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CONVERGENCE OF THE RELATIVE VALUE ITERATION FOR THE ERGODIC CONTROL PROBLEM OF NONDEGENERATE DIFFUSIONS UNDER NEAR-MONOTONE COSTS*

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Abstract. We study the relative value iteration for the ergodic control problem under a near-monotone running cost structure for a nondegenerate diffusion controlled through its drift. This algorithm takes the form of a quasi-linear parabolic Cauchy initial value problem in $\mathbb{R}^d$. We show that this Cauchy problem stabilizes or, in other words, that the solution of the quasi-linear parabolic equation converges for every bounded initial condition in $\mathcal{C}^2(\mathbb{R}^d)$ to the solution of the Hamilton–Jacobi–Bellman equation associated with the ergodic control problem.

Key words. controlled diffusions, ergodic control, Hamilton–Jacobi–Bellman equation, relative value iteration, parabolic Cauchy problem

AMS subject classifications. Primary, 93E15, 93E20; Secondary, 60J25, 60J60, 90C40

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1. Introduction. This paper is concerned with the time-asymptotic behavior of an optimal control problem for a nondegenerate diffusion controlled through its drift and described by an Itô stochastic differential equation (SDE) in $\mathbb{R}^d$ having the following form:

$$\mathrm{d}X_t = b(X_t, U_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t. \tag{1.1}$$

Here U_t is the control variable that takes values in some compact metric space. We impose standard assumptions on the data to guarantee the existence and uniqueness of solutions to (1.1). These are described in section 3.1. Let $r\colon \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}$ be a continuous function bounded from below, which without loss of generality we assume is nonnegative and is referred to as the *running cost*. As is well known, the ergodic control problem, in its *almost sure* (or *pathwise*) formulation, seeks to a.s. minimize over all admissible controls U the functional

$$\limsup_{t\to\infty}\ \frac{1}{t}\int_0^t r(X_s, U_s)\,\mathrm{d}s. \tag{1.2}$$

A weaker, *average* formulation seeks to minimize

$$\limsup_{t\to\infty}\ \frac{1}{t}\int_0^t \mathbb{E}^U\bigl[r(X_s, U_s)\bigr]\,\mathrm{d}s. \tag{1.3}$$

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Here $\mathbb{E}^U$ denotes the expectation operator associated with the probability measure on the canonical space of the process under the control U. We let ϱ be defined as

$$\varrho \triangleq \inf_U \limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}^U\bigl[r(X_s,U_s)\bigr]\,\mathrm{d}s, \tag{1.4}$$

i.e., the infimum of (1.3) over all admissible controls. (For the definition of admissible controls see section 3.1.) Under suitable hypotheses, solutions to the ergodic control problem can be constructed via the Hamilton–Jacobi–Bellman (HJB) equation

$$a^{ij}(x)\,\partial_{ij}V + H(x,\nabla V) = \varrho, \tag{1.5}$$

where $a = [a^{ij}]$ is the symmetric matrix $\frac{1}{2}\upsigma\upsigma^{\mathsf{T}}$ and

$$H(x,p) \triangleq \min_u \;\{b(x,u)\cdot p + r(x,u)\}\,.$$

The desired characterization is that a stationary Markov control v^* is optimal for the ergodic control problem if and only if it satisfies

$$H\bigl(x,\nabla V(x)\bigr) = b\bigl(x,v^*(x)\bigr)\cdot\nabla V(x) + r\bigl(x,v^*(x)\bigr) \tag{1.6}$$

a.e. in $\mathbb{R}^d$. Obtaining solutions to (1.5) is further complicated by the fact that ϱ is unknown. For controlled Markov chains the *relative value iteration* (RVI) originating in the work of White [23] provides an algorithm for solving the ergodic dynamic programming equation for the finite state, finite action case. Moreover, its ramifications have given rise to popular learning algorithms (Q-learning) [1].

In [3] we introduced a continuous time, continuous state space analogue of White's RVI given by the quasi-linear parabolic evolution equation

$$\partial_t\varphi(t,x) = a^{ij}(x)\,\partial_{ij}\varphi(t,x) + H(x,\nabla\varphi) - \varphi(t,0), \quad \varphi(0,x) = \varphi_0(x). \tag{1.7}$$

Under a uniform (geometric) ergodicity condition that ensures the well-posedness of the associated HJB equation we showed in [3] that the solution of (1.7) converges as $t\to\infty$ to a solution of (1.5), the limit being independent of the initial condition φ_0. In a related work we extended these results to zero-sum stochastic differential games and controlled diffusions under the risk sensitive criterion [5].

Even though the work in [3] was probably the first such study of convergence of a relative iteration scheme for continuous time and space Markov processes, the blanket stability hypothesis imposed weakens these results. Models of controlled diffusions enjoying a uniform geometric ergodicity do not arise often in applications. Rather, what we frequently encounter is a running cost which has a structure that penalizes unstable behavior and thus renders all stationary optimal controls stable. Such is the case for quadratic costs typically used in linear control models. A fairly general class of running costs of this type, which includes "norm-like" costs, consists of costs satisfying the *near-monotone* condition:

$$\liminf_{|x|\to\infty} \min_{u\in\mathbb{U}} r(x,u) > \varrho. \tag{1.8}$$

In this paper we relax the blanket geometric ergodicity assumption and study the RVI in (1.7) under the near-monotone hypothesis (1.8). It is well known that for near-monotone costs the HJB equation (1.5) possesses a unique up to a constant solution

V which is bounded below in $\mathbb{R}^d$ [4]. However, this uniqueness result is restricted. In general, for $\beta > \varrho$ the equation

$$a^{ij}(x)\,\partial_{ij}V + H(x,\nabla V) = \beta \tag{1.9}$$

can have a multitude of solutions which are bounded below [4, section 3.8.1]. As a result, the policy iteration algorithm (PIA) may fail to converge to an optimal value [2, 19]. In order to guarantee convergence of the PIA to an optimal control, in addition to the near-monotone assumption, a blanket Lyapunov condition is imposed in [19, Theorem 5.2] which renders all stationary Markov controls stable. In contrast, the RVI algorithm always converges to the optimal value function when initialized with some bounded initial value φ_0. The reason behind the difference in performance of the two algorithms can be explained as follows: First, recall that the PIA algorithm consists of the following steps:

1. Initialization. Set $k = 0$ and select some stationary Markov control v_0 which yields a finite average cost.
2. Value determination. Determine the average cost ϱ_{v_k} under the control v_k and obtain a solution V_k to the Poisson equation
$$a^{ij}(x)\,\partial_{ij}V_k + b^i\bigl(x,v_k(x)\bigr)\partial_i V_k(x) + r\bigl(x,v_k(x)\bigr) = \varrho_{v_k}, \quad x\in\mathbb{R}^d.$$
3. Termination. If
$$H(x,\nabla V_k) = \bigl[b\bigl(x,v_k(x)\bigr)\cdot\nabla V_k(x) + r\bigl(x,v_k(x)\bigr)\bigr] \quad \text{a.e.},$$
then return v_k.
4. Policy improvement. Select a stationary Markov control v_{k+1} which satisfies
$$v_{k+1}(x) \in \operatorname*{Arg\,min}_{u\in\mathbb{U}} \bigl[b(x,u)\cdot\nabla V_k(x) + r(x,u)\bigr], \qquad x\in\mathbb{R}^d.$$

It is straightforward to show that if $\widehat{V}$ is a solution to (1.9), which is bounded below in $\mathbb{R}^d$, and whose growth rate does not exceed the growth rate of an optimal value function V from (1.5), or in other words the weighted norm $\|\widehat{V}\|_V$ is finite (see Definition 3.6), then $\beta = \varrho$ and $\widehat{V}$ is an optimal value function. It turns out that if the value function V_0 determined at the first step $k = 0$ does not grow faster than an optimal value function V, then the algorithm will converge to an optimal value function. Otherwise, it might converge to a solution of (1.9) that is not optimal. However, if the growth rate of an optimal value function is not known, there is no simple way of selecting the initial control v_0 that will result in the right growth rate for V_0. To do so one must solve an HJB-type equation, which is precisely what the PIA algorithm tries to avoid. In contrast, as we show in this paper, the solution of the RVI algorithm has the property that $x \mapsto \varphi(t,x)$ has the same growth rate as the optimal value function V, asymptotically in t. This is an essential ingredient of the mechanism responsible for convergence.

The proof of convergence of (1.7) is facilitated by the study of the *value iteration* (VI) equation

$$\partial_t\overline{\varphi}(t,x) = a^{ij}(x)\,\partial_{ij}\overline{\varphi}(t,x) + H(x,\nabla\overline{\varphi}) - \varrho, \quad \overline{\varphi}(0,x) = \varphi_0(x). \tag{1.10}$$

The initial condition is the same as in (1.7). Also ϱ is as in (1.4), so it is assumed known. Note that if φ is a solution of (1.7), then

$$\overline{\varphi}(t,x) = \varphi(t,x) - \varrho\, t + \int_0^t \varphi(s,0)\,\mathrm{d}s, \qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}^d, \tag{1.11}$$

solves (1.10). We have in particular that

$$\overline{\varphi}(t,x) - \overline{\varphi}(t,0) = \varphi(t,x) - \varphi(t,0), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{1.12}$$

It follows that the function $f \triangleq \varphi - \overline{\varphi}$ does not depend on $x \in \mathbb{R}^d$ and satisfies

$$\frac{\mathrm{d}f}{\mathrm{d}t} + f = \varrho - \overline{\varphi}(t,0). \tag{1.13}$$

Conversely, if $\overline{\varphi}$ is a solution of (1.10), then solving (1.13) one obtains a corresponding solution of (1.7) that takes the form [3, Lemma 4.4]

$$\varphi(t,x) = \overline{\varphi}(t,x) - \int_0^t \mathrm{e}^{s-t}\, \overline{\varphi}(s,0)\,\mathrm{d}s + \varrho\,(1-\mathrm{e}^{-t}), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{1.14}$$

It also follows from (1.14) that if $t \mapsto \overline{\varphi}(t,x)$ is bounded for each $x \in \mathbb{R}^d$, then so is the map $t \mapsto \varphi(t,x)$, and if the former converges as $t \to \infty$, pointwise in x, then so does the latter.

We note here that we study solutions of the VI equation that have the stochastic representation

$$\overline{\varphi}(t,x) = \inf_U \; \mathbb{E}_x^U \left[\int_0^t r(X_s,U_s)\,\mathrm{d}s + \varphi_0(X_t)\right] - \varrho\, t, \tag{1.15}$$

where the infimum is over all admissible controls. These are called *canonical solutions* (see Definition 3.10). The first term in (1.15) is the total cost over the finite horizon $[0,t]$ with terminal penalty φ_0. Under the uniform geometric ergodicity hypothesis used in [3] it is straightforward to show that $t \mapsto \overline{\varphi}(t,x)$ is locally bounded in $x \in \mathbb{R}^d$. In contrast, under the near-monotone hypothesis alone, $t \mapsto \overline{\varphi}(t,x)$ may diverge for each $x \in \mathbb{R}^d$. To show convergence, we first identify a suitable region of attraction of the solutions of the HJB under the dynamics of (1.10) and then show that all ω-limit points of the semiflow of (1.7) lie in this region.

While we prefer to think of (1.7) as a continuous time and space RVI, it can also be viewed as a "stabilization of a quasi-linear parabolic PDE problem" analogous to the celebrated result of Hasminskiĭ (see [11]). Thus, the results in this paper are also likely to be of independent interest to the PDE community.

We summarize below the main result of the paper. We make one mild assumption: let v^* be some optimal stationary Markov control, i.e., a measurable function that satisfies (1.6). It is well known that under the near-monotone hypothesis the diffusion under the control v^* is positive recurrent. Let μ_{v^*} denote the unique invariant probability measure of the diffusion under the control v^*. We assume that the value function V in the HJB is integrable under μ_{v^*}.

THEOREM 1.1. *Suppose that the running cost is near-monotone and that the value function V of the HJB equation* (1.5) *for the ergodic control problem is integrable with respect to some optimal invariant probability distribution. Then for any bounded initial condition $\varphi_0 \in \mathcal{C}^2(\mathbb{R}^d)$ it holds that*

$$\lim_{t\to\infty} \varphi(t,x) = V(x) - V(0) + \varrho,$$

uniformly on compact sets of $\mathbb{R}^d$.

We also obtain a new stochastic representation for the value function of the HJB under near-monotone costs which we state as a corollary. This result is known to hold under uniform geometric ergodicity, but under the near-monotone cost hypothesis alone it is completely new.

COROLLARY 1.2. *Under the assumptions of Theorem* 1.1 *the value function V of the HJB for the ergodic control problem has the stochastic representation*

$$V(x) - V(y) = \lim_{t\to\infty} \left(\inf_U \mathbb{E}_x^U \left[\int_0^t r(X_s, U_s)\,\mathrm{d}s\right] - \inf_U \mathbb{E}_y^U \left[\int_0^t r(X_s, U_s)\,\mathrm{d}s\right]\right)$$

for all $x, y \in \mathbb{R}^d$.

We would like to note here that in [7] the authors study the VI algorithm for countable state controlled Markov chains with norm-like running costs, i.e., $\min_u\ r(x,u) \to \infty$ as $|x| \to \infty$. The initial condition φ_0 is chosen as a Lyapunov function corresponding to some stable control v_0. We leave it to the reader to verify that under these hypotheses $\|V\|_{\varphi_0} < \infty$. Moreover they assume that φ_0 is integrable with respect to the invariant probability distribution μ_{v^*}. (See the earlier discussion concerning the PIA algorithm.) Thus their hypotheses imply that the optimal value function V from (1.5) is also integrable with respect to μ_{v^*}.

Work related to this paper has appeared in [12,13,21]. In [12] the author considers a d-dimensional controlled diffusion governed by

$$\mathrm{d}X_t = U_t\,\mathrm{d}t + \mathrm{d}W_t, \tag{1.16}$$

where the control process also lives in $\mathbb{R}^d$ and the running cost is of the form

$$r(x,u) = l(x,u) + f(x), \tag{1.17}$$

and subject to the following assumption: The functions f and l are twice continuously differentiable, $u \mapsto l(x,u)$ is strictly convex, and there exist positive constants l_0, f_0, α, and $m^* > 1$ such that

$$l_0|u|^{m^*} \le l(x,u) \le l_0^{-1}|u|^{m^*}, \quad \bigl|D_x\, l(x,u)\bigr| \le l_0^{-1}\bigl(1+|u|^{m^*}\bigr) \quad \forall (x,u) \in \mathbb{R}^{2d},$$
$$f_0|x|^{\alpha} - f_0^{-1} \le f(x) \le f_0^{-1}\bigl(1+|x|^{\alpha}\bigr), \quad |Df(x)| \le f_0^{-1}\bigl(1+|x|^{\alpha-1}\bigr) \quad \forall x \in \mathbb{R}^d.$$

It is shown that (1.9) admits a unique solution (V^*, β) with V^* having polynomial growth and satisfying $\min_{\mathbb{R}^d}\ V^* = 1$. Moreover, $\beta = \varrho$. Provided that $\alpha \ge m^*$ and that the initial condition φ_0 is bounded below and has at most polynomial growth, it is shown that the solution of (1.10) converges uniformly on compacta and that $\overline{\varphi}(t,\cdot) - V^*(\cdot)$ tends to a constant as $t \to \infty$.

In [13] the authors consider the Cauchy problem

$$\partial_t f - \frac{1}{2}\Delta f + H(x, Df) = 0 \qquad \text{in}\ \ (0,\infty)\times\mathbb{R}^d,$$
$$f(0,\cdot) = f_0 \qquad \text{in}\ \ \mathbb{R}^d.$$

They assume that the Hamiltonian $H(x,p)$ has at most polynomial growth with respect to x and that it is convex and has at most quadratic growth in p. They also assume that the Hessian of H with respect to p is strictly positive definite and bounded. Additional assumptions which result in ergodicity are also employed. Then provided that the initial condition φ_0 has a certain minimal rate of growth, and grows at most at a polynomial rate, convergence of $\overline{\varphi}(t,\cdot) - V^*(\cdot)$ to a constant as $t \to \infty$ is established. The need for a minimal growth rate of the initial condition for convergence can be compared to [7] and Theorem 3.15 in section 3.4.

The paper is organized as follows. The next section introduces the notation used in the paper. Section 3 starts by describing in detail the model and the assumptions imposed. In section 3.2 we discuss some basic properties of the HJB equation for the ergodic control problem under near-monotone costs and the implications of the integrability of the value function under some optimal invariant distribution. In section 3.3 we address the issue of existence and uniqueness of solutions to (1.7) and (1.10) and describe some basic properties of these solutions. In section 3.4 we exhibit a region of attraction for the solutions of the VI. In section 4 we derive some essential growth estimates for the solutions of the VI and show that these solutions have locally bounded oscillation in $\mathbb{R}^d$, uniformly in $t \ge 0$. Section 5 is dedicated to the proof of convergence of the solutions of the RVI, while section 6 concludes with some pointers to future work.

2. Notation. The standard Euclidean norm in $\mathbb{R}^d$ is denoted by $|\cdot|$. The set of nonnegative real numbers is denoted by $\mathbb{R}_+$, $\mathbb{N}$ stands for the set of natural numbers, and $\mathbb{I}$ denotes the indicator function. We denote by $\uptau(A)$ the *first exit time* of a process $\{X_t,\ t \in \mathbb{R}_+\}$ from a set $A \subset \mathbb{R}^d$, defined by

$$\uptau(A) \triangleq \inf\,\{t > 0 : X_t \notin A\}.$$

The closure, the boundary, and the complement of a set $A \subset \mathbb{R}^d$ are denoted by $\overline{A}$, ∂A, and A^c, respectively. The open ball of radius R in $\mathbb{R}^d$, centered at the origin, is denoted by B_R, and we let $\uptau_R \triangleq \uptau(B_R)$ and $\breve{\uptau}_R \triangleq \uptau(B_R^c)$.

The term *domain* in $\mathbb{R}^d$ refers to a nonempty, connected open subset of the Euclidean space $\mathbb{R}^d$. For a domain $D \subset \mathbb{R}^d$, the space $\mathcal{C}^k(D)$ ($\mathcal{C}^\infty(D)$) refers to the class of all real-valued functions on D whose partial derivatives up to order k (of any order) exist and are continuous, and $\mathcal{C}_b(D)$ denotes the set of all bounded continuous real-valued functions on D.

We adopt the notation $\partial_t \triangleq \frac{\partial}{\partial t}$, and for $i, j \in \mathbb{N}$, $\partial_i \triangleq \frac{\partial}{\partial x_i}$ and $\partial_{ij} \triangleq \frac{\partial^2}{\partial x_i \partial x_j}$. We often use the standard summation rule that repeated subscripts and superscripts are summed from 1 through d. For example,

$$a^{ij}\partial_{ij}\varphi + b^i\partial_i\varphi \triangleq \sum_{i,j=1}^{d} a^{ij}\frac{\partial^2\varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i \frac{\partial \varphi}{\partial x_i}.$$

For a nonnegative multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ we let $D^\alpha \triangleq \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$. Let Q be a domain in $\mathbb{R}_+ \times \mathbb{R}^d$. Recall that $\mathcal{C}^{r,k+2r}(Q)$ stands for the set of continuous real-valued functions $\varphi(t,x)$ defined on Q such that the derivatives $D^\alpha\partial_t^\ell\varphi$ are bounded and continuous in Q for

$$|\alpha| + 2\ell \le k + 2r, \qquad \ell \le r.$$

By a slight abuse of notation, whenever the whole space $\mathbb{R}^d$ is concerned, we write $f \in \mathcal{C}^{r,k+2r}(I \times \mathbb{R}^d)$, where I is an interval in $\mathbb{R}_+$, whenever $f \in \mathcal{C}^{r,k+2r}(Q)$ for all bounded domains $Q \subset I \times \mathbb{R}^d$.

In general if $\mathcal{X}$ is a space of real-valued functions on Q, $\mathcal{X}_{\mathrm{loc}}$ consists of all functions f such that $f\varphi \in \mathcal{X}$ for every $\varphi \in \mathcal{C}_c^\infty(Q)$, the space of smooth functions on Q with compact support. In this manner we obtain, for example, the space $\mathcal{W}^{2,p}_{\mathrm{loc}}(Q)$.

We won't introduce here the parabolic Sobolev space $\mathcal{W}^{r,k+2r,p}(Q)$, since the solutions of (1.7) and (1.10) are in $\mathcal{C}^{1,2}\bigl((0,\infty) \times \mathbb{R}^d\bigr)$. The only exception is the function ψ in Theorem 4.7 and the function ψ_T used in the proof of Lemma 4.8. We refer the reader to [15] for definitions and properties of the parabolic Sobolev space.

3. Problem statement and preliminary results.

3.1. The model. The dynamics are modeled by a controlled diffusion process $X = \{X_t, t \geq 0\}$ taking values in the d-dimensional Euclidean space $\mathbb{R}^d$ and governed by the Itô SDE in (1.1). All random processes in (1.1) live in a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. The process W is a d-dimensional standard Wiener process independent of the initial condition X_0. The control process U takes values in a compact, metrizable set $\mathbb{U}$, and $U_t(\omega)$ is jointly measurable in $(t, \omega) \in [0, \infty) \times \Omega$. Moreover, it is *nonanticipative*: for $s < t$, $W_t - W_s$ is independent of

$$\mathfrak{F}_s \triangleq \text{the completion of } \sigma\{X_0, U_r, W_r, r \leq s\} \text{ relative to } (\mathfrak{F}, \mathbb{P}).$$

Such a process U is called an *admissible control*, and we let $\mathfrak{U}$ denote the set of all admissible controls.

We impose the following standard assumptions on the drift b and the diffusion matrix $\upsigma$ to guarantee existence and uniqueness of solutions to (1.1).

(A1) *Local Lipschitz continuity.* The functions

$$b = \left[b^1, \dots, b^d\right]^{\mathsf{T}} : \mathbb{R}^d \times \mathbb{U} \mapsto \mathbb{R}^d \quad \text{and} \quad \upsigma = \left[\upsigma^{ij}\right] : \mathbb{R}^d \mapsto \mathbb{R}^{d \times d}$$

are locally Lipschitz in x with a Lipschitz constant $\kappa_R > 0$ depending on $R > 0$. In other words, for all $x, y \in B_R$ and $u \in \mathbb{U}$,

$$|b(x,u) - b(y,u)| + \|\upsigma(x) - \upsigma(y)\| \leq \kappa_R |x - y|.$$

We also assume that b is continuous in (x, u).

(A2) *Affine growth condition.* b and $\upsigma$ satisfy a global growth condition of the form

$$|b(x,u)|^2 + \|\upsigma(x)\|^2 \leq \kappa_1 \left(1 + |x|^2\right) \qquad \forall (x,u) \in \mathbb{R}^d \times \mathbb{U},$$

where $\|\upsigma\|^2 \triangleq \operatorname{trace}\left(\upsigma\upsigma^{\mathsf{T}}\right)$.

(A3) *Local nondegeneracy.* For each $R > 0$, we have

$$\sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \geq \kappa_R^{-1}|\xi|^2 \qquad \forall x \in B_R$$

for all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$.

In integral form, (1.1) is written as

$$X_t = X_0 + \int_0^t b(X_s, U_s)\,\mathrm{d}s + \int_0^t \upsigma(X_s)\,\mathrm{d}W_s. \tag{3.1}$$

The second term on the right-hand side of (3.1) is an Itô stochastic integral. We say that a process $X = \{X_t(\omega)\}$ is a solution of (1.1) if it is $\mathfrak{F}_t$-adapted, continuous in t, defined for all $\omega \in \Omega$ and $t \in [0, \infty)$, and satisfies (3.1) for all $t \in [0, \infty)$ at once a.s.

We define the family of operators $L^u : \mathcal{C}^2(\mathbb{R}^d) \mapsto \mathcal{C}(\mathbb{R}^d)$, where $u \in \mathbb{U}$ plays the role of a parameter, by

$$L^u f(x) = a^{ij}(x)\,\partial_{ij} f(x) + b^i(x,u)\,\partial_i f(x), \quad u \in \mathbb{U}. \tag{3.2}$$

We refer to L^u as the *controlled extended generator* of the diffusion.

Of fundamental importance in the study of functionals of X is Itô's formula. For $f \in \mathcal{C}^2(\mathbb{R}^d)$ and with L^u as defined in (3.2), it holds that

$$f(X_t) = f(X_0) + \int_0^t L^{U_s} f(X_s)\,\mathrm{d}s + M_t \quad \text{a.s.,} \tag{3.3}$$

where

$$M_t \triangleq \int_0^t \big\langle \nabla f(X_s), \upsigma(X_s)\,\mathrm{d}W_s \big\rangle$$

is a local martingale. Krylov's extension of Itô's formula [14, p. 122] extends (3.3) to functions f in the local Sobolev space $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p \ge d$.

Recall that a control is called *Markov* if $U_t = v(t, X_t)$ for a measurable map $v : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{U}$, and it is called *stationary Markov* if v does not depend on t, i.e., $v : \mathbb{R}^d \mapsto \mathbb{U}$. Correspondingly, the equation

$$X_t = x_0 + \int_0^t b\big(X_s, v(s, X_s)\big)\,\mathrm{d}s + \int_0^t \upsigma(X_s)\,\mathrm{d}W_s \tag{3.4}$$

is said to have a *strong solution* if given a Wiener process $(W_t, \mathfrak{F}_t)$ on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, there exists a process X on $(\Omega, \mathfrak{F}, \mathbb{P})$, with $X_0 = x_0 \in \mathbb{R}^d$, which is continuous, $\mathfrak{F}_t$-adapted, and satisfies (3.4) for all t at once, a.s. A strong solution is called *unique* if any two such solutions X and X' agree $\mathbb{P}$-a.s. when viewed as elements of $\mathcal{C}\big([0,\infty), \mathbb{R}^d\big)$. It is well known that under assumptions (A1)–(A3), for any Markov control v, (3.4) has a unique strong solution [10].

Let $\mathfrak{U}_{\mathrm{SM}}$ denote the set of stationary Markov controls. Under $v \in \mathfrak{U}_{\mathrm{SM}}$, the process X is strong Markov, and we denote its transition function by $P^t_v(x, \cdot)$. It also follows from the work of [6, 22] that under $v \in \mathfrak{U}_{\mathrm{SM}}$, the transition probabilities of X have densities which are locally Hölder continuous. Thus L^v defined by

$$L^v f(x) = a^{ij}(x)\,\partial_{ij} f(x) + b^i\big(x, v(x)\big)\,\partial_i f(x), \quad v \in \mathfrak{U}_{\mathrm{SM}},$$

for $f \in \mathcal{C}^2(\mathbb{R}^d)$, is the generator of a strongly continuous semigroup on $\mathcal{C}_b(\mathbb{R}^d)$, which is strong Feller. We let $\mathbb{P}^v_x$ denote the probability measure and $\mathbb{E}^v_x$ the expectation operator on the canonical space of the process under the control $v \in \mathfrak{U}_{\mathrm{SM}}$, conditioned on the process X starting from $x \in \mathbb{R}^d$ at $t = 0$.

3.2. The ergodic control problem. We assume that the running cost function $r(x, u)$ is nonnegative, continuous, and locally Lipschitz in its first argument uniformly in $u \in \mathbb{U}$. Without loss of generality we let κ_R be a Lipschitz constant of $r(\cdot, u)$ over B_R. In summary, we assume the following:

(A4) $r\colon \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}_+$ is continuous and satisfies, for some constant $\kappa_R > 0$,

$$\big|r(x,u) - r(y,u)\big| \le \kappa_R |x - y| \qquad \forall x, y \in B_R,\ \forall u \in \mathbb{U},$$

and for all $R > 0$.

As mentioned in section 1, an important class of running cost functions arising in practice for which the ergodic control problem is well behaved are the near-monotone cost functions. Throughout this paper the near-monotone hypothesis (1.8) is in effect.

The ergodic control problem for near-monotone cost functions is characterized by the following theorem, which combines Theorems 3.4.7, 3.6.6, and 3.6.10 and

Lemmas 3.6.8 and 3.6.9 in [4]. Note that we choose to normalize the value function V^* differently here in order to facilitate the use of weighted norms.

THEOREM 3.1. *There exists a unique function $V^* \in \mathcal{C}^2(\mathbb{R}^d)$ which solves the HJB equation* (1.5) *and satisfies* $\min_{\mathbb{R}^d} V^* = 1$. *Also, a control $v^* \in \mathfrak{U}_{\mathrm{SM}}$ is optimal with respect to the criteria* (1.2) *and* (1.3) *if and only if it satisfies* (1.6) *a.e. in $\mathbb{R}^d$. Moreover, recalling that $\breve{\tau}_R = \tau(B_R^c)$, $R > 0$, we have*

$$V^*(x) = \inf_{v \in \mathfrak{U}_{\mathrm{SSM}}} \mathbb{E}_x^v \left[\int_0^{\breve{\tau}_R} \bigl(r\bigl(X_t, v(X_t)\bigr) - \varrho \bigr)\, \mathrm{d}t + V^*(X_{\breve{\tau}_R}) \right] \tag{3.5}$$
$$= \mathbb{E}_x^{v^*} \left[\int_0^{\breve{\tau}_R} \bigl(r\bigl(X_t, v^*(X_t)\bigr) - \varrho \bigr)\, \mathrm{d}t + V^*(X_{\breve{\tau}_R}) \right] \qquad \forall x \in B_R^c$$

for all $R > 0$.

Recall that control $v \in \mathfrak{U}_{\mathrm{SM}}$ is called *stable* if the associated diffusion is positive recurrent. We denote the set of such controls by $\mathfrak{U}_{\mathrm{SSM}}$ and let μ_v denote the unique invariant probability measure on $\mathbb{R}^d$ for the diffusion under the control $v \in \mathfrak{U}_{\mathrm{SSM}}$. Recall that $v \in \mathfrak{U}_{\mathrm{SSM}}$ if and only if there exists an inf-compact function $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$, a bounded domain $D \subset \mathbb{R}^d$, and a constant $\varepsilon > 0$ satisfying

$$L^v \mathcal{V}(x) \le -\varepsilon \qquad \forall x \in D^c.$$

It follows that the optimal control v^* in Theorem 3.1 is stable.

The technical assumption in Theorem 1.1 is the following.

Assumption 3.2. The value function V^* is integrable with respect to some optimal invariant probability distribution μ_{v^*}.

However, many results in this paper do not rely on Assumption 3.2.

Remark 3.3. Assumption 3.2 is equivalent to the following [4, Lemma 3.3.4]: there exists an optimal stationary control v^*, an inf-compact function $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$, and an open ball $B \subset \mathbb{R}^d$ such that

$$L^{v^*} \mathcal{V}(x) \le -V^*(x) \qquad \forall x \in B^c. \tag{3.6}$$

For the rest of the paper $v^* \in \mathfrak{U}_{\mathrm{SSM}}$ denotes some fixed control satisfying (1.6) and (3.6).

Remark 3.4. Assumption 3.2 is pretty mild. In the case that r is bounded it is equivalent to the statement that the mean hitting times to an open bounded set are integrable with respect to some optimal invariant probability distribution. In the case of one-dimensional diffusions, provided $\sigma(x) > \sigma_0$ for some constant $\sigma_0 > 0$, and $\limsup_{|x|\to\infty} \frac{x\, b(x)}{\sigma^2(x)} < -\frac{1}{2}$, the mean hitting time of $0 \in \mathbb{R}$ is bounded above by a second-degree polynomial in x [17, Theorem 5.6]. Therefore, in this case, the existence of second moments for μ_{v^*} implies Assumption 3.2. An example of this case, borrowed from [4, section 3.8.1], is the one-dimensional controlled diffusion

$$\mathrm{d}X_t = U_t\, \mathrm{d}t + \mathrm{d}W_t, \quad X_0 = x,$$

where $U_t \in [-1, 1]$ is the control variable. Let $r(x, u) = 1 - \mathrm{e}^{-|x|}$ be the running cost function. Clearly r is near-monotone. An optimal stationary Markov control is given by $v^*(x) = -\operatorname{sign}(x)$, and the corresponding invariant probability measure is

$$\mu_{v^*}(\mathrm{d}x) = \mathrm{e}^{-2|x|} \mathrm{d}x.$$

Also $\varrho = 1/3$. Solving the HJB we obtain

$$V^*(x) = \tfrac{2}{3}\bigl(\mathrm{e}^{-|x|} + |x| - 1\bigr),$$

which is clearly integrable with respect to μ_{v^*}.

Another class of problems for which Assumption 3.2 holds is linear controlled diffusions of the form

$$\mathrm{d}X_t = \bigl(AX_t + BU_t\bigr)\,\mathrm{d}t + \mathrm{d}W_t, \quad X_0 = x,$$

where $X_t \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{d\times m}$, and $U_t \in \mathbb{R}^m$. Let $r(x,u) = x^{\mathsf{T}}Rx + u^{\mathsf{T}}Su$ with R and S positive definitive square matrices. Suppose that the pair (A,B) is controllable. Then under any constant feedback control $v(x) = Zx$, $Z \in \mathbb{R}^{m\times d}$, such that the matrix $A+BZ$ is Hurwitz, the diffusion is positive recurrent and the average cost finite. Therefore $\varrho < \infty$ and r is clearly near-monotone. Since the action space is not compact this problem does not fit our model exactly. So we modify the model as follows. Let $\mathcal{Z} \subset \mathbb{R}^{m\times d}$ be a compact set that contains the optimal gain corresponding to the optimal linear feedback control for the linear quadratic problem above. We use the transformation $U_t = Z_tX_t$ with Z_t denoting the new control variable which lives in $\mathcal{Z}$. It is well known that the optimal invariant distribution is Gaussian and that V^* is quadratic in x. Integrability of V^* follows.

Assumption 3.2 is also implied by Assumption 3.18 in section 3.4, under which we obtain convergence of the VI algorithm (see Theorem 3.20).

We need the following lemma.

Lemma 3.5. *Under Assumption* 3.2*,*

$$\mathbb{E}^{v^*}_x\bigl[V^*(X_t)\bigr] \xrightarrow[t\to\infty]{} \mu_{v^*}[V^*] \triangleq \int_{\mathbb{R}^d} V^*(x)\,\mu_{v^*}(\mathrm{d}x) \qquad \forall x \in \mathbb{R}^d,$$

where, as defined earlier, μ_{v^*} is the invariant probability measure of the diffusion under the control v^*. Also there exists a constant m_r depending on r such that

$$\sup_{t\ge 0}\, \mathbb{E}^{v^*}_x\bigl[V^*(X_t)\bigr] \le m_r(V^*(x)+1) \qquad \forall x \in \mathbb{R}^d. \tag{3.7}$$

Proof. Since r is nonnegative, by Dynkin's formula we have

$$\mathbb{E}^{v^*}_x[V^*(X_t)] \le V^*(x) + \varrho\, t \qquad \forall t \ge 0,\ \forall x \in \mathbb{R}^d. \tag{3.8}$$

Therefore, since V^* is integrable with respect to μ_{v^*} by Assumption 3.2, the first result follows by [20, Theorem 5.3(i)]. The bound in (3.7) is the continuous time analogue of (14.5) in [18]. Recall that a skeleton of a continuous time Markov process is a discrete time Markov process with transition probability $\widehat{P} = \int_0^\infty \nu(\mathrm{d}t)P^t$, where ν is a probability measure on $(0,\infty)$. Since the diffusion is nondegenerate, any skeleton of the process is ϕ-irreducible, with an irreducibility measure absolutely continuous with respect to the Lebesgue measure. (For a definition of ϕ-irreducibility we refer the reader to [18, Chapter 4].) It is also straightforward to show that compact subsets of $\mathbb{R}^d$ are petite. Define the transition probability $\widetilde{P}$ by

$$\widetilde{P}f(x) = \int_{\mathbb{R}^d} \widetilde{P}(x,\mathrm{d}y)\,f(y) \triangleq \mathbb{E}^{v^*}_x[f(X_t)]|_{t=1}, \quad x \in \mathbb{R}^d,$$

for all functions $f \in \mathcal{C}_b(\mathbb{R}^d)$, and $g_r : \mathbb{R}^d \to \mathbb{R}_+$ by

$$g_r(x) \triangleq \mathbb{E}_x^{v^*}\left[\int_0^1 r\bigl(X_s, v^*(X_s)\bigr)\,\mathrm{d}s\right], \quad x \in \mathbb{R}^d.$$

Then (1.5) translates into the discrete time Poisson equation:

$$\widetilde{P}V^*(x) - V^*(x) = \varrho - g_r(x), \quad x \in \mathbb{R}^d. \tag{3.9}$$

It easily follows from the near-monotone hypothesis (1.8) that there exists a constant $\varepsilon_0 > 0$ and a ball $B_{R_0} \subset \mathbb{R}^d$, $R_0 > 0$, such that $g_r(x) - \varrho > \varepsilon_0$ for all $x \in B^c_{R_0}$. Since, in addition, $\int_{\mathbb{R}^d} V^*(x)\mu_{v^*}(\mathrm{d}x) < \infty$, it follows by (3.9) and [18, Theorem 14.0.1] that there exists a constant $\widetilde{m}$ such that

$$\sum_{n=0}^{\infty}\bigl|\widetilde{P}^n g_r(x) - \varrho\bigr| \le \widetilde{m}(V^*(x) + 1) \qquad \forall x \in \mathbb{R}^d. \tag{3.10}$$

By (3.9)–(3.10) we obtain

$$\begin{aligned}\widetilde{P}^n V^*(x) &= V^*(x) - \sum_{k=0}^{n-1}(\widetilde{P}^k g_r(x) - \varrho)\\ &\le (\widetilde{m} + 1)(V^*(x) + 1).\end{aligned} \tag{3.11}$$

By (3.8) and (3.11), writing the arbitrary $t \in \mathbb{R}_+$ as $t = n + \delta$, where n is the integer part of t and using the Markov property, we obtain

$$\begin{aligned}\mathbb{E}_x^{v^*}\bigl[V^*(X_t)\bigr] &= \mathbb{E}_x^{v^*}\Bigl[\mathbb{E}_{X_\delta}^{v^*}\bigl[V^*(X_{t-\delta})\bigr]\Bigr]\\ &= \mathbb{E}_x^{v^*}\bigl[\widetilde{P}^n V^*(X_\delta)\bigr]\\ &\le \mathbb{E}_x^{v^*}\bigl[(\widetilde{m}+1)(V^*(X_\delta)+1)\bigr]\\ &\le (\widetilde{m}+1)\,(V^*(x) + \varrho\,\delta + 1)\\ &\le (\widetilde{m}+1)\,(V^*(x) + \varrho + 1) \qquad \forall t \ge 0,\ \forall x \in \mathbb{R}^d,\end{aligned}$$

thus establishing (3.7). □

DEFINITION 3.6. *We let $\mathcal{C}_{V^*}(\mathbb{R}^d)$ denote the Banach space of functions $f \in \mathcal{C}(\mathbb{R}^d)$ with norm*

$$\|f\|_{V^*} \triangleq \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{V^*(x)}.$$

We also define

$$\mathcal{O}_{V^*} \triangleq \bigl\{f \in \mathcal{C}_{V^*}(\mathbb{R}^d) \cap \mathcal{C}^2(\mathbb{R}^d) : f \ge 0\bigr\}.$$

3.3. The RVI. The RVI and VI equations in (1.7) and (1.10) can also be written in the form

$$\partial_t\varphi(t,x) = \min_{u\in\mathbb{U}}\bigl[L^u\varphi(t,x) + r(x,u)\bigr] - \varphi(t,0), \qquad \varphi(0,x) = \varphi_0(x), \tag{3.12}$$

$$\partial_t\overline{\varphi}(t,x) = \min_{u\in\mathbb{U}}\bigl[L^u\overline{\varphi}(t,x) + r(x,u)\bigr] - \varrho, \qquad \overline{\varphi}(0,x) = \varphi_0(x). \tag{3.13}$$

DEFINITION 3.7. *Let $\hat{v} = \{\hat{v}_t,\ t \in \mathbb{R}_+\}$ denote a measurable selector from the minimizer in* (3.13) *corresponding to a solution $\overline{\varphi} \in \mathcal{C}\bigl([0,\infty)\times\mathbb{R}^d\bigr) \cap \mathcal{C}^{1,2}\bigl((0,\infty)\times\mathbb{R}^d\bigr)$. This is also a measurable selector from the minimizer in* (3.12), *provided $\overline{\varphi}$ and φ are related by* (1.11) *and* (1.14) *and vice versa. Note that the Markov control associated with $\hat{v}$ is computed "backward" in time (see* (1.15)*). Hence, for each $t \ge 0$ we define the (nonstationary) Markov control*

$$\hat{v}^t \triangleq \bigl\{\hat{v}^t_s = \hat{v}_{t-s}, s \in [0,t]\bigr\}.$$

Also, we adopt the simplifying notation

$$\overline{r}(x,u) \triangleq r(x,u) - \varrho.$$

In most of the statements of intermediary results the initial data φ_0 is assumed without loss of generality to be nonnegative. We start with a theorem that proves the existence of a solution to (3.13) that admits the stochastic representation in (1.15). This does not require Assumption 3.2.

First we need the following definition.

DEFINITION 3.8. *We define $\mathbb{R}^d_T \triangleq (0,T)\times\mathbb{R}^d$ and let $\overline{\mathbb{R}^d_T}$ denote its closure. We also let $\mathcal{C}^T_{V^*}(\mathbb{R}^d)$ denote the Banach space of functions in $\mathcal{C}(\overline{\mathbb{R}^d_T})$ with norm*

$$\|f\|_{V^*,T} \triangleq \sup_{(t,x)\in\overline{\mathbb{R}^d_T}} \frac{|f(t,x)|}{V^*(x)}.$$

THEOREM 3.9. *Provided $\varphi_0 \in \mathcal{O}_{V^*}$, then*

$$\overline{\varphi}(t,x) = \inf_{U\in\mathfrak{U}}\ \mathbb{E}^U_x \left[\int_0^t \overline{r}(X_s,U_s)\,\mathrm{d}s + \varphi_0(X_t)\right] \tag{3.14a}$$

is the minimal solution of (3.13) *in $\mathcal{C}\bigl([0,\infty)\times\mathbb{R}^d\bigr) \cap \mathcal{C}^{1,2}\bigl((0,\infty)\times\mathbb{R}^d\bigr)$ which is bounded below on $\mathbb{R}^d_T$ for any $T > 0$. With $\hat{v}^t$ as in Definition* 3.7, *it admits the representation*

$$\overline{\varphi}(t,x) = \mathbb{E}^{\hat{v}^t}_x \left[\int_0^t \overline{r}\bigl(X_s, \hat{v}^t_s(X_s)\bigr)\,\mathrm{d}s + \varphi_0(X_t)\right], \tag{3.14b}$$

and it holds that

$$\mathbb{E}^{\hat{v}^t}_x \bigl[\overline{\varphi}(t-\uptau_R\wedge t, X_{\uptau_R})\,\mathbb{I}\{\uptau_R < t\}\bigr] \xrightarrow[R\to\infty]{} 0 \tag{3.15}$$

for all $(t,x) \in \mathbb{R}_+\times\mathbb{R}^d$. Moreover $\overline{\varphi}(t,\cdot) \ge -\varrho\, t$ and satisfies the estimate

$$\|\overline{\varphi}\|_{V^*,T} \le (1+\varrho T)\ \max\bigl(1, \|\varphi_0\|_{V^*}\bigr) \qquad \forall T > 0. \tag{3.16}$$

Proof. Let $\varepsilon_0 > 0$ be such that

$$\liminf_{|x|\to\infty}\ \min_{u\in\mathbb{U}}\ r(x,u) > \varrho + \varepsilon_0.$$

With $\mathfrak{d}(x,B)$ denoting the Euclidean distance of the point $x \in \mathbb{R}^d$ from the set $B \subset \mathbb{R}^d$ and $B_n \subset \mathbb{R}^d$ denoting the ball of radius n centered at the origin, we define

$$r^n(x,u) = \frac{\mathfrak{d}(x,B^c_{n+1})\,r(x,u) + \mathfrak{d}(x,B_n)\,(\varrho+\varepsilon_0)}{\mathfrak{d}(x,B^c_{n+1}) + \mathfrak{d}(x,B_n)}.$$

Note that each r^n is Lipschitz in x, and by the near-monotone property of r, we have $r^n \le r^{n+1} \le r$ for all large enough n. In addition, $r^n \to r$ as $n \to \infty$. Let ϱ_n denote the optimal ergodic cost corresponding to r^n. Since $\varrho_n \le \varrho$, it follows that r^n is near-monotone for all n sufficiently large. Let also $\{\varphi_0^n : n \in \mathbb{N}\} \subset \mathcal{O}_{V^*}$ be a sequence satisfying $\varphi_0^n = 0$ on B_n^c and $\varphi_0^n \uparrow \varphi_0$ as $n \to \infty$. Without loss of generality we assume that $\|r^n\|_\infty \le n$ and $\|\varphi_0^n\|_\infty \le n$ for all $n \in \mathbb{N}$; otherwise we can slow down the growth of these sequences by repeating terms. The boundary value problem

$$\begin{aligned}
\partial_t \widehat{\varphi}_n^R(t,x) &= \min_{u \in \mathbb{U}} \bigl[L^u \widehat{\varphi}_n^R(t,x) + r^n(x,u)\bigr] \qquad \text{in } (0,T) \times B_R,\\
\widehat{\varphi}_n^R(0,x) &= \varphi_0^n(x) \quad \forall x \in B_R, \qquad \widehat{\varphi}_n^R(t,\cdot)|_{\partial B_R} = 0 \quad \forall t \in [0,T],
\end{aligned}$$

has a unique nonnegative solution in $\mathcal{C}\bigl([0,T] \times \overline{B}_R\bigr) \cap \mathcal{C}^{1,2}_{\mathrm{loc}}\bigl((0,T) \times B_R\bigr)$ for all $T > 0$ and $R > n$. This solution has the stochastic representation

$$\widehat{\varphi}_n^R(t,x) = \inf_{U \in \mathfrak{U}} \mathbb{E}_x^U \left[\int_0^{\tau_R \wedge t} r^n(X_s, U_s)\, \mathrm{d}s + \varphi_0^n(X_t)\, \mathbb{I}\{t < \tau_R\}\right], \tag{3.17}$$

where, as defined in section 2, τ_R denotes the first exit time from the ball B_R. By (3.17) we obtain

$$\begin{aligned}
\widehat{\varphi}_n^R(t,x) &\le \mathbb{E}_x^{v^*} \left[\int_0^{\tau_R \wedge t} r^n\bigl(X_s, v^*(X_s)\bigr)\, \mathrm{d}s + \varphi_0^n(X_t)\, \mathbb{I}\{t < \tau_R\}\right]\\
&\le \max\bigl(1, \|\varphi_0\|_{V^*}\bigr)\, \mathbb{E}_x^{v^*} \left[\int_0^{\tau_R \wedge t} r\bigl(X_s, v^*(X_s)\bigr)\, \mathrm{d}s + V^*(X_{\tau_R \wedge t})\right]\\
&\le \max\bigl(1, \|\varphi_0\|_{V^*}\bigr)\bigl(V^*(x) + \varrho\, t\bigr).
\end{aligned}$$

From [16, Theorem 6.2, p. 457] the derivatives $\bigl\{D^\alpha \partial_t^\ell \widehat{\varphi}_n^R : |\alpha| + 2\ell \le 2,\ R > n, n \in \mathbb{N}\bigr\}$ are locally Hölder equicontinuous in $\mathbb{R}^d_T$. Thus passing to the limit as $R \to \infty$ along a subsequence we obtain a nonnegative function $\widehat{\varphi}_n \in \mathcal{C}\bigl(\overline{\mathbb{R}^d_T}\bigr) \cap \mathcal{C}^{1,2}\bigl(\mathbb{R}^d_T\bigr)$ for all $T > 0$, which satisfies

$$\begin{aligned}
\partial_t \widehat{\varphi}_n(t,x) &= \min_{u \in \mathbb{U}} \bigl[L^u \widehat{\varphi}_n(t,x) + r^n(x,u)\bigr] \qquad \text{in } (0,\infty) \times \mathbb{R}^d,\\
\widehat{\varphi}_n(0,x) &= \varphi_0^n(x) \qquad \forall x \in \mathbb{R}^d.
\end{aligned} \tag{3.18}$$

By using Dynkin's formula on the cylinder $[0,t] \times B_R$, we obtain from (3.18) that

$$\widehat{\varphi}_n(t,x) = \inf_{U \in \mathfrak{U}} \mathbb{E}_x^U \left[\int_0^{\tau_R \wedge t} r^n(X_s, U_s)\, \mathrm{d}s + \widehat{\varphi}_n(t - \tau_R \wedge t, X_{\tau_R \wedge t})\right]. \tag{3.19}$$

It also follows by (3.17) that $\|\widehat{\varphi}_n(t,\cdot)\|_\infty \le n(t+1)$ for all $n \in \mathbb{N}$ and $t \ge 0$. By (3.19) we have the inequality

(3.20)

$$\begin{aligned}
\widehat{\varphi}_n(t,x) &\le \mathbb{E}_x^U \left[\int_0^{\tau_R \wedge t} r^n(X_s, U_s)\, \mathrm{d}s + \widehat{\varphi}_n(t - \tau_R \wedge t, X_{\tau_R \wedge t})\right]\\
&\le \mathbb{E}_x^U \left[\int_0^{\tau_R \wedge t} r^n(X_s, U_s)\, \mathrm{d}s + \varphi_0^n(X_t)\, \mathbb{I}\{\tau_R > t\}\right] + n(t+1)\, \mathbb{P}_x^U(\tau_R \le t)
\end{aligned}$$

for all $U \in \mathfrak{U}$. Taking limits as $R \to \infty$ in (3.20), using dominated convergence, we obtain

$$\widehat{\varphi}_n(t,x) \le \mathbb{E}_x^U \left[\int_0^t r^n(X_s,U_s)\,\mathrm{d}s + \varphi_0^n(X_t)\right] \qquad \forall\, U \in \mathfrak{U}. \tag{3.21}$$

Note that

$$0 \le \widehat{\varphi}_n(t,x) \le \limsup_{R\to\infty}\, \widehat{\varphi}_n^R(t,x) \le \max\bigl(1, \|\varphi_0\|_{V^*}\bigr)\bigl(V^*(x) + \varrho\, t\bigr). \tag{3.22}$$

Hence, as mentioned earlier, the derivatives $\bigl\{D^\alpha \partial_t^\ell \widehat{\varphi}_n : |\alpha| + 2\ell \le 2,\ n \in \mathbb{N}\bigr\}$ are locally Hölder equicontinuous in $(0,\infty)\times\mathbb{R}^d$. Also as shown in [4, p. 119] we have $\varrho_n \to \varrho$ as $n \to \infty$. Let $\{k_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ be an arbitrary increasing sequence. Then there exists some subsequence $\{k_n'\} \subset \{k_n\}$ such that $\widehat{\varphi}_{k_n'} \to \widehat{\varphi} \in \mathcal{C}\bigl(\overline{\mathbb{R}^d_T}\bigr) \cap \mathcal{C}^{1,2}\bigl(\mathbb{R}^d_T\bigr)$ for all $T > 0$, and $\widehat{\varphi}$ satisfies

$$\begin{aligned} \partial_t \widehat{\varphi}(t,x) &= \min_{u\in\mathbb{U}}\, [L^u \widehat{\varphi}(t,x) + r(x,u)] \qquad \text{in } (0,\infty)\times\mathbb{R}^d,\\ \widehat{\varphi}(0,x) &= \varphi_0(x) \qquad \forall x \in \mathbb{R}^d. \end{aligned} \tag{3.23}$$

Let $\hat{v}^t$ denote a stationary Markov control associated with the minimizer in (3.23) as in Definition 3.7. By using Dynkin's formula on the cylinder $[0,t]\times B_R$, we obtain from (3.23) that

$$\widehat{\varphi}(t,x) = \inf_{U\in\mathfrak{U}}\, \mathbb{E}_x^U \left[\int_0^{\uptau_R\wedge t} r(X_s,U_s)\,\mathrm{d}s + \widehat{\varphi}(t-\uptau_R\wedge t, X_{\uptau_R\wedge t})\right], \tag{3.24a}$$

$$\widehat{\varphi}(t,x) = \mathbb{E}_x^{\hat{v}^t} \left[\int_0^{\uptau_R\wedge t} r\bigl(X_s,\hat{v}_s^t(X_s)\bigr)\,\mathrm{d}s + \widehat{\varphi}(t-\uptau_R\wedge t, X_{\uptau_R\wedge t})\right]. \tag{3.24b}$$

Since $\widehat{\varphi}(t,\cdot)$ is nonnegative, letting $R\to\infty$ in (3.24b), by Fatou's lemma we obtain

$$\begin{aligned} \widehat{\varphi}(t,x) &\ge \mathbb{E}_x^{\hat{v}^t} \left[\int_0^t r\bigl(X_s,\hat{v}_s^t(X_s)\bigr)\,\mathrm{d}s + \varphi_0(X_t)\right]\\ &\ge \inf_{U\in\mathfrak{U}}\, \mathbb{E}_x^U \left[\int_0^t r(X_s,U_s)\,\mathrm{d}s + \varphi_0(X_t)\right]. \end{aligned} \tag{3.25}$$

Taking limits as $n\to\infty$ in (3.21), using monotone convergence for the first term on the right-hand side, we obtain

$$\widehat{\varphi}(t,x) \le \mathbb{E}_x^U \left[\int_0^t r(X_s,U_s)\,\mathrm{d}s + \varphi_0(X_t)\right] \qquad \forall\, U \in \mathfrak{U}. \tag{3.26}$$

By (3.25)–(3.26) we have

$$\widehat{\varphi}(t,x) = \inf_{U\in\mathfrak{U}}\, \mathbb{E}_x^U \left[\int_0^t r(X_s,U_s)\,\mathrm{d}s + \varphi_0(X_t)\right], \tag{3.27a}$$

$$\widehat{\varphi}(t,x) = \mathbb{E}_x^{\hat{v}^t} \left[\int_0^t r\bigl(X_s,\hat{v}_s^t(X_s)\bigr)\,\mathrm{d}s + \varphi_0(X_t)\right]. \tag{3.27b}$$

Let $\overline{\varphi}(t,x) \triangleq \widehat{\varphi}(t,x) - \varrho\, t$. Then $\overline{\varphi}$ solves (3.13) and (3.14a)–(3.14b) follow by (3.27a)–(3.27b). It is also clear that $\overline{\varphi}(t,x) \ge -\varrho\, t$, which together with (3.22) implies (3.16).

By (3.24b) we have

$$
\begin{aligned}
\widehat{\varphi}(t,x) = \mathbb{E}_x^{\hat{v}^t}\left[\int_0^{\tau_R\wedge t} r\bigl(X_s,\hat{v}_s^t(X_s)\bigr)\,\mathrm{d}s + \varphi_0(X_t)\,\mathbb{I}\{\tau_R \ge t\}\right]& \\
+\, \mathbb{E}_x^{\hat{v}^t}\bigl[\widehat{\varphi}(t-\tau_R\wedge t, X_{\tau_R})\,\mathbb{I}\{\tau_R < t\}\bigr]&.
\end{aligned}
\tag{3.28}
$$

The first term on the right-hand side of (3.28) tends to the right-hand side of (3.27b) by monotone convergence as $R\uparrow\infty$. Therefore (3.15) holds.

Suppose $\widetilde{\varphi}$ is a solution of (3.23) in $\mathcal{C}\bigl(\overline{\mathbb{R}_T^d}\bigr)\cap\mathcal{C}^{1,2}\bigl(\mathbb{R}_T^d\bigr)$ for some $T>0$, which is bounded below, and $\tilde{v}^t$ is a Markov control from the minimizer of (3.23). Applying Dynkin's formula on the cylinder $[0,t]\times B_R$ and letting $R\to\infty$ using Fatou's lemma, we obtain

$$
\begin{aligned}
\widetilde{\varphi}(t,x) &\ge \mathbb{E}_x^{\tilde{v}^t}\left[\int_0^t r\bigl(X_s,\tilde{v}_s^t(X_s)\bigr)\,\mathrm{d}s + \varphi_0(X_t)\right] \\
&\ge \inf_{U\in\mathfrak{U}}\ \mathbb{E}_x^U\left[\int_0^t r(X_s,U_s)\,\mathrm{d}s + \varphi_0(X_t)\right] \\
&\ge \widehat{\varphi}(t,x).
\end{aligned}
$$

Therefore $\overline{\varphi}(t,x)$ is the minimal solution of (3.13) in $\mathcal{C}\bigl([0,\infty)\times\mathbb{R}^d\bigr)\cap\mathcal{C}^{1,2}\bigl((0,\infty)\times\mathbb{R}^d\bigr)$ which is bounded below on $\mathbb{R}_T^d$ for each $T>0$. ∎

In the interest of economy of language we refer to the solution in (3.14a) as canonical. This is detailed in the following definition.

Definition 3.10. *Given an initial condition* $\varphi_0\in\mathcal{O}_{V^*}$ *we define the* canonical solution *to the VI in* (3.13) *as the solution which was constructed in the proof of Theorem* 3.9 *and was shown to admit the stochastic representation in* (3.14a). *In other words, this is the minimal solution of* (3.13) *in* $\mathcal{C}\bigl([0,\infty)\times\mathbb{R}^d\bigr)\cap\mathcal{C}^{1,2}\bigl((0,\infty)\times\mathbb{R}^d\bigr)$ *which is bounded below on* $\mathbb{R}_T^d$ *for any* $T>0$. *The canonical solution to the VI well defines the* canonical solution *to the RVI in* (3.12) *via* (1.14).

For the rest of the paper a solution to the RVI or VI is always meant to be a canonical solution. In summary, these are characterized by

$$
\begin{aligned}
\varphi(t,x) + \int_0^t \varphi(s,0)\,\mathrm{d}s &= \inf_{U\in\mathfrak{U}}\ \mathbb{E}_x^U\left[\int_0^t r(X_s,U_s)\,\mathrm{d}s + \varphi_0(X_t)\right] \\
&= \int_0^t \mathbb{E}_x^{\hat{v}^t}\bigl[r\bigl(X_s,\hat{v}_s^t(X_s)\bigr)\bigr]\,\mathrm{d}s + \mathbb{E}_x^{\hat{v}^t}\bigl[\varphi_0(X_t)\bigr].
\end{aligned}
\tag{3.29}
$$

Similarly

$$
\begin{aligned}
\overline{\varphi}(t,x) &= \inf_{U\in\mathfrak{U}}\ \mathbb{E}_x^U\left[\int_0^t \overline{r}(X_s,U_s)\,\mathrm{d}s + \varphi_0(X_t)\right] \\
&= \int_0^t \mathbb{E}_x^{\hat{v}^t}\bigl[\overline{r}\bigl(X_s,\hat{v}_s^t(X_s)\bigr)\bigr]\,\mathrm{d}s + \mathbb{E}_x^{\hat{v}^t}\bigl[\varphi_0(X_t)\bigr].
\end{aligned}
$$

The next lemma provides an important estimate for the canonical solutions of the VI.

Lemma 3.11. *Provided* $\varphi_0\in\mathcal{C}_{V^*}(\mathbb{R}^d)\cap\mathcal{C}^2(\mathbb{R}^d)$, *the canonical solution* $\overline{\varphi}\in\mathcal{C}^{1,2}(\mathbb{R}_+\times\mathbb{R}^d)$ *of* (3.13) *satisfies the bound*

$$
\mathbb{E}_x^{\hat{v}^t}\bigl[\varphi_0(X_t)-V^*(X_t)\bigr] \le \overline{\varphi}(t,x)-V^*(x) \le \mathbb{E}_x^{v^*}\bigl[\varphi_0(X_t)-V^*(X_t)\bigr]
\tag{3.30}
$$

for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$.

Proof. By (1.5) and (3.13) we obtain

$$-\partial_t(V^* - \overline{\varphi}) + L^{v^*}(V^* - \overline{\varphi}) \le 0$$

and

$$-\partial_t(V^* - \overline{\varphi}) + L^{\hat{v}_t}(V^* - \overline{\varphi}) \ge 0$$

from which, by an application of Itô's formula to $V^*(X_s) - \overline{\varphi}(t-s, X_s)$, $s \in [0,t]$, it follows that

$$\mathbb{E}_x^{v^*}\bigl[V^*(X_t) - \varphi_0(X_t)\bigr] \le V^*(x) - \overline{\varphi}(t,x)$$

and

$$\mathbb{E}_x^{\hat{v}^t}\bigl[V^*(X_t) - \varphi_0(X_t)\bigr] \ge V^*(x) - \overline{\varphi}(t,x),$$

respectively, and the estimate follows. ∎

Concerning the uniqueness of the canonical solution in a larger class of functions, this depends on the growth of V^* and the coefficients of the SDE in (1.1). Various such uniqueness results can be given based on different hypotheses on the growth of the data. The following result assumes that V^* has polynomial growth, which is the case in many applications.

THEOREM 3.12. *Let $\varphi_0 \in \mathcal{O}_{V^*}$ and suppose that for some constants c_1, c_2, and $m > 0$, $V^*(x) \le c_1 + c_2|x|^m$. Then any solution $\overline{\varphi}' \in \mathcal{C}\bigl(\overline{\mathbb{R}^d_T}\bigr) \cap \mathcal{C}^{1,2}\bigl(\mathbb{R}^d_T\bigr)$ of (3.13), for some $T > 0$, which is bounded below in $\overline{\mathbb{R}^d_T}$ and satisfies $\|\overline{\varphi}'\|_{V^*,T} < \infty$ agrees with the canonical solution $\overline{\varphi}$ on $\overline{\mathbb{R}^d_T}$.*

Proof. Let $\overline{\varphi}'$ be a solution satisfying the hypothesis in the theorem, and let $\overline{\varphi}$ be the canonical solution of (3.13) and $\hat{v}^t$ an associated Markov control as in Definition 3.7. Let $\overline{\varphi}_\varepsilon$, for $\varepsilon > 0$, denote the canonical solution of (3.13) with initial data $\varphi_0 + \varepsilon V^*$ and let $\{\hat{v}_{\varepsilon,t},\ t \in \mathbb{R}_+\}$ denote a measurable selector from the corresponding minimizer. By Theorem 3.9 for each $\varepsilon > 0$ we obtain

$$\begin{aligned}
\overline{\varphi}_\varepsilon(t,x) &= \inf_{U\in\mathfrak{U}}\ \mathbb{E}_x^U\left[\int_0^t \overline{r}\bigl(X_s, U_s\bigr)\,\mathrm{d}s + \varphi_0(X_t) + \varepsilon V^*(X_t)\right] \\
&\ge -\varrho\, t + \varepsilon \inf_{U\in\mathfrak{U}}\ \mathbb{E}_x^U\left[\int_0^t \overline{r}\bigl(X_s, U_s\bigr)\,\mathrm{d}s + V^*(X_t)\right] \\
&\ge \varepsilon V^*(x) - \varrho\, t.
\end{aligned}$$

Therefore by (3.15) for each $\varepsilon > 0$, we have

$$\mathbb{E}_x^{\hat{v}^t_\varepsilon}\bigl[V^*(X_{\tau_R})\,\mathbb{I}\{\tau_R < t\}\bigr] \xrightarrow[R\to\infty]{} 0 \qquad \forall (t,x) \in \mathbb{R}^d_T, \tag{3.31}$$

which in turn implies, since $\|\overline{\varphi}'\|_{V^*,T} < \infty$, that

$$\mathbb{E}_x^{\hat{v}^t_\varepsilon}\bigl[\overline{\varphi}'(t-\tau_R, X_{\tau_R})\,\mathbb{I}\{\tau_R < t\}\bigr] \xrightarrow[R\to\infty]{} 0 \qquad \forall (t,x) \in \mathbb{R}^d_T. \tag{3.32}$$

Since $-\partial_t\overline{\varphi}' + L^{\hat{v}_{\varepsilon,t}}\overline{\varphi}' + \overline{r}\bigl(x, \hat{v}_{\varepsilon,t}(x)\bigr) \ge 0$, we have that for all $(t,x) \in \mathbb{R}^d_T$,

$$\overline{\varphi}'(t,x) \le \mathbb{E}_x^{\hat{v}^t_\varepsilon}\left[\int_0^{\tau_R\wedge t} \overline{r}\bigl(X_s, \hat{v}^t_{\varepsilon,s}(X_s)\bigr)\,\mathrm{d}s + \overline{\varphi}'(t-\tau_R\wedge t, X_{\tau_R\wedge t})\right], \tag{3.33}$$

and taking limits as $R \to \infty$ in (3.33), using (3.32), it follows that $\overline{\varphi}' \le \overline{\varphi}_\varepsilon$ on $\overline{\mathbb{R}^d_T}$.

The polynomial growth of V^* implies that there exists a constant $m(x,T)$ such that $\mathbb{E}_x^U[V^*(X_t)] \le m(x,T)$ for all $(t,x) \in \mathbb{R}^d_T$ and $U \in \mathfrak{U}$ [4, Theorem 2.2.2]. Therefore, since

$$\begin{aligned}(3.34)\qquad \overline{\varphi}_\varepsilon(t,x) &\le \mathbb{E}_x^{\hat{v}^t}\left[\int_0^t \overline{r}\big(X_s, \hat{v}^t_s(X_s)\big)\,\mathrm{d}s + \varphi_0(X_t) + \varepsilon V^*(X_t)\right]\\ &\le \overline{\varphi}(t,x) + \varepsilon m(x,T) \qquad \forall (t,x) \in \mathbb{R}^d_T,\end{aligned}$$

and $\overline{\varphi}_\varepsilon \ge \overline{\varphi}$, it follows by (3.34) that $\overline{\varphi}_\varepsilon \to \overline{\varphi}$ on $\overline{\mathbb{R}^d_T}$ as $\varepsilon \downarrow 0$. Thus $\overline{\varphi}' \le \overline{\varphi}$ on $\overline{\mathbb{R}^d_T}$, and by the minimality of $\overline{\varphi}$ we must have equality. □

We can also obtain a uniqueness result on a larger class of functions that does not require V^* to have polynomial growth but assumes that the diffusion matrix is bounded in $\mathbb{R}^d$. This is given in Theorem 3.13 below, whose proof uses the technique in [8].

We define the following class of functions:

$$\mathfrak{G} \triangleq \Big\{f \in \mathcal{C}^2(\mathbb{R}^d) : \lim_{|x|\to\infty} f(x)\,\mathrm{e}^{-k|x|^2} = 0 \text{ for some } k > 0\Big\}.$$

THEOREM 3.13. *Suppose $V^* \in \mathfrak{G}$ and that $\|\upsigma\|$ is bounded in $\mathbb{R}^d$. Then, provided $\varphi_0 \in \mathcal{O}_{V^*}$, there exists a unique solution ψ to* (3.13) *such that* $\max_{t\in[0,T]}\ \psi(t,\cdot) \in \mathfrak{G}$ *for each $T > 0$.*

Proof. Let $\widehat{\varphi} \in \mathcal{C}\big([0,\infty)\times\mathbb{R}^d\big) \cap \mathcal{C}^{1,2}\big((0,\infty)\times\mathbb{R}^d\big)$ be the minimal nonnegative solution of

$$\begin{aligned}(3.35)\qquad \partial_t\widehat{\varphi}(t,x) &= \min_{u\in\mathbb{U}}\ [L^u\widehat{\varphi}(t,x) + r(x,u)] \qquad \text{in } (0,\infty)\times\mathbb{R}^d,\\ \widehat{\varphi}(0,x) &= \varphi_0(x) \qquad \forall x \in \mathbb{R}^d,\end{aligned}$$

and let $\{\hat{v}_t, t \in \mathbb{R}_+\}$ denote a measurable selector from the minimizer in (3.35). Suppose that $\widetilde{\varphi} \in \mathcal{C}\big([0,\infty)\times\mathbb{R}^d\big) \cap \mathcal{C}^{1,2}\big((0,\infty)\times\mathbb{R}^d\big)$ is any solution of (3.35) satisfying the hypothesis of the theorem, and let $\{\tilde{v}_t, t \in \mathbb{R}_+\}$ denote a measurable selector from the corresponding minimizer. Then $f \triangleq \widetilde{\varphi} - \widehat{\varphi}$ satisfies, for any $T > 0$,

$$(3.36)\qquad \partial_t f - L^{\hat{v}_t} f \le 0 \quad \text{and} \quad \partial_t f - L^{\tilde{v}_t} f \ge 0 \quad \text{in } (0,T]\times\mathbb{R}^d,$$

and $f(0,x) = 0$ for all $x \in \mathbb{R}^d$. By (3.16), the hypothesis that $V^* \in \mathfrak{G}$, and the hypothesis on the growth of f, it follows that for some $k = k(T) > 0$ large enough

$$(3.37)\qquad \lim_{|x|\to\infty}\ \max_{t\in[0,T]}\ |f(t,x)|\,\mathrm{e}^{-k|x|^2} = 0.$$

It is straightforward to verify by direct computation using the bounds on the coefficients of the SDE that there exists $\gamma = \gamma(k) > 1$ such that $g(t,x) \triangleq \mathrm{e}^{(1+\gamma t)(1+k|x|^2)}$ is a supersolution of

$$(3.38)\qquad \partial_t g - L^{v_t} g \ge 0 \quad \text{in } (0,T_0]\times\mathbb{R}^d \quad \text{with } T_0 \equiv \gamma^{-1}$$

under any Markov control $\{v_t\}$. By (3.37), for any $\varepsilon > 0$ we can select $R > 0$ large enough such that $|f(t,x)| \le \varepsilon g(t,x)$ for all $(t,x) \in [0,\gamma^{-1}]\times\partial B_R$. Using (3.36), (3.38), and Dynkin's formula on the strip $[0,\gamma^{-1}]\times\overline{B}_R$ it follows that $|f(t,x)| \le \varepsilon g(t,x)$ for all $(t,x) \in [0,\gamma^{-1}]\times\overline{B}_R$. Since $\varepsilon > 0$ was arbitrary this implies $f \equiv 0$, or equivalently that $\widetilde{\varphi} = \widehat{\varphi}$ on $[0,\gamma^{-1}]\times\mathbb{R}^d$.

Since, by (3.16), $\widehat{\varphi}(\gamma^{-1},\cdot)\in\mathcal{O}_{V^*}$, we can repeat the argument to show that $\widetilde{\varphi}=\widehat{\varphi}$ on $[\gamma^{-1},2\gamma^{-1}]\times\mathbb{R}^d$ and that the same holds by induction on $[n\gamma^{-1},(n+1)\gamma^{-1}]\times\mathbb{R}^d$, $n=2,3,\dots$, until we cover the interval $[0,T]$. This shows that $\widetilde{\varphi}=\widehat{\varphi}$ on $\overline{\mathbb{R}^d_T}$, and since $T>0$ was arbitrary the same holds on $[0,\infty)\times\mathbb{R}^d$. ∎

We do not enforce any of the assumptions of Theorems 3.12 or 3.13 for the main results of the paper. Rather our analysis is based on the canonical solution to the VI and RVI which is well defined (see Definition 3.10).

3.4. A region of attraction for the VI algorithm. In this section we describe a region of attraction for the VI algorithm. This is a subset of $\mathcal{C}^2(\mathbb{R}^d)$ which is positively invariant under the semiflow defined by (3.13) and all its points are convergent, i.e., converge to a solution of (1.5).

Definition 3.14. *We let $\overline{\Phi}_t[\varphi_0]:\mathcal{C}^2(\mathbb{R}^d)\to\mathcal{C}^2(\mathbb{R}^d)$, $t\in[0,\infty)$, denote the canonical solution (semiflow) of the VI in* (3.13) *starting from φ_0 and let $\Phi_t[\varphi_0]$ denote the corresponding canonical solution (semiflow) of the RVI in* (3.12)*. Let $\mathcal{E}$ denote the set of solutions of the HJB in* (1.5)*, i.e.,*

$$\mathcal{E}\triangleq\{V^*+c : c\in\mathbb{R}\}.$$

Also for $c\in\mathbb{R}$ we define the set $\mathcal{G}_c\subset\mathcal{C}^2(\mathbb{R}^d)$ by

$$\mathcal{G}_c\triangleq\bigl\{h\in\mathcal{C}^2(\mathbb{R}^d) : h-V^*\ge c,\ \|h\|_{V^*}<\infty\bigr\}.$$

Let Assumption 3.2 hold. We claim that for each $c\in\mathbb{R}$, $\mathcal{G}_c$ is positively invariant under the semiflow $\overline{\Phi}_t$. Indeed by (3.7) and (3.30), if $\varphi_0\in\mathcal{G}_c$, then we have that

$$\begin{aligned} c &\le \overline{\Phi}_t[\varphi_0](x)-V^*(x)\\ &\le \mathbb{E}^{v^*}_x\bigl[\varphi_0(X_t)-V^*(X_t)\bigr]\\ &\le m_r\|\varphi_0-V^*\|_{V^*}(V^*(x)+1)\qquad \forall (t,x)\in\mathbb{R}_+\times\mathbb{R}^d. \end{aligned}\tag{3.39}$$

Since translating φ_0 by a constant simply translates the orbit $\{\overline{\Phi}_t[\varphi_0], t\ge 0\}$ by the same constant, without loss of generality we let $c=0$, and we show that all the points of $\mathcal{G}_0$ are convergent.

Theorem 3.15. *Under Assumption* 3.2*, for each $\varphi_0\in\mathcal{G}_0$, the semiflow $\overline{\Phi}_t[\varphi_0]$ converges to $c_0+V^*\in\mathcal{E}$ as $t\to\infty$ for some $c_0\in\mathbb{R}$ that satisfies*

$$0\le c_0\le\int_{\mathbb{R}^d}\bigl(\varphi_0(x)-V^*(x)\bigr)\,\mu_{v^*}(\mathrm{d}x).\tag{3.40}$$

Also $\Phi_t[\varphi_0]$ converges to $V^*(\cdot)-V^*(0)+\varrho$ as $t\to\infty$.

Proof. Since, as we showed in the paragraph preceding the theorem, $\overline{\Phi}_t[\varphi_0]\in\mathcal{G}_0$ for all $t\ge 0$, by (3.14a) we have

$$\overline{\Phi}_t[\varphi_0](x)\le\mathbb{E}^{v^*}_x\left[\int_0^{t-\tau}\overline{r}\bigl(X_s,v^*(X_s)\bigr)\,\mathrm{d}s+\overline{\Phi}_\tau[\varphi_0](X_{t-\tau})\right]\qquad\forall\tau\in[0,t]\tag{3.41}$$

and for all $x\in\mathbb{R}^d$. Since $\overline{\Phi}_t[\varphi_0](x)-V^*(x)\ge 0$, and $\int_{\mathbb{R}^d}\overline{\Phi}_t[\varphi_0](x)\,\mu_{v^*}(\mathrm{d}x)$ is finite by Assumption 3.2, it follows by integrating (3.41) with respect to μ_{v^*} that the map

$$t\mapsto\int_{\mathbb{R}^d}\overline{\Phi}_t[\varphi_0](x)\,\mu_{v^*}(\mathrm{d}x)\tag{3.42}$$

is nonincreasing and bounded below. Hence it must be constant on the ω-limit set of φ_0 under the semiflow $\overline{\Phi}_t$ which is denoted by $\omega(\varphi_0)$. By (3.39) we have $\sup_{t>0} \left\|\overline{\Phi}_t[\varphi_0]\right\|_{V^*} < \infty$. Therefore by the interior estimates of solutions of (3.13) $\left\{\overline{\Phi}_t[\varphi_0], t > 0\right\}$ is locally precompact in $\mathcal{C}^2(\mathbb{R}^d)$. Hence $\omega(\varphi_0) \neq \varnothing$. Let $h \in \omega(\varphi_0)$ and define

$$f(t,x) \triangleq -\partial_t \overline{\Phi}_t[h](x) + L^{v^*}\bigl(\overline{\Phi}_t[h](x) - V^*(x)\bigr). \tag{3.43}$$

Then $f(t,x) \geq 0$ for all (t,x), and by applying Itô's formula to (3.43), we obtain

$$\overline{\Phi}_t[h](x) - V^*(x) - \mathbb{E}_x^{v^*}\left[h(X_t) - V^*(X_t)\right] = -\,\mathbb{E}_x^{v^*}\left[\int_0^t f(t-s, X_s)\,\mathrm{d}s\right]. \tag{3.44}$$

Integrating (3.44) with respect to the invariant distribution μ_{v^*} we obtain

$$\int_{\mathbb{R}^d} \bigl(\overline{\Phi}_t[h](x) - h(x)\bigr)\,\mu_{v^*}(\mathrm{d}x) = -\int_0^t \int_{\mathbb{R}^d} f(t-s, x)\,\mu_{v^*}(\mathrm{d}x)\,\mathrm{d}s \qquad \forall t \geq 0. \tag{3.45}$$

Since the term on the left-hand side of (3.45) equals 0, as we argued above, it follows that $f(t,x) = 0$, (t,x)-a.e., which in turn implies by (3.44) and Lemma 3.5 that

$$\lim_{t\to\infty} \overline{\Phi}_t[h](x) = V^*(x) - \int_{\mathbb{R}^d} \bigl(V^*(x) - h(x)\bigr)\,\mu_{v^*}(\mathrm{d}x).$$

It follows that $\omega(\varphi_0) \subset \mathcal{E} \cap \mathcal{G}_0$ and since the map in (3.42) is nonincreasing, it is straightforward to verify that $\omega(\varphi_0)$ must be a singleton and that (3.40) is satisfied. The convergence of $\Phi_t[\varphi_0]$ follows from the fact that by (1.14) $\Phi_t[\varphi_0]$ converges whenever $\overline{\Phi}_t[\varphi_0]$ does and also by observing that by (3.12) the only fixed point of Φ_t is the function $V^*(\cdot) - V^*(0) + \varrho$. □

Remark 3.16. It follows from Theorem 3.15 that the functional in (3.42) is strictly decreasing along the semiflow $\overline{\Phi}_t$ unless $\varphi_0 \in \mathcal{E}$. This is because if the map in (3.42) is constant on some interval $[t_0, t_1]$, with $t_1 > t_0$, then we must have

$$\overline{\Phi}_t[\varphi_0](x) = \mathbb{E}_x^{v^*}\left[\int_0^{t-t_0} \overline{r}\bigl(X_s, v^*(X_s)\bigr)\,\mathrm{d}s + \overline{\Phi}_{t_0}[\varphi_0](X_{t-t_0})\right] \qquad \forall t \in [t_0, t_1]. \tag{3.46}$$

But (3.46) implies that for some constant C_0, we must have $\Phi_t[\varphi_0](x) = C_0 + V^*(x)$ for all $t \in [t_0, t_1]$. As a result of this monotone property of the map in (3.42), if A is a bounded subset of $\mathcal{C}_{V^*}(\mathbb{R}^d)$, then the only subsets of $\mathcal{G}_c \cap A$, with $c \in \mathbb{R}$, which are invariant under the semiflow $\overline{\Phi}_t$ are the subsets of $\mathcal{E} \cap \mathcal{G}_c \cap A$. Similarly, the only subset of $\mathcal{G}_c \cap A$ which is invariant under the semiflow Φ_t is the singleton $\{V^*(\cdot) - V^*(0) + \varrho\}$, assuming of course that it is contained in $\mathcal{G}_c \cap A$. These facts are used later in the proof of Theorem 1.1.

We also have the following result, which does not require Assumption 3.2.

COROLLARY 3.17. *Suppose $\varphi_0 \in \mathcal{C}^2(\mathbb{R}^d)$ is such that $\varphi_0 - V^*$ is bounded. Then $\overline{\Phi}_t[\varphi_0]$ converges as $t \to \infty$ to a point in $\mathcal{E}$.*

Proof. By (3.30) and under the hypothesis, $x \mapsto \overline{\varphi}(t,x) - V^*(x)$ is bounded uniformly in t. Thus the result follows as in the proof of Theorem 3.15. □

An interesting class of problems are those for which V^* does not grow faster than $\min_{u\in\mathbb{U}} r(\cdot, u)$. More precisely, we consider the following property.

Assumption 3.18. There exist positive constants θ_1 and θ_2 such that

$$\min_{u\in\mathbb{U}} r(x,u) \geq \theta_1 V^*(x) - \theta_2 \qquad \forall x \in \mathbb{R}^d.$$

Remark 3.19. Assumption 3.18 is satisfied for linear systems with quadratic cost as described in Remark 3.4. It is also satisfied for the model in (1.16)–(1.17) if and only if $\alpha \ge m^*$. This is because as shown in [12, Theorem 2.2 and Proposition 4.2], there exist positive constants k_1, k_2, and k_3 such that

$$k_1|x|^{\alpha+1-\alpha/m^*} - k_2 \le V^*(x) \le k_3\bigl(1+|x|^{\alpha+1-\alpha/m^*}\bigr) \qquad \forall x\in\mathbb{R}^d.$$

Under Assumption 3.18 we have that

$$\begin{aligned} L^{v^*}V^*(x) &= \varrho - r\bigl(x,v^*(x)\bigr) \\ &\le \varrho+\theta_2-\theta_1 V^*(x), \end{aligned} \tag{3.47}$$

and it follows that under the control v^* the diffusion is geometrically ergodic with a Lyapunov function V^*. In particular Assumption 3.18 implies Assumption 3.2.

We have the following theorem.

THEOREM 3.20. *Suppose Assumption* 3.18 *and the hypotheses of either Theorem* 3.12 *or* 3.13 *hold. Then, provided $\varphi_0\in\mathcal{O}_{V^*}$, the semiflow $\overline{\Phi}_t[\varphi_0]$ converges, as $t\to\infty$, to a point $c_0+V^*\in\mathcal{E}$ satisfying*

$$-\frac{\varrho+\theta_2}{\theta_1} \le c_0 \le \frac{\varrho+\theta_2}{\theta_1}\|\varphi_0\|_{V^*}.$$

Therefore $\Phi_t[\varphi_0]$ converges to $V^*(\cdot)-V^*(0)+\varrho$ as $t\to\infty$.

Proof. We derive a lower bound for $\overline{\varphi}$ using the technique in [12, Proposition 5.5]. Let $\overline{\varphi}_\varepsilon$, for $\varepsilon>0$, denote the canonical solution of (3.13) with initial data $\varphi_0+\varepsilon V^*$ and let $\{\hat v_{\varepsilon,t}, t\in\mathbb{R}_+\}$ denote a measurable selector from the corresponding minimizer. Let

$$f_\varepsilon(t,x) \triangleq \overline{\varphi}_\varepsilon(t,x) - \bigl(1-\mathrm{e}^{-\theta_1 t}\bigr)\left(V^*(x)-\frac{\varrho+\theta_2}{\theta_1}\right).$$

Then, since $L^{\hat v_{\varepsilon,t}}V^*(x)\ge -\overline{r}\bigl(x,\hat v_{\varepsilon,t}(x)\bigr)$, using Assumption 3.18 we obtain that

$$\begin{aligned} F_\varepsilon(t,x) &\triangleq \partial_t f_\varepsilon(t,x) - L^{\hat v_{\varepsilon,t}} f_\varepsilon(t,x) \\ &= \overline{r}\bigl(x,\hat v_{\varepsilon,t}(x)\bigr) - \theta_1\mathrm{e}^{-\theta_1 t}\left(V^*(x)-\frac{\varrho+\theta_2}{\theta_1}\right) + \bigl(1-\mathrm{e}^{-\theta_1 t}\bigr)L^{\hat v_{\varepsilon,t}}V^*(x) \\ &\ge \overline{r}\bigl(x,\hat v_{\varepsilon,t}(x)\bigr) - \theta_1\mathrm{e}^{-\theta_1 t}\left(V^*(x)-\frac{\varrho+\theta_2}{\theta_1}\right) - \bigl(1-\mathrm{e}^{-\theta_1 t}\bigr)\overline{r}\bigl(x,\hat v_{\varepsilon,t}(x)\bigr) \\ &= \mathrm{e}^{-\theta_1 t}\bigl(r\bigl(x,\hat v_{\varepsilon,t}(x)\bigr)-\theta_1 V^*(x)+\theta_2\bigr) \\ &\ge 0 \qquad \forall (t,x)\in\mathbb{R}_+\times\mathbb{R}^d. \end{aligned} \tag{3.48}$$

By (3.16) we have

$$\|\overline{\varphi}_\varepsilon\|_{V^*,T} \le (1+\varrho T)\ \max\bigl(1,\varepsilon+\|\varphi_0\|_{V^*}\bigr) \qquad \forall T>0. \tag{3.49}$$

Therefore by (3.49) and (3.31), for all $T>0$, we have that

$$\mathbb{E}_x^{\hat v_\varepsilon^t}\bigl[\bigl|f_\varepsilon(t-\tau_R\wedge t, X_{\tau_R})\bigr|\,\mathbb{I}\{\tau_R<t\}\bigr] \xrightarrow[R\to\infty]{} 0 \qquad \forall (t,x)\in\mathbb{R}^d_T. \tag{3.50}$$

By using Dynkin's formula on the cylinder $[0,t] \times B_R$, we obtain from (3.48) that

$$
\begin{aligned}
f_\varepsilon(t,x) = \mathbb{E}_x^{\hat{v}_\varepsilon^t} \biggl[\int_0^{\tau_R \wedge t} F_\varepsilon(t-s, X_s)\,\mathrm{d}s + \bigl(\varphi_0(X_t) + \varepsilon V^*(X_t)\bigr)\,\mathbb{I}\{\tau_R \ge t\} \biggr] \\
+ \mathbb{E}_x^{\hat{v}_\varepsilon^t} \bigl[f_\varepsilon(t - \tau_R \wedge t, X_{\tau_R})\,\mathbb{I}\{\tau_R < t\} \bigr].
\end{aligned} \tag{3.51}
$$

Letting $R \to \infty$ in (3.51), by Fatou's lemma and (3.50) we obtain that $f_\varepsilon(t,x) \ge 0$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. By construction $\overline{\varphi}_\varepsilon$ satisfies

$$
\begin{aligned}
\partial_t \overline{\varphi}_\varepsilon(t,x) &= \min_{u \in \mathbb{U}} \bigl[L^u \overline{\varphi}_\varepsilon(t,x) + r(x,u) \bigr] - \varrho, \\
\overline{\varphi}_\varepsilon(0,x) &= \varphi_0(x) + \varepsilon V^*(x),
\end{aligned} \tag{3.52}
$$

and by (3.49) it is locally bounded uniformly in $\varepsilon \in (0,1)$. Therefore by the interior estimates of solutions of (3.52), as mentioned earlier, the derivatives $\bigl\{ D^\alpha \partial_t^\ell \overline{\varphi}_\varepsilon : |\alpha| + 2\ell \le 2, \varepsilon \in (0,1) \bigr\}$ are locally Hölder equicontinuous in $\mathbb{R}_+ \times \mathbb{R}^d$. Also by Theorem 3.9 $\overline{\varphi}_\varepsilon(t,\cdot) \ge -\varrho\, t$ for all $t > 0$. It follows that $\overline{\varphi}_\varepsilon \downarrow \overline{\varphi}_0 \in \mathcal{C}^{1,2}\bigl((0,\infty) \times \mathbb{R}^d\bigr)$ uniformly over compact subsets in $(0,\infty) \times \mathbb{R}^d$ and that the limit $\overline{\varphi}_0$ satisfies (3.13) and is bounded below in $\overline{\mathbb{R}^d_T}$ for all $T > 0$. By the uniqueness results of Theorem 3.12 or 3.13, $\overline{\varphi}_0$ agrees with the canonical solution $\overline{\varphi}$ of (3.13). It follows that

$$
\overline{\varphi}(t,x) - \bigl(1 - \mathrm{e}^{-\theta_1 t}\bigr) \left(V^*(x) - \frac{\varrho + \theta_2}{\theta_1} \right) = \lim_{\varepsilon \downarrow 0}\; f_\varepsilon(t,x) \ge 0 \tag{3.53}
$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. It is well known (see [4, Lemma 2.5.5]) that (3.47) implies that

$$
\mathbb{E}_x^{v^*} \left[V^*(X_t) \right] \le \frac{\varrho + \theta_2}{\theta_1} + \mathrm{e}^{-\theta_1 t} V^*(x) \qquad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad \forall U \in \mathfrak{U}.
$$

Therefore, the inequality on the right-hand side of (3.30) together with (3.53) imply that

$$
\bigl(1 - \mathrm{e}^{-\theta_1 t}\bigr) \left(V^*(x) - \frac{\varrho + \theta_2}{\theta_1} \right) \le \overline{\varphi}(t,x) \le V^*(x) + \|\varphi_0\|_{V^*} \left(\frac{\varrho + \theta_2}{\theta_1} + \mathrm{e}^{-\theta_1 t} V^*(x) \right). \tag{3.54}
$$

Since by (3.54) the orbit $\{\overline{\Phi}_t[\varphi_0], t \ge 0\}$ is bounded in $\mathcal{C}_{V^*}(\mathbb{R}^d)$ one may follow the proof of Theorem 3.15 to establish the result. Alternatively, we can use the following argument: By (3.54) every ω-limit point h of $\overline{\Phi}_t[\varphi_0]$ lies in the set

$$
G(\varphi_0) \triangleq \left\{ h \in \mathcal{C}^2(\mathbb{R}^d) : -\frac{\varrho + \theta_2}{\theta_1} \le h - V^* \le \frac{\varrho + \theta_2}{\theta_1} \|\varphi_0\|_{V^*} \right\}.
$$

Since the ω-limit set of φ_0 under $\overline{\Phi}_t$ is invariant and since by Remark 3.16 the only invariant subsets of $G(\varphi_0)$ are the subsets of $\mathcal{E} \cap G(\varphi_0)$, the result follows. ∎

4. Growth estimates for solutions of the VI. Most of the results of this section do not require Assumption 3.2. It is needed only for Lemma 4.10. Throughout this section and also in section 5 a solution $\overline{\varphi}$ (φ) always refers to the canonical solution of the VI (RVI) without further mention (see Definition 3.10).

LEMMA 4.1. *Suppose $\varphi_0 \in \mathcal{O}_{V^*}$. Then*

$$
\frac{1}{t}\, \overline{\varphi}(t,x) \xrightarrow[t \to \infty]{} 0.
$$

Proof. Since $\|\varphi_0\|_{V^*} < \infty$ it follows that $\frac{1}{t}\,\mathbb{E}_x^{v^*}[\varphi_0(X_t)] \to 0$ as $t \to \infty$ (see [4, Lemma 3.7.2(ii)]), and so we have

$$\begin{aligned} 0 &\le \liminf_{t\to\infty}\ \frac{1}{t}\ \inf_{U\in\mathfrak{U}}\ \mathbb{E}_x^U\left[\int_0^t \overline{r}(X_s,U_s)\,\mathrm{d}s + \varphi_0(X_t)\right] \\ &= \liminf_{t\to\infty}\ \frac{\overline{\varphi}(t,x)}{t} \le \limsup_{t\to\infty}\ \frac{\overline{\varphi}(t,x)}{t} \\ &\le \limsup_{t\to\infty}\ \frac{1}{t}\ \mathbb{E}_x^{v^*}\left[\int_0^t \overline{r}\bigl(X_s, v^*(X_s)\bigr)\,\mathrm{d}s + \varphi_0(X_t)\right] \\ &= 0. \end{aligned}$$

The first inequality above uses the fact that φ_0 is bounded below and that ϱ is the optimal ergodic cost. $\square$

LEMMA 4.2. *Provided $\|\varphi_0\|_\infty < \infty$, it holds that for all $t \ge 0$*

$$\overline{\varphi}(t-\tau,x) - \overline{\varphi}(t,x) \le \varrho\,\tau + \operatorname*{osc}_{\mathbb{R}^d}\ \varphi_0 \qquad \forall x\in\mathbb{R}^d, \quad \forall \tau\in[0,t].$$

Proof. We have

$$\begin{aligned} \overline{\varphi}(t-\tau,x) - \overline{\varphi}(t,x) &= \inf_{U\in\mathfrak{U}}\ \mathbb{E}_x^U\left[\int_0^{t-\tau} \overline{r}(X_s,U_s)\,\mathrm{d}s + \varphi_0(X_{t-\tau})\right] \\ &\qquad\qquad - \inf_{U\in\mathfrak{U}}\ \mathbb{E}_x^U\left[\int_0^t \overline{r}(X_s,U_s)\,\mathrm{d}s + \varphi_0(X_t)\right] \\ &\le -\inf_{U\in\mathfrak{U}}\ \mathbb{E}_x^U\left[\varphi_0(X_t) - \varphi_0(X_{t-\tau}) + \int_{t-\tau}^t \overline{r}(X_s,U_s)\,\mathrm{d}s\right] \\ &\le \varrho\,\tau + \operatorname*{osc}_{\mathbb{R}^d}\ \varphi_0. \qquad \square \end{aligned}$$

DEFINITION 4.3. *We define*

$$\mathcal{K} \triangleq \Bigl\{x\in\mathbb{R}^d : \min_{u\in\mathbb{U}}\ r(x,u) \le \varrho\Bigr\}.$$

Let B_0 be some open bounded ball containing $\mathcal{K}$ and define $\breve{\uptau} \triangleq \uptau(B_0^c)$. Also let $\delta_0 > 0$ be such that $r(x,u) \ge \varrho + \delta_0$ on B_0^c.

LEMMA 4.4. *Suppose $\varphi_0 \in \mathcal{O}_{V^*}$. Then it holds that*

$$\overline{\varphi}(t,x) \le \mathbb{E}_x^{v^*}\left[\int_0^{\breve{\uptau}\wedge t} \overline{r}\bigl(X_s, v^*(X_s)\bigr)\,\mathrm{d}s + \overline{\varphi}(t-\breve{\uptau}\wedge t, X_{\breve{\uptau}\wedge t})\right] \tag{4.1}$$

and

$$\overline{\varphi}(t,x) \ge \mathbb{E}_x^{\hat{v}^t}\left[\int_0^{\breve{\uptau}\wedge t} \overline{r}\bigl(X_s, \hat{v}_s^t(X_s)\bigr)\,\mathrm{d}s + \overline{\varphi}(t-\breve{\uptau}\wedge t, X_{\breve{\uptau}\wedge t})\right] \tag{4.2}$$

for all $x \in B_0^c$.

Proof. Let B_R be any ball that contains $\overline{B}_0$ and for $n\in\mathbb{N}$, let $\uptau_n$ denote the first exit time from B_{nR}. Using Dynkin's formula on (3.13), we obtain

$$\overline{\varphi}(t,x) = \inf_{U\in\mathfrak{U}}\ \mathbb{E}_x^U\left[\int_0^{\breve{\uptau}\wedge\uptau_n\wedge t} \overline{r}(X_s,U_s)\,\mathrm{d}s + \overline{\varphi}(t-\breve{\uptau}\wedge\uptau_n\wedge t, X_{\breve{\uptau}\wedge\uptau_n\wedge t})\right] \tag{4.3}$$

for $x \in B_R \cap B_0^c$. By (4.3) we have

$$\overline{\varphi}(t,x) \le \mathbb{E}_x^{v^*}\left[\int_0^{\breve{\tau}\wedge\tau_n\wedge t} r\bigl(X_s, v^*(X_s)\bigr)\,\mathrm{d}s\right] - \varrho\,\mathbb{E}_x^{v^*}[\breve{\tau}\wedge\tau_n\wedge t] + \mathbb{E}_x^{v^*}\bigl[\overline{\varphi}(t-\breve{\tau}\wedge\tau_n\wedge t, X_{\breve{\tau}\wedge\tau_n\wedge t})\bigr]. \tag{4.4}$$

We use the expansion

$$\mathbb{E}_x^{v^*}\bigl[\overline{\varphi}(t-\breve{\tau}\wedge\tau_n\wedge t, X_{\breve{\tau}\wedge\tau_n\wedge t})\bigr] = \mathbb{E}_x^{v^*}\bigl[\overline{\varphi}(t-\breve{\tau}\wedge t, X_{\breve{\tau}\wedge t})\,\mathbb{I}\{\tau_n > \breve{\tau}\wedge t\}\bigr] + \mathbb{E}_x^{v^*}\bigl[\overline{\varphi}(t-\tau_n\wedge t, X_{\tau_n})\,\mathbb{I}\{\tau_n \le \breve{\tau}\wedge t\}\bigr].$$

By (3.16) and the fact that, as shown in [4, Corollary 3.7.3],

$$\mathbb{E}_x^{v^*}\bigl[V^*(X_{\tau_n})\,\mathbb{I}\{\tau_n \le t\}\bigr] \xrightarrow[n\to\infty]{} 0,$$

we obtain

$$\mathbb{E}_x^{v^*}\bigl[\overline{\varphi}(t-\tau_n\wedge t, X_{\tau_n})\,\mathbb{I}\{\tau_n \le \breve{\tau}\wedge t\}\bigr] \xrightarrow[n\to\infty]{} 0.$$

Therefore by taking limits as $n \to \infty$ in (4.4) and also using monotone convergence for the first two terms on the right-hand side, we obtain (4.1).

To obtain a lower bound we start from

$$\overline{\varphi}(t,x) = \mathbb{E}_x^{\hat{v}^t}\left[\int_0^{\breve{\tau}\wedge\tau_n\wedge t} \overline{r}\bigl(X_s, \hat{v}_s^t(X_s)\bigr)\,\mathrm{d}s + \overline{\varphi}(t-\breve{\tau}\wedge\tau_n\wedge t, X_{\breve{\tau}\wedge\tau_n\wedge t})\right]. \tag{4.5}$$

Since for any fixed t the functions $\bigl\{\overline{\varphi}(t-s,x) : s \le t\bigr\}$ are uniformly bounded below, taking limits in (4.5) as $n \to \infty$, and using Fatou's lemma, we obtain (4.2). ∎

Lemma 4.5. *Suppose $\varphi_0 \in \mathcal{O}_{V^*}$. Then for any $t > 0$ we have*

$$\overline{\varphi}(t,x) > \min\left(\min_{[0,t]\times\overline{B}_0} \overline{\varphi},\ \min_{\mathbb{R}^d} \varphi_0\right) \qquad \forall x \in \mathbb{R}^d \setminus \overline{B}_0.$$

Proof. Let x be any point in the interior of B_0^c. By (4.2) we have

$$\begin{aligned}\overline{\varphi}(t,x) &\ge \mathbb{E}_x^{\hat{v}^t}\left[\int_0^{\breve{\tau}\wedge t} \overline{r}\bigl(X_s, \hat{v}_s^t(X_s)\bigr)\,\mathrm{d}s + \overline{\varphi}(t-\breve{\tau}\wedge t, X_{\breve{\tau}\wedge t})\right]\\ &\ge \delta_0\,\mathbb{E}_x^{\hat{v}^t}[\breve{\tau}\wedge t] + \mathbb{P}_x^{\hat{v}^t}(\breve{\tau}\le t)\,\min_{[0,t]\times\overline{B}_0}\overline{\varphi} + \mathbb{P}_x^{\hat{v}^t}(\breve{\tau} > t)\,\min_{\mathbb{R}^d}\varphi_0,\end{aligned}$$

and the result follows. ∎

Remark 4.6. By Lemma 4.5, if $\inf_{[0,\infty)\times\overline{B}_0} \overline{\varphi} > -\infty$, then $\overline{\varphi}$ is bounded below on $[0,\infty)\times\mathbb{R}^d$. If this is the case, the convergence of the VI and therefore also of the RVI follows as in the proof of Theorem 3.15. Therefore without loss of generality we assume for the remainder of the paper that

$$\inf_{[0,\infty)\times\overline{B}_0} \overline{\varphi} = -\infty.$$

It follows that there exists $T_0 > 0$ such that

$$\min_{[0,t]\times\overline{B}_0} \overline{\varphi} \le \min_{\mathbb{R}^d} \varphi_0 \qquad \forall t \ge T_0. \tag{4.6}$$

We use the parabolic Harnack inequality, which we quote in simplified form from the more general result in [9, Theorem 4.1] as follows.

THEOREM 4.7 (parabolic Harnack). *Let $B_{2R} \subset \mathbb{R}^d$ be an open ball and ψ be a nonnegative caloric function, i.e., a nonnegative solution of*

$$\partial_t \psi(t,x) - a^{ij}(t,x)\,\partial_{ij}\psi(t,x) + b^i(t,x)\,\partial_i\psi(t,x) = 0 \quad \text{on } [0,T]\times B_{2R}$$

with $a^{ij}(t,x)$ continuous in x and uniformly nondegenerate on $[0,T]\times B_{2R}$, and a^{ij} and b^i bounded on $[0,T]\times B_{2R}$. Then for any $\tau \in \bigl(0, {}^{T}\!/\!{}_{4}\bigr]$, there exists a constant C_H depending only on R, τ, and the ellipticity constant (and modulus of continuity) of a^{ij} and the bounds of a^{ij} and b^i on B_{2R} such that

$$\max_{[T-3\tau,T-2\tau]\times \overline{B}_R} \psi \le C_H \min_{[T-\tau,T]\times\overline{B}_R} \psi.$$

In the three lemmas that follow we apply Theorem 4.7 with $\tau \equiv 1$ and $B_0' = 2B_0$.

LEMMA 4.8. *There exists a constant M_0 such that for all $T \ge T_0 + 4$ it holds that*

$$\max_{[T-3,T-2]\times\overline{B}_0} \overline{\varphi} - \min_{[0,T]\times\overline{B}_0} \overline{\varphi} \le M_0 + C_H\biggl(\min_{[T-1,T]\times\overline{B}_0} \overline{\varphi} - \min_{[0,T]\times\overline{B}_0} \overline{\varphi}\biggr).$$

Proof. Let $\psi_T(t,x)$ be the unique solution in $\mathcal{W}^{1,2,p}_{\mathrm{loc}}\bigl((0,T)\times B_0'\bigr) \cap \mathcal{C}\bigl([0,T]\times\overline{B}_0'\bigr)$ of

$$\begin{aligned}
\partial_t\psi_T(t,x) - a^{ij}(x)\,\partial_{ij}\psi_T(t,x) - b^i\bigl(x,\hat{v}_t(x)\bigr)\,\partial_i\psi_T(t,x) = 0 \qquad &\text{on } [0,T]\times B_0',\\
\psi_T(t,x) = \overline{\varphi}(t,x) \qquad &\text{on } \bigl([0,T]\times\partial B_0'\bigr)\cup\bigl(\{0\}\times\overline{B}_0'\bigr)
\end{aligned}$$

with $\hat{v}^t$ as in Definition 3.7. By the maximum principle

$$\min_{[0,t]\times\overline{B}_0'} \psi_T \ge \min\biggl(\min_{[0,t]\times\partial B_0'} \overline{\varphi},\ \min_{\overline{B}_0'} \varphi_0\biggr). \tag{4.7}$$

Since $\overline{\psi}_T \triangleq \psi_T - \overline{\varphi}$ satisfies

$$\partial_t\overline{\psi}_T(t,x) - a^{ij}(x)\,\partial_{ij}\overline{\psi}_T(t,x) - b^i\bigl(x,\hat{v}_t(x)\bigr)\,\partial_i\overline{\psi}_T(t,x) + \overline{r}\bigl(x,\hat{v}_t(x)\bigr) = 0$$

on $[0,T]\times B_0'$, and

$$\overline{\psi}_T(t,x) = 0 \qquad \text{on } \bigl([0,T]\times\partial B_0'\bigr)\cup\bigl(\{0\}\times\overline{B}_0'\bigr),$$

it follows that there exists a constant $\overline{M}_0$ which depends only on B_0' (it is independent of T) such that

$$\max_{[0,T]\times\overline{B}_0'} \bigl|\overline{\psi}_T\bigr| \le \overline{M}_0 \qquad \forall T > 0. \tag{4.8}$$

Indeed a direct calculation yields

$$\begin{aligned}
\bigl|\overline{\psi}_T(t,x)\bigr| &= \left|\mathbb{E}_x^{\hat{v}^t}\left[\int_0^{t\wedge\uptau(B_0')} \overline{r}\bigl(X_s,\hat{v}_{t-s}(X_s)\bigr)\,\mathrm{d}s\right]\right|\\
&\le \sup_{U\in\mathfrak{U}} \mathbb{E}_x^U\left[\int_0^{\uptau(B_0')} \bigl|\overline{r}(X_s,U_s)\bigr|\,\mathrm{d}s\right]\\
&\le \|\overline{r}\|_{\infty,B_0'} \sup_{x\in B_0'} \sup_{U\in\mathfrak{U}} \mathbb{E}_x^U[\uptau(B_0')] < \infty
\end{aligned}$$

since the mean exit time from B_0' is upper bounded by a constant uniformly over all initial $x \in B_0'$ and all controls $U \in \mathfrak{U}$ by the weak maximum principle of Alexandroff.

Let $(\hat{t}, \hat{x})$ be a point at which $\overline{\varphi}$ attains its minimum on $[T-1, T] \times \overline{B}_0$. By Lemma 4.5, (4.6), and (4.7) the function $(t, x) \mapsto \psi_T(t, x) - \min_{[0,T]\times\overline{B}_0} \overline{\varphi}$ is nonnegative on $[T-4, T] \times B_0'$ for all $T \ge T_0 + 4$. Therefore by Theorem 4.7 we have

$$\begin{aligned}
\psi_T(t,x) - \min_{[0,T]\times\overline{B}_0} \overline{\varphi} &\le C_H \biggl(\psi_T(\hat{t},\hat{x}) - \min_{[0,T]\times\overline{B}_0} \overline{\varphi}\biggr) \\
&\le C_H \biggl(\overline{\psi}_T(\hat{t},\hat{x}) + \min_{[T-1,T]\times\overline{B}_0} \overline{\varphi} - \min_{[0,T]\times\overline{B}_0} \overline{\varphi}\biggr)
\end{aligned} \tag{4.9}$$

for all $t \in [T-3, T-2]$ and $x \in B_0$. Expressing the left-hand side of (4.9) as

$$\overline{\varphi}(t,x) - \min_{[0,T]\times\overline{B}_0} \overline{\varphi} + \overline{\psi}_T(t,x),$$

and using (4.8), Lemma 4.8 follows with

$$M_0 \triangleq (C_H + 1)\overline{M}_0. \qquad \square$$

LEMMA 4.9. *Provided $\varphi_0 \in \mathcal{C}^2(\mathbb{R}^d)$ is nonnegative and bounded, we have*

$$\overline{\varphi}(t,x) - \max_{\partial B_0} \overline{\varphi}(t,\cdot) \le 2\,\|\varphi_0\|_\infty + \bigl(1 + \varrho\,\delta_0^{-1}\bigr)V^*(x) \qquad \forall x \in B_0^c.$$

Proof. By Lemma 4.2

$$\overline{\varphi}(t-\tau, x) \le \overline{\varphi}(t,x) + \varrho\,\tau + \operatorname*{osc}_{\mathbb{R}^d} \varphi_0 \qquad \forall x \in \mathbb{R}^d, \quad 0 \le \tau \le t. \tag{4.10}$$

Therefore by (4.1) and (4.10), using the fact that $\overline{r} \ge 0$ on B_0^c, we obtain

$$\begin{aligned}
\overline{\varphi}(t,x) &\le \mathbb{E}_x^{v^*}\biggl[\int_0^{\breve{\tau}\wedge t} \overline{r}\bigl(X_s, v^*(X_s)\bigr)\,\mathrm{d}s + \overline{\varphi}(t - \breve{\tau}\wedge t, X_{\breve{\tau}\wedge t})\biggr] \\
&\le \mathbb{E}_x^{v^*}\biggl[\int_0^{\breve{\tau}} \overline{r}\bigl(X_s, v^*(X_s)\bigr)\,\mathrm{d}s\biggr] + \mathbb{E}_x^{v^*}\bigl[\overline{\varphi}(t - \breve{\tau}\wedge t, X_{\breve{\tau}})\,\mathbb{I}\{\breve{\tau} \le t\}\bigr] \\
&\quad + \mathbb{E}_x^{v^*}\bigl[\varphi_0(X_t)\,\mathbb{I}\{\breve{\tau} > t\}\bigr] \\
&\le V^*(x) + \mathbb{E}_x^{v^*}\bigl[\overline{\varphi}(t, X_{\breve{\tau}})\,\mathbb{I}\{\breve{\tau} \le t\}\bigr] \\
&\quad + \varrho\,\mathbb{E}_x^{v^*}\bigl[\breve{\tau}\,\mathbb{I}\{\breve{\tau} \le t\}\bigr] + \operatorname*{osc}_{\mathbb{R}^d} \varphi_0 + \|\varphi_0\|_\infty \\
&\le V^*(x) + \mathbb{P}_x^{v^*}\bigl(\{\breve{\tau} \le t\}\bigr)\Bigl(\max_{\partial B_0} \overline{\varphi}(t,\cdot)\Bigr) \\
&\quad + \varrho\,\mathbb{E}_x^{v^*}\bigl[\breve{\tau}\,\mathbb{I}\{\breve{\tau} \le t\}\bigr] + 2\,\|\varphi_0\|_\infty
\end{aligned} \tag{4.11}$$

for $x \in B_0^c$. Since $-\overline{\varphi}(t,x) \le \varrho\, t$, we have

$$\begin{aligned}
-\,\mathbb{P}_x^{v^*}\bigl(\{\breve{\tau} > t\}\bigr)\Bigl(\max_{\partial B_0} \overline{\varphi}(t,\cdot)\Bigr) &\le \varrho\,\mathbb{P}_x^{v^*}\bigl(\{\breve{\tau} > t\}\bigr)\,t \\
&\le \varrho\,\mathbb{E}_x^{v^*}\bigl[\breve{\tau}\,\mathbb{I}\{\breve{\tau} > t\}\bigr].
\end{aligned} \tag{4.12}$$

Hence subtracting $\max_{\partial B_0} \overline{\varphi}(t,\cdot)$ from both sides of (4.11) and using (4.12) together with the estimate $\mathbb{E}_x^{v^*}[\breve{\uptau}] \le \delta_0^{-1} V^*(x)$, which follows by (3.5), we obtain

$$\overline{\varphi}(t,x) - \max_{\partial B_0} \overline{\varphi}(t,\cdot) \le V^*(x) + \varrho\,\delta_0^{-1} V^*(x) + 2\,\|\varphi_0\|_\infty. \qquad \square$$

We define the set $\mathcal{T} \subset \mathbb{R}_+$ by

$$\mathcal{T} \triangleq \biggl\{ t \ge T_0 + 4 : \min_{[t-1,t]\times\overline{B}_0} \overline{\varphi} = \min_{[0,t]\times\overline{B}_0} \overline{\varphi} \biggr\},$$

where T_0 is as in Remark 4.6. By Remark 4.6, $\mathcal{T} \neq \varnothing$.

LEMMA 4.10. *Let Assumption* 3.2 *hold and suppose that the initial condition* $\varphi_0 \in \mathcal{C}^2(\mathbb{R}^d)$ *is nonnegative and bounded. Then there exists a constant* C_0 *such that*

$$\operatorname*{osc}_{B_0} \overline{\varphi}(t,\cdot) \le C_0 \qquad \forall t \ge 0.$$

Proof. Suppose $t \in \mathcal{T}$. Then, by Lemma 4.8,

$$\max_{x\in\overline{B}_0} \overline{\varphi}(t-2,x) - \min_{[0,t]\times\overline{B}_0} \overline{\varphi} \le M_0.$$

Therefore, by Lemma 4.9 we have

$$\overline{\varphi}(t-2,x) - \min_{[0,t]\times\overline{B}_0} \overline{\varphi} \le M_0 + 2\,\|\varphi_0\|_\infty + \bigl(1 + \varrho\,\delta_0^{-1}\bigr) V^*(x) \tag{4.13}$$

for all $(t,x) \in \mathcal{T} \times \mathbb{R}^d$. Next, fix any $t_0 \in \mathcal{T}$. It suffices to prove the result for $t \ge t_0$ since it trivially holds for t in the compact interval $[0,t_0]$. Given $t \ge t_0$ let $\tau \triangleq \sup\, \mathcal{T} \cap [t_0,t]$. Note then that

$$\min_{[0,\tau]\times\overline{B}_0} \overline{\varphi} = \min_{[0,t]\times\overline{B}_0} \overline{\varphi}. \tag{4.14}$$

By (4.13)–(4.14) we obtain

$$\begin{aligned}
\sup_{x\in B_0} \overline{\varphi}(t,x) &\le \sup_{x\in B_0} \mathbb{E}_x^{v^*}\biggl[\int_0^{t-\tau+2} \overline{r}\bigl(X_s, v^*(X_s)\bigr)\,\mathrm{d}s + \overline{\varphi}(\tau-2, X_{t-\tau+2})\biggr] \\
&\le \sup_{x\in B_0} \mathbb{E}_x^{v^*}\biggl[\int_0^{t-\tau+2} \overline{r}\bigl(X_s, v^*(X_s)\bigr)\,\mathrm{d}s + V^*(X_{t-\tau+2})\biggr] + M_0 \\
&\quad + \min_{[0,\tau]\times\overline{B}_0} \overline{\varphi} + 2\,\|\varphi_0\|_\infty + \varrho\,\delta_0^{-1} \sup_{x\in B_0} \mathbb{E}_x^{v^*}\bigl[V^*(X_{t-\tau+2})\bigr] \\
&\le \|V^*\|_{\infty,B_0} + M_0 + \min_{[0,t]\times\overline{B}_0} \overline{\varphi} + 2\,\|\varphi_0\|_\infty + \varrho\,\delta_0^{-1} K_0
\end{aligned} \tag{4.15}$$

with

$$K_0 \triangleq \sup_{t\ge 0}\, \sup_{x\in B_0}\, \mathbb{E}_x^{v^*}\bigl[V^*(X_t)\bigr].$$

By Lemma 3.5, K_0 is finite. Since

$$\operatorname*{osc}_{B_0} \overline{\varphi}(t,\cdot) \le \max_{x\in\overline{B}_0} \overline{\varphi}(t,x) - \min_{[0,t]\times\overline{B}_0} \overline{\varphi},$$

and $t \ge t_0$ was arbitrary, the result follows for all $t \ge t_0$ by (4.15). $\square$

5. Convergence of the RVI. We define the set $\mathcal{T}_0 \subset \mathbb{R}_+$ by

$$\mathcal{T}_0 \triangleq \bigl\{t \in \mathbb{R}_+ : \overline{\varphi}(t,0) \le \overline{\varphi}(t',0) \quad \forall t' \le t\bigr\}.$$

In the next lemma we use the variable

$$\Psi(t,x) \triangleq \overline{\varphi}(t,x) - \overline{\varphi}(t,0).$$

LEMMA 5.1. *Let Assumption* 3.2 *hold and also suppose that the initial condition* $\varphi_0 \in \mathcal{C}^2(\mathbb{R}^d)$ *is nonnegative and bounded. Then*

$$\Psi(t,x) \le C_0 + 2\,\|\varphi_0\|_\infty + \bigl(1 + \varrho\,\delta_0^{-1}\bigr)V^*(x) \qquad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{5.1a}$$

where C_0 *is the constant in Lemma* 4.10*, and there exists a constant* $\widehat{M}_0$ *such that*

$$\overline{\varphi}(t,0) - \overline{\varphi}(t',0) \le \widehat{M}_0 \qquad \forall t \ge t'. \tag{5.1b}$$

Proof. The estimate in (5.1a) follows by Lemmas 4.9 and 4.10. To show (5.1b) note that

$$\overline{\varphi}(t,0) - \overline{\varphi}(t',0) \le \overline{\varphi}(t,0) - \min_{s\in[0,t]}\,\overline{\varphi}(s,0) \qquad \forall t' \in [0,t]. \tag{5.2}$$

Let $t^* \in \operatorname{Arg\,min}_{s\in[0,t]}\,\overline{\varphi}(s,0)$ and define $T \triangleq t - t^*$. Clearly, $t^* = t - T \in \mathcal{T}_0$. We have

(5.3)
$$\begin{aligned}
\overline{\varphi}(t,0) - \overline{\varphi}(t-T,0) &\le \mathbb{E}_0^{v^*}\left[\int_0^T \overline{r}\bigl(X_s, v^*(X_s)\bigr)\,\mathrm{d}s + \overline{\varphi}(t-T, X_T)\right] - \overline{\varphi}(t-T,0)\\
&= \mathbb{E}_0^{v^*}\left[\int_0^T \overline{r}\bigl(X_s, v^*(X_s)\bigr)\,\mathrm{d}s + \Psi(t-T, X_T)\right]\\
&= V^*(0) - \mathbb{E}_0^{v^*}\,[V^*(X_T)] + \mathbb{E}_0^{v^*}\bigl[\Psi(t-T,X_T)\bigr]\\
&\le V^*(0) + C_0 + 2\,\|\varphi_0\|_\infty + \varrho\,\delta_0^{-1}\;\mathbb{E}_0^{v^*}\bigl[V^*(X_T)\bigr],
\end{aligned}$$

where the last inequality follows by (5.1a). It then follows by (5.3) that the map $t \mapsto \overline{\varphi}(t,0) - \overline{\varphi}(t-T,0)$ is bounded above by the constant

$$\widehat{M}_0 \triangleq C_0 + 2\,\|\varphi_0\|_\infty + \bigl(1 + \varrho\,\delta_0^{-1}\bigr)K_0.$$

The result then holds for any $t' < t$ by (5.2). ∎

The following corollary now follows by Lemmas 4.2, 4.10, and 5.1.

COROLLARY 5.2. *Under the hypotheses of Lemma* 5.1*, it holds that*

$$\operatorname*{osc}_{[t,t']\times B_0}\;\overline{\varphi} \le 2\,C_0 + \operatorname*{osc}_{\mathbb{R}^d}\;\varphi_0 + \widehat{M}_0 + \varrho(t'-t) \qquad \forall t' > t \ge 0.$$

Proof. By Lemmas 4.2 and 5.1

$$|\overline{\varphi}(s,0) - \overline{\varphi}(s',0)| \le \varrho\,\tau + \operatorname*{osc}_{\mathbb{R}^d}\;\varphi_0 + \widehat{M}_0 \qquad \forall x \in \mathbb{R}^d, \quad \forall s,s' \in [t, t+\tau]. \tag{5.4}$$

Hence the result follows by (5.4) and Lemma 4.10. ∎

LEMMA 5.3. *Under the hypotheses of Lemma* 5.1 *there exists a constant* $k_0 > 0$ *such that*

$$\mathbb{E}_x^{\hat{v}^t}[\breve{\tau} \wedge t] \le k_0 + 2\,\delta_0^{-1}\bigl(1 + \varrho\,\delta_0^{-1}\bigr)V^*(x) \qquad \forall x \in B_0^c.$$

Proof. Subtracting $\overline{\varphi}(t,0)$ from both sides of (4.2), we obtain

$$\begin{aligned}\Psi(t,x) \ge \mathbb{E}_x^{\hat{v}^t}\biggl[\int_0^{\breve{\tau}\wedge t} \overline{r}\bigl(X_s, \hat{v}_s^t(X_s)\bigr)\,\mathrm{d}s + \Psi(t - \breve{\tau}\wedge t, X_{\breve{\tau}\wedge t})\,\mathbb{I}\{\breve{\tau} \le t\} - \overline{\varphi}(t,0)\,\mathbb{I}\{\breve{\tau} > t\} \\ + \varphi_0(X_t)\,\mathbb{I}\{\breve{\tau} > t\} + \bigl(\overline{\varphi}(t - \breve{\tau}\wedge t, 0) - \overline{\varphi}(t,0)\bigr)\,\mathbb{I}\{\breve{\tau} \le t\}\biggr].\end{aligned}$$

We discard the nonnegative term $\varphi_0(X_t)\,\mathbb{I}\{\breve{\tau} > t\}$, and we use Lemma 4.10 and (5.1b) to write the above inequality as

$$\begin{aligned}\Psi(t,x) &\ge \mathbb{E}_x^{\hat{v}^t}\biggl[\int_0^{\breve{\tau}\wedge t} \overline{r}\bigl(X_s, \hat{v}_s^t(X_s)\bigr)\,\mathrm{d}s\biggr] - \sup_{0\le s\le t} \|\Psi(s,\cdot)\|_{\infty,B_0} \\ &\quad - \mathbb{E}_x^{\hat{v}^t}\bigl[\overline{\varphi}(t,0)\,\mathbb{I}\{\breve{\tau} > t\}\bigr] + \mathbb{E}_x^{\hat{v}^t}\bigl[\bigl(\overline{\varphi}(t - \breve{\tau}\wedge t, 0) - \overline{\varphi}(t,0)\bigr)\,\mathbb{I}\{\breve{\tau} \le t\}\bigr] \\ &\ge \mathbb{E}_x^{\hat{v}^t}\biggl[\int_0^{\breve{\tau}\wedge t} \overline{r}\bigl(X_s, \hat{v}_s^t(X_s)\bigr)\,\mathrm{d}s\biggr] - C_0 - \overline{\varphi}(t,0)\,\mathbb{P}_x^{\hat{v}^t}\bigl(\{\breve{\tau} > t\}\bigr) - \widehat{M}_0.\end{aligned} \tag{5.5}$$

By (5.1a) and (5.5) we obtain

$$\begin{aligned}C_0 + 2\,\|\varphi_0\|_\infty + \bigl(1 + \varrho\,\delta_0^{-1}\bigr)V^*(x) &\ge \delta_0\,\mathbb{E}_x^{\hat{v}^t}[\breve{\tau} \wedge t] - \overline{\varphi}(t,0)\,\mathbb{P}_x^{\hat{v}^t}\bigl(\{\breve{\tau} > t\}\bigr) - C_0 - \widehat{M}_0 \\ &\ge \biggl(\delta_0 - \frac{\overline{\varphi}(t,0)}{t}\biggr)\mathbb{E}_x^{\hat{v}^t}[\breve{\tau} \wedge t] - C_0 - \widehat{M}_0.\end{aligned}$$

The result then follows by Lemma 4.1. $\square$

LEMMA 5.4. *Under the hypotheses of Lemma* 5.1*,*

$$\overline{\varphi}(t,0)\,\mathbb{P}_x^{\hat{v}^t}\bigl(\{\breve{\tau} > t\}\bigr) \xrightarrow[t\to\infty]{} 0$$

uniformly on x *in compact sets of* $\mathbb{R}^d$*.*

Proof. By Lemmas 4.1 and 5.3 we have

$$\overline{\varphi}(t,0)\,\mathbb{P}_x^{\hat{v}^t}\bigl(\{\breve{\tau} > t\}\bigr) \le \frac{\overline{\varphi}(t,0)}{t}\bigl(k_0 + 2\,\delta_0^{-1}\bigl(1 + \varrho\,\delta_0^{-1}\bigr)V^*(x)\bigr) \xrightarrow[t\to\infty]{} 0$$

for all $x \in B_0^c$. $\square$

LEMMA 5.5. *Let Assumption* 3.2 *hold and also suppose the initial condition* $\varphi_0 \in \mathcal{C}^2(\mathbb{R}^d)$ *is nonnegative and bounded. Then the map* $t \mapsto \varphi(t,0)$ *is bounded on* $[0,\infty)$*, and it holds that*

$$-\operatorname*{osc}_{\mathbb{R}^d}\,\varphi_0 \le \liminf_{t\to\infty}\,\varphi(t,0) \le \limsup_{t\to\infty}\,\varphi(t,0) \le \widehat{M}_0 + \varrho.$$

Proof. Define

$$g(t) \triangleq \inf_{U\in\mathfrak{U}}\,\mathbb{E}_0^U\biggl[\int_0^t r(X_s, U_s)\,\mathrm{d}s + \varphi_0(X_t)\biggr].$$

By (3.29) we have

$$\int_0^t \varphi(s,0)\,\mathrm{d}s = \int_0^t \mathrm{e}^{s-t} g(s)\,\mathrm{d}s,$$

and hence

$$\begin{aligned}\varphi(t,0) &= g(t) - \int_0^t \mathrm{e}^{s-t} g(s)\,\mathrm{d}s \\ &= \bigl(1-\mathrm{e}^{-t}\bigr)^{-1}\int_0^t \mathrm{e}^{s-t}\bigl(g(t)-g(s)\bigr)\,\mathrm{d}s + \bigl(1-\mathrm{e}^{-t}\bigr)^{-1}\mathrm{e}^{-t}\int_0^t \mathrm{e}^{s-t} g(s)\,\mathrm{d}s\end{aligned} \tag{5.6}$$

for $t>0$. By Lemma 5.1, $g(t) \le \widehat{M}_0 + \varphi_0(0) + \varrho\, t$. Therefore the second term on the right-hand side of (5.6) vanishes as $t\to\infty$. By Lemma 4.2, $g(t)-g(s) \ge -\operatorname{osc}_{\mathbb{R}^d}\ \varphi_0$ for all $s\le t$. Also, by Lemma 5.1, $g(t)-g(s) \le \widehat{M}_0 + \varrho(t-s)$ for all $s\le t$. Evaluating the first integral on the right-hand side of (5.6) we obtain the bound

$$-\operatorname*{osc}_{\mathbb{R}^d}\ \varphi_0 \le \int_0^t \mathrm{e}^{s-t}\bigl(g(t)-g(s)\bigr)\,\mathrm{d}s \le \widehat{M}_0 + \varrho \qquad \forall t>0. \tag{5.7}$$

The result follows by (5.6)–(5.7). ∎

Combining Corollary 5.2, the boundedness of $t\mapsto\varphi(t,0)$ asserted in Lemma 5.5, and (1.12), it follows that $x\mapsto\varphi(t,x)$ is locally bounded in $\mathbb{R}^d$, uniformly in $t\ge 0$. Recall Definition 3.14. The standard interior estimates of the solutions of (3.12) provide us with the following regularity result.

THEOREM 5.6. *Under the hypotheses of Lemma* 5.5 *the closure of the orbit* $\{\Phi_t[\varphi_0], t\in\mathbb{R}_+\}$ *is locally compact in* $\mathcal{C}^2(\mathbb{R}^d)$.

Proof. By Lemma 4.9 and Corollary 5.2, the oscillation of $\overline{\varphi}$ is bounded on any cylinder $[n,n+1]\times B_R$ uniformly over $n\in\mathbb{N}$. This together with Lemma 5.5 implies that $\Phi_t[\varphi_0](x)$ is bounded on $(t,x)\in[n,n+1]\times B_R$ for any $R>0$, uniformly in $n\in\mathbb{N}$. It follows that the derivatives $\partial_{ij}\Phi_t[\varphi_0]$ are Hölder equicontinuous on every ball B_R uniformly in t. The result follows. ∎

We now turn to the proof of our main result.

Proof of Theorem 1.1. Let $\{t_n\}$ be any sequence tending to ∞ and let f be any limit in in the topology of Markov controls (see [4, section 2.4]) of $\{\hat{v}^{t_n}\}$ along some subsequence of $\{t_n\}$ also denoted as $\{t_n\}$. By Fatou's lemma and the stochastic representation of V^* in Theorem 3.1, we have

$$\begin{aligned}\liminf_{n\to\infty}\ \mathbb{E}_x^{\hat{v}^{t_n}}\biggl[\int_0^{\breve{\tau}\wedge t_n} \overline{r}\bigl(X_s,\hat{v}_s^{t_n}(X_s)\bigr)\,\mathrm{d}s\biggr] &\ge \mathbb{E}_x^f\biggl[\int_0^{\breve{\tau}} \overline{r}\bigl(X_s,f_s(X_s)\bigr)\,\mathrm{d}s\biggr] \\ &\ge \inf_{v\in\mathfrak{U}_{\mathrm{SSM}}}\ \mathbb{E}_x^v\biggl[\int_0^{\breve{\tau}} \overline{r}\bigl(X_s,v(X_s)\bigr)\,\mathrm{d}s\biggr] \\ &\ge V^*(x) - \|V^*\|_{\infty,B_0} \qquad \forall x\in B_0^c.\end{aligned} \tag{5.8}$$

The second inequality in (5.8) is due to the fact that the infimum of

$$\mathbb{E}_x^U\biggl[\int_0^{\breve{\tau}} \overline{r}\bigl(X_s,U_s\bigr)\,\mathrm{d}s\biggr]$$

over all $U \in \mathfrak{U}$ is realized at some $v \in \mathfrak{U}_{\mathrm{SSM}}$, while the third inequality follows by (3.5). Therefore, by (1.12), (5.5), (5.8), and Lemmas 5.4 and 5.5 we have that

$$\begin{aligned}\liminf_{t\to\infty}\ \varphi(t,x) &= \liminf_{t\to\infty}\ \bigl(\Psi(t,x)+\varphi(t,0)\bigr)\\ &\ge V^*(x) - \|V^*\|_{\infty,B_0} - C_0 - \widehat{M}_0 - \operatorname*{osc}_{\mathbb{R}^d}\ \varphi_0 \qquad \forall x\in B_0^c.\end{aligned}\tag{5.9}$$

Also, by (5.1a) and Lemma 5.5 we obtain

$$\begin{aligned}\limsup_{t\to\infty}\ \varphi(t,x) &= \limsup_{t\to\infty}\ \bigl(\Psi(t,x)+\varphi(t,0)\bigr)\\ &\le C_0 + 2\,\|\varphi_0\|_\infty + \bigl(1+\varrho\,\delta_0^{-1}\bigr)V^*(x) + \widehat{M}_0 + \varrho\end{aligned}\tag{5.10}$$

for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$.

Hence, by (5.9)–(5.10) if we select

$$c = -\bigl(\|V^*\|_{\infty,B_0} + C_0 + \widehat{M}_0 + \operatorname*{osc}_{\mathbb{R}^d}\ \varphi_0\bigr)$$

and define

$$A \triangleq \bigl\{h\in\mathcal{C}^2(\mathbb{R}^d) : \|h\|_{V^*} \le C_0 + \widehat{M}_0 + \varrho + 2\,\|\varphi_0\|_\infty + \bigl(1+\varrho\,\delta_0^{-1}\bigr)\bigr\},$$

then any ω-limit point of $\varphi(t,x)$ as $t\to\infty$ lies in $\mathcal{G}_c\cap A$ (see Definition 3.14). Since the ω-limit set of φ_0 is invariant under the semiflow Φ_t, and by Remark 3.16 the only invariant subset of $\mathcal{G}_c\cap A$ is the singleton $\{V^*-V^*(0)+\varrho\}$, the result follows. ∎

6. Concluding remarks. We have studied the RVI algorithm for an important class of ergodic control problems wherein instability is possible but is heavily penalized by the near-monotone structure of the running cost. The near-monotone cost structure plays a crucial role in the analysis and the proof of stabilization of the quasi-linear parabolic Cauchy initial value problem that models the algorithm.

We would like to conjecture that the RVI converges starting from any initial condition $\varphi_0\in\mathcal{O}_{V^*}$. It is only the estimate in Lemma 4.2 that restricts us to consider bounded initial conditions only. We want to mention here that a related such estimate can be obtained as follows:

$$\begin{aligned}\overline{\varphi}(t,x) &= \inf_{U\in\mathfrak{U}}\ \mathbb{E}_x^U\left[\int_0^t \overline{r}(X_s,U_s)\,\mathrm{d}s + \varphi_0(X_t)\right]\\ &= \inf_{U\in\mathfrak{U}}\ \mathbb{E}_x^U\left[\int_0^\tau \overline{r}(X_s,U_s)\,\mathrm{d}s + \overline{\varphi}(t-\tau,X_{t-\tau})\right]\\ &\ge -\varrho\,\tau + \min_{y\in\mathbb{R}^d}\ \overline{\varphi}(t-\tau,y) \qquad \forall\tau\in[0,t],\quad \forall x\in\mathbb{R}^d.\end{aligned}$$

In particular

$$\min_{\mathbb{R}^d}\ \overline{\varphi}(t-\tau,\cdot) - \min_{\mathbb{R}^d}\ \overline{\varphi}(t,\cdot) \le \varrho\,\tau \qquad \forall\tau\in[0,t],$$

and this estimate does not depend on the initial data φ_0. This suggests that it is probably worth studying the variation of the RVI algorithm that results by replacing $\varphi(t,0)$ by $\min_{\mathbb{R}^d}\ \varphi(t,\cdot)$ in (1.7).

Rate of convergence results and computational aspects of the algorithm are open issues.

REFERENCES

[1] J. Abounadi, D. P. Bertsekas, and V. S. Borkar, *Learning algorithms for Markov decision processes with average cost*, SIAM J. Control Optim., 40 (2001), pp. 681–698.

[2] A. Arapostathis, *On the policy iteration algorithm for nondegenerate controlled diffusions under the ergodic criterion*, in Optimization, Control, and Applications of Stochastic Systems, D. Hernández-Hernández and J. A. Minjárez-Sosa, eds., Systems Control Found. Appl., Birkhäuser Boston, Boston, MA, 2012, pp. 1–12.

[3] A. Arapostathis and V. S. Borkar, *A relative value iteration algorithm for nondegenerate controlled diffusions*, SIAM J. Control Optim., 50 (2012), pp. 1886–1902.

[4] A. Arapostathis, V. S. Borkar, and M. K. Ghosh, *Ergodic Control of Diffusion Processes*, Encyclopedia Math. Appl. 143, Cambridge University Press, Cambridge, UK, 2011.

[5] A. Arapostathis, V. S. Borkar, and K. S. Kumar, *Relative value iteration for stochastic differential games*, in Advances in Dynamical Games: Theory, Applications and Numerical Methods, V. Krivan and G. Zaccour, eds., Ann. Internat. Soc. Dynam. Games 13, Birkhäuser Boston, Boston, MA, 2013.

[6] V. I. Bogachev, N. V. Krylov, and M. Röckner, *On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions*, Comm. Partial Differential Equations, 26 (2001), pp. 2037–2080.

[7] R.-R. Chen and S. Meyn, *Value iteration and optimization of multiclass queueing networks*, Queueing Systems Theory Appl., 32 (1999), pp. 65–97.

[8] S. Eidelman, S. Kamin, and F. Porper, *Uniqueness of solutions of the Cauchy problem for parabolic equations degenerating at infinity*, Asymptot. Anal., 22 (2000), pp. 349–358.

[9] M. Gruber, *Harnack inequalities for solutions of general second order parabolic equations and estimates of their Hölder constants*, Math. Z., 185 (1984), pp. 23–43.

[10] I. Gyöngy and N. Krylov, *Existence of strong solutions for Itô's stochastic equations via approximations*, Probab. Theory Related Fields, 105 (1996), pp. 143–158.

[11] R. Z. Hasminskiĭ, *Ergodic properties of recurrent diffusion processes and stabilization of the solution of the Cauchy problem for parabolic equations*, Theory Probab. Appl., 5 (1960), pp. 179–196.

[12] N. Ichihara, *Large time asymptotic problems for optimal stochastic control with superlinear cost*, Stochastic Process. Appl., 122 (2012), pp. 1248–1275.

[13] N. Ichihara and S.-J. Sheu, *Large time behavior of solutions of Hamilton-Jacobi-Bellman equations with quadratic nonlinearity in gradients*, SIAM J. Math. Anal., 45 (2013), pp. 279–306.

[14] N. V. Krylov, *Controlled Diffusion Processes*, Appl. Math. 14, Springer-Verlag, New York, 1980.

[15] N. V. Krylov, *Lectures on Elliptic and Parabolic Equations in Sobolev Spaces*, Grad. Stud. Math. 96, American Mathematical Society, Providence, RI, 2008.

[16] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva, *Linear and quasi-linear equations of parabolic type*, Transl. Math. Monogr. 23, American Mathematical Society, Providence, RI, 1967.

[17] E. Löcherbach, D. Loukianova, and O. Loukianov, *Polynomial bounds in the ergodic theorem for one-dimensional diffusions and integrability of hitting times*, Ann. Inst. H. Poincaré Probab. Statist., 47 (2011), pp. 425–449.

[18] S. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, 2nd ed., Cambridge University Press, Cambridge, UK, 2009.

[19] S. P. Meyn, *The policy iteration algorithm for average reward Markov decision processes with general state space*, IEEE Trans. Automat. Control, 42 (1997), pp. 1663–1680.

[20] S. P. Meyn and R. L. Tweedie, *Stability of Markovian processes.* III. *Foster-Lyapunov criteria for continuous-time processes*, Adv. in Appl. Probab., 25 (1993), pp. 518–548.

[21] H. Nagai, *Downside risk minimization via a large deviations approach*, Ann. Appl. Probab., 22 (2012), pp. 608–669.

[22] W. Stannat, *(Nonsymmetric) Dirichlet operators on L^1: Existence, uniqueness and associated Markov processes*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 28 (1999), pp. 99–140.

[23] D. J. White, *Dynamic programming, Markov chains, and the method of successive approximations*, J. Math. Anal. Appl., 6 (1963), pp. 373–376.