open ball $D_3 \Supset D_2$ and let φ be the solution to the Dirichlet problem $L^v\varphi = -1$ in the annulus $D_3 \setminus \bar{D}_1$ and $\varphi = 0$ on $\partial D_1 \cup \partial D_3$. By Theorem A.8,

$$\inf_{v\in\mathfrak{U}_{\mathrm{SM}}} \left(\inf_{x\in\partial D_2} \varphi(x)\right) > 0\,,$$

and the result follows since $\mathbb{E}^v_x\bigl[\uptau(D_1^c)\bigr] > \varphi(x)$.

Let $n \in \mathbb{N}$ be large enough so that $D_2 \Subset B_n$, and let g_n be the solution of the Dirichlet problem $L^v g_n = -1$ in the annulus $B_n \setminus \bar{D}_1$, satisfying $g_n = 0$ on $\partial B_n \cup \partial D_1$. If $x_0 \in \partial D_2$ and $v \in \mathfrak{U}_{\mathrm{SSM}}$, then $\mathbb{E}^v_{x_0}[\uptau(D_1^c)] < \infty$. Since

$$g_n(x_0) = \mathbb{E}^v_{x_0}\left[\uptau(D_1^c) \wedge \uptau(B_n)\right] \le \mathbb{E}^v_{x_0}\left[\uptau(D_1^c)\right]\,,$$

by Harnack's inequality [17, Corollary 9.25, p. 250], the increasing sequence of L^v-harmonic functions $f_n = g_n - g_1$ is bounded locally in D_1^c, and hence approaches a limit as $n \to \infty$, which is an L^v-harmonic function on D_1^c. Therefore, $g = \lim_{n\to\infty} g_n$ is a bounded function on ∂D_2, and by monotone convergence, $g(x) = \mathbb{E}^v_x\left[\uptau(D_1^c)\right]$. Property (4.1c) follows.

Turning to (4.1d), let $\varphi_v(x) \triangleq \mathbb{P}^v_x\bigl(\uptau(D_2) > \uptau(D_1^c)\bigr)$. It follows from Theorems A.6 and A.11(i) that $\{\varphi_v\,,\, v \in \mathfrak{U}_{\mathrm{SM}}\}$ is equicontinuous on $\bar{D}_2 \setminus D_1$. We argue by contradiction. If $\varphi_{v_n}(x_n) \to 0$, as $n \to \infty$, for a pair of sequences $\{v_n\} \subset \mathfrak{U}_{\mathrm{SM}}$ and $\{x_n\} \subset \mathit{\Gamma}$, then Harnack's inequality implies that $\varphi_{v_n} \to 0$ uniformly over any compact subset of $D_2 \setminus \bar{D}_1$. Since $\varphi_{v_n} = 1$ on ∂D_2, this contradicts the equicontinuity of $\{\varphi_{v_n}\}$ and proves the claim. ∎

Proof of Theorem 4.2. The strong Markov property implies that $\{\tilde{X}_n\}_{n\in\mathbb{N}}$ is a Markov chain. Let R be large enough such that $D_2 \Subset B_R$. With $h \in \mathcal{C}(\partial D_1)$, $h \ge 0$, let ψ be the unique solution in

$$\mathscr{W}^{2,p}_{\mathrm{loc}}\bigl(B_R \cap \bar{D}_1^c\bigr) \cap \mathcal{C}\bigl(\bar{B}_R \cap D_1^c\bigr)\,, \quad p > 1\,,$$

of the Dirichlet problem $L^v\psi = 0$ in the annulus $B_R \cap \bar{D}_1^c$, with $\psi = h$ on ∂D_1 and $\psi = 0$ on ∂B_R. Then, for each $x \in \partial D_2$, the map $h \mapsto \psi(x)$, which by Itô's formula satisfies $\psi(x) = \mathbb{E}^v_x\bigl[h(X_{\uptau(D_1^c)\wedge\uptau_R})\bigr]$, defines a continuous linear functional on $\mathcal{C}(\partial D_1)$. By the Riesz representation theorem there exists a probability measure $q^v_{1,R}(x,\cdot) \in \mathfrak{P}(\partial D_1)$ such that

$$\psi(x) = \int_{\partial D_1} q^v_{1,R}(x, \mathrm{d}y) h(y)\,.$$

It is evident that for any $A \in \mathfrak{B}(\partial D_1)$, $q^v_{1,R}(x,A) \uparrow q^v_1(x,A)$, as $R \to \infty$, and that $q^v_1(x,A) = \mathbb{P}^v_x(X_{\uptau(D_1^c)} \in A)$. Similarly, the analogous Dirichlet problem on D_2 yields $q^v_2(x,\cdot) \in \mathfrak{P}(\partial D_2)$, satisfying $q^v_2(x,A) = \mathbb{P}^v_x(X_{\hat{\uptau}_1} \in A)$, and by Harnack's inequality, there exists a positive constant C_H such that, for all $x\,, x' \in \partial D_1$, and $A \in \mathfrak{B}(\partial D_2)$,

$$q^v_2(x,A) \le C_H q^v_2(x',A)\,. \tag{B.1}$$

Hence, the transition kernel

$$\tilde{P}_v(x,\cdot) = \int_{\partial D_2} q^v_2(x,\mathrm{d}y) q^v_1(y,\cdot)$$

of $\tilde{X}$ inherits Harnack's inequality in (B.1). Therefore for any fixed $x_0 \in \partial D_1$ we have $\tilde{P}_v(x,\cdot) \geq C_H^{-1}\tilde{P}_v(x_0,\cdot)$ for all $x \in \partial D_1$, which implies that $\tilde{P}_v$ is a contraction under the total variation norm and satisfies

$$\left\|\int_{\partial D_1} \bigl(\nu(\mathrm{d}x) - \nu'(\mathrm{d}x)\bigr)\tilde{P}_v(x,\cdot)\right\|_{\mathrm{TV}} \leq \bigl(1 - C_H^{-1}\bigr)\,\|\nu - \nu'\|_{\mathrm{TV}}$$

for all ν and ν' in $\mathfrak{P}(\partial D_1)$. Thus (4.2) holds with $\delta = \bigl(1 - C_H^{-1}\bigr)$. Since the fixed point of the contraction $\tilde{P}_v$ is unique, the chain is ergodic. This completes the proof of (i).

For part (ii) we first show that the maps $v \mapsto q_k^v$, $k = 1,2$, are continuous uniformly on ∂D_2 and ∂D_1, respectively. Indeed, as described above,

$$\varphi_v(x) = \int_{\partial D_1} q_1^v(x,\mathrm{d}y)h(y)\,, \quad v \in \mathfrak{U}_{\mathrm{SSM}}\,, \quad h \in \mathcal{C}(\partial D_1)\,,$$

is the unique bounded solution of the Dirichlet problem $L^v\varphi_v = 0$ in $\bar{D}_1^c$, and $\varphi_v = h$ on ∂D_1. Suppose $v_n \to v$ in $\mathfrak{U}_{\mathrm{SSM}}$ as $n \to \infty$. If G is a bounded $\mathcal{C}^2$ domain such that $\partial D_2 \subset G \Subset \bar{D}_1^c$, then by Lemma A.5, every subsequence of $\{\varphi_{v_n}\}$ contains a further subsequence also denoted as $\{\varphi_n\}$, which converges weakly in $\mathcal{W}^{2,p}(G)$, $p > 1$, to some L^v-harmonic function. Since G is arbitrary, we apply Lemma 3.5 to obtain a function $\tilde{\varphi} \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\bar{D}^c)$ that satisfies $L^v\tilde{\varphi} = 0$ on $\bar{D}_1^c$ and $\tilde{\varphi} = h$ on ∂D_1. By uniqueness $\tilde{\varphi} = \varphi_v$. Since the convergence is uniform on compact sets, we have

$$\sup_{\partial D_2} |\varphi_{v_n} - \varphi_v| \to 0\,.$$

In other words, $v \mapsto q_1^v$ is continuous uniformly on ∂D_2. Similarly, $v \mapsto q_2^v$ is continuous uniformly on ∂D_1. Thus their composition $v \mapsto \tilde{P}_v(x,\cdot)$ is continuous uniformly on $x \in \partial D_1$. Let $\{v_n\} \subset \mathfrak{U}_{\mathrm{SSM}}$ be any sequence converging to $v \in \mathfrak{U}_{\mathrm{SSM}}$ as $n \to \infty$. Since $\mathfrak{P}(\partial D_1)$ is compact, there exists a further subsequence also denoted as $\{v_n\}$ along which $\tilde{\mu}_{v_n} \to \tilde{\mu} \in \mathfrak{P}(\partial D_1)$. Hence, by the uniform convergence of

$$\int_{\partial D_1} \tilde{P}_{v_n}(x,\mathrm{d}y)f(y) \xrightarrow[n\to\infty]{} \int_{\partial D_1} \tilde{P}_v(x,\mathrm{d}y)f(y)$$

for any $f \in \mathcal{C}(\partial D_1)$, we obtain

$$\tilde{\mu}(\cdot) = \lim_{n\to\infty} \tilde{\mu}_{v_n}(\cdot) = \lim_{n\to\infty} \int_{\partial D_1} \tilde{\mu}_{v_n}(\mathrm{d}x)\tilde{P}_{v_n}(x,\cdot) = \int_{\partial D_1} \tilde{\mu}(\mathrm{d}x)\tilde{P}_v(x,\cdot)\,,$$

and by uniqueness $\tilde{\mu} = \tilde{\mu}_v$. Thus $v \mapsto \tilde{\mu}_v$ from $\mathfrak{U}_{\mathrm{SSM}}$ to $\mathfrak{P}(\partial D_1)$ is continuous.

Part (iii) is standard [19]. ◻

Proof of Lemma 4.4. Let $\{v_n\}$ be a sequence in $\mathcal{U}$ which converges (under the topology of Markov controls) to $v^* \in \bar{\mathcal{U}}$. Then by Lemma 3.4, for all $h \in \mathcal{C}_b(\mathbb{R}^d \times \mathbb{U})$ and $g \in \mathcal{L}^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d\times\mathbb{U}} g(x)\bigl(h_{v_n}(x) - h_{v^*}(x)\bigr)\,\mathrm{d}x \xrightarrow[n\to\infty]{} 0\,. \tag{B.2}$$

By the tightness assumption and Prohorov's theorem, $\mathcal{I}_{\mathcal{U}}$ is relatively compact in $\mathfrak{P}(\mathbb{R}^d)$, and thus $\{\mu_{v_n}\}$ has a limit point $\mu^* \in \mathfrak{P}(\mathbb{R}^d)$. Passing to a subsequence converging to this limit, which we also denote by $\{\mu_{v_n}\}$, and since by (4.6) and (4.7) the associated densities $\{\varphi_n\}$ are equibounded and Hölder equicontinuous on bounded subdomains of $\mathbb{R}^d$, it follows that $\{\varphi_n\}$ contains a subsequence (also denoted by $\{\varphi_n\}$) which converges to $\varphi^* \in \mathcal{C}(\mathbb{R}^d)$. Moreover, since $\mathcal{I}_{\mathcal{U}}$ is tight, $\{\varphi_n\}$ is uniformly integrable. It follows that $\{\varphi_n\}$ converges in $\mathcal{L}^1(\mathbb{R}^d)$ as well. Therefore $\int \varphi^* = 1$, $\varphi^* \geq 0$, and for $f \in \mathcal{C}_b(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} f(x)\varphi_n(x)\,\mathrm{d}x \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^d} f(x)\varphi^*(x)\,\mathrm{d}x\,.$$

This implies $\mu_{v_n} \to \mu^*$ in $\mathfrak{P}(\mathbb{R}^d)$ and, by Scheffé's theorem [5, p. 214], also in total variation. For $h \in \mathcal{C}_b(\mathbb{R}^d \times \mathbb{U})$, using the notation in (3.10), we form the triangle inequality

$$\text{(B.3)}\qquad \left|\int_{\mathbb{R}^d} h_{v_n}\,\mathrm{d}\mu_{v_n} - \int_{\mathbb{R}^d} h_{v^*}\,\mathrm{d}\mu^*\right| \leq \left|\int_{\mathbb{R}^d} h_{v^*}(x)\big(\varphi_n(x) - \varphi^*(x)\big)\,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^d} \big(h_{v_n}(x) - h_{v^*}(x)\big)\varphi^*(x)\,\mathrm{d}x\right|.$$

Since $\varphi_n \to \varphi^*$ in $\mathcal{L}^1(\mathbb{R}^d)$, the first term on the right-hand side of (B.3) converges to zero, as $n \to \infty$, and so does the second term by (B.2). Hence, by (B.3) and Lemma 3.4,

$$0 = \int_{\mathbb{R}^d} L^{v_n} f(x)\mu_{v_n}(\mathrm{d}x) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^d} L^{v^*} f(x)\mu^*(\mathrm{d}x) \quad \forall f \in \mathcal{C}_c^2(\mathbb{R}^d)\,,$$

implying, by Theorem 4.3, that $\mu^* = \mu_{v^*} \in \mathcal{I}_{\bar{\mathcal{U}}}$. This establishes (i). Since

$$\int_{\mathbb{R}^d} h_v(x)\mu_v(\mathrm{d}x) = \int_{\mathbb{R}^d\times\mathbb{U}} h(x,u)\pi_v(\mathrm{d}x,\mathrm{d}u)\,,$$

(B.3) also implies (ii). $\square$

Proof of Lemma 5.3. Let $\hat{\tau} \triangleq \tau(D^c)$ and

$$Z_t \triangleq \int_0^t \mathbb{I}_G(X_s)\,\mathrm{d}s\,, \quad t \geq 0\,.$$

Select $R' > R > 0$ such that $D \cup G \subset B_R$. Using the strong Markov property, and since $\mathbb{I}_G = 0$ on B_R^c,

$$\text{(B.4)}\qquad \begin{aligned} &\mathbb{E}_x^v\big[Z_{\hat{\tau}}\big] \leq \sup_{x'\in\partial B_R} \mathbb{E}_{x'}^v\big[Z_{\hat{\tau}}\big] \quad \forall x \in B_R^c,\\ &\mathbb{E}_x^v\big[Z_{\hat{\tau}}\big] \leq \mathbb{E}_x^v\big[Z_{\hat{\tau}\wedge\tau_R}\big] + \sup_{x'\in\partial B_R} \mathbb{E}_{x'}^v\big[Z_{\hat{\tau}}\big] \quad \forall x \in B_R \cap D^c\,. \end{aligned}$$

By (4.1a),

$$\text{(B.5)}\qquad \sup_{v\in\mathfrak{U}_{\mathrm{SM}}}\ \sup_{x\in B_R\cap D^c} \mathbb{E}_x^v\big[Z_{\hat{\tau}\wedge\tau_R}\big] \leq \sup_{v\in\mathfrak{U}_{\mathrm{SM}}}\ \sup_{x\in B_R\cap D^c} \mathbb{E}_x^v\big[\hat{\tau}\wedge\tau_R\big] < \infty.$$

By (B.4) and (B.5), it suffices to exhibit a uniform bound for $\mathbb{E}_x^v\big[Z_{\hat{\tau}}\big]$ on ∂B_R. By (4.1d), for some constant $\beta < 1$,

$$\sup_{v\in\mathfrak{U}_{\mathrm{SM}}}\ \sup_{x\in\partial B_R} \mathbb{P}_x^v(\hat{\tau} \geq \tau_{R'}) < \beta\,.$$

Set $\hat{\tau}(t) = \hat{\tau} \wedge t$. By conditioning first at $\tau_{R'}$, and using the fact that $\mathbb{I}_G = 0$ on B_R^c, we obtain, for $x \in \partial B_R$,

$$\begin{aligned}
\text{(B.6)} \qquad \mathbb{E}_x^v\left[Z_{\hat{\tau}(t)}\right] &= \mathbb{E}_x^v\left[Z_{\hat{\tau}(t)}\mathbb{I}(\hat{\tau}(t) < \tau_{R'})\right] + \mathbb{E}_x^v\left[Z_{\hat{\tau}(t)}\mathbb{I}(\hat{\tau}(t) \geq \tau_{R'})\right] \\
&\leq \mathbb{E}_x^v\left[Z_{\hat{\tau}\wedge\tau_{R'}}\right] + \left(\sup_{x\in\partial B_R} \mathbb{P}_x^v(\hat{\tau}(t) \geq \tau_{R'})\right)\left(\sup_{x\in\partial B_{R'}} \mathbb{E}_x^v\left[Z_{\hat{\tau}(t)}\right]\right) \\
&\leq \mathbb{E}_x^v\left[Z_{\hat{\tau}\wedge\tau_{R'}}\right] + \beta \sup_{x\in\partial B_R} \mathbb{E}_x^v\left[Z_{\hat{\tau}(t)}\right].
\end{aligned}$$

By (B.5) and (B.6), for all $v \in \mathfrak{U}_{\mathrm{SM}}$,

$$\text{(B.7)} \qquad \sup_{x\in\partial B_R} \mathbb{E}_x^v\left[Z_{\hat{\tau}(t)}\right] \leq (1-\beta)^{-1} \sup_{v\in\mathfrak{U}_{\mathrm{SM}}} \sup_{x\in\partial B_R} \mathbb{E}_x^v\left[Z_{\hat{\tau}\wedge\tau_{R'}}\right] < \infty\,.$$

Taking limits as $t \to \infty$ in (B.7), using monotone convergence, (i) follows.

Next we prove (ii). With $R > 0$ such that $B_R \Supset D$, let φ_R be the unique solution in $\mathcal{W}^{2,p}_{\mathrm{loc}}(B_R \cap \bar{D}^c) \cap \mathcal{C}(\bar{B}_R \cap D^c)$, $p > 1$, of the Dirichlet problem $L^v\varphi_R = -\mathbb{I}_G$ in the annulus $B_R \cap \bar{D}^c$, satisfying $\varphi_R = 0$ on the boundary. By Itô's formula, φ_R is dominated by $\xi^v_{D,G}$, and since it is nondecreasing in R, it converges uniformly over compact subsets of $\mathbb{R}^d$ as $R \uparrow \infty$ to some $\varphi \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\bar{D}^c) \cap \mathcal{C}_b(D^c)$, which solves

$$\text{(B.8)} \qquad L^v\varphi = -\mathbb{I}_G \quad \text{in } \bar{D}^c\,, \quad \varphi = 0 \quad \text{on } \partial D\,.$$

Since by hypothesis $\mathbb{P}^v(\tau(D^c) < \infty) = 1$, applying Itô's formula, we obtain $\varphi = \xi^v_{D,G}$. Hence $\xi^v_{D,G}$ is a bounded solution of (B.8). Suppose φ' is another bounded solution of (B.8). Then $\varphi - \varphi'$ is L^v-harmonic in $\bar{D}^c$ and equals zero on ∂D. However, it is well known that the process X governed by $v \in \mathfrak{U}_{\mathrm{SM}}$ is D-recurrent if and only if the Dirichlet problem $L^v\psi = 0$ in $\bar{D}^c$, $\psi = f$ on ∂D has a unique bounded solution ψ_f for all $f \in \mathcal{C}(\partial D)$ [19, Theorem 7.2, p. 100]. Hence $\varphi - \varphi'$ must be identically zero on D^c. Uniqueness follows.

To show (iii), let $v_n \to v$ in $\mathcal{U}$. By Lemmas A.5 and 3.5, every subsequence of $\{\xi^{v_n}_{D,G}\}$ contains a further subsequence converging weakly in $\mathcal{W}^{2,p}(D')$, $p > 1$, over any bounded domain $D' \Subset \bar{D}^c$ to some ψ satisfying $L^v\psi = -\mathbb{I}_G$ in D^c. By uniqueness of the solution to the Dirichlet problem this limit must be $\xi^v_{D,G}$, and since convergence is uniform over compact sets of $\bar{D}^c$, continuity of $(v, x) \mapsto \xi^v_{D,G}(x)$ follows. $\square$

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