

## ASPIRATION LEARNING IN COORDINATION GAMES\*

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**Abstract.** We consider the problem of distributed convergence to efficient outcomes in coordination games through dynamics based on aspiration learning. Our first contribution is the characterization of the asymptotic behavior of the induced Markov chain of the iterated process in terms of an equivalent finite-state Markov chain. We then characterize explicitly the behavior of the proposed aspiration learning in a generalized version of coordination games, examples of which include network formation and common-pool games. In particular, we show that in generic coordination games the frequency at which an efficient action profile is played can be made arbitrarily large. Although convergence to efficient outcomes is desirable, in several coordination games, such as common-pool games, attainability of fair outcomes, i.e., sequences of plays at which players experience highly rewarding returns with the same frequency, might also be of special interest. To this end, we demonstrate through analysis and simulations that aspiration learning also establishes fair outcomes in all symmetric coordination games, including common-pool games.

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**1. Introduction.** Distributed coordination is of particular interest in many engineering systems. Two examples are distributed overlay routing or network formation [7] and medium access control [12] in wireless communications. In either case, nodes need to utilize their resources *efficiently* so that a desirable global objective is achieved. For example, in network formation, nodes need to choose their immediate links so that connectivity is achieved with a minimum possible communication cost, i.e., minimum number of links. Similarly, in medium access control, users need to establish a *fair* scheduling of accessing a shared communication channel so that collisions (i.e., situations at which two or more users access the common resource) are avoided. In these scenarios, achieving *coordination* in a distributed and adaptive fashion to an *efficient* outcome is of special interest.

The distributed yet coupled nature of these problems, combined with a desire for online adaptation, motivates using models based on game theoretic learning [9, 24, 30]. In game theoretic learning, each agent is endowed with a set of actions and a utility/reward function that depends on that agent's and other agents' actions. Agents then learn which action to play based only on their own previous experience

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of the game (actions played and utilities received). A major challenge in this setting is that explicit utility function optimization may be impractical. This may be due to inherent complexity (e.g., a large number of players or actions), or the lack of any closed form expression for the utility function. Rather, rewards can be measured online. In terms of game theoretic learning, this eliminates adaptation based on an ability to compute a “best reply.” Another obstacle to utility maximization is that from any agent’s perspective, the environment includes other adapting agents, and hence is nonstationary. Consequently, actions that may have been effective in the past need not continue to be effective.

Motivated by these issues, this paper considers a form of distributed learning dynamics known as *aspiration learning*, where agents “satisfice” rather than “optimize.” The aspiration learning scheme is based on a simple principle of “win-stay, lose-shift” [23], according to which a successful action is repeated while an unsuccessful action is dropped. The success of an action is determined by a simple comparison test of its performance with the player’s desirable return (*aspiration level*). The aspiration level is updated to incorporate prior experience into the agent’s success criterion. Through this learning scheme agents *learn* to play their “best” action.

The history of aspiration learning schemes starts with the pioneering work of [26], where satisfaction seeking behavior was used to explain social decision making. A simple aspiration learning model is presented in [23], where games of two players and two actions are considered, and decisions are taken based on the “win-stay, lose-shift” rule. In the special case of two-player/two-action *mutual interest games* and *symmetric coordination games*, respectively, [22] and [15] show that the payoff-dominant action profile is selected with probability close to one. Similar are the results in [6, 14]. However, contrary to [22] and [15], both models incorporate a small perturbation in either the aspiration update [14] or the action update [6].

Recent research efforts on equilibrium selection in games have focused on achieving distributed convergence to *Pareto-efficient* payoff profiles, i.e., payoff profiles at which no action change can make a player better off while not making some other player worse off. For example, [18] introduced an aspiration learning algorithm that converges (in distribution) to action profiles that maximize social welfare in multiple player games. Some key characteristics of this algorithm is that agents keep track of their most recent satisfactory action and satisfactory payoff (benchmark action and payoff), and they update their actions by following a “win-stay lose-shift” rule, where the aspiration level is defined as the benchmark payoff. Convergence to the Pareto-efficient payoffs in two player games has also been investigated by [2]. The learning algorithm considered in [2] has two distinctive features: (a) agents commit on playing a series of actions for a  $k$ -period interval, and (b) agents make decisions according to a win-stay lose-shift rule, where aspiration levels are computed as the running average payoff over all the previous  $k$ -period intervals. It is shown that, in two player games, the agents’ payoffs converge to a small neighborhood of the set of the Pareto-efficient payoffs almost surely if  $k$  is sufficiently large.

In this paper, we also focus on achieving convergence to efficient payoff profiles (also part of the Pareto-efficient payoff profiles) in coordination games of large numbers of players and actions. Agents apply an aspiration learning scheme that is motivated by [14]. Our goal is to (a) characterize explicitly the asymptotic behavior of the process for generic games of multiple players and actions, and (b) derive conditions under which *efficient* payoffs are selected in large coordination games. Our main contribution is the characterization of the asymptotic behavior of the induced Markov chain by means of the invariant distributions of an equivalent finite-state

Markov chain, whenever the experimentation probability becomes sufficiently small. This equivalence simplifies the analysis of what would otherwise be an infinite state Markov process. These results extend prior analysis on this type of aspiration learning scheme to games of multiple players and actions. We also specialize the results for a class of games that is a generalized version of so-called *coordination games*. In particular, we show that, in these games, the unique invariant distribution of the equivalent finite-state Markov chain puts arbitrarily large weight on the *payoff-dominant* action profiles if the step size of the aspiration-level update becomes sufficiently small. We finally demonstrate the utility of the learning scheme to network formation games, which is of independent interest since prior learning schemes on network formation are primarily based on best-response dynamics, e.g., [3]. The above contributions generalize prior work of the same authors [5], which was restricted to a smaller family of coordination games.

While convergence to payoff-dominant action profiles in coordination games is desirable, another desirable property is a notion of fairness. In particular, for some coordination games where coincidence of interests is not so strong, such as the *Battle of the Sexes* (cf. [21, section 2.3]), convergence to a single action profile might not be *fair* for all agents that would probably rather be in a different action profile. Instead, an alternation between several action profiles might be more desirable, usually described through distributions in the *joint* action space. An example of a class of such coordination games is so-called common-pool games, where multiple users need to coordinate on utilizing a limited common resource. The proposed aspiration learning algorithm also may provide a distributed and adaptive approach for convergence to fair outcomes in such symmetric coordination games, such as common-pool games. This property is of independent interest, since it is relevant to several scenarios of distributed resource allocation, such as medium access control in wireless communications [12].

In comparison to prior and other current work, this paper develops (and corrects) the specific model of aspiration learning in [14] beyond two player games. This paper goes on to derive specialized results for coordination games involving convergence to efficient action profiles and fairness in symmetric games. The results in [18] use a simpler finite state model of aspiration learning and are applicable to almost all games. The results in [18] establish convergence to efficient action profiles, but as yet do not specify selection/fairness among these action profiles. The model of [2] is more closely related to the present model, but with a different definition of aspiration levels and a different mechanism to perturb aspirations. The results of convergence to efficiency in [2] extend beyond coordination games while requiring two player games and do not specify fairness/selection among efficient profiles.

The remainder of this paper is organized as follows. Section 2 defines coordination games and presents two special cases of coordination games, namely network formation and common-pool games. Section 3 presents the aspiration learning algorithm and its convergence properties in games of multiple players and actions. Section 4 specializes the convergence analysis to coordination games and establishes convergence to efficient outcomes. It also demonstrates the results through simulations in network formation games. Section 5 extends the convergence analysis to symmetric coordination games and establishes conditions under which convergence to fair outcomes is also established.

*Terminology.* We consider the standard setup of finite strategic-form games. There is a finite set of *agents* or *players*,  $\mathcal{I} = \{1, 2, \dots, n\}$ , and each agent has a finite set of actions, denoted by  $\mathcal{A}_i$ . The set of action profiles is the Cartesian product

$\mathcal{A} \triangleq \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ ;  $\alpha_i \in \mathcal{A}_i$  denotes an *action* of agent  $i$ ; and  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{A}$  denotes the *action profile* or *joint action* of all agents. The *payoff/utility function* of player  $i$  is a mapping  $u_i : \mathcal{A} \rightarrow \mathbb{R}$ . A strategic-form game, denoted  $\mathcal{G}$ , consists of the sets  $\mathcal{I}$ ,  $\mathcal{A}$  and the preference relation induced by the utility functions  $u_i$ ,  $i \in \mathcal{I}$ . An action profile  $\alpha^* \in \mathcal{A}$  is a (*pure*) *Nash equilibrium* if

$$(1.1) \quad u_i(\alpha_i^*, \alpha_{-i}^*) \geq u_i(\alpha'_i, \alpha_{-i}^*)$$

for all  $i \in \mathcal{I}$  and  $\alpha'_i \in \mathcal{A}_i$ , where  $-i$  denotes the complementary set  $\mathcal{I} \setminus \{i\}$ . We denote the set of pure Nash equilibria by  $\mathcal{A}^*$ . In case the inequality (1.1) is strict, the Nash equilibrium is called a *strict Nash equilibrium*. For the remainder of this paper, the term “Nash equilibrium” always refers to a “pure Nash equilibrium.”

## 2. Coordination games.

**2.1. Definitions.** Before defining coordination games, we first need to define the notion of *better reply*.

**DEFINITION 2.1** (better reply). *The better reply of agent  $i \in \mathcal{I}$  to an action profile  $\alpha = (\alpha_i, \alpha_{-i}) \in \mathcal{A}$  is a set valued map  $\text{BR}_i : \mathcal{A} \rightarrow 2^{\mathcal{A}_i}$  such that for any  $\alpha_i^* \in \text{BR}_i(\alpha)$  we have  $u_i(\alpha_i^*, \alpha_{-i}) > u_i(\alpha_i, \alpha_{-i})$ .*

A *coordination game* is defined as follows.

**DEFINITION 2.2** (coordination game). *A game of two or more agents is a coordination game if there exists  $\bar{\mathcal{A}} \subset \mathcal{A}$  such that the following conditions are satisfied:*

- (a) *for any  $\bar{\alpha} \in \bar{\mathcal{A}}$  and  $\alpha \notin \bar{\mathcal{A}}$ ,*

$$(2.1) \quad u_i(\bar{\alpha}) \geq u_i(\alpha) \quad \forall i \in \mathcal{I},$$

*i.e.,  $\bar{\mathcal{A}}$  payoff-dominates  $\mathcal{A} \setminus \bar{\mathcal{A}}$ ;*

- (b) *for any  $\alpha \in \mathcal{A} \setminus (\mathcal{A}^* \cup \bar{\mathcal{A}})$ , there exist  $i \in \mathcal{I}$  and action  $\alpha'_i \in \text{BR}_i(\alpha)$  such that*

$$(2.2) \quad u_j(\alpha'_i, \alpha_{-i}) \geq u_j(\alpha_i, \alpha_{-i}) \quad \forall j \neq i;$$

- (c) *for any  $\alpha^* \in \mathcal{A}^* \setminus \bar{\mathcal{A}}$  (if nonempty), there exist an action profile  $\tilde{\alpha} \in \mathcal{A}$  and a sequence of distinct agents  $j_1, \dots, j_{n-1} \in \mathcal{I}$ , such that*

$$u_i \left( \tilde{\alpha}_{j_1}, \dots, \tilde{\alpha}_{j_\ell}, \alpha_{-\{j_1, \dots, j_\ell\}}^* \right) < u_i(\alpha^*)$$

*for all  $i \in \{j_1, j_2, \dots, j_{\ell+1}\}$ ,  $\ell = 1, 2, \dots, n-1$ .*

A *strict coordination game* refers to a coordination game with the inequality (2.1) being strict.

The conditions of a coordination game establish a weak form of “coincidence of interests” and define a larger class of games than the ones traditionally considered as coordination games, e.g., [17, 28]. For example, according to [17], one of the conditions that a coordination game needs to satisfy is that payoff differences among players at any action profile are much smaller than payoff differences among different action profiles. This condition reflects a form of coincidence of interests. Definition 2.2(b) also establishes a similar form of coincidence of interests, but weaker in the sense that it holds for at least one direction of action change.

Note also that existence of Nash equilibria is not necessary for a game to be a coordination game. Thus, this definition results in a larger family of coordination games than the one introduced in earlier work [5]. Furthermore, if  $\mathcal{A}^* \subset \bar{\mathcal{A}}$ , then Definition 2.2 can be written solely with respect to the desirable set of profiles  $\bar{\mathcal{A}}$ . In that case, Definition 2.2(c) becomes vacuous since  $\mathcal{A}^* \setminus \bar{\mathcal{A}} = \emptyset$ .

TABLE 2.1  
The Stag-Hunt game.

	A	B
A	4, 4	0, 2
B	2, 0	3, 3

A trivial example of a coordination game is the Stag-Hunt game of Table 2.1.

First, there exists a payoff-dominant profile, namely  $(A, A)$ , that can be identified as the desirable set  $\bar{\mathcal{A}}$ , and satisfies Definition 2.2(a). Also, from any action profile outside  $\mathcal{A}^* \cup \bar{\mathcal{A}}$ , namely  $(A, B)$  or  $(B, A)$ , there is a better reply that improves the payoff for all agents (i.e., Definition 2.2(b) holds). Last, for any Nash equilibrium profile outside  $\bar{\mathcal{A}}$ , i.e.,  $(B, B)$ , there is a player (row or column) and an action which makes everyone worse off (i.e., Definition 2.2(c) holds). Thus, the Stag-Hunt game satisfies all of the conditions of Definition 2.2.

Note, finally, that in some games there might be multiple choices for the selection of the desirable set  $\bar{\mathcal{A}}$ . For example, in the Stag-Hunt game of Table 2.1, an alternative selection of  $\bar{\mathcal{A}}$  corresponds to the union of the action profiles  $(A, A)$  and  $(B, B)$ . In that case, both properties (a) and (b) of Definition 2.2 hold, while property (c) is vacuous. In other words, the Stag-Hunt game is also a coordination game with respect to the new selection of the desirable set  $\bar{\mathcal{A}}$ .

CLAIM 2.1. *In any coordination game and for any action profile  $\alpha \notin \mathcal{A}^* \cup \bar{\mathcal{A}}$  there exists a sequence of action profiles  $\{\alpha^k\}$ , such that  $\alpha^0 = \alpha$  and  $\alpha_i^k \in \text{BR}_i(\alpha^{k-1})$  for some  $i$ , terminates at an action profile in  $\mathcal{A}^* \cup \bar{\mathcal{A}}$ .*

*Proof.* By Definition 2.2(b) there exists an agent  $i \in \mathcal{I}$  and an action  $\alpha_i^1 \in \text{BR}_i(\alpha^0)$ , such that  $u_i(\alpha_i^1, \alpha_{-i}^0) > u_i(\alpha_i^0, \alpha_{-i}^0)$  and  $u_s(\alpha_i^1, \alpha_{-i}^0) \geq u_s(\alpha_i^0, \alpha_{-i}^0)$  for all  $s \neq i$ . Define  $\alpha^1 \triangleq (\alpha_i^1, \alpha_{-i}^0)$ . Unless  $\alpha^1 \in \mathcal{A}^* \cup \bar{\mathcal{A}}$ , we can repeat the same argument to generate an action profile  $\alpha^2$  and so on. Thus, we construct a sequence  $(\alpha^0, \alpha^1, \alpha^2, \dots)$  along which the map  $\alpha \mapsto \sum_{i \in \mathcal{I}} u_i(\alpha)$  is strictly monotone. However, since  $\mathcal{A}$  is finite, the sequence must necessarily terminate at some  $\alpha^k \in \mathcal{A}^* \cup \bar{\mathcal{A}}$  for  $k < |\mathcal{A}|$ .  $\square$

Note that when  $\bar{\mathcal{A}} \subseteq \mathcal{A}^*$ , then a direct consequence of Claim 2.1 is that coordination games are weakly acyclic games (cf. [30]).

**2.2. Network formation games.** Network formation games are of particular interest in wireless communications due to their utility in modeling distributed topology control [25] and overlay routing [7]. Recent developments in distributed learning dynamics, e.g., [4], have also provided the tools for computing efficient solutions for these games in a distributed manner.

To illustrate how a network formation game can be modeled as a coordination game, we introduce a simple network formation game motivated by [13]. Let us consider  $n$  nodes deployed on the plane and assume that the set of actions of each agent  $i$ ,  $\mathcal{A}_i$ , contains all possible combinations of neighbors of  $i$ , denoted  $\mathcal{N}_i$ , with which a link can be established, i.e.,  $\mathcal{A}_i = 2^{\mathcal{N}_i}$ . Links are considered unidirectional, and a link established by node  $i$  with node  $s$ , denoted  $(s, i)$ , starts at  $s$  with the arrowhead pointing to  $i$ . A graph  $G$  is defined as a collection of nodes and directed links. Define also a path from  $s$  to  $i$  as a sequence of nodes and directed links that starts at  $s$  and ends to  $i$  following the orientation of the graph, i.e.,

$$(s \rightarrow i) = \{s = s_0, (s_0, s_1), s_1, \dots, (s_{m-1}, s_m), s_m = i\}$$

for some positive integer  $m$ . In a *connected* graph, there is a path from any node to any other node.

Let us consider the utility function  $u_i : \mathcal{A} \rightarrow \mathbb{R}$ ,  $i \in \mathcal{I}$ , defined by

$$(2.3) \quad u_i(\alpha) \triangleq \sum_{s \in \mathcal{I} \setminus \{i\}} \chi_\alpha(s \rightarrow i) - \nu |\alpha_i|,$$

where  $|\alpha_i|$  denotes the number of links corresponding to  $\alpha_i$  and  $\nu$  is a constant in  $(0, 1)$ . Also,

$$\chi_\alpha(s \rightarrow i) \triangleq \begin{cases} 1 & \text{if } (s \rightarrow i) \subseteq G_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where  $G_\alpha$  denotes the graph induced by joint action  $\alpha$ . The resulting Nash equilibria are usually called *Nash networks* [3]. As was shown in Proposition 4.2 in [4], a network  $G^*$  is a Nash network if and only if it is *critically connected*, i.e., (i) it is connected, and (ii) for any  $(s, i) \in G$ ,  $(s \rightarrow i)$  is the unique path from  $s$  to  $i$ . For example, the resulting Nash networks for  $n = 3$  agents and unconstrained neighborhoods are shown in Figure 2.1.

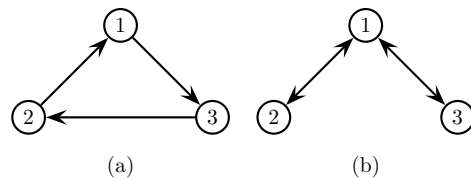


FIG. 2.1. *Nash networks in the case of  $n = 3$  agents and  $0 < \nu < 1$ .*

Let us define  $\bar{\mathcal{A}}$  to be the set of action profiles

$$\bar{\mathcal{A}} \triangleq \{\alpha^* \in \mathcal{A} : u_i(\alpha^*) = \max_{\alpha \in \mathcal{A}} u_i(\alpha) \forall i \in \mathcal{I}\},$$

which corresponds to the set of *payoff-dominant* networks. Note that payoff-dominant networks (if they exist) are connected with minimum number of links. Also, *not* all Nash networks are necessarily payoff-dominant. For example, in Figure 2.1(a), assuming that  $0 < \nu < 1$ , all players realize the same utility, which is equal to  $2 - \nu$ . This is a strict Nash network since each agent can only be worse off by unilaterally changing its links. It is also the payoff-dominant network. On the other hand, Figure 2.1(b) is a nonstrict Nash network and is payoff-dominated by Figure 2.1(a).

The utility function (2.3) corresponds to the *connections model* of [13] and has been used to describe various economic and social contexts such as transmission of information. It has also been applied for distributed topology control in wireless networks [16]. Practically, it constitutes a measure of network connectivity, since the maximum utility for node  $i$  is achieved when there is a path from any other node to  $i$ .

**CLAIM 2.2.** *The network formation game defined by (2.3) is a coordination game, provided the set of payoff-dominant networks is nonempty.*

*Proof.* For a joint action  $\alpha \notin \mathcal{A}^*$ , suppose that an agent  $i$  picks the *best reply* in  $\text{BR}_i(\alpha) \neq \emptyset$  (i.e., the most profitable better reply). Then no other agent becomes worse off, since a best reply for  $i$  always retains connectivity. Note that this is not

necessarily true for any other better reply. Thus, Definition 2.2(b) is satisfied. In order to show property (c), consider any joint action  $\alpha$  that is a Nash network. If any one agent  $j_1$  selects the action  $\tilde{\alpha}_{j_1}$  of establishing “no links,” then there exists at least one other agent  $j_2 \neq j_1$  whose payoff becomes strictly less than the equilibrium payoff (e.g., pick  $j_2$  such that  $(j_1, j_2) \in G_\alpha$ ). This is due to the fact that  $\alpha$  is critically connected. Continue in the same manner by selecting  $\tilde{\alpha}_{j_2}$  to be the action of establishing “no links,” and so on. This way, we may construct a sequence of agents and an action profile which satisfies Definition 2.2(c) of a coordination game.  $\square$

The condition that payoff-dominant networks exist is not restrictive. For example, if  $\mathcal{N}_i = \mathcal{I} \setminus \{i\}$  for all  $i$ , then the set of *wheel networks* (cf. [4]) is payoff dominant.

In a forthcoming section, we present a distributed optimization approach for achieving convergence to payoff-dominant networks through aspiration learning, which is of independent interest.

**2.3. Common-pool games.** Common-pool games refer to strategic interactions where two or more agents need to decide unilaterally whether or not to utilize a limited common resource. In such interactions, each agent would rather use the common resource by itself than share it with another agent, which is usually penalizing for both.

We define common-pool games as follows.

DEFINITION 2.3 (common-pool game). *A common-pool game is a strategic-form game such that for each agent  $i \in \mathcal{I}$ ,  $\mathcal{A}_i = \{p_0, p_1, \dots, p_{m-1}\}$ , with  $0 \leq p_0 < p_1 < \dots < p_{m-1}$ , and*

$$u_i(\alpha) \triangleq \begin{cases} 1 - c_j & \text{if } \alpha_i = p_j \text{ and } \alpha_i > \max_{\ell \neq i} \alpha_\ell, \\ -c_j + \tau_j & \text{if } \alpha_i = p_j \text{ and } \exists s \in \mathcal{I} \setminus \{i\} \text{ such that (s.t.) } \alpha_s > \max_{\ell \neq s} \alpha_\ell, \\ -c_j & \text{if } \alpha_i = p_j \text{ and } \nexists s \in \mathcal{I} \text{ s.t. } \alpha_s > \max_{\ell \neq s} \alpha_\ell, \end{cases}$$

where  $0 \leq c_0 < \dots < c_{m-1} < 1$ ,  $\tau_j > 0$  for all  $j = 0, 1, \dots, m - 2$ , and

$$-c_0 < -c_{m-1} + \tau_{m-1} < \dots < -c_0 + \tau_0 < 1 - c_{m-1}.$$

This definition of a common-pool game can be viewed as a finite-action analog of continuous-action common-pool games defined in [20]. Table 2.2 presents an example of a common-pool game of two players and three actions.

TABLE 2.2  
A common-pool game of two players and three actions.

	$p_0$	$p_1$	$p_2$
$p_0$	$-c_0, -c_0$	$-c_0 + \tau_0, 1 - c_1$	$-c_0 + \tau_0, 1 - c_2$
$p_1$	$1 - c_1, -c_0 + \tau_0$	$-c_1, -c_1$	$-c_1 + \tau_1, 1 - c_2$
$p_2$	$1 - c_2, -c_0 + \tau_0$	$1 - c_2, -c_1 + \tau_1$	$-c_2, -c_2$

We call “successful” any action profile in which one player’s action is strictly greater than any other player’s action. Any other situation corresponds to a “failure.” In common-pool games, we define the set of desirable action profiles  $\bar{\mathcal{A}}$ , as the set of successful action profiles, i.e.,

$$(2.4) \quad \bar{\mathcal{A}} \triangleq \left\{ \alpha \in \mathcal{A} : \exists i \in \mathcal{I} \text{ s.t. } \alpha_i > \max_{\ell \neq i} \alpha_\ell \right\}.$$

For example, this set of joint actions corresponds to the off-diagonal action profiles in Table 2.2. Moreover, the set  $\bar{\mathcal{A}}$  payoff-dominates the set  $\mathcal{A} \setminus \bar{\mathcal{A}}$ .

**CLAIM 2.3.** *Any common-pool game is a strict coordination game.*

*Proof.* Let  $\bar{\mathcal{A}}$  be defined as in (2.4). Note first that for any  $\alpha^* \in \bar{\mathcal{A}}$  and  $\alpha \in \mathcal{A} \setminus \bar{\mathcal{A}}$ , we have  $u_i(\alpha^*) > u_i(\alpha)$  for all  $i \in \mathcal{I}$ . In other words, Definition 2.2(a) is satisfied.

Moreover, note that any  $\alpha \notin \bar{\mathcal{A}}$  is not a Nash equilibrium. For any action profile  $\alpha \notin \bar{\mathcal{A}}$ , pick an agent  $i$  such that  $i \in \arg \max_{s \in \mathcal{I}} \alpha_s$ . Let us also assume that  $\alpha_i = p_j$  for some  $j \in \{0, 1, \dots, m-1\}$ . If  $j > 0$ , then agent  $i$  can increase its utility by selecting action  $p_k$  for any  $k < j$ . In that case, the utility of any other agent either increases or remains the same. If, instead,  $j = 0$ , then agent  $i$  can increase its utility by selecting action  $p_k$  for any  $k > j$ . In this case, the utility of any other agent increases. Thus, Definition 2.2(b) is also satisfied.

Last, note that  $\mathcal{A}^* \subseteq \bar{\mathcal{A}}$ . To check this, consider any  $\alpha \notin \bar{\mathcal{A}}$ . As the previous discussion revealed, there always exist an agent and a better reply for that agent, i.e.,  $\mathcal{A}^* \subseteq \bar{\mathcal{A}}$ . Thus, Definition 2.2(c) is trivially satisfied.  $\square$

If we imagine that a common-pool game is played repeatedly over time, it would be desirable that (i) failures are avoided, and (ii) agents manage to equally share the time they succeed (i.e., access the common resource). In other words, convergence to a successful state may not be sufficient. Instead, a (possibly time-dependent) solution that equally divides the time-slots that each user utilizes the common resource would seem more appropriate.

Distributed convergence to such solutions is currently an open issue in packet radio multiple-access protocols (see, e.g., [10, Chapter 5]). In these scenarios, there are multiple users that compete for access to a single communication channel. Each user needs to decide whether or not to occupy the channel in a given time-slot based only on *local* information. If more than one user is occupying the channel, then a *collision* occurs and the user needs to resubmit the data. An example of such multiple-access protocol is the Aloha protocol [1], where users decide on transmitting a packet according to a probabilistic pattern. In this line of work, the action space of each user consists of multiple power levels of transmission [27]. If a user transmits with a power level that is strictly larger than the power level of any other user, then it is able to transmit successfully, otherwise a collision occurs and transmission is not possible. This game can be formulated in a straightforward manner as a common-pool game.

In a forthcoming section we provide a distributed solution to this problem using aspiration learning which is of independent interest.

**3. Aspiration learning.** In this section, we define aspiration learning, motivated by [14]. For some constants  $\zeta > 0$ ,  $\epsilon > 0$ ,  $\lambda \geq 0$ ,  $c > 0$ ,  $0 < h < 1$ , and  $\underline{\rho}, \bar{\rho} \in \mathbb{R}$ , such that

$$-\infty < \underline{\rho} < \min_{\alpha \in \mathcal{A}, i \in \mathcal{I}} u_i(\alpha) \leq \max_{\alpha \in \mathcal{A}, i \in \mathcal{I}} u_i(\alpha) < \bar{\rho} < \infty,$$

the aspiration learning iteration initialized at  $(\alpha(0), \rho(0))$  is described in Table 3.1.

According to this algorithm, each agent  $i$  keeps track of an *aspiration level*  $\rho_i$ , which measures player  $i$ 's desirable return and is defined as a perturbed fading memory average of its payoffs throughout the history of play.

Given the current aspiration level  $\rho_i(t)$ , agent  $i$  selects a new action  $\alpha_i(t+1)$ . If the previous action  $\alpha_i(t)$  provided utility at least  $\rho_i(t)$ , then the agent is "satisfied" and repeats the same action, i.e.,  $\alpha_i(t+1) = \alpha_i(t)$ . Otherwise,  $\alpha_i(t+1)$  is selected



TABLE 3.1  
Aspiration learning.

<p>At every <math>t = 0, 1, \dots</math>, and for each <math>i \in \mathcal{I}</math>,</p> <ol style="list-style-type: none"> <li>1. Agent <math>i</math> plays <math>\alpha_i(t)</math> and measures utility <math>u_i(\alpha(t))</math>.</li> <li>2. Agent <math>i</math> updates its aspiration level according to                     <math display="block">\rho_i(t+1) = \text{sat}[\rho_i(t) + \epsilon[u_i(\alpha(t)) - \rho_i(t)] + r_i(t)],</math>                     where                     <math display="block">r_i(t) \triangleq \begin{cases} 0 &amp; \text{with probability (w.p.) } 1 - \lambda, \\ \text{rand}[-\zeta, \zeta] &amp; \text{w.p. } \lambda, \end{cases}</math>                     and                     <math display="block">\text{sat}[\rho] \triangleq \begin{cases} \bar{\rho} &amp; \text{if } \rho &gt; \bar{\rho}, \\ \rho &amp; \text{if } \rho \in [\underline{\rho}, \bar{\rho}], \\ \underline{\rho} &amp; \text{if } \rho &lt; \underline{\rho}. \end{cases}</math> </li> <li>3. Agent <math>i</math> updates its action:                     <math display="block">\alpha_i(t+1) = \begin{cases} \alpha_i(t) &amp; \text{w.p. } \phi(u_i(\alpha(t)) - \rho_i(t)), \\ \text{rand}(\mathcal{A}_i \setminus \alpha_i(t)) &amp; \text{w.p. } 1 - \phi(u_i(\alpha(t)) - \rho_i(t)), \end{cases}</math>                     where                     <math display="block">\phi(z) \triangleq \begin{cases} 1 &amp; \text{if } z \geq 0, \\ \max(h, 1 + cz) &amp; \text{if } z &lt; 0. \end{cases}</math> </li> <li>4. Agent <math>i</math> updates the time and repeats.</li> </ol>
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randomly over all available actions, where the probability of selecting again  $\alpha_i(t)$  depends on the level of discontent measured by the difference  $u_i(\alpha(t)) - \rho_i(t) < 0$ . The random variables  $\{r_i(t) : t \geq 0, i \in \mathcal{I}\}$  are independent and identically distributed and are referred to as the “tremble.”

Let  $\mathcal{X} \triangleq \mathcal{A} \times [\underline{\rho}, \bar{\rho}]^n$ , i.e., pairs of joint actions  $\alpha$  and vectors of aspiration levels,  $\rho_i$ ,  $i \in \mathcal{I}$ . The set  $\mathcal{A}$  is endowed with the product topology,  $[\underline{\rho}, \bar{\rho}]$  with its usual Euclidean topology, and  $\mathcal{X}$  with the corresponding product topology. We also let  $\mathcal{B}(\mathcal{X})$  denote the Borel  $\sigma$ -field of  $\mathcal{X}$ , and  $\mathcal{P}(\mathcal{X})$  the set of probability measures on  $\mathcal{B}(\mathcal{X})$  endowed with the Prohorov topology, i.e., the topology of weak convergence. The algorithm in Table 3.1 defines an  $\mathcal{X}$ -valued Markov chain. Let  $P_\lambda : \mathcal{X} \times \mathcal{B}(\mathcal{X}) \rightarrow [0, 1]$  denote its transition probability function, parameterized by  $\lambda > 0$ . We refer to the process with  $\lambda > 0$  as the *perturbed process*.

We let  $\mathcal{C}(\mathcal{X})$  denote the Banach space of real-valued continuous functions on  $\mathcal{X}$  under the sup-norm (denoted by  $\|\cdot\|_\infty$ ) topology. For  $f \in \mathcal{C}(\mathcal{X})$  we define

$$P_\lambda f(x) \triangleq \int_{\mathcal{X}} P_\lambda(x, dy) f(y) \quad \text{and} \quad \mu[f] \triangleq \int_{\mathcal{X}} \mu(dx) f(x), \quad \mu \in \mathcal{P}(\mathcal{X}).$$

It is straightforward to verify that  $P_\lambda$  has the Feller property, i.e.,  $P_\lambda f \in \mathcal{C}(\mathcal{X})$  for all  $f \in \mathcal{C}(\mathcal{X})$ . Recall that  $\mu_\lambda \in \mathcal{P}(\mathcal{X})$  is called an invariant probability measure for  $P_\lambda$  if

$$(\mu_\lambda P_\lambda)(A) \triangleq \int_{\mathcal{X}} \mu_\lambda(dx) P_\lambda(x, A) = \mu_\lambda(A) \quad \forall A \in \mathcal{B}(\mathcal{X}).$$

Since  $\mathcal{X}$  is a compact metric space and  $P_\lambda$  has the Feller property, it admits an invariant probability measure  $\mu_\lambda$  [11, Theorem 7.2.3].

We are interested in the asymptotic behavior of the aspiration learning algorithm as the “experimentation probability”  $\lambda$  approaches zero. We say that a state  $x \in \mathcal{X}$  is *stochastically stable* if any collection of invariant probability measures  $\{\mu_\lambda \in \mathcal{P}(\mathcal{X}) : \mu_\lambda P_\lambda = \mu_\lambda, \lambda > 0\}$  satisfies  $\liminf_{\lambda \downarrow 0} \mu_\lambda(x) > 0$ . It turns out that the stochastically stable states comprise a finite subset of  $\mathcal{X}$  which is defined next.

**DEFINITION 3.1.** *A pure strategy state is a state  $s = (\alpha, \rho) \in \mathcal{X}$  such that for all  $i \in \mathcal{I}$ ,  $u_i(\alpha) = \rho_i$ . The set of pure strategy states is denoted by  $\mathcal{S}$  and  $|\mathcal{S}|$  denotes its cardinality.*

Note that the set  $\mathcal{S}$  is isomorphic to  $\mathcal{A}$  and can be identified as such.

As is customary, the Dirac measure in  $\mathcal{P}(\mathcal{X})$  supported at  $x \in \mathcal{X}$  is denoted by  $\delta_x$ . The objective in this section is to characterize the set of stochastically stable states. Our main result is summarized in the following theorem.

**THEOREM 3.2.** *There exists a unique probability vector  $\pi = (\pi_1, \dots, \pi_{|\mathcal{S}|})$  such that for any collection of invariant probability measures  $\{\mu_\lambda \in \mathcal{P}(\mathcal{X}) : \mu_\lambda P_\lambda = \mu_\lambda, \lambda > 0\}$ , we have*

$$\lim_{\lambda \downarrow 0} \mu_\lambda(\cdot) = \hat{\mu}(\cdot) \triangleq \sum_{s \in \mathcal{S}} \pi_s \delta_s(\cdot),$$

where convergence is in the weak\* sense.

As we show later,  $\pi$  in Theorem 3.2 is the unique invariant distribution of a finite-state Markov chain.

*Remark 3.1.* The expected asymptotic behavior of aspiration learning can be characterized by  $\hat{\mu}$  and, therefore,  $\pi$ . In particular, by Birkhoff’s individual ergodic theorem, e.g., [11, Theorem 2.3.4], and the weak convergence of  $\mu_\lambda$  to  $\hat{\mu}$ , the expected percentage of time that the process spends in any  $B \in \mathcal{B}(\mathcal{X})$  such that  $\partial B \cap \mathcal{S} \neq \emptyset$  is given by  $\hat{\mu}(B)$  as the experimentation probability  $\lambda$  approaches zero and time increases, i.e.,

$$\lim_{\lambda \downarrow 0} \left( \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{k=0}^{t-1} P_\lambda^k(x, B) \right) = \hat{\mu}(B).$$

The proof of Theorem 3.2 requires a series of propositions, which comprise the remaining of this section.

Let  $P(\cdot, \cdot)$  denote the transition probability function on  $\mathcal{X} \times \mathcal{B}(\mathcal{X})$  corresponding to  $\lambda = 0$ . We refer to the process  $\{X_t : t \geq 0\}$  governed by  $P$  as the *unperturbed process*. Let  $\Omega \triangleq \mathcal{X}^\infty$  denote the canonical path space, i.e., an element  $\omega \in \Omega$  is a sequence  $\{\omega(0), \omega(1), \dots\}$ , with  $\omega(t) = (\alpha(t), \rho(t)) \in \mathcal{X}$ . We use the same notation for the elements  $(\alpha, \rho)$  of the space  $\mathcal{X}$  and for the coordinates of the process  $X_t = (\alpha(t), \rho(t))$ . Also let  $\mathbb{P}_x$  denote the unique probability measure induced by  $P$  on the product  $\sigma$ -algebra of  $\mathcal{X}^\infty$ , initialized at  $x = (\alpha, \rho)$ , and  $\mathbb{E}_x$  the corresponding expectation operator. Also let  $\mathfrak{F}_t \triangleq \sigma(X_\tau, \tau \leq t)$ ,  $t \geq 0$ , denote the  $\sigma$ -algebra generated by  $\{X_\tau, \tau \leq t\}$ .

For  $t \geq 0$  define the sets

$$A_t \triangleq \{\omega \in \Omega : \alpha(\tau) = \alpha(t) \forall \tau \geq t\},$$

$$B_t \triangleq \{\omega \in \Omega : \alpha(\tau) = \alpha(0) \forall 0 \leq \tau \leq t\}.$$

Note that  $\{B_t : t \geq 0\}$  is a nonincreasing sequence, i.e.,  $B_{t+1} \subseteq B_t$ , while  $\{A_t : t \geq 0\}$  is nondecreasing. Let

$$A_\infty \triangleq \bigcup_{t=0}^\infty A_t, \quad B_\infty \triangleq \bigcap_{t=1}^\infty B_t.$$

The set  $A_\infty$  is the event that agents eventually play the same action profile, while  $B_\infty$  is the event that agents never change their actions. Recall that the shift operator  $\theta_t : \Omega \rightarrow \Omega$ ,  $t \geq 0$ , satisfies  $X_s(\theta_t(\omega)) = X_{s+t}(\omega)$ . Therefore  $A_t = \theta_t^{-1}(B_\infty)$ .

For  $D \in \mathcal{B}(\mathcal{X})$  we let  $\tau(D)$  denote the first hitting time of  $D$ , i.e.,

$$(3.1) \quad \tau(D) \triangleq \inf \{t \geq 0 : X_t \in D\}.$$

PROPOSITION 3.3. *It holds that*

$$\inf_{x \in \mathcal{X}} \mathbb{P}_x(B_\infty) > 0 \quad \text{and} \quad \inf_{x \in \mathcal{X}} \mathbb{P}_x(A_\infty) = 1.$$

*Proof.* Assume that the process is initialized at  $X_0 = x = (\alpha, \rho)$ . Note that  $B_t$  consists of those sample paths which satisfy

$$\rho_i(\tau) = u_i(\alpha) - (1 - \epsilon)^\tau (u_i(\alpha) - \rho), \quad 0 \leq \tau < t, \quad i \in \mathcal{I}.$$

Therefore, we have

$$(3.2) \quad \mathbb{P}_x(B_t) = \prod_{0 \leq \tau < t} \prod_{i \in \mathcal{I}} \max \left\{ h, 1 - c(1 - \epsilon)^\tau (\rho_i - u_i(\alpha))^+ \right\},$$

where

$$(x)^+ \triangleq \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $T_0$  satisfy  $c(1 - \epsilon)^{T_0}(\bar{\rho} - \underline{\rho}) \leq \min \{1 - h, \epsilon\}$ . Then

$$\begin{aligned} \mathbb{P}_x(B_t) &\geq h^{nT_0} \prod_{i \in \mathcal{I}} \prod_{T_0 < \tau < t} \left( 1 - c(1 - \epsilon)^\tau (\rho_i - u_i(\alpha))^+ \right) \\ &\geq h^{nT_0} \prod_{i \in \mathcal{I}} \left( 1 - c(\rho_i - u_i(\alpha))^+ \sum_{\tau=T_0+1}^t (1 - \epsilon)^\tau \right) \\ &\geq h^{nT_0} \prod_{i \in \mathcal{I}} \left( 1 - (1 - \epsilon) \frac{(\rho_i - u_i(\alpha))^+}{\bar{\rho} - \underline{\rho}} \right) \quad \forall t > T_0, \end{aligned}$$

and since the sequence  $\{B_t\}$  is nonincreasing, it also is for all  $t \geq 0$ . Therefore, by continuity from above, we obtain  $\inf_{x \in \mathcal{X}} \mathbb{P}_x(B_\infty) \geq \epsilon^n h^{nT_0}$ , which proves the first claim.

Next, define the set

$$D_\ell \triangleq \{(\alpha, \rho) \in \mathcal{X} : \rho_i - u_i(\alpha) \leq (1 - \epsilon)^\ell (\bar{\rho} - \underline{\rho}) \forall i \in \mathcal{I}\}, \quad \ell \geq 0,$$

and note that  $\mathbb{P}_x(B_\ell) \leq P^\ell(x, D_\ell)$ , where  $P^t$ ,  $t \geq 0$ , denotes the multistage transition probability function defined by the recursion  $P^t = P^{t-1}P$  and  $P^0 = I$ . Thus, using the Markov property over  $k$  time blocks of length  $\ell$ , we obtain the rough estimate

$$\begin{aligned} \mathbb{P}_x(\tau(D_\ell) > k\ell) &\leq \mathbb{P}_x(X_{j\ell} \in D_\ell^c, j = 1, \dots, k) \\ &\leq \mathbb{P}_x(X_{j\ell} \in D_\ell^c, j = 1, \dots, k-1) \left( \sup_{z \in D_\ell^c} P^\ell(z, D_\ell^c) \right) \\ (3.3) \quad &\leq \left( 1 - \inf_{z \in \mathcal{X}} \mathbb{P}_z(B_\ell) \right) \mathbb{P}_x(X_{j\ell} \in D_\ell^c, j = 1, \dots, k-1). \end{aligned}$$

Let  $q_0 \triangleq 1 - \inf_{z \in \mathcal{X}} \mathbb{P}_z(B_\infty)$ . We have already shown that  $q_0 < 1$ . Finite induction on (3.3) yields

$$\mathbb{P}_x(\tau(D_\ell) > k\ell) \leq \left( 1 - \inf_{z \in \mathcal{X}} \mathbb{P}_z(B_\ell) \right)^k \leq q_0^k.$$

We have

$$\mathbb{P}_x(A_{k\ell}) \geq \sum_{t=1}^{k\ell} \mathbb{P}_x(\tau(D_\ell) = t, X \circ \theta_t \in B_\infty),$$

and thus using the Markov property together with the fact that  $X_{\tau(D_\ell)} \in D_\ell$  almost surely (a.s.) on  $\{\tau(D_\ell) < \infty\}$ , and setting  $k = \ell$ , we obtain

$$\begin{aligned} \mathbb{P}_x(A_{\ell^2}) &\geq \sum_{t=1}^{\ell^2} \mathbb{P}_x(\tau(D_\ell) = t) \inf_{y \in D_\ell} \mathbb{P}_y(B_\infty) \\ &\geq \left( 1 - \mathbb{P}_x(\tau(D_\ell) > \ell^2) \right) \inf_{y \in D_\ell} \mathbb{P}_y(B_\infty) \\ (3.4) \quad &\geq (1 - q_0^\ell) \inf_{y \in D_\ell} \mathbb{P}_y(B_\infty). \end{aligned}$$

It is clear by (3.2) that  $\inf_{x \in D_\ell} \mathbb{P}_x(B_\infty) \rightarrow 1$  as  $\ell \rightarrow \infty$ . Therefore, both terms on the right-hand side of (3.4) converge to 1 as  $\ell \rightarrow \infty$ , and the proof is complete.  $\square$

**PROPOSITION 3.4.** *There exists a transition probability function  $\Pi$  on  $\mathcal{X} \times \mathcal{P}(\mathcal{X})$  that has the Feller property, and  $\Pi(x, \cdot)$  is supported on  $\mathcal{S}$  for all  $x \in \mathcal{X}$ , such that the following hold.*

- (i) For all  $f \in \mathcal{C}(\mathcal{X})$ ,  $\lim_{t \rightarrow \infty} \|P^t f - \Pi f\|_\infty = 0$ .
- (ii) If  $R_\lambda$  is a resolvent of  $P$ , defined by

$$R_\lambda \triangleq \varphi(\lambda) \sum_{t=0}^{\infty} (1 - \varphi(\lambda))^t P^t,$$

where  $\varphi(\lambda) \in (0, 1)$ ,  $\lambda > 0$ , and  $\lim_{\lambda \rightarrow 0} \varphi(\lambda) = 0$ , then

$$\lim_{\lambda \rightarrow 0} \|R_\lambda f - \Pi f\|_\infty = 0 \quad \forall f \in \mathcal{C}(\mathcal{X}).$$

*Proof.* For  $f \in \mathcal{C}(\mathcal{X})$  and  $x \in \mathcal{X}$ , we have  $\mathbb{E}_x[f(X_t)] = P^t f(x)$ . Since  $A_t = \theta_t^{-1}(B_\infty)$ , then using the Markov property we obtain that, for any positive  $t$  and  $t'$ ,

$$\begin{aligned}
 |P^{2t} f(x) - P^{2t+t'} f(x)| &= |\mathbb{E}_x[f(X_{2t}) - f(X_{2t+t'})]| \\
 &= |\mathbb{E}_x[(f(X_{2t}) - f(X_{2t+t'}))\mathbf{1}_{A_t}]| \\
 &\quad + |\mathbb{E}_x[(f(X_{2t}) - f(X_{2t+t'}))\mathbf{1}_{A_t^c}]| \\
 &\leq \left| \mathbb{E}_x \left[ \mathbb{E}[(f(X_{2t}) - f(X_{2t+t'}))\mathbf{1}_{A_t} \mid \mathfrak{F}_t] \right] \right| + 2\mathbb{P}_x(A_t^c)\|f\|_\infty \\
 &\leq \mathbb{E}_x \left[ \mathbb{E}_{X_t} [|f(X_{2t}) - f(X_{2t+t'})| \mathbf{1}_{A_t}] \right] + 2\mathbb{P}_x(A_t^c)\|f\|_\infty \\
 (3.5) \quad &\leq \sup_{z \in \mathcal{X}} \mathbb{E}_z [|f(X_t) - f(X_{t+t'})| \mathbf{1}_{B_\infty}] + 2\mathbb{P}_x(A_t^c)\|f\|_\infty.
 \end{aligned}$$

Since for any initial condition  $x = (\alpha, \rho)$  the dynamics on  $B_\infty$  evolve according to

$$\rho(t) = \varrho(t; \alpha, \rho) \triangleq u(\alpha) - (1 - \epsilon)^t(u(\alpha) - \rho),$$

the continuity of  $f$  (which is necessarily uniform since  $\mathcal{X}$  is compact) yields

$$\begin{aligned}
 (3.6) \quad &\sup_{t' \geq 0} \sup_{(\alpha, \rho) \in \mathcal{X}} \mathbb{E}_{(\alpha, \rho)} [|f(X_t) - f(X_{t+t'})| \mathbf{1}_{B_\infty}] \\
 &= \sup_{t' \geq 0} \sup_{(\alpha, \rho) \in \mathcal{X}} |f(\alpha, \varrho(t; \alpha, \rho)) - f(\alpha, \varrho(t+t'; \alpha, \rho))| \xrightarrow{t \rightarrow \infty} 0.
 \end{aligned}$$

By (3.5)–(3.6) and Proposition 3.3 we obtain

$$\sup_{t' > 0} \|P^{2t} f - P^{2t+t'} f\|_\infty \xrightarrow{t \rightarrow \infty} 0.$$

Therefore, the sequence  $\{P^t f, t \in \mathbb{N}\}$  is Cauchy in  $(\mathcal{C}(\mathcal{X}), \|\cdot\|_\infty)$ , and hence converges in  $\mathcal{C}(\mathcal{X})$ . Let  $\varphi(f)(x) \triangleq \lim_{t \rightarrow \infty} P^t f(x)$ . Then for each  $x, f \mapsto \varphi(f)(x)$  defines a bounded linear functional on  $\mathcal{C}(\mathcal{X})$ . It is a positive functional since  $\varphi(f)(x) \geq 0$ , for  $f \geq 0$ , and if  $\mathbf{1}$  denotes the constant function equal to 1,  $\varphi(\mathbf{1})(x) = 1$ . Then, by the Riesz representation theorem,  $\varphi(f)(x)$  is a Borel probability measure on  $\mathcal{X}$  for each  $x$ . Denote this by  $\Pi(x, \cdot)$ . Since  $\varphi : \mathcal{C}(\mathcal{X}) \rightarrow \mathcal{C}(\mathcal{X})$ , it follows that  $\Pi$  has the Feller property. Also, by the definition of  $\Pi$ , we have

$$(3.7) \quad \|P^t f - \Pi f\|_\infty \xrightarrow{t \rightarrow \infty} 0 \quad \forall f \in \mathcal{C}(\mathcal{X}).$$

This proves Proposition 3.4(i).

Next, using a triangle inequality, we have for each  $T > 0$ ,

$$\|R_\lambda f - \Pi f\|_\infty \leq \varphi(\lambda) \sum_{t=0}^{T-1} (1 - \varphi(\lambda))^t \|P^t f - \Pi f\|_\infty + (1 - \varphi(\lambda))^T \sup_{t \geq T} \|P^t f - \Pi f\|_\infty.$$

Letting  $\lambda \downarrow 0$ , we obtain

$$\|R_\lambda f - \Pi f\|_\infty \leq \sup_{t \geq T} \|P^t f - \Pi f\|_\infty \quad \forall T > 0,$$

and Proposition 3.4(ii) follows by (3.7).  $\square$

We can decompose the transition probability function of the perturbed process as

$$(3.8) \quad P_\lambda = (1 - \varphi(\lambda))P + \varphi(\lambda)Q_\lambda, \quad \varphi(\lambda) \triangleq 1 - (1 - \lambda)^n,$$

where  $\varphi(\lambda)$  is the probability that at least one agent trembles, and satisfies  $\varphi(\lambda) \downarrow 0$  as  $\lambda \downarrow 0$ . Also, define the “lifted” transition probability function:

$$P_\lambda^L \triangleq \varphi(\lambda) \sum_{t=0}^{\infty} (1 - \varphi(\lambda))^t Q_\lambda P^t = Q_\lambda R_\lambda,$$

where  $R_\lambda$  was defined in Proposition 3.4 (the equality on the right-hand side is evident by Fubini). Similarly we decompose  $Q_\lambda$  as

$$Q_\lambda = (1 - \psi(\lambda))Q + \psi(\lambda)Q^*, \quad \psi(\lambda) \triangleq 1 - \frac{n\lambda(1 - \lambda)^{n-1}}{1 - (1 - \lambda)^n}.$$

Here  $Q$  is the transition probability function induced by aspiration learning where exactly one player trembles, and  $Q^*$  is the transition probability function where at least two players tremble simultaneously.

We have the following proposition.

PROPOSITION 3.5. *The following hold:*

- (i) For  $f \in \mathcal{C}(\mathcal{X})$ ,  $\lim_{\lambda \rightarrow 0} \|P_\lambda^L f - Q\Pi f\|_\infty = 0$ .
- (ii) Any invariant distribution  $\mu_\lambda$  of  $P_\lambda$  is also an invariant distribution of  $P_\lambda^L$ .
- (iii) Any weak limit point in  $\mathcal{P}(\mathcal{X})$  of  $\mu_\lambda$ , as  $\lambda \downarrow 0$ , is an invariant probability measure of  $Q\Pi$ .

*Proof.* (i) We have

$$(3.9) \quad \begin{aligned} \|P_\lambda^L f - Q\Pi f\|_\infty &\leq \|Q_\lambda(R_\lambda f - \Pi f)\|_\infty + \|Q_\lambda \Pi f - Q\Pi f\|_\infty \\ &\leq \|R_\lambda f - \Pi f\|_\infty + \|Q_\lambda \Pi f - Q\Pi f\|_\infty. \end{aligned}$$

The first term on the right-hand side of (3.9) tends to 0 as  $\lambda \downarrow 0$  by Proposition 3.4, while the second term does the same by the definition of  $Q_\lambda$ .

(ii) Multiplying both sides of (3.8) by  $R_\lambda$ , we have

$$(3.10) \quad P_\lambda R_\lambda = R_\lambda - \varphi(\lambda)I + \varphi(\lambda)P_\lambda^L,$$

where  $I$  denotes the identity operator. Let  $\mu_\lambda$  denote an invariant distribution of  $P_\lambda$ . Hence, by (3.10), we have

$$\mu_\lambda R_\lambda = \mu_\lambda R_\lambda - \varphi(\lambda)\mu_\lambda + \varphi(\lambda)\mu_\lambda P_\lambda^L,$$

and the second claim follows.

(iii) Let  $\hat{\mu}$  be a limit point of  $\mu_\lambda$  as  $\lambda \downarrow 0$ . For any  $f \in \mathcal{C}(\mathcal{X})$ , we have

$$\hat{\mu}[f] - (\hat{\mu}Q\Pi)[f] = (\hat{\mu}[f] - \mu_\lambda[f]) + \mu_\lambda[P_\lambda^L f - Q\Pi f] + (\mu_\lambda[Q\Pi f] - \hat{\mu}[Q\Pi f]).$$

The first and third terms on the right-hand side tend to 0 as  $\lambda \downarrow 0$  along some sequence, by the weak convergence  $\mu_\lambda$  to  $\hat{\mu}$ , while the second term is dominated by  $\|P_\lambda^L f - Q\Pi f\|_\infty$  that also tends to 0 by part (i).  $\square$

For  $s \in \mathcal{S}$  let  $N_\varepsilon(s)$  denote the open  $\varepsilon$ -neighborhood of  $s$  in  $\mathcal{X}$ . For any two pure strategy states,  $s, s' \in \mathcal{S}$ , define

$$\hat{P}_{ss'} \triangleq \lim_{t \rightarrow \infty} QP^t(s, N_\varepsilon(s'))$$

for some  $\varepsilon > 0$  sufficiently small. By Proposition 3.3,  $\hat{P}_{ss'}$  is independent of the selection of  $\varepsilon$ . Define also the  $|\mathcal{S}| \times |\mathcal{S}|$  stochastic matrix  $\hat{P} \triangleq [\hat{P}_{ss'}]$ .

PROPOSITION 3.6. *There exists a unique invariant probability measure  $\hat{\mu}$  of  $Q\Pi$ . It satisfies*

$$(3.11) \quad \hat{\mu}(\cdot) = \sum_{s \in \mathcal{S}} \pi_s \delta_s(\cdot)$$

for some constants  $\pi_s \geq 0, s \in \mathcal{S}$ . Moreover,  $\pi = (\pi_1, \dots, \pi_{|\mathcal{S}|})$  is an invariant distribution of  $\hat{P}$ , i.e.,  $\pi = \pi \hat{P}$ .

*Proof.* By Proposition 3.4, the support of  $\Pi$  is  $\mathcal{S}$ , and so is the support of  $Q\Pi$ . Thus, for any sufficiently small  $\varepsilon > 0$ ,  $Q\Pi(s, s') = Q\Pi(s, N_\varepsilon(s'))$ . Since  $Q\Pi$  is a Feller transition function, it admits an invariant probability measure, say  $\hat{\mu}$ . The support of  $\hat{\mu}$  is also  $\mathcal{S}$ , and, therefore, it has the form of (3.11) for some constants  $\pi_s \geq 0, s \in \mathcal{S}$ .

Note also that  $N_\varepsilon(s')$  is a continuity set of  $Q\Pi(s, \cdot)$ , i.e.,  $Q\Pi(s, \partial N_\varepsilon(s')) = 0$ . Therefore, by the Portmanteau theorem,

$$Q\Pi(s, N_\varepsilon(s')) = \lim_{t \rightarrow \infty} QP^t(s, N_\varepsilon(s')) = \hat{P}_{ss'}.$$

If we also define  $\pi_s \triangleq \hat{\mu}(N_\varepsilon(s))$ , then

$$\pi_{s'} = \hat{\mu}(N_\varepsilon(s')) = \sum_{s \in \mathcal{S}} \pi_s Q\Pi(s, N_\varepsilon(s')) = \sum_{s \in \mathcal{S}} \pi_s \hat{P}_{ss'},$$

which shows that  $\pi$  is an invariant distribution of  $\hat{P}$ , i.e.,  $\pi = \pi \hat{P}$ .

To establish the uniqueness of the invariant distribution of  $Q\Pi$ , recall the definition of  $Q$ . Since  $\mathcal{S}$  is isomorphic with  $\mathcal{A}$ , we can identify  $s \in \mathcal{S}$  with an element  $\alpha \in \mathcal{A}$ . If agent  $i$  trembles, then all actions in  $\mathcal{A}_i$  have positive probability of being selected, i.e.,  $Q(\alpha, (\alpha'_i, \alpha_{-i})) > 0$  for all  $\alpha'_i \in \mathcal{A}_i$  and  $i \in \mathcal{I}$ . It follows by Proposition 3.3 that  $Q\Pi(\alpha, (\alpha'_i, \alpha_{-i})) > 0$  for all  $\alpha'_i \in \mathcal{A}_i$  and  $i \in \mathcal{I}$ . Finite induction then shows that  $(Q\Pi)^n(\alpha, \alpha') > 0$  for all  $\alpha, \alpha' \in \mathcal{A}$ . It follows that if we restrict the domain of  $Q\Pi$  to  $\mathcal{S}$ , then  $Q\Pi$  defines an irreducible stochastic matrix. Therefore,  $Q\Pi$  has a unique invariant distribution.  $\square$

Theorem 3.2 follows from Propositions 3.5 and 3.6. Moreover, Proposition 3.6 shows that the unique invariant probability measure of  $Q\Pi$  agrees with the unique invariant probability distribution of the finite stochastic matrix  $\hat{P}$ .

Remark 3.2. A similar result to Proposition 3.5(i), based on which Theorem 3.2 was shown, has also been derived in [14, Theorem 2]. The result in [14] though assumes incorrectly that the process  $Q$  satisfies the strong Feller property. Note that the proof of Proposition 3.5 does not make use of any such assumption and provides a corrected analysis for the asymptotic behavior of the aspiration learning scheme presented in [14].

In the forthcoming sections, we demonstrate the importance of Theorem 3.2 in characterizing the asymptotic behavior of aspiration learning in large coordination games. Note that prior analysis of this type of aspiration learning, e.g., in [6, 14], was only restricted to two player and two action games.

**4. Efficiency in coordination games.** In this section, we study the asymptotic behavior of the invariant distribution  $\pi$  of  $\hat{P}$  in strict coordination games when the step size  $\epsilon$  approaches zero. The aim is to characterize the states in  $\mathcal{S}$  that are stochastically stable with respect to the parameter  $\epsilon$ . To this end, first denote  $\bar{\mathcal{S}}$  as the set of pure strategy states that correspond to  $\bar{\mathcal{A}}$ . Clearly,  $\bar{\mathcal{S}}$  is isomorphic to  $\bar{\mathcal{A}}$ . Also, denote by  $\mathcal{S}^*$  the set of pure strategy states that correspond to the set of Nash action profiles  $\mathcal{A}^*$ .

We define two constants that are important in the analysis:

$$\Delta_{\min} \triangleq \min_{i \in \mathcal{I}} \min_{\alpha \in \bar{\mathcal{A}}, \alpha' \notin \bar{\mathcal{A}}} \{u_i(\alpha) - u_i(\alpha')\},$$

$$\Delta_{\max} \triangleq \max_{i \in \mathcal{I}} \max_{\alpha \neq \alpha'} |u_i(\alpha') - u_i(\alpha)|.$$

For strict coordination games  $\Delta_{\min} > 0$ , and it is the smallest possible payoff decrease from the dominant payoff due to any deviation from the set of actions in  $\bar{\mathcal{A}}$ .

To facilitate the analysis we let  $\mathbb{P}_x$  and  $\mathbb{E}_x$  denote the probability and expectation operator, respectively, on the path space of a Markov process  $X_t$  starting at  $x \in \mathcal{X}$  at  $t = 0$ , and governed by the family of transition probabilities  $\{QP^t : t \geq 0\}$ . In other words  $\mathbb{P}_x(X_t \in A) = QP^{t-1}(x, A)$  for any  $A \in \mathcal{B}(\mathcal{X})$ .

**4.1. Two technical lemmas.** Lemma 4.1 below introduces two new hypotheses. The first hypothesis corresponds to the case at which payoff differences within the same action profile are smaller than payoff differences between dominant and non-dominant action profiles. The second hypothesis corresponds to the case where each player receives a unique payoff within  $\bar{\mathcal{A}}$ .

**LEMMA 4.1.** *Let  $\mathcal{G}$  be a strict coordination game satisfying either one of the following two hypotheses:*

(H1)  $\delta^* \triangleq \max_{i \neq j} \max_{\alpha \in \mathcal{A}} |u_i(\alpha) - u_j(\alpha)| < \Delta_{\min}$ .

(H2)  $\bar{\mathcal{A}} \equiv \{\bar{\alpha} \in \bar{\mathcal{A}} : u_i(\bar{\alpha}) = \max_{\alpha \in \bar{\mathcal{A}}} u_i(\alpha) \quad \forall i \in \mathcal{I}\}$ .

*Then, there exists a constant  $C_0 = C_0(\delta^*, \Delta_{\min}, \Delta_{\max})$  such that if  $\zeta < C_0$ , then*

$$\hat{P}_{\bar{s}s} \xrightarrow{\epsilon \downarrow 0} 0 \quad \forall \bar{s} \in \bar{\mathcal{S}}, s \in \mathcal{S} \setminus \bar{\mathcal{S}}.$$

*Proof.* Suppose (H1) holds. Select  $\zeta < \frac{1}{2}(\Delta_{\min} - \delta^*)$ . Let  $x(0) = \bar{s} \equiv (\bar{\alpha}, \bar{\rho}) \in \bar{\mathcal{S}}$ . Without loss of generality suppose agent 1 trembles. If  $r_1(0) < 0$ , the process clearly converges to  $\bar{s}$  as  $t \rightarrow \infty$  with probability 1. Therefore, suppose  $r_1(0) > 0$ . Note that for  $t \geq 0$  we have

$$\begin{aligned} |\rho_i(t+1) - \rho_j(t+1)| &\leq (1-\epsilon)|\rho_i(t) - \rho_j(t)| + \epsilon|u_i(\alpha(t)) - u_j(\alpha(t))| \\ (4.1) \qquad \qquad \qquad &\leq (1-\epsilon)|\rho_i(t) - \rho_j(t)| + \epsilon\delta^* \quad \forall i, j \in \mathcal{I}, \end{aligned}$$

and since  $\zeta < \frac{1}{2}(\Delta_{\min} - \delta^*)$  by a straightforward induction argument using (4.1) we obtain

$$(4.2) \qquad \max_{i, j \in \mathcal{I}} |\rho_i(t) - \rho_j(t)| \leq \frac{\Delta_{\min} + \delta^*}{2} \quad \forall t \geq 0.$$

For  $i \in \mathcal{I}$  define

$$\check{\rho}_i \triangleq \min_{\bar{\alpha} \in \bar{\mathcal{A}}} u_i(\bar{\alpha}) \quad \text{and} \quad \hat{\rho}_i \triangleq \max_{\alpha \in \bar{\mathcal{A}} \setminus \bar{\mathcal{A}}} u_i(\alpha),$$



and for  $k = 0, 1$  define the sets

$$D_k \triangleq \left\{ (\alpha, \rho) \in \mathcal{X} : \rho_i \leq \frac{\check{\rho}_i + (2k + 1)\hat{\rho}_i}{2k + 2} + \frac{\Delta_{\min} + \delta^*}{4}, i \in \mathcal{I} \right\}.$$

Also let

$$\Gamma \triangleq \left\{ (\alpha, \rho) \in \mathcal{X} : \min(\check{\rho}_i - \rho_i, \rho_i - \hat{\rho}_i) \geq \frac{1}{4}(\Delta_{\min} - \delta^*), i \in \mathcal{I} \right\}$$

and

$$\bar{\Gamma} \triangleq \{ (\alpha, \rho) \in \Gamma : \alpha \in \bar{\mathcal{A}} \}.$$

Recall the definition of  $\tau$  in (3.1), and in order to simplify the notation, let  $\tau_k \triangleq \tau(D_k)$  for  $k = 0, 1$ . Note the following: First, using (4.2), we obtain

$$(4.3) \quad \Gamma \subset D_0 \setminus D_1.$$

Second, since  $|\rho_i(t + 1) - \rho_i(t)| \leq \epsilon \Delta_{\max}$ , we obtain

$$(4.4) \quad \left( \tau_1 - \tau_0 - \frac{\Delta_{\min}}{4\epsilon \Delta_{\max}} \right) \mathbf{1}_{\{\tau_0 < \infty\}} \geq 0 \quad \tilde{\mathbb{P}}_{\bar{s}}\text{-a.s.}$$

It is also evident that

$$(4.5) \quad \left\{ \limsup_{t \rightarrow \infty} d_{\mathcal{S}}(X_t, \mathcal{S} \setminus \bar{\mathcal{S}}) = 0 \right\} \subset \{ \tau_1 < \infty \} \quad \tilde{\mathbb{P}}_{\bar{s}}\text{-a.s.},$$

where  $d_{\mathcal{S}}$  is a metric in  $\mathcal{S}$ . It is clear from the definition of  $P$  that if  $x \in \Gamma$ , there are two possibilities: If a profile  $\alpha \in \mathcal{A} \setminus \bar{\mathcal{A}}$  is played, then  $\rho_i$  decreases in value for all  $i \in \mathcal{I}$ , or, in other words,  $P(x, \Gamma) = 1$  for all  $x \in (\Gamma \cap D_1^c) \setminus \bar{\Gamma}$ . Otherwise, if a profile in  $\bar{\mathcal{A}}$  is played, then the sample path gets trapped in the domain of attraction of  $\bar{\mathcal{S}}$ . This means that if  $x \in \bar{\Gamma}$ , then  $\mathbb{P}_x(\tau_1 < \infty) = 0$ , where  $\mathbb{P}_x$  is the probability measure induced by  $P$  defined in section 3. In this case, and by (4.3), we also have

$$P(x, \bar{\Gamma}) \geq \min \left\{ \frac{c}{4}(\Delta_{\min} - \delta^*), 1 - h \right\} \triangleq \gamma \quad \forall x \in \Gamma \cap D_1^c.$$

Thus, using the Markov property we obtain, with  $t_0 \triangleq \left\lfloor \frac{\Delta_{\min}}{4\epsilon \Delta_{\max}} \right\rfloor$ ,

$$(4.6) \quad P^{t_0}(x, \Gamma \setminus \bar{\Gamma}) \leq (1 - \gamma)^{t_0} \quad \forall x \in \Gamma \cap D_1^c.$$

Conditioning on  $\mathfrak{F}_{\tau_0}$  and using the strong Markov property, (4.4), (4.6), and the foregoing, we obtain

$$(4.7) \quad \begin{aligned} \tilde{\mathbb{P}}_{\bar{s}}(\tau_1 < \infty) &\leq \tilde{\mathbb{E}}_{\bar{s}} \left[ \tilde{\mathbb{E}}_{\bar{s}} \left[ \mathbf{1}_{\{\tau_1 < \infty\}} \mid \mathfrak{F}_{\tau_0} \right] \right] \\ &\leq \tilde{\mathbb{E}}_{\bar{s}} \left[ \mathbb{P}_{X_{\tau_0}}(\tau_1 < \infty) \right] \\ &\leq \sup_{x \in \Gamma \cap D_1^c} \mathbb{P}_x(\tau_1 < \infty) \\ &\leq \sup_{x \in \Gamma \cap D_1^c} P^{t_0}(x, \Gamma \setminus \bar{\Gamma}) \\ &\leq \exp \left( \left\lfloor \frac{\Delta_{\min}}{4\epsilon \Delta_{\max}} \right\rfloor \log(1 - \gamma) \right). \end{aligned}$$

The result then follows by (4.5) and (4.7).

Next, suppose (H2) holds. Note that in this case  $\check{\rho}_i \equiv u_i(\bar{\alpha})$  for all  $\bar{\alpha} \in \bar{\mathcal{A}}$ . Pick any  $\zeta < \frac{\Delta_{\min}^2}{4\Delta_{\max}}$ . As before we may suppose that agent 1 trembles. Let  $N^*(\epsilon) \triangleq \lfloor \zeta/\epsilon\Delta_{\min} \rfloor$ . Let  $\check{\tau}$  be the first time that an action profile in  $\mathcal{A} \setminus \bar{\mathcal{A}}$  has been played at least  $N^*(\epsilon)$  times. Then, at time  $\check{\tau}$  the aspiration level of the initially perturbed agent 1 satisfies

$$\rho_1(\check{\tau}) \leq \check{\rho}_1 + \zeta - \epsilon\Delta_{\min}N^*(\epsilon) \leq \check{\rho}_1,$$

while the aspiration level of any agent  $i \in \mathcal{I}$  satisfies

$$\rho_i(\check{\tau}) \geq \check{\rho}_i - \epsilon\Delta_{\max} \left\lfloor \frac{\zeta}{\epsilon\Delta_{\min}} \right\rfloor \geq \check{\rho}_i - \epsilon\Delta_{\max} \frac{\zeta}{\epsilon\Delta_{\min}} > \check{\rho}_i - \frac{\Delta_{\min}}{4}.$$

For  $k = 0, 1$  define the sets

$$\tilde{D}_k \triangleq \left\{ (\alpha, \rho) \in \mathcal{X} : \rho_i \leq \frac{\check{\rho}_i + (2k + 1)\hat{\rho}_i}{2k + 2}, i \in \mathcal{I} \right\},$$

and let  $\check{\tau}_k \triangleq \tau(\tilde{D}_k)$  for  $k = 0, 1$ . Also define

$$\tilde{\Gamma} \triangleq \left\{ (\alpha, \rho) \in \mathcal{X} : \rho_i \leq \check{\rho}_i - \frac{\Delta_{\min}^2}{4\Delta_{\max}}, i \in \mathcal{I} \right\}.$$

It is straightforward to show that  $\tilde{\mathbb{P}}_{\bar{s}}(X_{\check{\tau}_0} \in \tilde{\Gamma}) = 1$ . From this point on, we proceed as in the previous case.  $\square$

For the lemma that follows we need to define the following constant. For each  $\alpha^* \in \mathcal{A}^* \setminus \bar{\mathcal{A}}$ , select any  $\tilde{\alpha} \in \mathcal{A}$  and  $\{j_1, \dots, j_{n-1}\} \subset \mathcal{I}$  which satisfy Definition 2.2(c), and define

$$\Delta_0 \triangleq \frac{1}{2} \min_{\alpha^* \in \mathcal{A}^* \setminus \bar{\mathcal{A}}} \min_{1 \leq \ell \leq n-1} \min_{i \in \{j_1, \dots, j_{\ell+1}\}} \left\{ u_i(\alpha^*) - u_i(\tilde{\alpha}_{j_1}, \dots, \tilde{\alpha}_{j_\ell}, \alpha^*_{-\{j_1, \dots, j_\ell\}}) \right\}.$$

By Definition 2.2(c),  $\Delta_0 > 0$ .

LEMMA 4.2. *Suppose*

$$(4.8) \quad \epsilon < \frac{\Delta_0 \wedge \Delta_{\min}}{n\Delta_{\max}}.$$

*Then, for any strict coordination game  $\mathcal{G}$  for which  $\mathcal{A}^* \setminus \bar{\mathcal{A}} \neq \emptyset$ , there exists a constant  $M_0 = M_0(h, |\mathcal{A}|) > 0$  such that*

$$\hat{P}_{s^*\bar{s}} \geq \frac{M_0}{c\zeta \wedge (1-h)} \quad \forall s^* \in \mathcal{S}^* \setminus \bar{\mathcal{S}}, \bar{s} \in \bar{\mathcal{S}}.$$

*Proof.* Let  $s^* = (\alpha^*, \rho^*) \in \mathcal{S}^* \setminus \bar{\mathcal{S}}$ ,  $\bar{s} = (\bar{\alpha}, \bar{\rho}) \in \bar{\mathcal{S}}$ . Suppose  $\tilde{\alpha} \in \mathcal{A}$  and  $\{j_1, \dots, j_{n-1}\} \subset \mathcal{I}$  are the action profile and sequence of agents, respectively, corresponding to  $\alpha^*$  used in the calculation of  $\Delta_0$ . Consider the set of sample paths  $s(t) = (\alpha(t), \rho(t))$  satisfying  $s(0) = s^*$ ,  $\rho_{j_1}(1) \in (\rho_{j_1}^*, \rho_{j_1}^* + \zeta)$ ,  $\rho_{-j_1}(1) = \rho_{-j_1}^*$ , and  $\alpha(t) = (\tilde{\alpha}_{j_1}, \dots, \tilde{\alpha}_{j_t}, \alpha^*_{-\{j_1, \dots, j_t\}})$  for  $0 < t < n$ . We have

$$(4.9) \quad Q(s(0), s(1)) \geq \frac{1}{2n} \frac{(c\zeta \wedge (1-h))}{|\mathcal{A}_{j_1}|}.$$

By (4.8),  $\rho_i^* - \rho_i(t) \leq \Delta_0$  for all  $i \in \mathcal{I}$  and  $t \leq n$ . Therefore,

$$\rho_i(t) - u_i(\alpha(t)) \geq \Delta_0 \quad \forall i \in \{j_1, \dots, j_{t+1}\}$$

for  $0 \leq t < n$ , and hence we obtain

$$(4.10) \quad P(s(t-1), s(t)) \geq h^{n-1} \frac{(c\Delta_0 \wedge (1-h))}{|\mathcal{A}_{j_{t+1}}|}, \quad 1 < t < n,$$

and

$$(4.11) \quad P(s(n-1), \bar{s}) \geq \frac{(c\Delta_0 \wedge (1-h))^n}{|\mathcal{A}|}.$$

By (4.8), we have

$$(4.12) \quad \bar{\rho}_i - \rho_i(n) \geq \Delta_{\min} + \rho_i^* - \rho_i(n) > 0 \quad \forall i \in \mathcal{I}.$$

By (4.12),  $\Pi(s(n-1), \bar{s}) \geq P(s(n-1), \bar{s})$ . Consequently, the result follows by (4.9)–(4.11).  $\square$

**4.2. Main result.** We define inductively the collection of sets

$$\mathcal{S}_k \triangleq \left\{ s = (\alpha, \rho) \in \bigcup_{j=0}^{k-1} (\mathcal{S}_j)^c : \exists i \in \mathcal{I}, \alpha'_i \in \text{BR}_i(\alpha) \text{ satisfying (2.2)} \right. \\ \left. \text{and } (\alpha'_i, \alpha_{-i}) \in \mathcal{S}_{k-1} \right\}$$

for  $\mathcal{S}_0 = \mathcal{S}^* \cup \bar{\mathcal{S}}$ . For example,  $\mathcal{S}_1$  includes all pure strategy states for which there exist an agent  $i$  and an action  $\alpha'_i \in \text{BR}_i(\alpha)$  which satisfies (2.2) (i.e., makes no other player worse off) and also  $\alpha' = (\alpha'_i, \alpha_{-i}) \in \mathcal{S}_0$ . Also let  $K$  denote the maximum  $k$  for which  $\mathcal{S}_k$  is nonempty, i.e.,  $K \triangleq \max \{k \in \mathbb{N} : \mathcal{S}_k \neq \emptyset\}$ . Such  $K$  is well-defined since the set of action profiles  $\mathcal{A}$  is finite.

**LEMMA 4.3.** *In any coordination game, the collection of sets  $\{\mathcal{S}_k\}_{k=0}^K$  forms a partition of  $\mathcal{S}$ .*

*Proof.* By definition of the collection  $\{\mathcal{S}_k\}_{k=0}^K$ , the sets  $\mathcal{S}_k$  are mutually disjoint. It remains to show that their union coincides with  $\mathcal{S}$ . Assume not, i.e., assume that there exists  $s \in \mathcal{S}$  such that  $s = (\alpha, \rho) \notin \bigcup_{k=1}^K \mathcal{S}_k$ . According to the definition of a coordination game and Claim 2.1, there exists a sequence of action profiles  $\{\alpha^j\}$ , such that  $\alpha^0 = \alpha$  and  $\alpha^j = \text{BR}_i(\alpha^{j-1})$  for some  $i \in \mathcal{I}$  terminates in  $\mathcal{A}^* \cup \bar{\mathcal{A}}$ . Let  $\{s^j\}$  denote the sequence of pure strategy states which corresponds to  $\{\alpha^j\}$ . Then, for some  $j^*$  we have  $s^{j^*} \in \mathcal{S}^* \cup \bar{\mathcal{S}}$ , i.e.,  $s^{j^*} \in \mathcal{S}_0$ . Since  $s^{j^*} \in \mathcal{S}_0$ , then we should also have that  $s^{j^*-1} \in \mathcal{S}_1, \dots, s^0 = s \in \mathcal{S}_{j^*}$ . However, this conclusion contradicts our assumption that  $s \notin \bigcup_{k=1}^K \mathcal{S}_k$ . Thus,  $\bigcup_{k=1}^K \mathcal{S}_k = \mathcal{S}$ , and therefore, the collection of sets  $\{\mathcal{S}_k\}_{k=0}^K$  defines a partition for  $\mathcal{S}$ .  $\square$

**THEOREM 4.4.** *Let  $\mathcal{G}$  be a strict coordination game that satisfies either one of the hypotheses (H1) or (H2) in Lemma 4.1, and suppose that  $\zeta < C_0$ . Then  $\pi_{s_i} \rightarrow 0$  as  $\epsilon \downarrow 0$  for all  $s_i \notin \bar{\mathcal{S}}$ .*

*Proof.* Consider the partition of  $\mathcal{S}$  defined by the family of sets  $\{\mathcal{S}_k\}_{k=0}^K$ . Let  $\hat{P}_{\mathcal{S}_i \mathcal{S}_j}$  denote the substochastic matrix composed of the transition probabilities  $\hat{P}_{s_i s_j}$  for  $s_i \in \mathcal{S}_i$  and  $s_j \in \mathcal{S}_j$ . In other words  $[\hat{P}_{\mathcal{S}_i \mathcal{S}_j}]$  is the block decomposition of

$\hat{P}$  subordinate to the partition  $\{\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_K\}$ . Similarly, we define  $\tilde{\mathcal{S}}^* \triangleq \mathcal{S}^* \setminus \bar{\mathcal{S}}$ , and let

$$\begin{pmatrix} \hat{P}_{\bar{\mathcal{S}}\bar{\mathcal{S}}} & \hat{P}_{\bar{\mathcal{S}}\tilde{\mathcal{S}}^*} \\ \hat{P}_{\tilde{\mathcal{S}}^*\bar{\mathcal{S}}} & \hat{P}_{\tilde{\mathcal{S}}^*\tilde{\mathcal{S}}^*} \end{pmatrix}$$

denote the block decomposition of  $\hat{P}_{\mathcal{S}_0\mathcal{S}_0}$  subordinate to the partition  $(\bar{\mathcal{S}}, \tilde{\mathcal{S}}^*)$  of  $\mathcal{S}_0$ . From

$$\pi_{\bar{\mathcal{S}}} = \pi_{\bar{\mathcal{S}}}\hat{P}_{\bar{\mathcal{S}}\bar{\mathcal{S}}} + \pi_{\bar{\mathcal{S}}^c}\hat{P}_{\bar{\mathcal{S}}^c\bar{\mathcal{S}}},$$

we obtain

$$\pi_{\bar{\mathcal{S}}}(I - \hat{P}_{\bar{\mathcal{S}}\bar{\mathcal{S}}}) = \pi_{\bar{\mathcal{S}}}\hat{P}_{\bar{\mathcal{S}}\bar{\mathcal{S}}^c} = \pi_{\bar{\mathcal{S}}^c}\hat{P}_{\bar{\mathcal{S}}^c\bar{\mathcal{S}}}.$$

By Lemma 4.1,  $\hat{P}_{\bar{\mathcal{S}}\bar{\mathcal{S}}^c} \rightarrow 0$  as  $\epsilon \rightarrow 0$ , while by Lemma 4.2 for some positive constant  $\tilde{\delta}$ , which does not depend on  $\epsilon$ , we have  $\hat{P}_{\tilde{\mathcal{S}}^*\bar{\mathcal{S}}}\mathbf{1} \geq \tilde{\delta}\mathbf{1}$ . Thus,

$$\tilde{\delta}\pi_{\tilde{\mathcal{S}}^*}\mathbf{1} \leq \pi_{\tilde{\mathcal{S}}^*}\hat{P}_{\tilde{\mathcal{S}}^*\bar{\mathcal{S}}}\mathbf{1} \leq \pi_{\bar{\mathcal{S}}}\hat{P}_{\bar{\mathcal{S}}\bar{\mathcal{S}}^c}\mathbf{1} = \pi_{\bar{\mathcal{S}}^c}\hat{P}_{\bar{\mathcal{S}}^c\bar{\mathcal{S}}}\mathbf{1} \xrightarrow{\epsilon \rightarrow 0} 0,$$

and we obtain

$$(4.13) \quad \pi_{\tilde{\mathcal{S}}^*} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Similarly, from the equation  $\pi_{\mathcal{S}_0} = \pi_{\mathcal{S}_0}\hat{P}_{\mathcal{S}_0\mathcal{S}_0} + \pi_{\mathcal{S}_0^c}\hat{P}_{\mathcal{S}_0^c\mathcal{S}_0}$ , we obtain  $\pi_{\mathcal{S}_0}\hat{P}_{\mathcal{S}_0\mathcal{S}_0^c}\mathbf{1} = \pi_{\mathcal{S}_0^c}\hat{P}_{\mathcal{S}_0^c\mathcal{S}_0}\mathbf{1}$ . It is straightforward to show, using Definition 2.2(b), that for some positive constant  $\hat{\delta}$ , which does not depend on  $\epsilon$ , we have  $\hat{P}_{\mathcal{S}_k\mathcal{S}_{k+1}}\mathbf{1} \geq \hat{\delta}\mathbf{1}$  for all  $k \geq 0$ . Combining the equations above we get

$$\begin{aligned} \hat{\delta}\pi_{\mathcal{S}_0}\mathbf{1} &\leq \pi_{\mathcal{S}_0}\hat{P}_{\mathcal{S}_0\mathcal{S}_1}\mathbf{1} \leq \pi_{\mathcal{S}_0}\hat{P}_{\mathcal{S}_0\mathcal{S}_0^c}\mathbf{1} \\ &= \pi_{\mathcal{S}_0^c}\hat{P}_{\mathcal{S}_0^c\mathcal{S}_0}\mathbf{1} \\ &= \pi_{\bar{\mathcal{S}}}\hat{P}_{\bar{\mathcal{S}}\bar{\mathcal{S}}}\mathbf{1} + \pi_{\tilde{\mathcal{S}}^*}\hat{P}_{\tilde{\mathcal{S}}^*\bar{\mathcal{S}}}\mathbf{1} \xrightarrow{\epsilon \rightarrow 0} 0, \end{aligned}$$

where in the last line we used Lemma 4.1 and (4.13). Thus, we have shown that  $\pi_{\mathcal{S}_0} \rightarrow 0$  as  $\epsilon \rightarrow 0$ . We proceed by induction. Suppose  $\pi_{\mathcal{S}_k} \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Then,

$$\hat{\delta}\pi_{\mathcal{S}_{k+1}}\mathbf{1} \leq \pi_{\mathcal{S}_{k+1}}\hat{P}_{\mathcal{S}_{k+1}\mathcal{S}_k}\mathbf{1} \leq \pi_{\mathcal{S}_k}\mathbf{1} \xrightarrow{\epsilon \rightarrow 0} 0,$$

which shows that  $\pi_{\mathcal{S}_{k+1}} \rightarrow 0$  as  $\epsilon \rightarrow 0$ . By Lemma 4.3, the proof is complete.  $\square$

Theorem 4.4 combined with Theorem 3.2 provides a complete characterization of the time average asymptotic behavior of aspiration learning in strict coordination games.

**4.3. Simulations in network formation games.** In this section, we demonstrate the asymptotic behavior of aspiration learning in coordination games as described by Theorems 3.2 and 4.4. Consider the network formation game of section 2.2 which, according to Claim 2.2, is a (nonstrict) coordination game. Although Theorem 4.4 was only shown for strict coordination games, our intention here is to demonstrate that it also applies to the larger class of (nonstrict) coordination games.

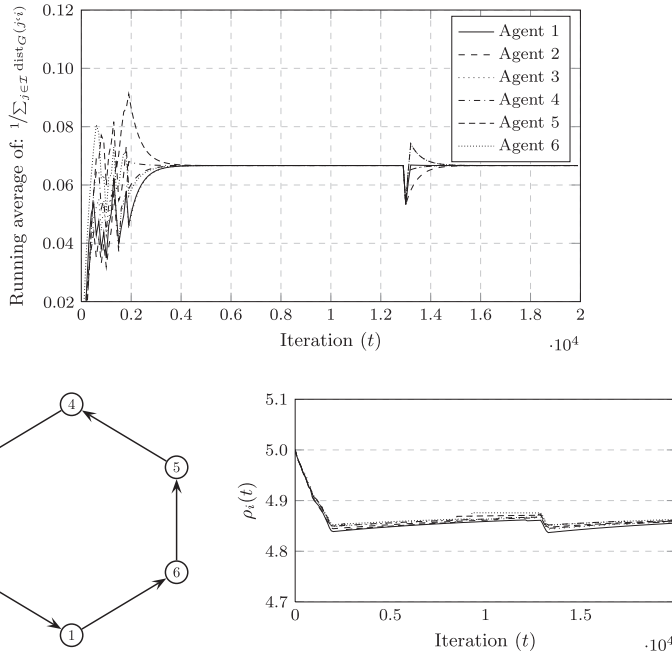


FIG. 4.1. A typical response of aspiration learning in the network formation game.

We consider a set of six nodes deployed on the plane, so that the neighbors of each node are the two immediate nodes (e.g.,  $\mathcal{N}_1 = \{2, 6\}$ ). Note that a payoff-dominant set of networks exists and corresponds to the wheel networks, where each node has a single link. We pick the set  $\bar{\mathcal{A}}$  of desirable networks as the set of wheel networks. Note that the set  $\bar{\mathcal{A}}$  satisfies hypothesis (H2) of Lemma 4.1.

In order for the average behavior to be observed,  $\lambda$  and  $\epsilon$  need to be sufficiently small. We choose  $h = 0.01$ ,  $c = 0.2$ ,  $\zeta = 0.01$ ,  $\epsilon = \lambda = 0.0001$ , and  $\nu = 1/8$ . In Figure 4.1, we have plotted a typical response of aspiration learning for this setup, where the final graph and the aspiration level as a function of time are shown.

To better illustrate the response of aspiration learning, define the distance from node  $j$  to node  $i$ , denoted  $\text{dist}_G(j, i)$ , as the minimum number of hops from  $j$  to  $i$ . We also adopt the convention  $\text{dist}_G(i, i) = 0$  and  $\text{dist}_G(j, i) = \infty$  if there is no path from  $j$  to  $i$  in  $G$ . Figure 4.1 also plots, for each node, the running average of the inverse total distance from all other nodes, i.e.,  $1/\sum_{j \in \mathcal{I}} \text{dist}_G(j, i)$ . This number is zero if the node is disconnected from any of its immediate neighbors.

We observe that the payoff-dominant profile (wheel network) is played with frequency that approaches one. In fact, the aspiration level converges to  $(n - 1) - \nu = 4.875$  and the inverse total distance converges to  $1/15 \approx 0.067$ , both of which correspond to the wheel network of Figure 4.1.

**5. Fairness in symmetric and coordination games.** In several coordination games, establishing convergence (in the way defined by Theorem 3.2) to the set of desirable states  $\bar{\mathcal{S}}$  (as Theorem 4.4 showed) may not be sufficient. For example, in common-pool games of section 2.3, convergence to  $\bar{\mathcal{S}}$  does not guarantee that all agents get access to the common resource in a *fair* schedule. In the remainder of this section, we establish conditions under which *fairness* is also established.

**5.1. A property of finite Markov chains.** In this section, we provide an approach on characterizing explicitly the invariant distribution of a finite-state, irreducible and aperiodic Markov chain. We use a characterization introduced by [8], which has been extensively used for showing stochastic stability arguments for several learning dynamics; see, e.g., [19, 29]. In particular, for finite Markov chains an invariant distribution can be expressed as the ratio of sums of products consisting of transition probabilities. These products can be described conveniently by means of graphs on the set of states of the chain.

Let  $\mathcal{S}$  be a finite set of states, whose elements are denoted by  $s_k, s_\ell$ , etc., and let  $\mathcal{W}$  be a subset of  $\mathcal{S}$ .

**DEFINITION 5.1** ( $\mathcal{W}$ -graph). *A graph consisting of arrows  $s_k \rightarrow s_\ell$  ( $s_k \in \mathcal{S} \setminus \mathcal{W}, s_\ell \in \mathcal{S}, s_\ell \neq s_k$ ) is called a  $\mathcal{W}$ -graph if it satisfies the following conditions:*

1. *every point  $k \in \mathcal{S} \setminus \mathcal{W}$  is the initial point of exactly one arrow;*
2. *there are no closed cycles in the graph; or, equivalently, for any point  $s_k \in \mathcal{S} \setminus \mathcal{W}$  there exists a sequence of arrows leading from it to some point  $s_\ell \in \mathcal{W}$ .*

We denote by  $\mathcal{G}\{\mathcal{W}\}$  the set of  $\mathcal{W}$ -graphs; we shall use the letter  $g$  to denote graphs.

If  $\hat{P}_{s_k s_\ell}$  are nonnegative numbers, where  $s_k, s_\ell \in \mathcal{S}$ , define the product

$$\varpi(g) \triangleq \prod_{(s_k \rightarrow s_\ell) \in g} \hat{P}_{s_k s_\ell}.$$

The following lemma holds.

**LEMMA 5.2** (see Lemma 6.3.1 in [8]). *Let us consider a Markov chain with a finite set of states  $\mathcal{S}$  and transition probabilities  $\{\hat{P}_{s_k s_\ell}\}$  and assume that every state can be reached from any other state in a finite number of steps. Then the stationary distribution of the chain is  $\pi = [\pi_s]$ , where*

$$\pi_s = \frac{R_s}{\sum_{s_i \in \mathcal{S}} R_{s_i}}, \quad s \in \mathcal{S},$$

and  $R_s \triangleq \sum_{g \in \mathcal{G}\{s\}} \varpi(g)$ .

**5.2. Fairness in symmetric games.** In this section, using Theorem 3.2 and Lemma 5.2 we establish fairness in *symmetric* games, defined as follows.

**DEFINITION 5.3** (symmetric game). *A game  $\mathcal{G}$  characterized by the action profile set  $\mathcal{A}$  is symmetric if, for any two agents  $i, j \in \mathcal{I}$  and any action profile  $\alpha \in \mathcal{A}$ , the following hold: (a) if  $\alpha_i = \alpha_j$ , then  $u_i(\alpha) = u_j(\alpha)$ , and (b) if  $\alpha_i \neq \alpha_j$ , then there exists an action profile  $\alpha' \in \mathcal{A} \setminus \{\alpha\}$ , such that the following two conditions are satisfied:*

1.  $\alpha'_i = \alpha_j$ ,  $\alpha_i = \alpha'_j$ , and  $\alpha'_k = \alpha_k$  for all  $k \neq i, j$ ;
2.  $u_i(\alpha') = u_j(\alpha)$ ,  $u_i(\alpha) = u_j(\alpha')$ , and  $u_k(\alpha') = u_k(\alpha)$  for any  $k \neq i, j$ .

Define the following equivalence relation between states in  $\mathcal{S}$ .

**DEFINITION 5.4** (state equivalence). *For any two pure-strategy states  $s, s' \in \mathcal{S}$  such that  $s \neq s'$ , let  $\alpha$  and  $\alpha'$  denote the corresponding action profiles. We write  $s \sim s'$  if there exist  $i, j \in \mathcal{I}, i \neq j$ , such that the following two conditions are satisfied:*

1.  $\alpha'_i = \alpha_j$ ,  $\alpha_i = \alpha'_j$ , and  $\alpha'_k = \alpha_k$  for all  $k \neq i, j$ ;
2.  $u_i(\alpha') = u_j(\alpha)$ ,  $u_i(\alpha) = u_j(\alpha')$ , and  $u_k(\alpha') = u_k(\alpha)$  for any  $k \neq i, j$ .

Since there is a one-to-one correspondence between  $\mathcal{S}$  and  $\mathcal{A}$ , we also say that two action profiles  $\alpha$  and  $\alpha'$  are equivalent, if the conditions of Definition 5.4 are satisfied.

**LEMMA 5.5.** *For any symmetric game and for any two pure-strategy states  $s, s' \in \mathcal{S}$  such that  $s \sim s'$ ,  $\pi_s = \pi_{s'}$ .*

*Proof.* Let us consider any two pure-strategy states  $s, s' \in \mathcal{S}$  such that  $s \sim s'$ . Let us also consider any  $\{s\}$ -graph  $g$ , i.e.,  $g \in \mathcal{G}\{s\}$ . Such a graph can be identified as a collection of paths, i.e., for some  $M \geq 1$ , we have  $g = \bigcup_{m=1}^M g_m$ , where

$$g_m = \bigcup_{\ell=1}^{L(m)-1} (s_{\kappa_m(\ell)} \rightarrow s_{\kappa_m(\ell+1)})$$

for some  $L(m) \geq 1$ . In the above expression, the function  $\kappa_m$  provides an enumeration of the states that belong to the path  $g_m$ . Note that due to the definition of  $\mathcal{G}\{s\}$ -graphs, we should have that  $s_{\kappa_m(L(m))} = s$  for all  $m = 1, \dots, M$ . Moreover, if  $M > 1$ , we should also have

$$\bigcap_{m=1}^M \{s_{\kappa_m(1)}, \dots, s_{\kappa_m(L(m)-1)}\} = \emptyset,$$

i.e., the paths  $\{g_m\}$  do not cross each other, except at node  $s$ .

Let us consider any other state  $s' \in \mathcal{S}$  such that  $s' \sim s$ . Since the game is symmetric, for any graph  $g \in \mathcal{G}\{s\}$ , there exists a unique graph  $g' \in \mathcal{G}\{s'\}$  which satisfies  $g' = \bigcup_{m=1}^M g'_m$ , where

$$g'_m = \bigcup_{\ell=1}^{L(m)-1} (s'_{\kappa_m(\ell)} \rightarrow s'_{\kappa_m(\ell+1)})$$

and  $s_{\kappa_m(\ell)} \sim s'_{\kappa_m(\ell)}$ ,  $\ell = 1, \dots, L(m)$  for all  $m \in \{1, \dots, M\}$ .

The transition probability between any two states is a sum of probabilities of sequences of action profiles. Since the game is symmetric, for any such sequence of action profiles which leads, for instance, from  $s_{\kappa_m(\ell)}$  to  $s_{\kappa_m(\ell+1)}$ , there exists an equivalent sequence of action profiles which leads from  $s'_{\kappa_m(\ell)}$  to  $s'_{\kappa_m(\ell+1)}$ . Therefore, we should have that

$$\hat{P}_{s_{\kappa_m(\ell)} s_{\kappa_m(\ell+1)}} = \hat{P}_{s'_{\kappa_m(\ell)} s'_{\kappa_m(\ell+1)}}$$

for any  $m = 1, \dots, M$ , and hence,  $\varpi(g') = \varpi(g)$ . In other words, there exists an isomorphism between the graphs in the sets  $\mathcal{G}\{s\}$  and  $\mathcal{G}\{s'\}$ , such that any two isomorphic graphs have the same transition probability. Thus, we have  $\pi_s = \pi_{s'}$  for any two states  $s, s'$  such that  $s \sim s'$ .  $\square$

Lemma 5.5 can be used to provide a more explicit characterization of the invariant distribution  $\pi$  in several classes of coordination games which are also symmetric, e.g., common-pool games.

**5.3. Fairness in common-pool games.** First, recall that in common-pool games we define the set of “desirable” or “successful” action profiles  $\bar{\mathcal{A}}$  as in (2.4). To characterize more explicitly the invariant distribution  $\pi$ , we define the subset of pure-strategy states  $\bar{\mathcal{S}}_i$  that correspond to “successful” states for agent  $i$  by

$$\bar{\mathcal{S}}_i \triangleq \{s \in \mathcal{S} : \alpha_i > \alpha_j \ \forall j \neq i\}.$$

In other words,  $\bar{\mathcal{S}}_i$  corresponds to the set of pure-strategy states in which the action of agent  $i$  is strictly larger than the action of any other agent  $j \neq i$ . We also define  $\bar{\mathcal{S}} \triangleq \bigcup_{i \in \mathcal{I}} \bar{\mathcal{S}}_i$ .

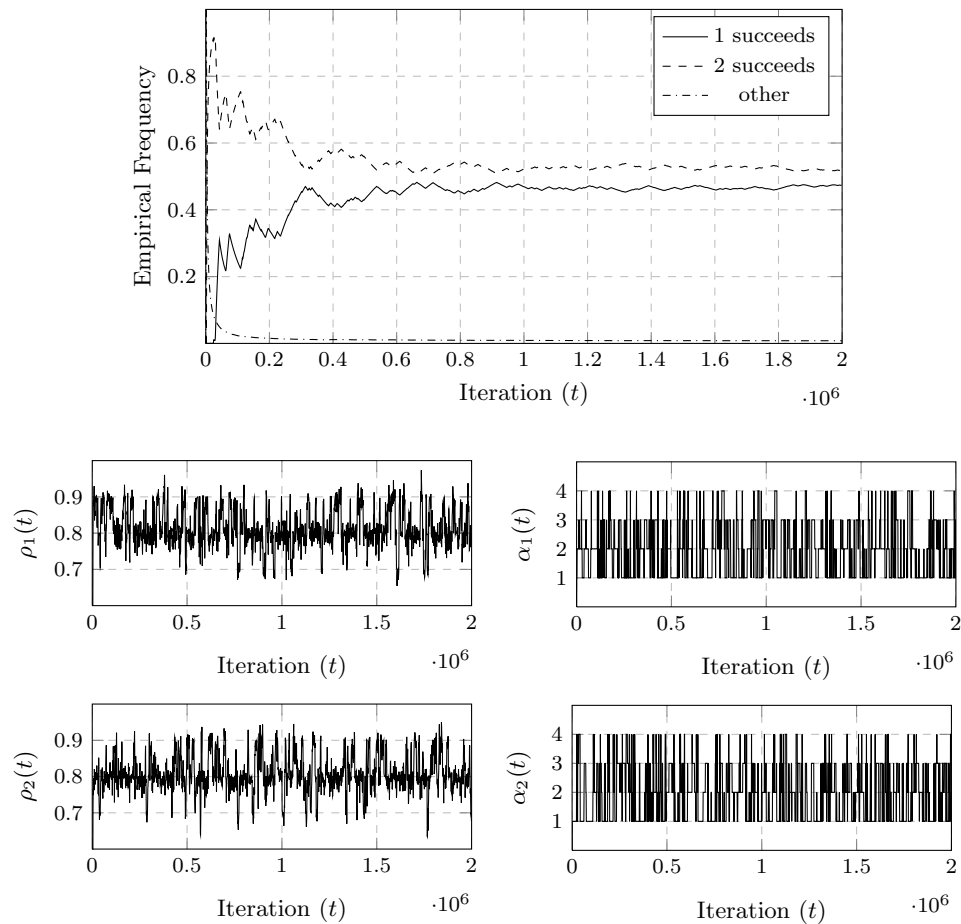


FIG. 5.1. A typical response of aspiration learning in a common-pool game with two players and four actions.

Note that the equivalence relation  $\sim$  defines an isomorphism among the states of any two sets  $\bar{\mathcal{S}}_i$  and  $\bar{\mathcal{S}}_j$  for any  $i \neq j$ . This is due to the fact that for any state  $s_i \in \bar{\mathcal{S}}_i$ , there exists a unique state  $s_j \in \bar{\mathcal{S}}_j$  such that  $s_i \sim s_j$ .

LEMMA 5.6. For any common-pool game,  $\pi_{\bar{\mathcal{S}}_1} = \dots = \pi_{\bar{\mathcal{S}}_n}$ .

*Proof.* As already mentioned, for any  $i, j \in \mathcal{I}$  such that  $i \neq j$  and for any state  $s_i \in \bar{\mathcal{S}}_i$ , there exists a unique state  $s_j \in \bar{\mathcal{S}}_j$  such that  $s_j \sim s_i$ . Therefore, the sets  $\bar{\mathcal{S}}_i$  and  $\bar{\mathcal{S}}_j$  are isomorphic with respect to the equivalence relation  $\sim$ . Since a common-pool game is symmetric, from Lemma 5.5 we conclude that  $\pi_{\bar{\mathcal{S}}_1} = \dots = \pi_{\bar{\mathcal{S}}_n}$ .  $\square$

THEOREM 5.7. Let  $\mathcal{G}$  be a common-pool game which satisfies hypothesis (H1) of Lemma 4.1. There exists a constant  $C_0 > 0$  such that for any  $\zeta < C_0$ ,

$$\pi_{\bar{\mathcal{S}}_i} \xrightarrow{\epsilon \downarrow 0} \frac{1}{n}$$

for all  $i \in \mathcal{I}$ .



*Proof.* First, we recognize that the sets  $\{\bar{\mathcal{S}}_i\}$  are mutually disjoint, and  $\bigcup_{i=1}^n \bar{\mathcal{S}}_i = \bar{\mathcal{S}}$ . Then, by Theorem 4.4, and for any  $\zeta < \frac{1}{2}(\Delta_{\min} - \delta^*)$ , we have  $\pi_{\bar{\mathcal{S}}} = \sum_{i=1}^n \pi_{\bar{\mathcal{S}}_i} \rightarrow 1$  as  $\epsilon \rightarrow 0$ . Last, by Lemma 5.6, the conclusion follows.  $\square$

In other words, we have shown that the invariant distribution  $\pi$  puts equal weight on either agent “succeeding,” which establishes a form of *fairness* over time. Moreover, it puts zero weight on states outside  $\bar{\mathcal{S}}$  (i.e., states which correspond to “failures”) as  $\epsilon \rightarrow 0$ .

**5.4. Simulations in common-pool games.** Theorems 3.2 and 5.7 provide a characterization of the asymptotic behavior of aspiration learning in common-pool games as  $\lambda$  and  $\epsilon$  approach zero. In fact, according to Remark 3.1, the expected percentage of time that the aspiration learning spends in any one of the pure-strategy sets  $\bar{\mathcal{S}}_i$  should be equal as the perturbation probability  $\lambda \downarrow 0$  and  $t \rightarrow \infty$  (i.e., fairness is established). Moreover, the expected percentage of “failures” (i.e., states outside  $\bar{\mathcal{S}}$ ) approaches zero as  $\epsilon \downarrow 0$ .

We consider the following setup for aspiration learning:  $\lambda = \epsilon = 0.001$ ,  $h = 0.01$ ,  $c = 0.05$ , and  $\zeta = 0.05$ . Also, we consider a common-pool game of two players and four actions, where  $c_0 = 0$ ,  $c_1 = 0.1$ ,  $c_2 = 0.2$ ,  $c_3 = 0.3$ , and  $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0.8$ . Note that the maximum payoff difference within the same action profile is  $\delta^* = 0.1$ , and the minimum payoff difference between  $\bar{\mathcal{A}}$  and  $\mathcal{A} \setminus \bar{\mathcal{A}}$  is  $\Delta_{\min} = 0.6$ . Therefore, the hypotheses of Theorem 5.7 are clearly satisfied since  $\delta^* < \Delta_{\min}$  and  $\zeta < \frac{1}{2}(\Delta_{\min} - \delta^*)$ . Under this setup, Figure 5.1 demonstrates the response of aspiration learning. We observe, as Theorem 5.7 predicts, that the frequency with which either agent succeeds approaches  $1/2$  as time increases. Also, the frequency of collisions (i.e., the joint actions in which neither agent succeeds) approaches zero as time increases.

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