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The Dirichlet problem for stable-like operators and related probabilistic representations

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ABSTRACT
We study stochastic differential equations with jumps with no diffusion part, governed by a large class of stable-like operators, which may contain a drift term. For this class of operators, we establish the regularity of solutions to the Dirichlet problem up to the boundary as well as the usual stochastic characterization of these solutions. We also establish key connections between the recurrence properties of the jump process and the associated nonlocal partial differential operator. Provided that the process is positive (Harris) recurrent, we also show that the mean hitting time of a ball is a viscosity solution of an exterior Dirichlet problem.

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1. Introduction

Stochastic differential equations (SDEs) with jumps have received wide attention in stochastic analysis as well as in the theory of differential equations. Unlike continuous diffusion processes, SDEs with jumps have long range interactions and therefore the generators of such processes are nonlocal in nature. These processes arise in various applications, for instance, in mathematical finance and control [23, 37] and image processing [26]. There have been various studies on such processes from a stochastic analysis viewpoint concentrating on existence, uniqueness, and stability properties of the solution of the SDE [1, 9, 21, 22, 31, 33] as well as from a differential equation viewpoint focusing on the existence and regularity of viscosity solutions [6, 7, 18]. One of our objectives in this paper is to establish stochastic representations of solutions of SDEs with jumps through the associated integro-differential operator.

Let us consider a Markov process X in $\mathbb{R}^d$ with generator $\mathcal{I}$. Let D be a smooth bounded domain in $\mathbb{R}^d$. We denote the first exit time of the process X from D by $\tau(D) = \inf\{t \geq 0 : X_t \notin D\}$. One can formally say that

$$u(x) := \mathbb{E}_x\left[\int_0^{\tau(D)} f(X_s)\,\mathrm{d}s\right] \tag{1.1}$$

satisfies the equation

$$\mathcal{I}u = -f \quad \text{in } D, \quad u = 0 \quad \text{in } D^c, \tag{1.2}$$

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where $\mathbb{E}_x$ denotes the expectation operator on the canonical space of the process starting at x when $t = 0$. An important question is when can we actually identify the solution of (1.2) as the right-hand side of (1.1). When $\mathcal{I} = \Delta + b$, i.e., X is a drifted Brownian notion, one can use the regularity of the solution and Itô's formula to establish (1.1). Clearly, one standard method to obtain a representation of the mean first exit time from D is to find a classical solution of (1.2) for nonlocal operators. This is related to the work in [10] where estimates of classical solutions for stable-like operators are obtained when $D = \mathbb{R}^d$. A future research direction mentioned in [10] concerns the existence and regularity of solutions to the Dirichlet problem for stable-like operators. We provide an answer to some of these questions in Theorems 3.1 and 3.2.

One of the main results of this paper is the existence of a classical solution of (1.2) for a fairly large class of nonlocal operators. We study operators of the form

$$\mathcal{I}u(x) = b(x) \cdot \nabla u(x) + \int_{\mathbb{R}^d} \mathfrak{d}u(x; z)\, \pi(x, z)\, \mathrm{d}z, \tag{1.3}$$

where

$$\mathfrak{d}u(x; z) := u(x + z) - u(x) - \mathbf{1}_{\{|z|\le 1\}} \nabla u(x) \cdot z, \tag{1.4}$$

with $\mathbf{1}_A$ denoting the indicator function of a set A. Throughout the paper, we use the symbol π to denote the "kernel" of the operator. We primarily focus on operators for which π takes the form $\pi(x, z) = \frac{k(x,z)}{|z|^{d+\alpha}}$, with $\alpha \in (1, 2)$, and b and k are locally Hölder in x with exponent β, and $k(x, \cdot) - k(x, 0)$ satisfies the integrability condition in (3.1). This class of operators, without the drift term, is essentially the one considered by Bass in [10], and he referred to them as *stable-like*, a term which we adopt. Some of the future research directions mentioned in [10] concern the existence and regularity of solutions to the Dirichlet problem for stable-like operators. We provide an answer to some of these questions in Theorem 3.1, Corollary 3.1 and Theorem 3.2. We show in Theorem 3.2 that u defined by (1.1) is the unique solution of (1.2) in $C^{2s+\beta}_{\mathrm{loc}}(D) \cap C(\mathbb{R}^d)$. This result can be extended to include nonzero boundary conditions provided that the boundary data are regular enough. The proof is based on various regularity results concerning the Dirichlet problem, including optimal regularity up to the boundary, which comprise Section 3. We also wish to bring to the attention of the reader two recent papers [27, 35] which are closely related to our work.

To help the reader, we summarize here the different classes of operators used in the paper. The most general class considered denoted by $\mathfrak{L}_\alpha$ consists of operators as in (1.3) with $\pi(x, z) = \frac{k(x,z)}{|z|^{d+\alpha}}$, $\alpha \in (1, 2)$, and with $b : \mathbb{R}^d \to \mathbb{R}^d$ and $k : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)$ Borel measurable and locally bounded. The subclass of $\mathfrak{L}_\alpha$ with symmetric kernels, i.e., $k(x, z) = k(x, -z)$ is denoted by $\mathfrak{L}^{\mathsf{sym}}_\alpha$ (Definition 2.2). Results concerning these classes are in Lemma 2.3. A subclass of these denoted by $\mathfrak{L}_\alpha(\lambda)$, where λ is a parameter that controls the growth of b and k, is studied in Sections 4 and 5.1 (Definition 4.1). The main results of the paper in Section 3 hold over the class of stable-like operators mentioned earlier, which is denoted by $\mathfrak{I}_\alpha(\beta, \theta, \lambda)$. Here β, θ, and λ are parameters (Definition 3.1). This class is then studied further in Section 5.3. The kernels in this class are not assumed to be symmetric.

Recall that a function h is said to be *harmonic* with respect to X in D if $h(X_{t\wedge\tau(D)})$ is a martingale. One of the important properties of non-negative harmonic functions for nondegenerate continuous diffusions is the Harnack inequality, which plays a crucial role in various regularity and stability estimates. The work in [13] proves the Harnack inequality for a class of pure jump processes, and this is further generalized in [11] for nonsymmetric kernels that may have variable order. A parabolic Harnack inequality is obtained in [8] for symmetric jump processes associated with the Dirichlet form, with a symmetric kernel. Sufficient conditions on Markov processes to satisfy the Harnack inequality are identified [38]. The Harnack property is also established for jump processes with a nondegenerate diffusion part in [5, 24, 39]. A Harnack-type estimate for harmonic functions that are not necessarily non-negative in all of $\mathbb{R}^d$ is established in [29]. Nevertheless, the Harnack property is quite delicate for nonlocal operators, and important counterexamples can be found in [17, 28].

In this paper, we prove a Harnack inequality for harmonic functions relative to the operator $\mathcal{I}$ in (1.3) when k and b are locally bounded and measurable, and either $k(x,z) = k(x,-z)$, or k satisfies (3.1) (Theorem 4.1). The proof is based on verifying the sufficient conditions [38], through a series of lemmas. So, even though in a sense it lacks novelty, we include the proof in the paper since we use the Harnack property in Section 5. Let us also mention that the estimates obtained in Section 4 may also be used to establish Hölder continuity for harmonic functions by following a similar method as in [12]. However, we do not pursue this here.

In Section 5, we study the ergodic properties of the Markov process such as positive (Harris) recurrence, invariant probability measures, etc. We provide a sufficient condition for positive recurrence and the existence of an invariant probability measure (Theorems 5.1 and 5.2). This is done through imposing a *Lyapunov stability* condition on the generator. Following Has'minskiĭ's method, we establish the existence of an invariant probability measure for a fairly large class of processes. We also show that one may obtain a positive recurrent process using a nonsymmetric kernel and no drift (Theorem 5.3). In this case, the nonsymmetric part of the kernel plays the role of the drift. Let us mention here that in [41] the author provides sufficient conditions for positive recurrence of a class of jump diffusions and this is accomplished by constructing suitable Lyapunov-type functions. However, the class of kernels considered in [41] satisfies a different set of hypotheses than those assumed in this paper and in a certain way lies in the complement of the class of Lévy kernels that we consider. Stability of one-dimensional processes is discussed in [40] under the assumption of Lebesgue-irreducibility. Last, we want to point out one of the interesting results of this paper, and this is the characterization of the mean hitting time of a bounded domain as a viscosity solution of an exterior Dirichlet problem (Theorem 5.4). This is established for the class of operators in Definition 3.1 and can be viewed as a partial converse to Theorem 5.1. Therefore, provided that the drift $b(x)$ and the numerator $k(x,z)$ of the kernel have at most affine growth in x (2.5), Theorems 5.1 and 5.4 imply that a Markov process with generator in the class of stable-like operators studied in Section 3 is positive recurrent if and only if the Lyapunov criterion in Definition 5.1 holds. For nondegenerate diffusions, this is of course a well-known result due to Has'minskiĭ.

The organization of the paper is as follows. In Section 1.1, we introduce the notation used in the paper. In Section 2, we introduce the model and derive some basic results. Section 3 is devoted to the regularity of solutions to the Dirichlet problem. In Section 4, we establish the Harnack property as mentioned earlier. Section 5 establishes connections between the recurrence properties of the process and the solutions of the nonlocal equations.

1.1. Notation

The standard norm in the d-dimensional Euclidean space $\mathbb{R}^d$ is denoted by $|\cdot|$, and let $\mathbb{R}^d_* := \mathbb{R}^d \setminus \{0\}$. The set of nonnegative real numbers is denoted by $\mathbb{R}_+$, $\mathbb{N}$ stands for the set of natural numbers, and $\mathbf{1}_A$ denotes the indicator function of a set A. For vectors $a, b \in \mathbb{R}^d$, we denote the scalar product by $a \cdot b$. We denote the maximum (minimum) of two real numbers a and b by $a \vee b$ ($a \wedge b$). Let $a^+ := a \vee 0$ and $a^- := (-a) \vee 0$. By $\lfloor a \rfloor$ ($\lceil a \rceil$), we denote the largest (least) integer less than (greater than) or equal to the real number a. For $x \in \mathbb{R}^d$ and $r \geq 0$, we denote by $B_r(x)$ the open ball of radius r around x in $\mathbb{R}^d$, while B_r without an argument denotes the ball of radius r around the origin. Also in the interest of simplifying the notation, we use $B \equiv B_1$, i.e., the unit ball centered at 0.

Given a metric space $\mathcal{S}$, we denote by $\mathcal{B}(\mathcal{S})$ and $B_b(\mathcal{S})$ the Borel σ-algebra of $\mathcal{S}$ and the set of bounded Borel measurable functions on $\mathcal{S}$, respectively. The set of Borel probability measures on $\mathcal{S}$ is denoted by $\mathcal{P}(\mathcal{S})$, $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation norm on $\mathcal{P}(\mathcal{S})$, and δ_x the Dirac mass at x. For any function $g : \mathcal{S} \to \mathbb{R}^d$, we define $\|g\|_\infty := \sup_{x \in \mathcal{S}} |g(x)|$.

The closure and the boundary of a set $A \subset \mathbb{R}^d$ are denoted by $\overline{A}$ and ∂A, respectively, and $|A|$ denotes the Lebesgue measure of A. We also define

$$\tau(A) := \inf\,\{s \geq 0 : X_s \notin A\}.$$

Therefore, $\tau(A)$ denotes the first exit time of the process X from A. For $R > 0$, we often use the abbreviated notation $\tau_R := \tau(B_R)$.

We introduce the following notation for spaces of real-valued functions on a set $A \subset \mathbb{R}^d$. The space $L^p(A)$, $p \in [1, \infty)$, stands for the Banach space of (equivalence classes) measurable functions g satisfying $\int_A |g(x)|^p \,\mathrm{d}x < \infty$, and $L^\infty(A)$ is the Banach space of functions that are essentially bounded in A. For an integer $k \geq 0$, the space $C^k(A)$ ($C^\infty(A)$) refers to the class of all functions whose partial derivatives up to order k (of any order) exist and are continuous, $C^k_c(A)$ is the space of functions in $C^k(A)$ with compact support, and $C^k_b(A)$ is the subspace of $C^k(A)$ consisting of those functions whose derivatives up to order k are bounded. Also, the space $C^{k,r}(A)$, $r \in (0, 1]$, is the class of all functions whose partial derivatives up to order k are Hölder continuous of order r. For simplicity, we write $C^{0,r}(A) = C^r(A)$. For any $\gamma > 0$, $C^\gamma(A)$ denotes the space $C^{\lfloor\gamma\rfloor, \gamma - \lfloor\gamma\rfloor}(A)$, under the convention $C^{k,0}(A) = C^k(A)$.

In general, if $\mathcal{X}$ is a space of real-valued functions on a domain D, $\mathcal{X}_{\mathrm{loc}}$ consists of all functions g such that $g\varphi \in \mathcal{X}$ for every $\varphi \in C^\infty_c(D)$.

For a non-negative multiindex $\beta = (\beta_1, \dots, \beta_d)$, let $|\beta| := \beta_1 + \cdots + \beta_d$ and $D^\beta := \partial_1^{\beta_1} \cdots \partial_d^{\beta_d}$, where $\partial_i := \frac{\partial}{\partial x_i}$.

Given a domain D with a C^2 boundary, we define $d_x := \operatorname{dist}(x, \partial D)$ and $d_{xy} := \min(d_x, d_y)$, for $x, y \in D$. For $u \in C(D)$ and $r \in \mathbb{R}$, we introduce the weighted norm

$$[\![u]\!]^{(r)}_{0;D} := \sup_{x \in D}\, d_x^r\, |u(x)|,$$

and, for $k \in \mathbb{N}$ and $\delta \in (0, 1]$, the seminorms

$$[\![u]\!]^{(r)}_{k;D} := \sup_{|\beta|=k}\, \sup_{x \in D}\, d_x^{k+r} \big|D^\beta u(x)\big|,$$

$$[\![u]\!]^{(r)}_{k,\delta;D} := \sup_{|\beta|=k}\, \sup_{x,y \in D}\, \left(d_{xy}^{k+\delta+r}\, \frac{\big|D^\beta u(x) - D^\beta u(y)\big|}{|x-y|^\delta} \right).$$

For $r \in \mathbb{R}$ and $\gamma \geq 0$, with $\gamma + r \geq 0$, we define the space

$$\mathscr{C}^{(r)}_\gamma(D) := \big\{u \in C^\gamma(D) \cap C(\mathbb{R}^d) : u(x) = 0 \text{ for } x \in D^c,\ \|u\|^{(r)}_{\gamma;D} < \infty\big\},$$

where

$$\|u\|_{\gamma;D}^{(r)} := \sum_{k=0}^{\lceil\gamma\rceil-1} [\![u]\!]_{k,D}^{(r)} + [\![u]\!]_{\lceil\gamma\rceil-1,\,\gamma+1-\lceil\gamma\rceil;\,D}^{(r)},$$

under the convention $\|u\|_{0;D}^{(r)} = [\![u]\!]_{0;D}^{(r)}$. We also use the notation $\|u\|_{k,\delta;D}^{(r)} = \|u\|_{k+\delta;D}^{(r)}$ for $\delta \in (0,1]$. It is straightforward to verify that $\|u\|_{\gamma;D}^{(r)}$ is a norm, under which $\mathscr{C}_{\gamma}^{(r)}(D)$ is a Banach space.

If the distance functions d_x or d_{xy} are not included in the above definitions, we denote the corresponding seminorms by $[\,\cdot\,]_{k;D}$ or $[\,\cdot\,]_{k,\delta;D}$ and define

$$\|u\|_{C^{k,\delta}(D)} := \sum_{\ell=0}^{k} [u]_{\ell;D} + [u]_{k,\delta;D}.$$

Thus, $\|u\|_{C^{\gamma}(D)}$ is well defined for any $\gamma > 0$, by the identification $C^{\gamma}(D) = C^{\lfloor\gamma\rfloor,\gamma-\lfloor\gamma\rfloor}(A)$.

We recall the well-known interpolation inequalities [25, Lemma 6.32, p. 30]. Let $u \in C^{2,\beta}(D)$. Then for any ε there exists a constant $C = C(\varepsilon, j, k, r)$ such that

$$\begin{aligned} &[\![u]\!]_{j,\gamma;D}^{(0)} \le C\,\|u\|_{0;D}^{(0)} + \varepsilon\, [\![u]\!]_{k,\beta;D}^{(0)} \\ &\|u\|_{j,\gamma;D}^{(0)} \le C\,\|u\|_{0;D}^{(0)} + \varepsilon\, [\![u]\!]_{k,\beta;D}^{(0)} \end{aligned} \qquad j = 0,1,2,\ 0 \le \beta,\ \gamma \le 1,\ j+\gamma < k+\beta.$$

Throughout the paper $s \in (1/2, 1)$ is a parameter, and $\alpha = 2s$.

2. Preliminary results

Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\pi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ be two given measurable functions. We define the nonlocal operator $\mathcal{I}$ as follows:

$$\mathcal{I}u(x) := b(x) \cdot \nabla u(x) + \int_{\mathbb{R}^d} \mathfrak{d}u(x; z)\, \pi(x, z)\, \mathrm{d}z, \tag{2.1}$$

with $\mathfrak{d}u$ as in (1.4). We always assume that

$$\int_{\mathbb{R}^d} (|z|^2 \wedge 1)\, \pi(x, z)\, \mathrm{d}z \;<\; \infty \qquad \forall\, x \in \mathbb{R}^d.$$

Note that (2.1) is well defined for any $u \in C_b^2(\mathbb{R}^d)$. Let $\Omega = \mathcal{D}([0,\infty), \mathbb{R}^d)$ denotes the space of all right-continuous functions mapping $[0,\infty)$ to $\mathbb{R}^d$, having finite left limits (cádlág). Define $X_t = \omega(t)$ for $\omega \in \Omega$ and let $\{\mathcal{F}_t\}$ be the right-continuous filtration generated by the process $\{X_t\}$. In this paper, we always assume that given any initial distribution ν_0, there exists a strong Markov process $(X, \mathbb{P}_{\nu_0})$ that satisfies the martingale problem corresponding to $\mathcal{I}$, i.e., $\mathbb{P}_{\nu_0}(X_0 \in A) = \nu_0(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$ and for any $g \in C_b^2(\mathbb{R}^d)$,

$$g(X_t) - g(X_0) - \int_0^t \mathcal{I}g(X_s)\, \mathrm{d}s$$

is a martingale with respect to the filtration $\{\mathcal{F}_t\}$. We denote the law of the process by $\mathbb{P}_x$ when $\nu_0 = \delta_x$. Sufficient conditions on b and π to ensure the existence of such processes are available in the literature. Unfortunately, the available sufficient conditions do not cover a wide class of operators $\mathcal{I}$. We refer the reader to [9] for the available results in this direction as well as to [2, 14, 21, 22, 31, 33]. When $b \equiv 0$, well posedness of the martingale problem is obtained under some regularity assumptions on π in [1].

Let us mention once more that our goal here is not to study the existence of a solution to the martingale problem. Therefore, we do not assume any regularity conditions on the coefficients, unless otherwise stated.

We recall the definition of a viscosity solution [6, 18].

Definition 2.1. Let D be a domain with C^2 boundary. A function $u : \mathbb{R}^d \to \mathbb{R}$ which is upper (lower) semicontinuous on $\overline{D}$ is said to be a subsolution (supersolution) to

$$\begin{aligned} \mathcal{I}u &= -f \quad \text{in } D,\\ u &= g \quad \text{in } D^c, \end{aligned}$$

where $\mathcal{I}$ is given by (2.1), if for any $x \in \overline{D}$ and a function $\varphi \in C^2(\mathbb{R}^d)$ such that $\varphi(x) = u(x)$ and $\varphi(z) > u(z)$ $\big(\varphi(z) < u(z)\big)$ on $\mathbb{R}^d \setminus \{x\}$, it holds that

$$\mathcal{I}\varphi(x) \geq -f(x) \quad \big(\mathcal{I}\varphi(x) \leq -f(x)\big), \quad \text{if } x \in D,$$

while, if $x \in \partial D$, then

$$\max\,(\mathcal{I}\varphi(x) + f(x), g(x) - u(x)) \geq 0 \quad \big(\min\,(\mathcal{I}\varphi(x) + f(x), g(x) - u(x)) \leq 0\big).$$

A function u is said to be a viscosity solution if it is both a sub and a supersolution.

In Definition 2.1, we may assume that φ is bounded, provided u is bounded. Otherwise, we may modify the function φ by replacing it with u outside a small ball around x. It is evident that every classical solution is also a viscosity solution.

2.1. Three lemmas concerning operators with measurable kernels

Lemma 2.1. *Let D be a bounded domain. Suppose X is a strong Markov process associated with $\mathcal{I}$ in (2.1), with b locally bounded, and that the integrability conditions*

$$\sup_{x\in K} \int_{\{|z|>1\}} |z|\, \pi(x,z)\, \mathrm{d}z < \infty, \quad \text{and} \quad \inf_{x\in K} \int_{\mathbb{R}^d} |z|^2 \pi(x,z)\, \mathrm{d}z = \infty \tag{2.2}$$

hold for any compact set K. Then $\sup_{x\in D} \mathbb{E}_x[(\tau(D))^m] < \infty$, for any positive integer m.

Proof. Without loss of generality, we assume that $0 \in D$. Otherwise we inflate the domain to include 0. Let $\bar{d} = \operatorname{diam}(D)$ and $M_D = \sup_{x\in \bar{D}} |b(x)|$. Recall that B_R denotes the ball of radius R around the origin. We choose $R > 1 \vee 2(\bar{d} \vee M_D)$, and large enough so as to satisfy the inequality

$$\inf_{x\in D} \int_{B_R} |z|^2\, \pi(x,z)\, \mathrm{d}z > 1 + 2\bar{d}M_D + 2\bar{d} \sup_{x\in D} \int_{\{1<|z|\leq R\}} |z|\, \pi(x,z)\, \mathrm{d}z.$$

Let $\varphi \in C_b^2(\mathbb{R}^d)$ be a radially increasing function such that $\varphi(x) = |x|^2$ for $|x| \leq 2R$ and $\varphi(x) = 8R^2$ for $|x| \geq 2R + 1$. Then, for any $x \in D$, we have

$$\begin{aligned} \mathcal{I}\varphi(x) &= b(x) \cdot \nabla\varphi(x) + \int_{\mathbb{R}^d} \mathfrak{d}\varphi(x;z)\, \pi(x,z)\, \mathrm{d}z \\ &\geq -2\bar{d}M_D + \int_{B_R} \big(\varphi(x+z) - \varphi(x) - \nabla\varphi(x)\cdot z\big)\, \pi(x,z)\, \mathrm{d}z \\ &\quad + \int_{\{1<|z|\leq R\}} \nabla\varphi(x)\cdot z\, \pi(x,z)\, \mathrm{d}z + \int_{B_R^c} \big(\varphi(x+z) - \varphi(x)\big)\, \pi(x,z)\, \mathrm{d}z. \end{aligned}$$

Also, for any $|z| \geq R$, it holds that $|x+z| \geq \bar{d} \geq |x|$. Therefore $\varphi(x+z) \geq \varphi(x)$. Hence

$$\begin{aligned}
\mathcal{I}\varphi(x) &\geq -2\bar{d}M_D + \int_{\{1<|z|\leq R\}} \big(\nabla\varphi(x)\cdot z\big)\,\pi(x,z)\,\mathrm{d}z \\
&\quad + \int_{B_R} \big(\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z\big)\,\pi(x,z)\,\mathrm{d}z \\
&\geq -2\bar{d}M_D - 2\bar{d}\int_{\{1<|z|\leq R\}} |z|\,\pi(x,z)\,\mathrm{d}z + \int_{B_R} |z|^2\,\pi(x,z)\,\mathrm{d}z \\
&\geq 1.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}_x[\varphi(X_{\tau(D)\wedge t})] - \varphi(x) &= \mathbb{E}_x\left[\int_0^{\tau(D)\wedge t} \mathcal{I}\varphi(X_s)\,\mathrm{d}s\right] \\
&\geq \mathbb{E}_x[\tau(D)\wedge t] \quad \forall\, x\in D.
\end{aligned}$$

Letting $t\to\infty$, we obtain $\mathbb{E}_x[\tau(D)] \leq 8R^2$. Since $x\in D$ is arbitrary, this shows that

$$\sup_{x\in D}\, \mathbb{E}_x[\tau(D)] \leq 8R^2.$$

We continue using the method of induction. We have proved the result for $m=1$. Assume that it is true for m, i.e., $M_m := \sup_{x\in D} \mathbb{E}_x[(\tau(D))^m] < \infty$. Let $h(x) = M_m\varphi(x)$ where φ is defined above. Then from the calculations above, we obtain

$$\mathbb{E}_x[h(X_{\tau(D)\wedge t})] - h(x) \geq \mathbb{E}_x[M_m(\tau(D)\wedge t)] \quad \forall\, x\in D. \tag{2.3}$$

Denoting $\tau(D)$ by τ, we have

$$\begin{aligned}
\mathbb{E}_x[\tau^{m+1}] &= \mathbb{E}_x\left[\int_0^\infty (m+1)(\tau-t)^m\,\mathbf{1}_{\{t<\tau\}}\,\mathrm{d}t\right] \\
&= \mathbb{E}_x\left[\int_0^\infty (m+1)\mathbb{E}_x\big[(\tau-t)^m\,\mathbf{1}_{\{t<\tau\}} \,\big|\, \mathcal{F}_{t\wedge\tau}\big]\,\mathrm{d}t\right] \\
&= \mathbb{E}_x\left[\int_0^\infty (m+1)\mathbf{1}_{\{t\wedge\tau<\tau\}}\mathbb{E}_{X_{t\wedge\tau}}[\tau^m]\,\mathrm{d}t\right] \\
&\leq \sup_{x\in D}\, \mathbb{E}_x[\tau^m]\,\mathbb{E}_x\left[\int_0^\infty (m+1)\mathbf{1}_{\{t\wedge\tau<\tau\}}\,\mathrm{d}t\right] \\
&\leq M_m(m+1)\,\mathbb{E}_x[\tau],
\end{aligned}$$

and in view of (2.3), the proof is complete. □

Boundedness of solutions to the Dirichlet problem on bounded domains and with zero boundary data is asserted in the following lemma.

Lemma 2.2. *Let b and f be locally bounded functions and D a bounded domain. Suppose π satisfies (2.2). Then there exists a constant C, depending on* diam(D), $\sup_{x\in D} |b(x)|$ *and π, such that any viscosity solution u to the equation*

$$\begin{aligned}
\mathcal{I}u &= f \quad \text{in } D,\\
u &= 0 \quad \text{in } D^c
\end{aligned}$$

satisfies $\|u\|_\infty \leq C\,\sup_{x\in D} |f(x)|$.

Proof. As shown in the proof of Lemma 2.1, there exists a non-negative, radially nondecreasing function $\xi \in C_b^2(\mathbb{R}^d)$ satisfying $\mathcal{I}\xi(x) > \sup_{x\in D} |f(x)|$ for all $x \in \bar{D}$. Let $M > 0$ be the smallest number such that $M - \xi$ touches u from above at least at one point. We claim that $M \leq \|\xi\|_\infty$. If not, then $M - \xi(x) > 0$ for all $x \in D^c$. Therefore $M - \xi$ touches u in D from above. Hence by the definition of a viscosity solution, we have $\mathcal{I}(M - \xi(x)) \geq f(x)$, or equivalently, $\mathcal{I}\xi(x) \leq -f(x)$, where $x \in D$ is a point of contact from above. But this contradicts the definition of ξ. Thus $M \leq \|\xi\|_\infty$. Also by the definition of M, we have

$$\sup_{x\in D} u(x) \leq \sup_{x\in D} (M - \xi(x)) \leq M \leq \|\xi\|_\infty.$$

The result then follows by applying the same argument to $-u$. □

Definition 2.2. Let $\mathfrak{L}_\alpha$ denotes the class of operators $\mathcal{I}$ of the form

$$\mathcal{I}u(x) := b(x)\cdot\nabla u(x) + \int_{\mathbb{R}^d} \mathfrak{d}u(x;z)\,\frac{k(x,z)}{|z|^{d+\alpha}}\,\mathrm{d}z, \quad u \in C_b^2(\mathbb{R}^d), \tag{2.4}$$

with $b : \mathbb{R}^d \to \mathbb{R}^d$ and $k : \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ Borel measurable and locally bounded, and $\alpha \in (1,2)$. We also assume that $x \mapsto \sup_{z\in\mathbb{R}^d} k^{-1}(x,z)$ is locally bounded. The subclass of $\mathfrak{L}_\alpha$ consisting of those $\mathcal{I}$ satisfying $k(x,z) = k(x,-z)$ is denoted by $\mathfrak{L}_\alpha^{\mathsf{sym}}$.

Consider the following *growth condition*: There exists a constant K_0 such that

$$x\cdot b(x) \,\vee\, |x|\,k(x,z) \leq K_0\,(1+|x|^2) \quad \forall\, x,z \in \mathbb{R}^d. \tag{2.5}$$

It turns out that under (2.5), the Markov process associated with $\mathcal{I}$ does not have finite explosion time, as the following lemma shows.

Lemma 2.3. *Let $\mathcal{I} \in \mathfrak{L}_\alpha$ and suppose that for some constant $K_0 > 0$, the data satisfy the growth condition in (2.5). Let X be a Markov process associated with $\mathcal{I}$. Then*

$$\mathbb{P}_x\left(\sup_{s\in[0,T]} |X_s| < \infty\right) = 1 \quad \forall\, T > 0.$$

Proof. Let $\delta \in (0, \alpha-1)$ and $\varphi \in C^2(\mathbb{R}^d)$ be a nondecreasing, radial function satisfying

$$\varphi(x) = \left(1+|x|^\delta\right) \quad \text{for } |x| \geq 1, \quad \text{and} \quad \varphi(x) \geq 1 \quad \text{for } |x| < 1.$$

We claim that

$$\left|\int_{\mathbb{R}^d} \mathfrak{d}\varphi(x;z)\,\frac{k(x,z)}{|z|^{d+\alpha}}\,\mathrm{d}z\right| \leq \kappa_0\,(1+|x|^\delta) \qquad \forall x \in \mathbb{R}^d, \tag{2.6}$$

for some constant κ_0. To prove (2.6), first note that since the second partial derivatives of φ are bounded over $\mathbb{R}^d$, it follows that $\left|\int_{|z|\leq 1} \mathfrak{d}\varphi(x;z)\,\frac{k(x,z)}{|z|^{d+\alpha}}\,\mathrm{d}z\right|$ is bounded by some constant. It is easy to verify that, provided $z \neq 0$, then

$$\begin{aligned} \left||x+z|^\delta - |x|^\delta\right| &\leq 2\delta|z|\,|x|^{\delta-1}, \quad \text{if } |x| \geq 2|z|, \\ \left||x+z|^\delta - |x|^\delta\right| &\leq 8|z|^\delta, \quad \text{if } |x| < 2|z|, \end{aligned} \tag{2.7}$$

for some constant κ. By the hypothesis in (2.5), for some constant c, we have

$$k(x,z) \leq c\,(1+|x|) \quad \forall x \in \mathbb{R}^d. \tag{2.8}$$

Combining (2.7)–(2.8), we obtain, for $|x| > 1$,

$$\begin{aligned}\left|\int_{|z|>1} \mathfrak{d}\varphi(x;z)\,\frac{k(x,z)}{|z|^{d+\alpha}}\,\mathrm{d}z\right| &\le \int_{1<|z|\le\frac{|x|}{2}} 2\delta\, c\,(1+|x|)\,|x|^{\delta-1}\,|z|\,\frac{1}{|z|^{d+\alpha}}\,\mathrm{d}z\\ &\quad+\int_{|z|>\frac{|x|}{2}} 8c\,(1+|x|)\,|z|^{\delta}\,\frac{1}{|z|^{d+\alpha}}\,\mathrm{d}z\\ &\le \kappa(d)\left(\frac{2\,\delta\,c}{\alpha-1}\,(1+|x|)\,|x|^{\delta-1}+\frac{2^{3+\alpha-\delta}c}{\alpha-\delta}\,(1+|x|)\,|x|^{\delta-\alpha}\right)\end{aligned}$$

for some constant $\kappa(d)$, thus establishing (2.6).

By (2.6) and the assumption on the growth of b in (2.5), we obtain

$$|\mathcal{I}\varphi(x)| \le K_1\,\varphi(x) \qquad \forall x\in\mathbb{R}^d,$$

for some constant K_1. Then, by Dynkin's formula, we have,

$$\begin{aligned}\mathbb{E}_x\big[\varphi(X_{t\wedge\tau_n})\big] &= \varphi(x)+\mathbb{E}_x\left[\int_0^{t\wedge\tau_n}\mathcal{I}\varphi(X_s)\,\mathrm{d}s\right]\\ &\le \varphi(x)+K_1\,\mathbb{E}_x\left[\int_0^{t\wedge\tau_n}\varphi(X_s)\,\mathrm{d}s\right]\\ &\le \varphi(x)+K_1\int_0^t \mathbb{E}_x\left[\varphi(X_{s\wedge\tau_n})\right]\mathrm{d}s,\end{aligned}$$

where in the second inequality, we use the property that φ is radial and nondecreasing. Hence, by the Gronwall inequality, we have

$$\mathbb{E}_x\big[\varphi(X_{t\wedge\tau_n})\big] \le \varphi(x)\,\mathrm{e}^{K_1 t} \quad \forall\, t>0,\ \forall\, n\in\mathbb{N}. \tag{2.9}$$

Since $\mathbb{E}_x\big[\varphi(X_{t\wedge\tau_n})\big] \ge \varphi(n)\,\mathbb{P}_x(\tau_n\le t)$, we obtain by (2.9) that

$$\begin{aligned}\mathbb{P}_x\left(\sup_{s\in[0,T]}|X_s|\ge n\right) &= \mathbb{P}_x(\tau_n\le T)\\ &\le \frac{\varphi(x)}{1+n^{\delta}}\,\mathrm{e}^{K_1 T} \quad \forall\, T>0,\ \forall\, n\in\mathbb{N},\end{aligned}$$

from which the conclusion of the lemma follows. □

3. The Dirichlet problem for a class of stable-like operators

The class of operators studied in this section is defined as follows.

Definition 3.1. Let $\lambda : [0,\infty)\to(0,\infty)$ be a nondecreasing function that plays the role of a parameter. For a bounded domain D define $\lambda_D := \sup\{\lambda(R) : D\subset B_{R+1}\}$. Let $\mathfrak{I}_\alpha(\beta,\theta,\lambda)$, where $\beta\in(0,1]$, $\theta\in(0,1)$, denotes the class of operators $\mathcal{I}$ as in (2.4) that satisfy, on each bounded domain D, the following properties:

(a) $\alpha\in(1,2)$.

(b) b is locally Hölder continuous with exponent β and satisfies

$$|b(x)|\le\lambda_D \quad\text{and}\quad |b(x)-b(y)|\le\lambda_D\,|x-y|^{\beta} \quad \forall\, x,y\in D.$$

(c) The map $k(x,z)$ is continuous in x and measurable in z and satisfies

$$|k(x,z) - k(y,z)| \le \lambda_D\, |x-y|^{\beta} \qquad \forall\, x,\, y \in D, \quad \forall\, z \in \mathbb{R}^d$$
$$\lambda_D^{-1} \le k(x,z) \le \lambda_D \quad \forall\, x \in D,\ \forall\, z \in \mathbb{R}^d.$$

(d) For any $x \in D$, we have

$$\int_{\mathbb{R}^d} \left(|z|^{\alpha-\theta} \wedge 1\right) \frac{|k(x,z) - k(x,0)|}{|z|^{d+\alpha}}\, \mathrm{d}z \le \lambda_D. \tag{3.1}$$

Remark 3.1. It is evident that if $|k(x,z) - k(x,0)| \le \tilde{\lambda}_D |z|^{\theta'}$ for some $\theta' > \theta$, then property (d) of Definition 3.1 is satisfied.

We study the Dirichlet problem

$$\begin{aligned} \mathcal{I}u &= f \quad \text{in } D,\\ u &= 0 \quad \text{in } D^c, \end{aligned} \tag{3.2}$$

where $\mathcal{I} \in \mathfrak{I}_\alpha(\beta, \theta, \lambda)$, f is Hölder continuous with exponent β, and D is a bounded open set with a C^2 boundary.

In this section, it is convenient to use $s \equiv \frac{\alpha}{2}$ as the parameter reflecting the order of the kernel. Throughout this section, we assume $s > 1/2$.

We may view $\mathcal{I}$ as the sum of the operator $\mathcal{I}_0$ defined by

$$\mathcal{I}_0\, u(x) := b(x) \cdot \nabla u(x) + \int_{\mathbb{R}^d} \mathfrak{d}u(x;z)\, \frac{k(x,0)}{|z|^{d+2s}}\, \mathrm{d}z,$$

which is uniformly elliptic on every bounded domain, and a perturbation that takes the form

$$\widetilde{\mathcal{I}}\, u(x) := \int_{\mathbb{R}^d} \mathfrak{d}u(x;z)\, \frac{k(x,z) - k(x,0)}{|z|^{d+2s}}\, \mathrm{d}z.$$

We are not assuming that the numerator k is symmetric, as in the approximation techniques in [15, 19, 32]. Moreover, these operators are not addressed in [20] due to the presence of the drift term.

Recall the definition of weighted Hölder norms in Section 1.1. We start with the following lemma.

Lemma 3.1. *Let D be a C^2 bounded domain in $\mathbb{R}^d$ and $r \in (0,s]$. Suppose $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and the constants $\beta \in (0,1)$, $\theta \in \big(0, (2s-1) \wedge \beta\big)$, and $\lambda_D > 0$ satisfy parts* (c) *and* (d) *of Definition 3.1. We define*

$$\tilde{k}(x,z) := c(d,2s) \left(\frac{k(x,z)}{k(x,0)} - 1\right),$$
$$\mathcal{H}[v](x) := \int_{\mathbb{R}^d} \mathfrak{d}v(x;z)\, \frac{\tilde{k}(x,z)}{|z|^{d+2s}}\, \mathrm{d}z,$$

where $c(d,2s) = c(d,\alpha)$ is the normalization constant of the fractional Laplacian.

Suppose that either of the following assumptions hold:

(i) $\beta \le r$.

(ii) $\beta \in (r, 1)$ *and* $\frac{\tilde{k}(x,z)}{|z|^\theta}$ *is bounded on* $(x, z) \in D \times \mathbb{R}^d$, *or, equivalently, it satisfies*

$$|k(x,z) - k(x,0)| \le \tilde{\lambda}_D\, |z|^\theta \quad \forall x \in D,\ \forall z \in \mathbb{R}^d, \tag{3.3}$$

for some positive constant $\tilde{\lambda}_D$.

Then, if $v \in \mathscr{C}^{(-r)}_{2s-\theta}(D)$, *we have*

$$[\![\mathcal{H}[v]]\!]^{(2s-r-\theta)}_{0;D} \le M_0\, \|v\|^{(-r)}_{2s-\theta;D},$$

and if $v \in \mathscr{C}^{(-r)}_{2s+\beta-\theta}(D)$, *it holds that* $\mathcal{H}[v] \in \mathscr{C}^{(2s-r-\theta)}_{\beta}(D)$, *and*

$$\left\|\mathcal{H}[v]\right\|^{(2s-r-\theta)}_{\beta;D} \le M_1\, \|v\|^{(-r)}_{2s+\beta-\theta;D} \tag{3.4}$$

for some constants M_0 *and* M_1 *which depend only on* d, s, β, r, *and* D.

Moreover, over a set of parameters of the form $\{(r, \beta) : r \in (\varepsilon, 1),\ \beta \in (0, 1)\}$, *constants* M_0 *and* M_1 *can be selected which do not depend on* β *or* r, *but only on* $\varepsilon > 0$.

Proof. Let $x \in D$ and define $R = \frac{d_x}{4}$. We suppose that $R < 1$. It is clear that $\tilde{k}$ satisfies (3.1), and that it is Hölder continuous. Abusing the notation, we will use the same symbol λ_D as a constant in the estimates. We have,

$$\left|\mathfrak{d}v(x;z)\right| \le |z|^{2s-\theta}\, R^{r+\theta-2s}\, [\![v]\!]^{(-r)}_{2s-\theta;D} \qquad \forall\, z \in B_R. \tag{3.5}$$

Also, since $|z| \ge R$ on B_R^c, we obtain

$$\begin{aligned} \left|\mathfrak{d}v(x;z)\right| &\le \Big(|z|^r\, [\![v]\!]^{(-r)}_{r;D} + |z|\, R^{r-1}\, [\![v]\!]^{(-r)}_{1;D}\Big)\, \mathbf{1}_{\{|z|\le 1\}} + 2\, \|v\|_{C(D)}\, \mathbf{1}_{\{|z|>1\}} \\ &\le \big(|z| \wedge 1\big)^{2s-\theta}\, R^{r+\theta-2s}\Big([\![v]\!]^{(-r)}_{r;D} + [\![v]\!]^{(-r)}_{1;D}\Big) + 2\, \|v\|_{C(D)}\, \mathbf{1}_{\{|z|>1\}} \end{aligned} \tag{3.6}$$

for all $z \in B_R^c$. Integrating, using (3.1), and (3.5)–(3.6), as well as the Hölder interpolation inequalities, we obtain

$$|\mathcal{H}[v](x)| \le c_1\, (4\, d_x)^{r+\theta-2s}\, \|v\|^{(-r)}_{2s-\theta;D} \quad \forall x \in D,$$

for some constant c_1. Therefore, for some constant M_0, we have

$$[\![\mathcal{H}[v]]\!]^{(2s-r-\theta)}_{0;D} \le M_0\, \|v\|^{(-r)}_{2s-\theta;D}. \tag{3.7}$$

Next consider two points $x, y \in D$. If $|x-y| \ge 4d_{xy}$, then (3.7) provides a suitable estimate. Indeed, if x, $y \in D$ are such $4d_{xy} \le |x-y|$, then, for any r, we have

$$\begin{aligned} d_{xy}^{2s-r-\theta}\, d_{xy}^{\beta}\, \frac{|\mathcal{H}[v](x) - \mathcal{H}[v](y)|}{|x-y|^\beta} &\le \frac{1}{4^\beta}\, d_{xy}^{2s-r-\theta}\, |\mathcal{H}[v](x) - \mathcal{H}[v](y)| \\ &\le \frac{1}{4^\beta}\, d_x^{2s-r-\theta}\, |\mathcal{H}[v](x)| + \frac{1}{4^\beta}\, d_y^{2s-r-\theta}\, |\mathcal{H}[v](y)| \\ &\le \frac{2M_0}{4^\beta}\, \|v\|^{(-r)}_{2s-\theta;D}. \end{aligned}$$

So it suffices to consider the case $|x-y| < 4d_{xy}$. Therefore, we may suppose that x is as above and that $y \in B_R(x)$. Then $d_{xy} \le 4R$. With $\widetilde{\pi}(x,z) := \frac{\tilde{k}(x,z)}{|z|^{d+2s}}$, we write

$$\begin{aligned} F(x,y;z) &:= \mathfrak{d}v(x;z)\, \widetilde{\pi}(x,z) - \mathfrak{d}v(y;z)\, \widetilde{\pi}(y,z) \\ &= F_1(x,y;z) + F_2(x,y;z), \end{aligned}$$

with

$$F_1(x,y;z) := \Big(\mathfrak{d}v(x;z) + \mathfrak{d}v(y;z)\Big)\,\frac{\widetilde{\pi}(x,z) - \widetilde{\pi}(y,z)}{2},$$
$$F_2(x,y;z) := \Big(\mathfrak{d}v(x;z) - \mathfrak{d}v(y;z)\Big)\,\frac{\widetilde{\pi}(x,z) + \widetilde{\pi}(y,z)}{2}.$$

We modify the estimate in (3.5), and write

$$\Big|\mathfrak{d}v(x;z) + \mathfrak{d}v(y;z)\Big| \le 2\,|z|^{\gamma_0}\,R^{r-\gamma_0}\,[\![v]\!]^{(-r)}_{\gamma_0;D}, \quad \text{if } z \in B_R,$$

with $\gamma_0 = (2s+\beta-\theta)\wedge(s+1)$, and

$$\Big|\mathfrak{d}v(x;z) + \mathfrak{d}v(y;z)\Big| \le 2\Big(|z|^r\,[\![v]\!]^{(-r)}_{r;D} + |z|\,R^{r-1}\,[\![v]\!]^{(-r)}_{1;D}\Big)\,\mathbf{1}_{\{|z|\le 1\}} + 4\|v\|_{C(D)}\,\mathbf{1}_{\{|z|>1\}},$$

if $z \in B^c_R$. We use the Hölder continuity of $x \mapsto \tilde{k}(x,\,\cdot\,)$ to obtain

$$\int_{\mathbb{R}^d} F_1(x,y;z)\,\mathrm{d}z \le c_2\,R^{r-2s}\,|x-y|^{\beta}\,\|v\|^{(-r)}_{\gamma_0;D}$$

for some constant c_2. We write this as

$$\begin{aligned} R^{2s-r-\theta}\,R^{\beta}\,\frac{\int_{\mathbb{R}^d} F_1(x,y;z)\,\mathrm{d}z}{|x-y|^{\beta}} &\le R^{2s-r-\beta}\,R^{\beta}\,\frac{\int_{\mathbb{R}^d} F_1(x,y;z)\,\mathrm{d}z}{|x-y|^{\beta}} \\ &\le c_2\,\|v\|^{(-r)}_{\gamma_0;D}. \end{aligned} \tag{3.8}$$

For F_2, we use

$$\mathfrak{d}v(x;z) = z\cdot\int_0^1 \big(\nabla v(x+tz) - \nabla v(x)\big)\,\mathrm{d}t,$$

combined with the following fact: If $\varphi \in C^{\gamma}(B)$ for $\gamma \in (0,1]$ and $x, y, x+z, y+z$ are points in B and $\delta \in (0,\gamma)$, then adopting the notation $\Delta\varphi_x(z) := \varphi(x+z) - \varphi(x)$, we obtain by Young's inequality, that

$$\begin{aligned} \frac{|\Delta\varphi_x(z) - \Delta\varphi_y(z)|}{|z|^{\gamma-\delta}|x-y|^{\delta}} &\le \frac{\gamma-\delta}{\gamma}\,\frac{\Delta\varphi_x(z)| + |\Delta\varphi_y(z)|}{|z|^{\gamma}} + \frac{\delta}{\gamma}\,\frac{|\Delta\varphi_{x+z}(y-x)| + |\Delta\varphi_x(y-x)|}{|x-y|^{\gamma}} \\ &\le 2[f]_{\gamma;B}. \end{aligned}$$

The same inequality also holds for $\gamma \in (1,2)$ and $\delta \in (\gamma-1,1)$. For this, we use

$$\begin{aligned} &\frac{|\Delta\varphi_x(z) - \Delta\varphi_y(z)|}{|z|^{\gamma-\delta}|x-y|^{\delta}} \\ &\quad \le \frac{1-\delta}{2-\gamma}\,\frac{|z|\,\Big|\int_0^1 \big(\nabla\varphi(x+tz) - \nabla\varphi(y+tz)\big)\,\mathrm{d}t\Big|}{|x-y|^{\gamma-1}\,|z|} \\ &\qquad + \frac{1+\delta-\gamma}{2-\gamma}\,\frac{|x-y|\,\Big|\int_0^1 \big(\nabla\varphi(y+z+t(x-y)) - \nabla\varphi(y+t(x-y))\big)\,\mathrm{d}t\Big|}{|z|^{\gamma-1}\,|x-y|} \end{aligned}$$

Therefore, in either of the cases (i) or (ii), we obtain

$$\Big|\nabla v(x+tz) - \nabla v(x) - \nabla v(y+tz) + \nabla v(y)\Big| \le 2|tz|^{2s-\theta-1}|x-y|^{\beta}\,[\nabla v]_{2s-\theta-1+\beta;\,B_{2R}(x)}$$

for $t \in [0,1]$ and

$$\big|\eth v(x;z) - \eth v(y;z)\big| \le \frac{2}{2s-\theta}\, |z|^{2s-\theta}\, |x-y|^{\beta}\, R^{r+\theta-\beta-2s}\, [\![v]\!]^{(-r)}_{2s+\beta-\theta;D} \quad \forall\, z \in B_R. \tag{3.9}$$

Concerning the integration on B_R^c, we use

$$\begin{aligned}
&\big|v(x) - v(y) - z\cdot\big(\nabla v(x) - \nabla v(y)\big)\mathbf{1}_{\{|z|\le 1\}}\big| \\
&\quad\le |x-y|^{\beta\vee r}\, d_{xy}^{r-\beta\vee r}\, [\![v]\!]^{(-r)}_{\beta\vee r;D} + \big(|z|\wedge 1\big)\, |x-y|^{\beta}\, d_{xy}^{r-\beta-1}\, [\![v]\!]^{(-r)}_{1+\beta;D} \\
&\quad\le c_3\, \big(|z|\wedge 1\big)^{2s-\theta}\, |x-y|^{\beta}\, R^{r+\theta-\beta-2s}\, \|v\|^{(-r)}_{1+\beta;D} \quad \forall\, z\in B_R^c,
\end{aligned} \tag{3.10}$$

for some constant c_3, and

$$|v(x+z) - v(y+z)| \le |x-y|^{\beta\vee r}\, (d_{x+z}\wedge d_{y+z})^{r-\beta\vee r}\, [\![v]\!]^{(-r)}_{\beta\vee r;D} \quad \forall\, z\in B_R^c. \tag{3.11}$$

Integrating the terms on the right-hand side of (3.9)–(3.10) is straightforward. Doing so, and using the fact that $1+\beta < 2s+\beta-\theta$, one obtains the desired estimate.

Concerning the integral of $|v(x+z) - v(y+z)|$ on B_R^c, we distinguish between the cases (i) and (ii). Let $\widetilde{\pi}(z) := \frac{|\widetilde{\pi}(x,z)+\widetilde{\pi}(y,z)|}{2}$. In case (i), we have

$$\begin{aligned}
&\int_{B_R^c} |v(x+z) - v(y+z)|\, \widetilde{\pi}(z)\, \mathrm{d}z \\
&\quad\le |x-y|^{r}\, [\![v]\!]^{(-r)}_{r;D} \int_{B_R^c} \widetilde{\pi}(z)\, \mathrm{d}z \\
&\quad\le |x-y|^{\beta}\, R^{r-\beta}\, R^{\theta-2s}\, [\![v]\!]^{(-r)}_{r;D} \int_{\mathbb{R}^d} \big(|z|\wedge \operatorname{diam}(D)\big)^{2s-\theta}\, \widetilde{\pi}(z)\, \mathrm{d}z,
\end{aligned} \tag{3.12}$$

where we use the fact that $|z| > R$ on B_R^c. In case (ii), the integral is estimated over disjoint sets. We define

$$\mathcal{Z}_{xy}(a) := \{z\in\mathbb{R}^d : d_{x+z}\wedge d_{y+z} < a\} \quad \text{for } a\in(0,R).$$

Since $d_{x+z}\wedge d_{y+z} \in [R, \operatorname{diam}(D)]$ for $x \in \mathcal{Z}^c_{xy}(R)$, integration is straightforward, after replacing $(d_{x+z}\wedge d_{y+z})^{r-\beta}$ in (3.11) with $R^{r-\beta}$. Thus, similarly to (3.12), we obtain

$$\begin{aligned}
&\int_{B_R^c\cap\,\mathcal{Z}^c_{xy}(R)} |v(x+z) - v(y+z)|\, \widetilde{\pi}(z)\, \mathrm{d}z \\
&\quad\le |x-y|^{\beta}\, R^{r-\beta}\, [\![v]\!]^{(-r)}_{\beta;D} \int_{B_R^c\cap\,\mathcal{Z}^c_{xy}(R)} \widetilde{\pi}(z)\, \mathrm{d}z \\
&\quad\le |x-y|^{\beta}\, R^{r+\theta-\beta-2s}\, [\![v]\!]^{(-r)}_{\beta;D} \int_{\mathbb{R}^d} \big(|z|\wedge \operatorname{diam}(D)\big)^{2s-\theta}\, \widetilde{\pi}(z)\, \mathrm{d}z.
\end{aligned} \tag{3.13}$$

Since $\mathcal{Z}_{xy}(R) \subset B_R^c$, it remains to compute the integral on $\mathcal{Z}_{xy}(R)$. For $\varepsilon > 0$, we denote by D_ε the ε-neighborhood of D, i.e.,

$$D_\varepsilon := \{z\in\mathbb{R}^d : \operatorname{dist}(z,D) < \varepsilon\}. \tag{3.14}$$

We also define

$$\widetilde{D}(\varepsilon) := \{z\in D : \operatorname{dist}(z,\partial D) \ge \varepsilon\}.$$

In other words, $\widetilde{D}(\varepsilon) = (D^c)^c_\varepsilon$. We will make use of the following simple fact: There exists a constant C_0, such that for all $x \in D$ and positive constants R and ε which satisfy $0 < \varepsilon \le R$ and $d_x \ge 3R$, it holds that

$$\int_{x+z\in D_\varepsilon\setminus\widetilde{D}(\varepsilon)} \frac{\mathrm{d}z}{|z|^d} \le \frac{C_0\,\varepsilon}{R}. \tag{3.15}$$

Observe that the support of $|v(x+z)-v(y+z)|$ in $\mathcal{Z}_{xy}(R)$ is contained in the disjoint union of the sets

$$\widetilde{\mathcal{Z}}_{xy}(R) := \bigl\{z \in \mathcal{Z}_{xy}(R) : d_{x+z} \wedge d_{y+z} > 0\bigr\},$$

and

$$\widehat{\mathcal{Z}}_{xy} := \bigl\{z \in \mathbb{R}^d : x+z \in D_{|x-y|} \setminus D \text{ or } y+z \in D_{|x-y|} \setminus D\bigr\}.$$

We also have the bound $|v(x+z)-v(y+z)| \le |x-y|^r [\![v]\!]^{(-r)}_{r;D}$ for $z \in \widehat{\mathcal{Z}}_{xy}$. Therefore, using (3.15), we obtain

$$\begin{aligned}
\int_{\widehat{\mathcal{Z}}_{xy}} |v(x+z)-v(y+z)|\,\widetilde{\pi}(z)\,\mathrm{d}z &\le |x-y|^r [\![v]\!]^{(-r)}_{r;D}\, R^{\theta-2s} \int_{\widehat{\mathcal{Z}}_{xy}} |z|^{2s-\theta}\,\widetilde{\pi}(z)\,\mathrm{d}z \\
&\le |x-y|^r [\![v]\!]^{(-r)}_{r;D}\, R^{\theta-2s} \int_{\widehat{\mathcal{Z}}_{xy}} \frac{\mathrm{d}z}{|z|^d} \\
&\le 2\,\tilde{\lambda}_D\, C_0\, |x-y|^{r+1} [\![v]\!]^{(-r)}_{r;D}\, R^{\theta-2s}\, R^{-1} \\
&\le 2\,\tilde{\lambda}_D\, C_0\, |x-y|^{\beta}\, R^{r+\theta-\beta-2s}\, [\![v]\!]^{(-r)}_{r;D}.
\end{aligned} \tag{3.16}$$

To evaluate the integral over $\widetilde{\mathcal{Z}}_{xy}(R)$, we define

$$G(z) := \frac{|v(x+z)-v(y+z)|}{|x-y|^\beta\, [\![v]\!]^{(-r)}_{\beta;D}}.$$

By (3.11), we have

$$\bigl\{z \in \widetilde{\mathcal{Z}}_{xy}(R) : G(z) > h\bigr\} \subset \bigl\{z \in \mathbb{R}^d : x+z \in \widetilde{D}^c\bigl(h^{\frac{-1}{\beta-r}}\bigr)\bigr\} \cup \bigl\{z \in \mathbb{R}^d : y+z \in \widetilde{D}^c\bigl(h^{\frac{-1}{\beta-r}}\bigr)\bigr\}.$$

Therefore, by (3.15), we obtain

$$\begin{aligned}
\widetilde{\pi}\bigl(\bigl\{z \in \widetilde{\mathcal{Z}}_{xy}(R) : G(z) > h\bigr\}\bigr) &\le 2R^{\theta-2s} \int_{\widehat{\mathcal{Z}}_{xy}} |z|^{2s-\theta}\,\widetilde{\pi}(z)\,\mathrm{d}z \\
&\le 2\,\tilde{\lambda}_D\, C_0\, R^{\theta-2s-1} h^{\frac{-1}{\beta-r}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_{\widetilde{\mathcal{Z}}_{xy}(R)} G(z)\,\widetilde{\pi}(z)\,\mathrm{d}z &= \int_0^\infty \widetilde{\pi}\bigl(\bigl\{z \in \widetilde{\mathcal{Z}}_{xy}(R) : G(z) > h\bigr\}\bigr)\,\mathrm{d}h \\
&\le 2\,\tilde{\lambda}_D\, C_0\, R^{\theta-2s-1} \int_{R^{r-\beta}}^\infty h^{\frac{-1}{\beta-r}}\,\mathrm{d}h \\
&\le \frac{2\,(\beta-r)}{1+r-\beta}\,\tilde{\lambda}_D\, C_0\, R^{\theta-2s-1}\, R^{1+r-\beta}.
\end{aligned} \tag{3.17}$$

Thus, combining (3.9)–(3.10) with (3.12) in case (i), or with (3.13), (3.16), and (3.17) in case (ii), and using the Hölder interpolation inequalities, we obtain

$$R^{2s-r-\theta}\, R^{\beta}\, \frac{\int_{\mathbb{R}^d} F_2(x,y;z)\, \mathrm{d}z}{|x-y|^{\beta}} \le c_4\, [\![v]\!]^{(-r)}_{2s+\beta-\theta;D} \tag{3.18}$$

for some constant c_4.

Therefore, by (3.7), (3.8), and (3.18), we obtain (3.4), and the proof is complete. □

Remark 3.2. It is evident from the proof of Lemma 3.1 that the assumption in (3.3) may be replaced by the following: There exists a constant M_D, such that for all $x \in D$ and positive constants R and ε which satisfy $0 < \varepsilon \le R$ and $d_x \ge 3R$, it holds that

$$\int_{x+z\in D_{\varepsilon}\setminus \widetilde{D}(\varepsilon)} \frac{\tilde{k}(x,z)}{|z|^{d-\theta}}\, \mathrm{d}z \le M_D\, \frac{\varepsilon}{R}.$$

The same applies to Theorems 3.1 and 3.2 which appear later in this section.

Recall that the fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) = c(d,2s)\ \mathsf{PV}\!\int_{\mathbb{R}^d} \frac{u(x)-u(z)}{|z|^{d+2s}}\, \mathrm{d}z$$

where PV denotes the Cauchy principal value. To proceed, we need certain properties of solutions of $(-\Delta)^s u = f$ in a bounded domain D, and $u = 0$ on D^c, with f not necessarily in $L^{\infty}(D)$. We start by exhibiting a suitable supersolution.

Lemma 3.2 (Supersolution). *For any $q \in \big(s-1/2, s\big)$, there exists a constant $c_0 > 0$ and a radial continuous function φ such that*

$$\begin{cases} (-\Delta)^s\varphi(x) \ge d_x^{q-2s}, & \text{in } B_4 \setminus \overline{B}_1,\\ \varphi = 0 & \text{in } \overline{B}_1,\\ 0 \le \varphi \le c_0(|x|-1)^q & \text{in } B_4 \setminus B_1,\\ 1 \le \varphi \le c_0 & \text{in } \mathbb{R}^d \setminus B_4, \end{cases}$$

where $d_x = \mathrm{dist}(x, \partial B_1)$.

Proof. In view of the Kelvin transform [34, Proposition A.1], it is enough to prove the following: for $q \in \big(s-1/2, s\big)$, and with $\psi(x) := [(1-|x|)^+]^q$, we have

$$(-\Delta)^s \psi(x) \ge c_1\,(1-|x|)^{q-2s}, \quad \text{for all } x \in B_1, \tag{3.19}$$

for some positive constant c_1. To prove (3.19), let $x_0 \in B_1$. Due to the rotational symmetry, we may assume $x_0 = re_1$ for some $r \in (0,1)$. Let ϖ_1 denotes the projection onto the first coordinate in $\mathbb{R}^d$, i.e., $\varpi_1(z_1,\dots,z_d) = (z_1, 0, \dots, 0)$. Then, using the fact that $(1-|z|)^+ \le$

$(1-|\varpi_1(z)|)^+$, we obtain

$$\begin{aligned}
-(-\Delta)^s\psi(x_0) &= c(d,2s)\ \mathsf{PV}\!\int_{\mathbb{R}^d}\big(\psi(x_0+z)-\psi(x_0)\big)\frac{1}{|z|^{d+2s}}\,\mathrm{d}z\\
&= c(d,2s)\ \mathsf{PV}\!\int_{\mathbb{R}^d}\Big(\big[\big(1-|re_1+z|\big)^+\big]^q-(1-r)^q\Big)\frac{1}{|z|^{d+2s}}\,\mathrm{d}z\\
&\le c(d,2s)\ \mathsf{PV}\!\int_{\mathbb{R}^d}\Big(\big[\big(1-|re_1+\varpi_1(z)|\big)^+\big]^q-(1-r)^q\Big)\frac{1}{|z|^{d+2s}}\,\mathrm{d}z.
\end{aligned}$$

Note that for $y\in\mathbb{R}$, $y\neq 0$, we have

$$\int_{\mathbb{R}^{d-1}}\frac{\mathrm{d}\tilde{z}}{\big(y^2+|\tilde{z}|^2\big)^{\frac{d+2s}{2}}}=\frac{1}{|y|^{1+2s}}\int_{\mathbb{R}^{d-1}}\frac{\mathrm{d}\tilde{z}}{\big(1+|\tilde{z}|^2\big)^{\frac{d+2s}{2}}}$$

by a straightforward change of variables. Therefore, integrating with respect to $(z_2,\dots,z_d)$, we obtain, for some positive constant c_2, that

$$\begin{aligned}
-(-\Delta)^s\psi(x_0) &\le c_2\ \mathsf{PV}\!\int_{\mathbb{R}}\Big(\big[(1-|r+y|)^+\big]^q-(1-r)^q\Big)\frac{1}{|y|^{1+2s}}\,\mathrm{d}y\\
&\le c_2\ \mathsf{PV}\!\int_{\mathbb{R}}\Big(\big[(1-r-y)^+\big]^q-(1-r)^q\Big)\frac{1}{|y|^{1+2s}}\,\mathrm{d}y\\
&= c_2(1-r)^{q-2s}\ \mathsf{PV}\!\int_{\mathbb{R}}\Big(\big[(1-\tilde{y})^+\big]^q-1\Big)\frac{1}{|\tilde{y}|^{1+2s}}\,\mathrm{d}\tilde{y}.
\end{aligned}$$

In the inequality above, we have used $1-|y|\le 1-y$, and in the last equality, the change of variables $y=(1-r)\tilde{y}$. Define

$$A(q):=\mathsf{PV}\!\int_{\mathbb{R}}\Big([(1-y)^+]^q-1\Big)\frac{1}{|y|^{1+2s}}\,\mathrm{d}y=\mathsf{PV}\!\int_0^\infty\frac{y^q-1}{|1-y|^{1+2s}}\,\mathrm{d}y-\int_{-\infty}^0\frac{1}{|1-y|^{1+2s}}\,\mathrm{d}y,$$

$$B(q):=\mathsf{PV}\!\int_0^\infty\frac{y^q-1}{|1-y|^{1+2s}}\,\mathrm{d}y.$$

We need to show that $A(q)<0$ for q close to s. It is known that $A(s)=0$ [34, Proposition 3.1]. Therefore, it is enough to show that $B(q)$ is strictly increasing for $q\in\big(s-1/2,s\big)$. We have

$$B(q)=\lim_{\epsilon\searrow 0}\left[\int_0^{1-\epsilon}\frac{y^q-1}{|1-y|^{1+2s}}\,\mathrm{d}y+\int_{1+\epsilon}^\infty\frac{y^q-1}{|1-y|^{1+2s}}\,\mathrm{d}y\right]. \tag{3.20}$$

It is straightforward to show that

$$\lim_{\epsilon\searrow 0}\int_{1-\epsilon}^{\frac{1}{1+\epsilon}}\frac{y^q-1}{|1-y|^{1+2s}}\,\mathrm{d}y=0.$$

and using this, we can combine the integrals in (3.20) to write

$$B(q)=\lim_{\epsilon\searrow 0}\int_0^{\frac{1}{1+\epsilon}}\frac{(y^q-1)(1-y^{2s-1-q})}{|1-y|^{1+2s}}\,\mathrm{d}y=\int_0^1\frac{(y^q-1)(1-y^{2s-1-q})}{|1-y|^{1+2s}}\,\mathrm{d}y.$$

Since $|2s-1-q|\le 1$, it follows that $B(q)$ is finite. Direct differentiation then shows that, provided $q>2s-1-q$, we have

$$\frac{\mathrm{d}}{\mathrm{d}q}\big[(y^q-1)(1-y^{2s-1-q})\big]=(y^q-y^{2s-1-q})\log y>0\quad\forall y\in(0,1),$$

and hence $B(q)$ is strictly increasing on $q\in\big(s-1/2,s\big)$. This completes the proof. $\square$

In the lemma that follows $d_x = \mathrm{dist}(x, \partial D)$, as defined in Section 1.1.

Lemma 3.3. *Let D be a C^2 bounded domain in $\mathbb{R}^d$, and $f\colon D \to \mathbb{R}$ be a continuous map satisfying $\sup_{x\in D} |f(x)|\, d_x^{\delta} < \infty$ for some $\delta < s$. Then there exists a viscosity solution $u \in C(\mathbb{R}^d)$ to*

$$\begin{aligned} (-\Delta)^s u &= -f \quad \text{in } D, \\ u &= 0 \quad \text{in } D^c. \end{aligned} \tag{3.21}$$

Also, for every $q < s$, we have

$$|u(x)| \le C_1 \, [\![f]\!]_{0;D}^{(\delta)} \, d_x^{q} \quad \forall\, x \in \overline{D}, \tag{3.22a}$$

$$\|u\|_{C^q(\overline{D})} \le C_1 \, [\![f]\!]_{0;D}^{(\delta)} \tag{3.22b}$$

for some constant C_1 that depends only on s, δ, q and the domain D. Moreover, since $u = 0$ on D^c, it follows that $\|u\|_{C^q(\mathbb{R}^d)} < \infty$ for all $q < s$.

Proof. By Corollary 4 in [36] for each $f \in C^2(\mathbb{R}^d)$, there exists a viscosity solution $u \in C(\mathbb{R}^d)$ to (3.21). Therefore, the same is the case for $f \in C(D) \cap L^{\infty}(D)$ by [34, Remark 2.11]. Given f as in the statement of the lemma, let $f_n := (f \wedge n) \vee (-n)$, for $n \in \mathbb{N}$, and u_n be the corresponding viscosity solution to (3.21).

Comparing u_n (and $-u_n$) to the supersolution in Lemma 3.2, we deduce that there exists a compact set $K_1 \subset D$ such that

$$|u_n(x)| \le \kappa_1 \left(\sup_{x\in K_1} |u_n(x)| + [\![f_n]\!]_{0;D}^{(\delta)} \right) d_x^{q} \quad \forall\, x \in K_1^c,\ \forall\, n \in \mathbb{N}, \tag{3.23}$$

where the constant κ_1 depends only on K_1 and D. Also, using the same argument as in Lemma 2.2, we can show that for any compact $K_2 \subset D$, there exists a constant κ_2, depending on D, and satisfying

$$\sup_{x\in K_2} |u_n(x)| \le \kappa_2 \left(\sup_{x\in K_2} |f_n(x)| + \sup_{x\in D\setminus K_2} |u_n(x)| \right) \quad \forall\, n \in \mathbb{N}. \tag{3.24}$$

We choose K_2 and $K_1 \subset K_2$ such that $\sup_{x\in K_2^c\cap D} |d_x^{q}| < \frac{1}{2\kappa_1\kappa_2}$. Then from (3.23)–(3.24), we obtain

$$\sup_{x\in K_2} |u_n(x)| \le \kappa_3 \, [\![f_n]\!]_{0;D}^{(\delta)} \quad \forall\, n \in \mathbb{N}, \tag{3.25}$$

for some constant κ_3. Combining (3.23) and (3.25), we obtain

$$|u_n(x)| \le C_1 \, [\![f_n]\!]_{0;D}^{(\delta)} \, d_x^{q} \quad \forall\, x \in \overline{D}\ \forall\, n \in \mathbb{N}. \tag{3.26}$$

Also, by following the argument in the proof of [34, Proposition 1.1], we obtain

$$\|u_n\|_{C^q(\overline{D})} \le C_1 \sup_{x\in D} d_x^{\delta} |f_n(x)| \quad \forall\, n \in \mathbb{N}. \tag{3.27}$$

Since the right-hand side of (3.27) is bounded uniformly in $n \in \mathbb{N}$, we may select a subsequence, also denoted as n, along which u_n converges to some function $u \in C^q(\overline{D})$ for any $q < s$. Taking limits as $n \to \infty$ in (3.26) and (3.27), we obtain (3.22a) and (3.22b), respectively. The stability property of viscosity solutions [18, Lemma 4.5] implies that u is a viscosity solution. This completes the proof. □

Our main result in this section is the following.

Theorem 3.1. *Let $\mathcal{I} \in \mathfrak{I}_{2s}(\beta, \theta, \lambda)$, f be locally Hölder continuous in $\mathbb{R}^d$ with exponent β, and D be a bounded domain with a C^2 boundary. We assume that neither β nor $2s + \beta$ are integers and that either $\beta < s$ or that $\beta \geq s$ and*

$$|k(x,z) - k(x,0)| \leq \tilde{\lambda}_D\, |z|^\theta \quad \forall x \in D,\ \forall z \in \mathbb{R}^d,$$

for some positive constant $\tilde{\lambda}_D$. Then the Dirichlet problem in (3.2) has a unique solution in $C^{2s+\beta}_{\mathrm{loc}}(D) \cap C(\overline{D})$. Moreover, for any $r < s$, we have the estimate

$$\|u\|^{(-r)}_{2s+\beta;D} \leq C_0\, \|f\|_{C^\beta(\overline{D})}$$

for some constant C_0 that depends only on d, β, r, s, and the domain D.

Proof. Consider the case $\beta \geq s$. We write (3.2) as

$$\begin{aligned} (-\Delta)^s u(x) = \mathcal{T}[u](x) &:= \frac{c(d,2s)}{k(x,0)} \big(-f(x) + b(x) \cdot \nabla u(x) \big) + \mathcal{H}[u](x) \quad \text{in } D, \\ u = 0 \quad &\text{in } D^c, \end{aligned} \tag{3.28}$$

and we apply the Leray–Schauder fixed point theorem. Also, without loss of generality, we assume $\theta < 2s - 1$. We choose any $r \in (0,s)$ which satisfies

$$r > \Big(s - \frac{\theta}{2}\Big) \vee \Big(1 - s + \frac{\theta}{2}\Big),$$

and let $v \in \mathscr{C}^{(-r)}_{2s+\beta-\theta}(D)$. Then $\mathcal{H}[v] \in \mathscr{C}^{(2s-r-\theta)}_{\beta}(D)$ by Lemma 3.1. Since $\nabla v \in \mathscr{C}^{(1-r)}_{2s+\beta-\theta-1}(D)$ and $(1-r) \wedge (2s - r - \theta) < s$ by hypothesis, then applying Lemma 3.3, we conclude that there exists a solution u to $(-\Delta)^s u = \mathcal{T}[v]$ on D, with $u = 0$ on D^c, such that $u \in \mathscr{C}^{(-q)}_0(D)$ for any $q < s$.

Next we obtain some estimates that are needed to apply the Leray–Schauder fixed point theorem. By Lemma 3.1, we obtain

$$\big\|\mathcal{H}[v]\big\|^{(2s-r-\theta/2)}_{0;D} = \big\|\mathcal{H}[v]\big\|^{(2s-(r-\theta/2)-\theta)}_{0;D} \leq \kappa_1\, \|v\|^{(-r+\theta/2)}_{2s-\theta;D},$$

and similarly,

$$\big\|\mathcal{H}[v]\big\|^{(2s-r-\theta/2)}_{\beta;D} \leq \kappa_1\, \|v\|^{(-r+\theta/2)}_{2s+\beta-\theta;D}, \tag{3.29}$$

for some constant κ_1 which does not depend on θ or r. Thus, since by hypothesis $2s - r - \theta/2 < s$ and $1 - r + \theta/2 < s$, we obtain by Lemma 3.3 that

$$\|u\|_{C^r(\mathbb{R}^d)} \leq \kappa_1' \Big(\|f\|_{C(\overline{D})} + \|\nabla v\|^{(1-r+\theta/2)}_{0;D} + \|v\|^{(-r+\theta/2)}_{2s-\theta;D} \Big) \tag{3.30}$$

for some constant κ_1'. Also, by Lemma 2.10 in [34], there exists a constant κ_2, depending only on β, s, r, and d, such that

$$\|u\|_{2s+\beta;D}^{(-r)} \le \kappa_2 \left(\|u\|_{C^r(\mathbb{R}^d)} + \left\|\mathcal{T}[v]\right\|_{\beta;D}^{(2s-r)}\right). \tag{3.31}$$

It follows by (3.30)–(3.31) that $v \mapsto u$ is a continuous map from $\mathscr{C}_{2s+\beta-\theta}^{(-r)}$ to itself. Moreover, since $\mathscr{C}_{2s+\beta}^{(-r)}(D)$ is precompact in $\mathscr{C}_{2s+\beta-\theta}^{(-r)}(D)$, it follows that $v \mapsto u$ is compact.

Next we obtain a bound for $\|u\|_{2s+\beta;D}^{(-r)}$. By (3.29), we have

$$\begin{aligned}\left\|\mathcal{H}[v]\right\|_{\beta;D}^{(2s-r)} &\le \big(\operatorname{diam}(D)\big)^{\theta/2}\left\|\mathcal{H}[v]\right\|_{\beta;D}^{(2s-r-\theta/2)} \\ &\le \kappa_1 \big(\operatorname{diam}(D)\big)^{\theta/2} \|v\|_{2s+\beta-\theta;D}^{(-r+\theta/2)}\Big).\end{aligned}$$

Therefore, since also $2s - r > 1 - r + \theta/2$, we obtain

$$\left\|\mathcal{T}[v]\right\|_{\beta;D}^{(2s-r)} \le \kappa_3 \left(\|f\|_{C^\beta(\bar{D})} + [\![v]\!]_{1;D}^{(-r+\theta/2)} + \|v\|_{2s+\beta-\theta;D}^{(-r+\theta/2)}\right) \tag{3.32}$$

for some constant κ_3. By the Hölder interpolation inequalities, for any $\varepsilon > 0$, there exists $\widetilde{C}(\varepsilon) > 0$ such that

$$[\![v]\!]_{1;D}^{(-r+\theta/2)} + \|v\|_{2s+\beta-\theta;D}^{(-r+\theta/2)} \le \widetilde{C}(\varepsilon)\, [\![v]\!]_{0;D}^{(-r+\theta/2)} + \varepsilon\, \|v\|_{2s+\beta;D}^{(-r+\theta/2)}. \tag{3.33}$$

Combining (3.30), (3.31), and (3.32), and then using (3.33) and the inequality

$$[\![v]\!]_{2s+\beta;D}^{(-r+\theta/2)} \le \big(\operatorname{diam}(D)\big)^{\theta/2} \|v\|_{2s+\beta;D}^{(-r)}$$

we obtain

$$\|u\|_{2s+\beta;D}^{(-r)} \le \kappa_4(\varepsilon) \left(\|f\|_{C^\beta(\bar{D})} + \|v\|_{0;D}^{(-r+\theta/2)}\right) + \varepsilon\, \|v\|_{2s+\beta;D}^{(-r)}. \tag{3.34}$$

To apply the Leray–Schauder fixed point theorem, it suffices to show that the set of solutions $u \in \mathscr{C}_{2s+\beta}^{(-r)}(D)$ of $(-\Delta)^s u(x) = \xi\, \mathcal{T}[u](x)$, for $\xi \in [0,1]$, with $u = 0$ on D^c, is bounded in $\mathscr{C}_{2s+\beta}^{(-r)}(D)$. However, from the above calculations, any such solution u satisfies (3.34) with $v \equiv u$. Moreover by Lemma 2.2,

$$\sup_{x\in D} |u(x)| \le \kappa_5 \sup_{x\in D} |f(x)| \tag{3.35}$$

for some constant κ_5. We also have that

$$\begin{aligned}\|u\|_{0;D}^{(-r+\theta/2)} &\le \varepsilon^{-r+\theta/2} \sup_{x\in D,\, d_x \ge \varepsilon} |u(x)| + \varepsilon^{\theta/2} \sup_{x\in D,\, d_x<\varepsilon} d_x^{-r}\, |u(x)| \\ &\le \varepsilon^{-r+\theta/2} \sup_{x\in D} |u(x)| + \varepsilon^{\theta/2}\, \|u\|_{0;D}^{(-r)}.\end{aligned} \tag{3.36}$$

Choosing $\varepsilon > 0$ small enough, and using (3.35)–(3.36) on the right-hand side of (3.34) with $v \equiv u$, we obtain

$$\|u\|_{2s+\beta;D}^{(-r)} \le \kappa_6\, \|f\|_{C^\beta(\bar{D})} \tag{3.37}$$

for some constant κ_6. Hence by the Leray–Schauder fixed point theorem, the map $v \mapsto u$ given by (3.28) has a fixed point $u \in \mathscr{C}_{2s+\beta}^{(-r)}(D)$, i.e.,

$$(-\Delta)^s u(x) = \mathcal{T}[u](x).$$

Hence, this is a solution to (3.2). Uniqueness is obvious as u is a classical solution. The bound in (3.37) then applies and the proof is complete. The proof in the case $\beta < s$ is completely analogous. □

Optimal regularity up to the boundary can be obtained under additional hypotheses. The following result is a modest extension of the results in [34, Proposition 1.1].

Corollary 3.1. *Let $\mathcal{I} \in \mathfrak{I}_{2s}(\beta,\theta,\lambda)$ with $\theta > s$, f be locally Hölder continuous with exponent β, and D be a bounded domain with a C^2 boundary. Suppose in addition that $b = 0$ and that k is symmetric, i.e., $k(x,z) = k(x,-z)$. Then the solution of the Dirichlet problem in (3.2) is in $C^s(\mathbb{R}^d)$. Moreover, for any $\beta < s$, we have $u \in \mathscr{C}^{(-s)}_{2s+\beta}(D)$.*

Proof. By Theorem 3.1, the Dirichlet problem in (3.2) has a unique solution in $C^{2s+\rho}_{\mathrm{loc}}(D) \cap C(\overline{D})$, for any $\rho < \beta \wedge s$. Moreover, for any $r < s$, we have the estimate

$$\|u\|^{(-r)}_{2s+\rho;D} \le C_0\, \|f\|_{C^\beta(\overline{D})}.$$

Fix $r = 2s - \theta$. Then

$$\int_{R<|z|<1} |z|^r \,\frac{\tilde{k}(x,z)}{|z|^{d+2s}}\,\mathrm{d}z = \int_{R<|z|<1} |z|^{2s-\theta}\,\frac{\tilde{k}(x,z)}{|z|^{d+2s}}\,\mathrm{d}z \le \lambda_D.$$

By (3.6) and the symmetry of the kernel, it follows that

$$\left|\int_{R<|z|} \mathfrak{d}u(x;z)\,\frac{\tilde{k}(x,z)}{|z|^{d+2s}}\,\mathrm{d}z\right| \le \kappa_1\Big([\![u]\!]^{(-r)}_{r;D} + \|u\|_{C(\overline{D})}\Big) \quad \forall x \in D,$$

for some constant κ_1. Combining this with the estimate in Lemma 3.1 we obtain

$$[\![\mathcal{H}[u]]\!]^{(0)}_{0;D} \le M_0\, \|u\|^{(-r)}_{r;D} < \infty,$$

implying that $\mathcal{H}[u] \in L^\infty(D)$. It then follows by [34, Proposition 1.1] that $u \in C^s(\mathbb{R}^d)$, and that for some constant C depending only on s, we have

$$\begin{aligned}
\|u\|_{C^s(\mathbb{R}^d)} &\le C\,\|\mathcal{T}[u]\|_{L^\infty(D)}\\
&\le C\lambda_D^{-1}\,c(d,2s)\Big(\|f\|_{L^\infty(D)} + \|\mathcal{H}[u]\|_{L^\infty(D)}\Big)\\
&\le C\lambda_D^{-1}\,c(d,2s)\Big(\|f\|_{L^\infty(D)} + M_0\,\|u\|^{(-r)}_{r;D}\Big).
\end{aligned}$$

Using the Hölder interpolation inequalities, we obtain from the preceding estimate that

$$\|u\|_{C^s(\mathbb{R}^d)} \le \tilde{C}\,\|f\|_{L^\infty(D)}$$

for some constant $\tilde{C}$ depending only on s, θ, and λ_D.

Applying Lemma 3.1 once more, we conclude that $\mathcal{H}[u] \in \mathscr{C}^{(s)}_{\beta'}(D)$ for any $\beta' \le r$, and that

$$\big\|\mathcal{H}[u]\big\|^{(s)}_{\beta';D} \le M_1\,\|u\|^{(-r)}_{2s+\beta'-\theta;D}.$$

Hence, applying [34, Proposition 1.4], we obtain

$$\|u\|^{(-s)}_{2s+\beta';D} \le C_1\Big(\|u\|_{C^s(\mathbb{R}^d)} + \big\|\mathcal{T}[u]\big\|^{(s)}_{\beta';D}\Big)$$

for some constant C_1, and we can repeat this procedure to reach $u \in \mathscr{C}^{(-s)}_{2s+\beta}(D)$. □

Concerning the stochastic representation of the solutions to the Dirichlet problem in (3.2), we have the following.

Theorem 3.2. *Let $\mathcal{I} \in \mathfrak{I}_{2s}(\beta,\theta,\lambda)$, D be a bounded domain with C^2 boundary, and $f \in C^{\beta}(\overline{D})$. We assume that neither β nor $2s+\beta$ are integers and that either $\beta < s$ or that $\beta \geq s$ and*

$$|k(x,z) - k(x,0)| \leq \tilde{\lambda}_D\, |z|^{\theta} \quad \forall x \in D,\ \forall z \in \mathbb{R}^d,$$

for some positive constant $\tilde{\lambda}_D$. Let $\mathbb{E}_x$ denotes the expectation operator corresponding to the Markov process X with generator given by $\mathcal{I}$. Then $u(x) := \mathbb{E}_x\bigl[\int_0^{\tau(D)} f(X_t)\,\mathrm{d}t\bigr]$ is the unique solution in $C^{2s+\beta}(D) \cap C(\overline{D})$ to (3.2).

Proof. Recall the definition of D_ε in (3.14). Note that for ε small enough, D_ε has a C^2 boundary. Let

$$\tilde{f}(x) := \inf_{y\in D} \Bigl(f(y) + \|f\|_{C^{\beta}(\overline{D})}\, |x-y|^{\beta}\Bigr), \quad x \in \overline{D}_\varepsilon,$$

i.e., $\tilde{f}$ is a β-Hölder extension of f. Then by Theorem 3.1, there exists $u_\varepsilon \in C^{2s+\beta}(D_\varepsilon) \cap C(\overline{D}_\varepsilon)$ satisfying

$$\begin{aligned} \mathcal{I} u_\varepsilon &= -\tilde{f} \quad \text{in } D_\varepsilon,\\ u_\varepsilon &= 0 \quad \text{in } D_\varepsilon^c. \end{aligned}$$

We also have the estimate (recall the definition of $\|\cdot\|^{(r)}_{\beta;D}$ in Section 1.1)

$$\|u_\varepsilon\|^{(-r)}_{2s+\beta;D_\varepsilon} \leq C_0\, \|\tilde{f}\|_{C^{\beta}(\overline{D}_\varepsilon)},$$

with r some fixed constant in $(0,s)$. As can be seen from the Lemma 2.2 and the proof of Theorem 3.1, we may select a constant C_0, that does not depend on ε, for ε small enough. Since $u_\varepsilon = 0$ in D_ε^c, it follows that

$$\|u_\varepsilon\|_{C^{r}(\mathbb{R}^d)} \leq c_1\, \|u_\varepsilon\|^{(-r)}_{2s+\beta;D_\varepsilon}$$

for some constant c_1, independent of ε, for all small enough ε. Hence $u_\varepsilon \to u$ as $\varepsilon \to 0$, along some subsequence, and $u \in C^{2s+\beta}(D) \cap C(\overline{D})$ by Theorem 3.1. By Itô's formula, we obtain

$$u_\varepsilon(x) = \mathbb{E}_x\bigl[u_\varepsilon(X_{\tau(D)})\bigr] + \mathbb{E}_x\biggl[\int_0^{\tau(D)} f(X_t)\,\mathrm{d}t\biggr].$$

Letting $\varepsilon \searrow 0$, we obtain the result. Uniqueness follows from Theorem 3.1. □

Theorem 3.2 can be extended to account for nonzero boundary conditions, provided the boundary data are regular enough, say in $C^3(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$.

4. The Harnack property for operators containing a drift term

In this section, we prove a Harnack inequality for harmonic functions. Throughout Sections 4 and 5, we use the parameter $\alpha = 2s$. The classes of operators considered are summarized in the following definition.

Definition 4.1. With λ as in Definition 3.1, let $\mathfrak{L}_\alpha(\lambda)$ denotes the class of operators $\mathcal{I} \in \mathfrak{L}_\alpha$ satisfying

$$|b(x)| \le \lambda_D, \quad \text{and} \quad \lambda_D^{-1} \le k(x,z) \le \lambda_D \quad \forall\, x \in D,\ z \in \mathbb{R}^d,$$

for a bounded domain D. As in Definition 2.2, the subclass of $\mathfrak{L}_\alpha(\lambda)$ consisting of those $\mathcal{I}$ satisfying $k(x,z) = k(x,-z)$ is denoted by $\mathfrak{L}_\alpha^{\mathsf{sym}}(\lambda)$. Also by $\mathfrak{L}_{\alpha,\theta}(\lambda)$, we denote the subset of $\mathfrak{L}_\alpha(\lambda)$ satisfying

$$\int_{\mathbb{R}^d} \left(|z|^{\alpha-\theta} \wedge 1\right) \frac{|k(x,z) - k(x,0)|}{|z|^{d+\alpha}}\, \mathrm{d}z \le \lambda_D \quad \forall x \in D,$$

for any bounded domain D.

A measurable function $h : \mathbb{R}^d \to \mathbb{R}$ is said to be *harmonic* with respect to $\mathcal{I}$ in a domain D if for any bounded subdomain $G \subset D$, it satisfies

$$h(x) = \mathbb{E}_x[h(X_{\tau(G)})] \quad \forall\, x \in G,$$

where $(X, \mathbb{P}_x)$ is a strong Markov process associated with $\mathcal{I}$.

Theorem 4.1. *Let D be a bounded domain of $\mathbb{R}^d$ and $K \subset D$ be compact. Then there exists a constant C_H depending on K, D, and λ, such that any bounded, non-negative function which is harmonic in D with respect to an operator $\mathcal{I} \in \mathfrak{L}_\alpha^{\mathsf{sym}}(\lambda) \cup \mathfrak{L}_{\alpha,\theta}(\lambda)$, $\theta \in (0,1)$, satisfies*

$$h(x) \le C_H\, h(y) \quad \textit{for all } x, y \in K.$$

We prove Theorem 4.1 by verifying the conditions in [38] where a Harnack inequality is established for a general class of Markov processes. We accomplish this through Lemmas 4.1–4.4 which follow. Let us also mention that some of the proof techniques are standard, but we still add them for clarity. In fact, the Harnack property with nonsymmetric kernel is also discussed in [38] under some regularity condition on $k(\cdot,\cdot)$ and under the assumption of the existence of a harmonic measure. The proof of Lemma 4.2 (b) below holds under very general conditions and does not rely on the existence of a harmonic measure.

The following lemma is a careful modification of [39, Lemma 2.1] (for the proof see Lemma 3.5 and Remark 3.2 in [3]).

Lemma 4.1. *Let $(X, \mathbb{P}_x)$ be a strong Markov process associated with $\mathcal{I} \in \mathfrak{L}_\alpha$, and D be a given bounded domain. There exits a constant $\kappa_1 > 0$ such that for any $x \in D$ and $r \in (0,1)$ it holds that*

$$\mathbb{P}_x\left(\sup_{0\le s\le t} |X_s - x| > r\right) \le \kappa_1 t\, r^{-\alpha} \quad \forall\, x \in D,$$

where $X_0 = x$.

In Lemmas 4.2–4.4 which follow, $(X, \mathbb{P}_x)$ is a strong Markov process associated with $\mathcal{I} \in \mathfrak{L}_\alpha^{\mathsf{sym}}(\lambda) \cup \mathfrak{L}_{\alpha,\theta}(\lambda)$, and D is a bounded domain.

Lemma 4.2. *Let D be a bounded domain. There exist positive constants κ_2 and r_0 such that for any $x \in D$ and $r \in (0, r_0)$,*

(a) $\inf_{z\in B_{\frac{r}{2}}(x)} \mathbb{E}_z[\tau(B_r(x))] \geq \kappa_2^{-1} r^\alpha$,
(b) $\sup_{z\in B_r(x)} \mathbb{E}_z[\tau(B_r(x))] \leq \kappa_2\, r^\alpha$.

Proof. By Lemma 4.1 there exists a constant κ_1 such that

$$\mathbb{P}_x(\tau(B_r(x)) \leq t) \leq \kappa_1 t r^{-\alpha}, \tag{4.1}$$

for all $t \geq 0$, and all $x \in D_2 := \{y : \operatorname{dist}(y, D) < 2\}$. We choose $t = \frac{r^\alpha}{2\kappa_1}$. Then for $z \in B_{\frac{r}{2}}(x)$, we obtain by (4.1) that

$$\begin{aligned}
\mathbb{E}_z[\tau(B_r(x))] &\geq \mathbb{E}_z[\tau(B_{\frac{r}{2}}(z))] \\
&\geq \frac{r^\alpha}{2\kappa_1}\, \mathbb{P}_z\Big(\tau(B_{\frac{r}{2}}(z)) > \frac{r^\alpha}{2\kappa_1}\Big) \\
&\geq \frac{r^\alpha}{4\kappa_1}.
\end{aligned}$$

This proves the part (a).

To prove part (b), we consider a radially nondecreasing function $\varphi \in C_b^2(\mathbb{R}^d)$, which is convex in B_4 and satisfies

$$\varphi(x+z) - \varphi(x) - z\cdot\nabla\varphi(x) \geq c_1|z|^2 \qquad \text{for } |x| \leq 1,\ |z| \leq 3,$$

for some positive constant c_1. For an arbitrary point $x_0 \in D$, define $g_r(x) := \varphi(\frac{x-x_0}{r})$. Then for $x \in B_r(x_0)$ and $\mathcal{I} \in \mathfrak{L}_\alpha^{\mathsf{sym}}(\lambda)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathfrak{d}g_r(x;z)\, \frac{k(x,z)}{|z|^{\alpha+d}}\, \mathrm{d}z &= \int_{|z|\leq 3r} \big(g_r(x+z) - g_r(x) - z\cdot\nabla g_r(x)\big)\frac{k(x,z)}{|z|^{\alpha+d}}\, \mathrm{d}z \\
&\quad + \int_{|z|>3r} \big(g_r(x+z) - g_r(x)\big)\frac{k(x,z)}{|z|^{\alpha+d}}\, \mathrm{d}z \\
&\geq \frac{c_1}{r^2}\, \lambda_D^{-1} \int_{|z|\leq 3r} |z|^{2-d-\alpha}\, \mathrm{d}z \\
&= c_2\, \frac{3^{2-\alpha}}{2-\alpha}\, \lambda_D^{-1}\, r^{-\alpha}
\end{aligned}$$

for some constant $c_2 > 0$, where in the first equality, we use the fact that $k(x,z) = k(x,-z)$, and for the second inequality, we use the property that $g(x+z) \geq g(x)$ for $|z| \geq 3r$. It follows that we may choose r_0 small enough such that

$$\mathcal{I}g_r(x) \geq c_3 r^{-\alpha} \quad \text{for all } r \in (0, r_0),\ x \in B_r(x_0),\ \text{and } x_0 \in D,$$

with $c_3 := \frac{c_2}{2}\, \frac{3^{2-\alpha}}{2-\alpha}\, \lambda_D^{-1}$.

To obtain a similar estimate for $\mathcal{I} \in \mathfrak{L}_{\alpha,\theta}(\lambda)$, we fix some $\theta_1 \in (0, \theta \wedge (\alpha-1))$. Let $\hat{k}(x,z) := k(x,z) - k(x,0)$. We have

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathfrak{d}g_r(x;z)\, \frac{k(x,z)}{|z|^{\alpha+d}}\, \mathrm{d}z &= \int_{|z|\leq 3r} \mathfrak{d}g_r(x;z)\frac{k(x,z)}{|z|^{\alpha+d}}\, \mathrm{d}z - \int_{3r<|z|<1} z\cdot\nabla g_r(x)\frac{k(x,z)-k(x,0)}{|z|^{d+\alpha}}\, \mathrm{d}z \\
&\quad + \int_{|z|>3r} \big(g_r(x+z) - g_r(x)\big)\frac{k(x,z)}{|z|^{\alpha+d}}\, \mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{c_1}{\lambda_D\, r^2}\int_{|z|\leq 3r}|z|^{2-d-\alpha}\,\mathrm{d}z - \frac{\|\nabla\varphi\|_\infty}{r}\int_{3r<|z|<1}|z|\frac{|\hat{k}(x,z)|}{|z|^{d+\alpha}}\,\mathrm{d}z\\
&\geq c_2\,\frac{3^{2-\alpha}}{(2-\alpha)\,\lambda_D\, r^\alpha} - \frac{\|\nabla\varphi\|_\infty}{r}\int_{3r<|z|<1}|z|^{\alpha-\theta_1}(3r)^{-\alpha+\theta_1+1}\frac{|\hat{k}(x,z)|}{|z|^{d+\alpha}}\,\mathrm{d}z\\
&\geq c_2\,\frac{3^{2-\alpha}}{(2-\alpha)\,\lambda_D\, r^\alpha} - \frac{\|\nabla\varphi\|_\infty}{r}\int_{3r<|z|<1}|z|^{\alpha-\theta}(3r)^{-\alpha+\theta_1+1}\frac{|\hat{k}(x,z)|}{|z|^{d+\alpha}}\,\mathrm{d}z\\
&\geq c_2\,\frac{3^{2-\alpha}}{(2-\alpha)\,\lambda_D\, r^\alpha} - \kappa(d)3^{\alpha-\theta_1+1}r^{-\alpha+\theta_1}\,\lambda_D\,\|\nabla\varphi\|_\infty\\
&\geq c_4\, r^{-\alpha}\quad \forall\, x\in B_r(x_0),
\end{aligned}
$$

for some constant $c_4 > 0$ and r small, where in the third inequality, we used the fact that $\theta_1 < \alpha - 1$. Thus by Itô's formula we obtain

$$
\mathbb{E}_x\big[\tau(B_r(x_0))\big] \leq c_4^{-1} r^\alpha\,\|\varphi\|_\infty \quad \forall\, x\in B_r(x_0).
$$

This completes the proof. □

Lemma 4.3. *There exists a constant $\kappa_3 > 0$ such that for any $r\in(0,1)$, $x\in D$, and $A\subset B_r(x)$ we have*

$$
\mathbb{P}_z\big(\tau(A^c) < \tau(B_{3r}(x))\big) \geq \kappa_3\,\frac{|A|}{|B_r(x)|}\quad \forall\, z\in B_{2r}(x).
$$

Proof. Let $\hat{\tau} := \tau(B_{3r}(x))$. Suppose $\mathbb{P}_z(\tau(A^c) < \hat{\tau}) < 1/4$ for some $z\in B_{2r}(x)$. Otherwise there is nothing to prove as $\frac{|A|}{|B_r(x)|}\leq 1$. By Lemma 4.1, there exists $t>0$ such that $\mathbb{P}_y(\hat{\tau}\leq tr^\alpha)\leq 1/4$ for all $y\in B_{2r}(x)$. Hence using the Lévy-system formula, we obtain

$$
\begin{aligned}
\mathbb{P}_y(\tau(A^c) < \hat{\tau}) &\geq \mathbb{E}_y\Big[\sum_{s\leq\tau(A^c)\wedge\hat{\tau}\wedge tr^\alpha}\mathbf{1}_{\{X_{s-}\neq X_s, X_s\in A\}}\Big]\\
&= \mathbb{E}_y\Big[\int_0^{\tau(A^c)\wedge\hat{\tau}\wedge tr^\alpha}\int_A\frac{k(X_s, z-X_s)}{|z-X_s|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}s\Big]\\
&\geq \mathbb{E}_y\Big[\int_0^{\tau(A^c)\wedge\hat{\tau}\wedge tr^\alpha}\int_A\frac{\lambda_D^{-1}}{(4r)^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}s\Big]\\
&\geq \kappa_3'\, r^{-\alpha}\,\frac{|A|}{|B_r(x)|}\mathbb{E}_y[\tau(A^c)\wedge\hat{\tau}\wedge tr^\alpha]
\end{aligned}
\tag{4.2}
$$

for some constant $\kappa_3' > 0$, where in the third inequality, we use the fact that $|X_s - z|\leq 4r$ for $s<\hat{\tau}$, $z\in A$. On the other hand, we have

$$
\begin{aligned}
\mathbb{E}_y[\tau(A^c)\wedge\hat{\tau}\wedge tr^\alpha] &\geq t\,r^\alpha\,\mathbb{P}_y(\tau(A^c)\geq\hat{\tau}\geq tr^\alpha)\\
&= t\,r^\alpha\left[1-\mathbb{P}_y(\tau(A^c)<\hat{\tau})-\mathbb{P}_y(\hat{\tau}<tr^\alpha)\right]\\
&\geq \frac{t}{2}\,r^\alpha.
\end{aligned}
\tag{4.3}
$$

Therefore combining (4.2)–(4.3), we obtain $\mathbb{P}_z(\tau(A^c) < \hat{\tau}) \geq \frac{t\kappa_3'}{2}\frac{|A|}{|B_r(x)|}$. □

Lemma 4.4. *There exists positive constants κ_i, $i = 4, 5$, such that if $x \in D$, $r \in (0, 1)$, $z \in B_r(x)$, and H is a bounded non-negative function with support in $B^c_{2r}(x)$, then*

$$\mathbb{E}_z\big[H(X_{\tau(B_r(x))})\big] \le \kappa_4\,\mathbb{E}_z\big[\tau(B_r(x))\big]\int_{\mathbb{R}^d} H(y)\frac{k(x, y-x)}{|y-x|^{d+\alpha}}\,\mathrm{d}y,$$

and

$$\mathbb{E}_z\big[H(X_{\tau(B_r(x))})\big] \ge \kappa_5\,\mathbb{E}_z\big[\tau(B_r(x))\big]\int_{\mathbb{R}^d} H(y)\frac{k(x, y-x)}{|y-x|^{d+\alpha}}\,\mathrm{d}y.$$

The proof follows using the same argument as in [38, Lemma 3.5].

Proof of Theorem 4.1. By Lemmas 4.2, 4.3, and 4.4, the hypotheses (A1)–(A3) in [38] are satisfied. Hence the proof follows from [38, Theorem 2.4]. □

5. Positive recurrence and invariant probability measures

In this section, we study the recurrence properties for a Markov process with generator $\mathcal{I} \in \mathfrak{L}_\alpha$ (Definitions 2.2 and 4.1). Many of the results of this section are based on the assumption of the existence of a *Lyapunov function*.

Definition 5.1. We say that the operator $\mathcal{I}$ of the form (2.4) satisfies the Lyapunov stability condition if there exists a $\mathcal{V} \in C^2(\mathbb{R}^d)$ such that $\inf_{x\in\mathbb{R}^d}\mathcal{V}(x) > -\infty$, and for some compact set $\mathcal{K} \subset \mathbb{R}^d$ and $\varepsilon > 0$, we have

$$\mathcal{I}\,\mathcal{V}(x) \le -\varepsilon \quad \forall\, x \in \mathcal{K}^c. \tag{5.1}$$

It is straightforward to verify that if $\mathcal{V}$ satisfies (5.1) for $\mathcal{I} \in \mathfrak{L}_\alpha$, then

$$\int_{|z|\ge 1} |\mathcal{V}(z)|\frac{1}{|z|^{d+\alpha}}\,\mathrm{d}z < \infty. \tag{5.2}$$

Proposition 5.1. *If there exists a constant $\gamma \in (1, \alpha)$ such that*

$$\frac{b(x)\cdot x}{|x|^{2-\gamma}\,\sup_{z\in\mathbb{R}^d} k(x,z) \vee 1} \xrightarrow[|x|\to\infty]{} -\infty,$$

then the operator $\mathcal{I}$ satisfies the Lyapunov stability condition.

Proof. Consider a non-negative function $\varphi \in C^2(\mathbb{R}^d)$ such that $\varphi(x) = |x|^\gamma$ for $|x| \ge 1$, and let $\overline{k}(x) := \sup_{z\in\mathbb{R}^d} k(x, z)$. Since the second derivatives of φ are bounded in $\mathbb{R}^d$, and k is also bounded, it follows that

$$\left|\int_{|z|\le 1} \mathfrak{d}\varphi(x; z)\,\frac{k(x,z)}{|z|^{d+\alpha}}\,\mathrm{d}z\right| \le \kappa_1\,\overline{k}(x)$$

for some constant κ_1 which depends on the bound of the trace of the Hessian of φ. Following the same steps as in the proof of (2.6), and using the fact that k is bounded in $\mathbb{R}^d \times \mathbb{R}^d$, we obtain

$$\left|\int_{|z|>1} (|x+z|^\gamma - |x|^\gamma)\,\frac{k(x,z)}{|z|^{d+\alpha}}\,\mathrm{d}z\right| \le \kappa_2\,\overline{k}(x)\,(1 + |x|^{\gamma-\alpha}) \qquad \text{if } |x| > 1, \tag{5.3}$$

for some constant $\kappa_2 > 0$. Since also,

$$\left| \int_{\mathbb{R}^d} \mathbf{1}_{B_1}(x+z)\, \frac{k(x,z)}{|z|^{d+\alpha}}\, \mathrm{d}z \right| \le \kappa_3\, \overline{k}(x)\, (|x|-1)^{-\alpha} \quad \text{for } |x| > 2, \tag{5.4}$$

for some constant κ_3, it follows by the above that

$$\left| \int_{\mathbb{R}^d} \mathfrak{d}\varphi(x;z)\, \frac{k(x,z)}{|z|^{d+\alpha}}\, \mathrm{d}z \right| \le \kappa_4\, \overline{k}(x)\, (1 + |x|^{\gamma-\alpha}) \qquad \forall x \in \mathbb{R}^d, \tag{5.5}$$

for some constant κ_4. Therefore by the hypothesis and (5.5), it follows that $\mathcal{I}\varphi(x) \to -\infty$ as $|x| \to \infty$. □

Lemma 5.1. *Let X be the Markov process associated with a generator $\mathcal{I} \in \mathfrak{L}_\alpha(\lambda)$, and suppose that $\mathcal{I}$ satisfies the Lyapunov stability hypothesis (5.1) and the growth condition in (2.5). Then for any $x \in \mathcal{K}^c$, we have*

$$\mathbb{E}_x[\tau(\mathcal{K}^c)] \le \frac{2}{\varepsilon}\big(\mathcal{V}(x) + (\inf \mathcal{V})^-\big).$$

Proof. Let $R_0 > 0$ be such that $\mathcal{K} \subset B_{R_0}$. We choose a cut-off function χ which equals 1 on B_{R_1}, with $R_1 > 2R_0$, vanishes outside of B_{R_1+1}, and $\|\chi\|_\infty = 1$. Then $\varphi := \chi\mathcal{V}$ is in $C_b^2(\mathbb{R}^d)$. Clearly if $|x| \le R_0$ and $|x+z| \ge R_1$, then $|z| > R_0$, and thus $|x+z| \le 2|z|$. Therefore, for large enough R_1, we obtain

$$\begin{aligned}
\left| \int_{\mathbb{R}^d} \big(\varphi(x+z) - \mathcal{V}(x+z)\big) \frac{k(x,z)}{|z|^{d+\alpha}}\, \mathrm{d}z \right| &\le 2 \int_{\{|x+z|\ge R_1\}} |\mathcal{V}(x+z)|\, \frac{k(x,z)}{|z|^{d+\alpha}}\, \mathrm{d}z \\
&\le 2^{d+\alpha+1} \lambda_{B_{R_0}} \int_{\{|x+z|\ge R_1\}} |\mathcal{V}(x+z)|\, \frac{1}{|x+z|^{d+\alpha}}\, \mathrm{d}z \\
&\le \frac{\varepsilon}{2} \qquad \forall\, x \in B_{R_0}.
\end{aligned}$$

Hence, for all R_1 large enough, we have

$$\mathcal{I}\varphi(x) \le -\frac{\varepsilon}{2} \quad \forall x \in B_{R_0} \setminus \mathcal{K}.$$

Let $\widetilde{\tau}_R = \tau(\mathcal{K}^c) \wedge \tau(B_R)$. Then applying Itô's formula, we obtain

$$\mathbb{E}_x\big[\mathcal{V}(X_{\widetilde{\tau}_{R_0}})\big] - \mathcal{V}(x) \le -\frac{\varepsilon}{2}\, \mathbb{E}_x[\widetilde{\tau}_{R_0}] \quad \forall\, x \in B_{R_0} \setminus \mathcal{K},$$

implying that

$$\mathbb{E}_x[\widetilde{\tau}_{R_0}] \le \frac{2}{\varepsilon}\big(\mathcal{V}(x) + (\inf \mathcal{V})^-\big). \tag{5.6}$$

By the growth condition and Lemma 2.3, $\tau(B_R) \to \infty$ as $R \to \infty$ with probability 1. Hence the result follows by applying Fatou's lemma to (5.6). □

5.1. Existence of invariant probability measures

Recall that a Markov process is said be to positive (Harris) recurrent if for any compact set G with positive Lebesgue measure it holds that $\mathbb{E}_x[\tau(G^c)] < \infty$ for any $x \in \mathbb{R}^d$.

We recall the Lévy-system formula, the proof of which is a straightforward adaptation of the proof for a purely nonlocal operator and can be found in [13, Proposition 2.3 and Remark 2.4] [21, 24].

Proposition 5.2. *If A and B are disjoint Borel sets in $\mathcal{B}(\mathbb{R}^d)$, then for any $x \in \mathbb{R}^d$,*

$$\sum_{s\le t} \mathbf{1}_{\{X_{s-}\in A,\, X_s\in B\}} - \int_0^t \int_B \mathbf{1}_{\{X_s\in A\}} \frac{k(X_s, z-X_s)}{|X_s - z|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}s$$

is a $\mathbb{P}_x$-martingale.

We have the following theorem.

Theorem 5.1. *If $\mathcal{I} \in \mathfrak{L}_\alpha(\lambda)$ satisfies the Lyapunov stability hypothesis, and the growth condition in (2.5), then the associated Markov process is positive recurrent.*

Proof. First we note that if the Lyapunov condition is satisfied for some compact set $\mathcal{K}$, then it is also satisfied for any compact set containing $\mathcal{K}$. Hence we may assume that $\mathcal{K}$ is a closed ball centered at origin. Let D be an open ball with center at origin and containing $\mathcal{K}$. We define

$$\hat\tau_1 := \inf\,\{t \ge 0 \,:\, X_t \notin D\}, \qquad \hat\tau_2 := \inf\,\{t > \tau \,:\, X_t \in \mathcal{K}\}.$$

Therefore for $X_0 = x \in \mathcal{K}$, $\hat\tau_2$ denotes the first return time to $\mathcal{K}$ after hitting D^c. First we prove that

$$\sup_{x\in\mathcal{K}}\ \mathbb{E}_x[\hat\tau_2] < \infty. \tag{5.7}$$

By Lemma 5.1, we have $\mathbb{E}_x[\tau(\mathcal{K}^c)] \le \frac{2}{\varepsilon}[\mathcal{V}(x) + (\inf \mathcal{V})^-]$ for $x \in \mathcal{K}^c$. By Lemma 2.1, we have $\sup_{x\in\mathcal{K}} \mathbb{E}_x[\hat\tau_1] < \infty$. Let $\mathscr{P}_{\hat\tau_1}(x,\cdot\,)$ denotes the exit distribution of the process X starting from $x \in \mathcal{K}$. To prove (5.7), it suffices to show that

$$\sup_{x\in\mathcal{K}} \int_{D^c} \big(\mathcal{V}(y) + (\inf\mathcal{V})^-\big)\,\mathscr{P}_{\hat\tau_1}(x, \mathrm{d}y) < \infty,$$

and since $\mathcal{V}$ is locally bounded, it is enough that

$$\sup_{x\in\mathcal{K}} \int_{B_R^c} \big(\mathcal{V}(y) + (\inf\mathcal{V})^-\big)\,\mathscr{P}_{\hat\tau_1}(x, \mathrm{d}y) < \infty \tag{5.8}$$

for some ball B_R. To accomplish this, we choose R large enough, so that

$$\frac{|x-z|}{|z|} > \frac{1}{2} \qquad \text{for } |z| \ge R,\ x \in D.$$

Then, for any Borel set $A \subset B_R^c$, by Proposition 5.2, we have that

$$\begin{aligned}
\mathbb{P}_x(X_{\hat\tau_1\wedge t} \in A) &= \mathbb{E}_x\Bigg[\sum_{s\le\hat\tau_1\wedge t} \mathbf{1}_{\{X_{s-}\in D,\, X_s\in A\}}\Bigg]\\
&= \mathbb{E}_x\Bigg[\int_0^{\hat\tau_1\wedge t} \mathbf{1}_{\{X_s\in D\}} \int_A \frac{k(X_s, z-X_s)}{|X_s - z|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}s\Bigg]
\end{aligned}$$

$$\begin{aligned}&\leq 2^{d+\alpha}\lambda_D\,\mathbb{E}_x\Big[\int_0^{\hat{\tau}_1\wedge t}\int_A \frac{1}{|z|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}s\Big]\\&= 2^{d+\alpha}\lambda_D\,\mathbb{E}_x[\hat{\tau}_1\wedge t]\,\mu(A),\end{aligned}$$

where μ is the σ-finite measure on $\mathbb{R}^d_*$ with density $\frac{1}{|z|^{d+\alpha}}$. Thus letting $t \to \infty$, we obtain

$$\mathscr{P}_{\hat{\tau}_1}(x, A) \leq 2^{d+\alpha}\lambda_D\left(\sup_{x\in\mathcal{K}}\ \mathbb{E}_x[\hat{\tau}_1]\right)\mu(A).$$

Therefore, using a standard approximation argument, we deduce that for any non-negative function g, it holds that

$$\int_{B^c_R} g(y)\,\mathscr{P}_{\hat{\tau}_1}(x, \mathrm{d}y) \leq \tilde{\kappa}\int_{B^c_R} g(y)\mu(\mathrm{d}y)$$

for some constant $\tilde{\kappa}$. This proves (5.8) since $\mathcal{V}$ is integrable on B^c_R with respect to μ and $\mu(B^c_R) < \infty$.

Next we prove that the Markov process is positive recurrent. We need to show that for any compact set G with positive Lebesgue measure, $\mathbb{E}_x[\tau(G^c)] < \infty$ for any $x \in \mathbb{R}^d$. Given a compact G and $x \in G^c$, we choose a closed ball $\mathcal{K}$, which satisfies the Lyapunov condition relative to $\mathcal{V}$, and such that $G \cup \{x\} \subset \mathcal{K}$. Let D be an open ball containing $\mathcal{K}$. We define a sequence of stopping times $\{\hat{\tau}_k,\ k = 0, 1, \dots\}$ as follows:

$$\begin{aligned}\hat{\tau}_0 &= 0\\ \hat{\tau}_{2n+1} &= \inf\{t > \hat{\tau}_{2n} : X_t \notin D\},\\ \hat{\tau}_{2n+2} &= \inf\{t > \hat{\tau}_{2n+1} : X_t \in \mathcal{K}\}, \quad n = 0, 1, \dots .\end{aligned}$$

Using the strong Markov property and (5.8), we obtain $\mathbb{E}_x[\hat{\tau}_n] < \infty$ for all $n \in \mathbb{N}$. From Lemma 4.1, there exist positive constants t and r such that

$$\sup_{x\in\mathcal{K}}\ \mathbb{P}_x(\tau(D) < t) \leq \sup_{x\in\mathcal{K}}\ \mathbb{P}_x(\tau(B_r(x)) < t) \leq \frac{1}{4}.$$

Therefore, using a similar argument as in Lemma 4.3, we can find a constant $\delta > 0$ such that

$$\inf_{x\in\mathcal{K}}\ \mathbb{P}_x(\tau(G^c) < \tau(D)) > \delta.$$

Hence

$$p := \sup_{x\in\mathcal{K}}\ \mathbb{P}_x(\tau(D) < \tau(G^c)) \leq 1 - \delta < 1.$$

Thus by the strong Markov property, we obtain

$$\mathbb{P}_x(\tau(G^c) > \hat{\tau}_{2n}) \leq p\,\mathbb{P}_x(\tau(G^c) > \hat{\tau}_{2n-2}) \leq \cdots \leq p^n \quad \forall\, x \in \mathcal{K}.$$

This implies $\mathbb{P}_x(\tau(G^c) < \infty) = 1$. Hence, for $x \in \mathcal{K}$, we obtain

$$\begin{aligned}\mathbb{E}_x[\tau(G^c)] &\leq \sum_{n=1}^{\infty}\mathbb{E}_x\big[\hat{\tau}_{2n}\mathbf{1}_{\{\hat{\tau}_{2n-2}<\tau(G^c)\leq\hat{\tau}_{2n}\}}\big]\\ &= \sum_{n=1}^{\infty}\sum_{l=1}^{n}\mathbb{E}_x\big[(\hat{\tau}_{2l} - \hat{\tau}_{2l-2})\mathbf{1}_{\{\hat{\tau}_{2n-2}<\tau(G^c)\leq\hat{\tau}_{2n}\}}\big]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=1}^{\infty} \sum_{n=l}^{\infty} \mathbb{E}_x\big[(\hat{\tau}_{2l} - \hat{\tau}_{2l-2}) \mathbf{1}_{\{\hat{\tau}_{2n-2} < \tau(G^c) \le \hat{\tau}_{2n}\}}\big] \\
&= \sum_{l=1}^{\infty} \mathbb{E}_x\big[(\hat{\tau}_{2l} - \hat{\tau}_{2l-2}) \mathbf{1}_{\{\hat{\tau}_{2l-2} < \tau(G^c)\}}\big] \\
&\le \sum_{l=1}^{\infty} p^{l-1} \sup_{x\in\mathcal{K}} \mathbb{E}_x[\hat{\tau}_2] \\
&= \frac{1}{1-p} \sup_{x\in\mathcal{K}} \mathbb{E}_x[\hat{\tau}_2] < \infty.
\end{aligned}
$$

Since also $\mathbb{E}_x[\tau(\mathcal{K}^c)] < \infty$ for all $x \in \mathbb{R}^d$, this completes the proof. □

Theorem 5.2. *Let X be a Markov process associated with a generator $\mathcal{I} \in \mathfrak{L}_\alpha^{\mathrm{sym}}(\lambda) \cup \mathfrak{L}_{\alpha,\theta}(\lambda)$, and suppose that the Lyapunov stability hypothesis (5.1) and the growth condition in (2.5) hold. Then X has an invariant probability measure.*

Proof. The proof is based on Has′minskii's construction. Let $\mathcal{K}, D, \hat{\tau}_1$, and $\hat{\tau}_2$ be as in the proof of Theorem 5.1. Let $\hat{X}$ be a Markov process on $\mathcal{K}$ with transition kernel given by

$$
\hat{\mathbb{P}}_x(\mathrm{d}y) = \mathbb{P}_x(X_{\hat{\tau}_2} \in \mathrm{d}y).
$$

Let φ be any bounded, non-negative measurable function on D. Define $Q_\varphi(x) = \mathbb{E}_x[\varphi(X_{\hat{\tau}_2})]$. We claim that Q_φ is harmonic in D. Indeed if we define $\tilde{\varphi}(x) = \mathbb{E}_x[\varphi(X_{\tau(\mathcal{K}^c)})]$ for $x \in D^c$, then by the strong Markov property, we obtain $Q_\varphi(x) = \mathbb{E}_x[\tilde{\varphi}(X_{\hat{\tau}_1})]$, and the claim follows. By Theorem 4.1, there exists a positive constant C_H, independent of φ, satisfying

$$
Q_\varphi(x) \le C_H Q_f(y) \quad \forall\, x, y \in \mathcal{K}. \tag{5.9}
$$

We note that $Q_{\mathbf{1}_\mathcal{K}} \equiv 1$. Let $Q(x, A) := Q_{\mathbf{1}_A}(x)$, for $A \subset \mathcal{K}$. For any pair of probability measures μ and μ' on $\mathcal{K}$, we claim that

$$
\left\| \int_\mathcal{K} \big(\mu(\mathrm{d}x) - \mu'(\mathrm{d}x)\big) Q(x,\cdot) \right\|_{\mathrm{TV}} \le \frac{C_H - 1}{C_H} \|\mu - \mu'\|_{\mathrm{TV}}. \tag{5.10}
$$

This implies that the map $\mu \to \int_\mathcal{K} Q(x,\cdot)\mu(\mathrm{d}x)$ is a contraction, and hence it has a unique fixed point $\hat{\mu}$ satisfying $\hat{\mu}(A) = \int_\mathcal{K} Q(x, A)\hat{\mu}(\mathrm{d}x)$ for any Borel set $A \subset \mathcal{K}$. In fact, $\hat{\mu}$ is the invariant probability measure of the Markov chain $\hat{X}$. Next we prove (5.10). Given any two probability measure μ, μ' on $\mathcal{K}$, we can find subsets F and G of $\mathcal{K}$ such that

$$
\left| \int_\mathcal{K} \big(\mu(\mathrm{d}x) - \mu'(\mathrm{d}x)\big) Q(x,\cdot) \right|_{\mathrm{TV}} = 2 \int_\mathcal{K} \big(\mu(\mathrm{d}x) - \mu'(\mathrm{d}x)\big)\, Q(x, F),
$$

$$
\|\mu - \mu'\|_{\mathrm{TV}} = 2(\mu - \mu')(G).
$$

In fact, the restriction of $(\mu - \mu')$ to G is a non-negative measure and its restriction to G^c is nonpositive measure. By (5.9), we have

$$
\inf_{x\in G^c} Q(x, F) \ge \sup_{x\in G} Q(x, F) \tag{5.11}
$$

Hence, using (5.11), we obtain

$$\begin{aligned}
&\left\| \int_{\mathcal{K}} \big(\mu(\mathrm{d}x) - \mu'(\mathrm{d}x)\big) Q(x,\cdot) \right\|_{\mathrm{TV}} \\
&\quad = 2\int_G \big(\mu(\mathrm{d}x) - \mu'(\mathrm{d}x)\big) Q(x,F) + 2\int_{G^c} \big(\mu(\mathrm{d}x) - \mu'(\mathrm{d}x)\big) Q(x,F) \\
&\quad \le 2(\mu-\mu')(G) \sup_{x\in G} Q(x,F) + 2(\mu-\mu')(G^c) \inf_{x\in G^c} Q(x,F) \\
&\le 2(\mu-\mu')(G) \sup_{x\in G} Q(x,F) - \frac{2}{C_H}(\mu-\mu')(G) \sup_{x\in G} Q(x,F) \\
&\quad \le \big(1 - C_H^{-1}\big) \|\mu - \mu'\|_{\mathrm{TV}}.
\end{aligned}$$

This proves (5.10).

We define a probability measure ν on $\mathbb{R}^d$ as follows.

$$\int_{\mathbb{R}^d} \varphi(x)\,\nu(\mathrm{d}x) = \frac{\int_{\mathcal{K}} \mathbb{E}_x\big[\int_0^{\hat{\tau}_2} \varphi(X_s)\,\mathrm{d}s\big]\hat{\mu}(\mathrm{d}x)}{\int_{\mathcal{K}} \mathbb{E}_x[\hat{\tau}_2]\hat{\mu}(\mathrm{d}x)}, \quad \varphi \in C_b(\mathbb{R}^d).$$

It is straight forward to verify that ν is an invariant probability measure of X [4, Theorem 2.6.9]. □

Remark 5.1. If $k(\cdot,\cdot) = 1$ and the drift b belongs to certain Kato class, in particular bounded, [16] then the transition probability has a continuous density, and therefore any invariant probability measure has a continuous density. Since any two distinct ergodic measures are mutually singular, this implies the uniqueness of the invariant probability measure. As shown later in Proposition 5.4, open sets have strictly positive mass under any invariant measure.

The following result is fairly standard.

Proposition 5.3. *Let $\mathcal{I} \in \mathfrak{L}_\alpha$ and $\mathcal{V} \in C^2(\mathbb{R}^d)$ be a non-negative function satisfying $\mathcal{V}(x) \to \infty$ as $|x| \to \infty$, and $\mathcal{I}\mathcal{V} \le 0$ outside some compact set $\mathcal{K}$. Let ν be an invariant probability measure of the Markov process associated with the generator $\mathcal{I}$. Then*

$$\int_{\mathbb{R}^d} |\mathcal{I}\mathcal{V}(x)|\,\nu(\mathrm{d}x) \le 2\int_{\mathcal{K}} |\mathcal{I}\mathcal{V}(x)|\,\nu(\mathrm{d}x).$$

Proof. Let $\varphi_n : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth nondecreasing, concave, function such that

$$\varphi_n(x) = \begin{cases} x & \text{for } x \le n, \\ n + 1/2 & \text{for } x \ge n+1. \end{cases}$$

Due to concavity, we have $\varphi_n(x) \le |x|$ for all $x \in \mathbb{R}_+$. Then $\mathcal{V}_n(x) := \varphi_n(\mathcal{V}(x))$ is in $C_b^2(\mathbb{R}^d)$, and it also follows that $\mathcal{I}\mathcal{V}_n(x) \to \mathcal{I}\mathcal{V}(x)$ as $n \to \infty$. Since ν is an invariant probability measure, it holds that

$$\int_{\mathbb{R}^d} \mathcal{I}\mathcal{V}_n(x)\,\nu(\mathrm{d}x) = 0. \tag{5.12}$$

By concavity, $\varphi_n(y) \le \varphi_n(x) + (y-x)\cdot\varphi_n'(x)$ for all $x, y \in \mathbb{R}_+$. Hence

$$\begin{aligned}
\mathcal{I}\,\mathcal{V}_n(x) &= \int_{\mathbb{R}^d} \mathfrak{d}\mathcal{V}_n(x;z)\,\frac{k(x,z)}{|z|^{d+\alpha}}\,\mathrm{d}z + \varphi_n'(\mathcal{V}(x))\,b(x)\cdot\nabla\mathcal{V}(x)\\
&\le \int_{\mathbb{R}^d} \varphi_n'(\mathcal{V}(x))\,\mathfrak{d}\mathcal{V}(x;z)\,\frac{k(x,z)}{|z|^{d+\alpha}}\,\mathrm{d}z + \varphi_n'(\mathcal{V}(x))\,b(x)\cdot\nabla\mathcal{V}(x)\\
&= \varphi_n'(\mathcal{V}(x))\,\mathcal{I}\,\mathcal{V}(x),
\end{aligned}$$

which is negative for $x \in \mathcal{K}^c$. Therefore using (5.12) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} |\mathcal{I}\,\mathcal{V}_n(x)|\,\nu(\mathrm{d}x) &= \int_{\mathcal{K}} |\mathcal{I}\,\mathcal{V}_n(x)|\,\nu(\mathrm{d}x) - \int_{\mathcal{K}^c} \mathcal{I}\,\mathcal{V}_n(x)\,\nu(\mathrm{d}x)\\
&= \int_{\mathcal{K}} |\mathcal{I}\,\mathcal{V}_n(x)|\,\nu(\mathrm{d}x) + \int_{\mathcal{K}} \mathcal{I}\,\mathcal{V}_n(x)\,\nu(\mathrm{d}x)\\
&\le 2\int_{\mathcal{K}} |\mathcal{I}\,\mathcal{V}_n(x)|\,\nu(\mathrm{d}x). \qquad (5.13)
\end{aligned}$$

On the other hand, with $A_n := \{y \in \mathbb{R}^d : \mathcal{V}(y) \ge n\}$, and provided $\mathcal{V}(x) < n$, we have

$$\begin{aligned}
|\mathcal{I}\,\mathcal{V}_n(x)| &\le |\mathcal{I}\,\mathcal{V}(x)| + \int_{x+z\in A_n} |\mathcal{V}(x+z) - \mathcal{V}_n(x+z)|\,\frac{k(x,z)}{|z|^{d+\alpha}}\,\mathrm{d}z\\
&\le |\mathcal{I}\,\mathcal{V}(x)| + \int_{x+z\in A_n} |\mathcal{V}(x+z)|\,\frac{k(x,z)}{|z|^{d+\alpha}}\,\mathrm{d}z.
\end{aligned}$$

This together with (5.2) imply that there exists a constant κ such that

$$|\mathcal{I}\,\mathcal{V}_n(x)| \le \kappa + |\mathcal{I}\,\mathcal{V}(x)| \quad \forall\, x \in \mathcal{K},$$

and all large enough n. Therefore, letting $n \to \infty$ and using Fatou's lemma for the term on the left-hand side of (5.13), and the dominated convergence theorem for the term on the right-hand side, we obtain the result. □

5.2. A class of operators with variable order kernels

It is quite evident from Theorem 5.2 that the Harnack inequality plays a crucial role in the analysis. Therefore, one might wish to establish positive recurrence for an operator with a variable order kernel and deploy the Harnack inequality from [11] to prove a similar result as in Theorem 5.2.

Theorem 5.3. *Let $\pi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be a non-negative measurable function satisfying the following properties, for $1 < \alpha' < \alpha < 2$:*

(a) *There exists a constant $c_1 > 0$ such that $\mathbf{1}_{\{|z|>1\}}\pi(x,z) \le \frac{c_1}{|z|^{d+\alpha'}}$ for all $x \in \mathbb{R}^d$;*

(b) *For some constant $c_2 > 0$, we have*

$$\pi(x, z-x) \le c_2\,\pi(y, y-z), \quad \textit{whenever} \quad |z-x| \wedge |z-y| \ge 1,\ |x-y| \le 1;$$

(c) *For each $R > 0$, there exists $q_R > 0$ such that*

$$\frac{q_R^{-1}}{|z|^{d+\alpha'}} \le \pi(x,z) \le \frac{q_R}{|z|^{d+\alpha}} \quad \forall x \in \mathbb{R}^d,\ \forall z \in B_R;$$

(d) *For each $R > 0$, there exist constants $R_1 > 0$, $\sigma \in (1, 2)$, and $\kappa_\sigma = \kappa_\sigma(R, R_1) > 0$ such that*

$$\frac{\kappa_\sigma^{-1}}{|z|^{d+\sigma}} \le \pi(x, z) \le \frac{\kappa_\sigma}{|z|^{d+\sigma}} \qquad \forall x \in B_R,\ \forall z \in B_{R_1}^c;$$

(e) *There exists $\mathcal{V} \in C^2(\mathbb{R}^d)$ that is bounded from below in $\mathbb{R}^d$, a compact set $\mathcal{K} \subset \mathbb{R}^d$ and a constant $\varepsilon > 0$, satisfying*

$$\int_{\mathbb{R}^d} \mathfrak{d}\mathcal{V}(x; z)\, \pi(x, z)\, \mathrm{d}z \;<\; -\varepsilon \quad \forall x \in \mathcal{K}^c.$$

Then the Markov process associated with the above kernel has an invariant probability measure.

The first three assumptions guarantee the Harnack property for associated harmonic functions [11]. Then the conclusion of Theorem 5.3 follows using an argument similar to the one used in the proof of Theorem 5.2.

Next we present an example of a kernel π that satisfies the conditions in Theorem 5.3. We accomplish this by adding a nonsymmetric bump function to a symmetric kernel.

Example 5.1. Let $\varphi : \mathbb{R}^d \to [0, 1]$ be a smooth function such that

$$\varphi(x) = \begin{cases} 1 & \text{for } |x| \le \dfrac{1}{2}, \\ 0 & \text{for } |x| \ge 1. \end{cases}$$

Define for $1 < \alpha' < \beta' < \alpha < 2$,

$$\gamma(x, z) := \varphi\left(2\frac{x + z}{1 + |x|}\right)(1 - \varphi(4x))(\alpha' - \beta'),$$

and let

$$\tilde{\pi}(x, z) := \frac{1}{|z|^{d+\beta'+\gamma(x,z)}},$$

$$\pi(x, z) := \frac{1}{|z|^{d+\alpha}} + \tilde{\pi}(x, z).$$

We prove that π satisfies the conditions of Theorem 5.3. Let us also mention that there exists a unique solution to the martingale problem corresponding to the kernel π [30, 31]. We only show that conditions (b) and (e) hold. It is straightforward to verify (a), (c), and (d).

Note that $\alpha' - \beta' \le \gamma(x, z) \le 0$ for all x, z. Let $x,\ y,\ z \in \mathbb{R}^d$ such that $|x - z| \wedge |y - z| \ge 1$ and $|x - y| \le 1$. Then $|z - y| \le 1 + |z - x|$. By a simple calculation, we obtain

$$\begin{aligned} \tilde{\pi}(x, z - x) &\le \left(1 + \frac{1}{|z - x|}\right)^{d+\beta'+\gamma(x,z-x)} \frac{1}{|z - y|^{d+\beta'+\gamma(x,z-x)}} \\ &\le 2^{d+\beta'} \frac{1}{|z - y|^{d+\beta'+\gamma(y,z-y)}} |z - y|^{-\beta'(x,z-x)+\gamma(y,z-y)}. \end{aligned}$$

Hence it is enough to show that

$$|z - y|^{-\gamma(x,z-x)+\gamma(y,z-y)} < \varrho \tag{5.14}$$

for some constant ϱ which does not depend on x, y, and z. Note that if $|x| \le 2$, which implies that $|y| \le 3$, then for $|z| \ge 4$ we have $\gamma(x, z-x) = 0 = \gamma(y, z-y)$. Therefore, for $|x| \le 2$, it holds that

$$|z-y|^{-\gamma(x,z-x)+\gamma(y,z-y)} \le 7^{\beta'-\alpha'}. \tag{5.15}$$

Suppose that $|x| \ge 2$, then $|y| \ge 1$. Since we only need to consider the case where $\gamma(x, z-x) \ne \gamma(y, z-y)$, we restrict our attention to $z \in \mathbb{R}^d$ such that $|z| \le 2(1+|x|)$. We obtain

$$\begin{aligned}\log(|z-y|)(-\gamma(x,z-x)+\gamma(y,z-y)) &\le \log\big(3(1+|x|)\big)\,\|\varphi'\|_\infty\,\frac{2|z|(\beta'-\alpha')}{(1+|x|)(1+|y|)}\\ &\le \log\big(3(1+|x|)\big)\,\|\varphi'\|_\infty\,\frac{4(1+|x|)(\beta'-\alpha')}{(1+|x|)|x|}.\end{aligned} \tag{5.16}$$

Since the term on the right-hand side of (5.16) is bounded in $\mathbb{R}^d$, the bound in (5.14) follows by (5.15)–(5.16).

Next we prove the Lyapunov property. We fix a constant $\eta \in (\alpha', \beta')$, and choose some function $\mathcal{V} \in C^2(\mathbb{R}^d)$ such that $\mathcal{V}(x) = |x|^\eta$ for $|x| > 1$. Since $\widetilde{\pi}(x,z) \le \frac{1}{|z|^{d+\alpha'}}$ for all $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^d_*$, it follows that

$$x \mapsto \left|\int_{|z|\le 1} \mathfrak{d}\mathcal{V}(x;z)\,\widetilde{\pi}(x,z)\,\mathrm{d}z\right|$$

is bounded by some constant on $\mathbb{R}^d$. By (5.5),

$$\left|\int_{\mathbb{R}^d} \mathfrak{d}\mathcal{V}(x;z)\,\pi(x,z)\,\mathrm{d}z\right| \le c_0\,(1+|x|^{\eta-\alpha}) \quad \forall x \in \mathbb{R}^d,$$

for some constant c_0. Therefore, in view of (5.4), it is enough to show that, for $|x| \ge 4$, there exist positive constants c_1 and c_2 such that

$$\int_{|z|>1} \big(|x+z|^\eta - |x|^\eta\big)\,\widetilde{\pi}(x,z)\,\mathrm{d}z \le c_1 - c_2|x|^{\eta-\alpha'}. \tag{5.17}$$

By the definition of, γ it holds that

$$\widetilde{\pi}(x,z) = \frac{1}{|z|^{d+\beta'}}, \quad \text{if } |x+z| \ge \frac{3}{4}|x|, \text{ and } |x| \ge 2, \tag{5.18}$$

while

$$\widetilde{\pi}(x,z) = \frac{1}{|z|^{d+\alpha'}}, \quad \text{if } |x+z| \le \frac{|x|}{4}. \tag{5.19}$$

Suppose that $|x| > 2$. Since $|x+z| \le \frac{|x|}{4}$ implies that $\frac{3}{4}|x| \le |z| \le \frac{5}{4}|x|$, we obtain by (5.19) that

$$\begin{aligned}\int_{|x+z|\le\frac{|x|}{4},\,|z|>1} \big(|x+z|^\eta - |x|^\eta\big)\,\widetilde{\pi}(x,z)\,\mathrm{d}z &\le -\int_{|x+z|\le\frac{|x|}{4}} \big(1-\tfrac{1}{4^\eta}\big)\,|x|^\eta\,\big(\tfrac{4}{5}\big)^{d+\alpha'}\frac{1}{|x|^{d+\alpha'}}\,\mathrm{d}z\\ &\le -\big(1-\tfrac{1}{4^\eta}\big)\,\big(\tfrac{4}{5}\big)^{d+\alpha'}\,|x|^{\eta-\alpha'}\int_{|x+z|\le\frac{|x|}{4}} \frac{\mathrm{d}z}{|x|^d}\\ &\le -m_1\,|x|^{\eta-\alpha'}, \quad \text{if } |x| > 2,\end{aligned} \tag{5.20}$$

for some constant $m_1 > 0$, where we use the fact that the integral in the second inequality is independent of x due to rotational invariance. Also, $|x+z| \leq \frac{3}{4}|x|$ implies $\frac{1}{4}|x| \leq |z| \leq \frac{7}{4}|x|$, and in a similar manner, using (5.18), we obtain

$$\int_{|x+z|\leq \frac{3|x|}{4},\, |z|>1} \left(|x+z|^\eta - |x|^\eta\right) \frac{1}{|x|^{d+\beta'}}\, \mathrm{d}z \geq -\int_{\frac{1}{4}|x|\leq|z|\leq\frac{7}{4}|x|} |x|^\eta\, 4^{d+\beta'} \frac{1}{|x|^{d+\beta'}}\, \mathrm{d}z$$
$$\geq -m_2\, |x|^{\eta-\beta'}, \quad \text{if } |x| > 2, \tag{5.21}$$

for some constant $m_2 > 0$. Let $A_1 := \left\{z : \frac{1}{4}|x| \leq |x+z| \leq \frac{3}{4}|x|\right\}$. Since η is positive, we have

$$\int_{\{|z|\geq 1\}\cap A_1} \left(|x+z|^\eta - |x|^\eta\right) \widetilde{\pi}(x,z)\, \mathrm{d}z \leq 0.$$

Thus, combining this observation with (5.3) and (5.21), we obtain

$$\begin{aligned}
\int_{|x+z|>\frac{|x|}{4},\, |z|>1} \left(|x+z|^\eta - |x|^\eta\right) \widetilde{\pi}(x,z)\, \mathrm{d}z &\leq \int_{|x+z|>\frac{3}{4}|x|,\, |z|>1} \left(|x+z|^\eta - |x|^\eta\right) \frac{1}{|z|^{d+\beta'}}\, \mathrm{d}z \\
&= \int_{|z|>1} \left(|x+z|^\eta - |x|^\eta\right) \frac{1}{|z|^{d+\beta'}}\, \mathrm{d}z \\
&\quad - \int_{|x+z|\leq\frac{3|x|}{4},\, |z|>1} \left(|x+z|^\eta - |x|^\eta\right) \frac{1}{|x|^{d+\beta'}}\, \mathrm{d}z \\
&\leq m_3\, (1 + |x|^{\eta-\beta'})
\end{aligned} \tag{5.22}$$

for some constant $m_3 > 0$. Combining (5.20) and (5.22), we obtain

$$\int_{|z|>1} \left(|x+z|^\eta - |x|^\eta\right) \widetilde{\pi}(x,z)\, \mathrm{d}z \leq m_3\, (1 + |x|^{\eta-\beta'}) - m_1\, |x|^{\eta-\alpha'}, \quad \text{if } |x| > 2. \tag{5.23}$$

Therefore, (5.17) follows by (5.23), and the Lyapunov property holds.

Proposition 5.4. *Let D be any bounded open set in $\mathbb{R}^d$ and X be a Markov process associated with either $\mathcal{I} \in \mathfrak{L}_\alpha$, or a generator with kernel π as in Theorem 5.3. Suppose that for any compact set K and any open set G, it holds that $\sup_{x\in K}\mathbb{P}_x(\tau(G^c) > T) \to 0$ as $T \to \infty$. Then for any invariant probability measure ν of X, we have $\nu(D) > 0$.*

Proof. We argue by contradiction. Suppose $\nu(D) = 0$. Let $x_0 \in D$ and $r \in (0,1)$ be such that $B_{2r}(x_0) \subset D$. By Lemma 4.1 [11, Proposition 3.1], we have

$$\sup_{x\in B_r(x_0)} \mathbb{P}_x\big(\tau(B_r(x)) \leq t\big) \leq \kappa\, t, \quad t > 0,$$

for some constant κ which depends on r. Therefore, there exists $t_0 > 0$ such that

$$\inf_{x\in B_r(x_0)} \mathbb{P}_x\big(\tau(B_r(x)) \geq t_0\big) \geq \frac{1}{2}.$$

Let K be a compact set satisfying $\nu(K) > \frac{1}{2}$. By the hypothesis there exists $T_0 > 0$ such that $\sup_{x\in K} \mathbb{P}_x(\tau(B_r^c(x_0)) > T) \le 1/2$ for all $T \ge T_0$. Hence

$$
\begin{aligned}
0 = \nu(D) &\ge \frac{1}{T_0+t_0}\int_0^{T_0+t_0}\int_{\mathbb{R}^d} \nu(\mathrm{d}x)P(t,x;B_{2r}(x_0))\,\mathrm{d}t\\
&= \frac{1}{T_0+t_0}\int_{\mathbb{R}^d}\nu(\mathrm{d}x)\,\mathbb{E}_x\left[\int_0^{T_0+t_0}\mathbf{1}_{\{B_{2r}(x_0)\}}(X_s)\,\mathrm{d}t\right]\\
&\ge \frac{1}{T_0+t_0}\int_K \nu(\mathrm{d}x)\,\mathbb{E}_x\Bigg[\mathbf{1}_{\{\tau(B_r^c(x_0))\le T_0\}}\mathbb{E}_{X_{\tau(B_r^c(x_0))}}\Bigg[\mathbf{1}_{\{\tau(B_{2r}(x_0))\ge t_0\}}\\
&\qquad \int_{\tau(B_r^c(x_0))}^{T_0+t_0}\mathbf{1}_{\{B_{2r}(x_0)\}}(X_s)\,\mathrm{d}t\Bigg]\Bigg]\\
&\ge \frac{1}{T_0+t_0}\nu(K)\inf_{x\in K}\mathbb{P}_x\big(\tau(B_r^c(x_0))\le T_0\big)\inf_{x\in B_r(x_0)}\mathbb{P}_x\big(\tau(B_{2r}(x_0))\ge t_0\big)\,t_0\\
&\ge \frac{1}{T_0+t_0}\frac{\nu(K)}{2}\inf_{x\in B_r(x_0)}\mathbb{P}_x\big(\tau(B_{2r}(x))\ge t_0\big)\,t_0\\
&\ge \frac{t_0}{4(T_0+t_0)} > 0.
\end{aligned}
$$

But this is a contradiction. Hence $\nu(D) > 0$. □

5.3. Mean recurrence times

This section is devoted to the characterization of the mean hitting time of bounded open sets for Markov processes with generators studied in Section 3. The results hold for any bounded domain D with C^2 boundary, but for simplicity, we state them for the unit ball centered at 0. As introduced earlier, we use the notation $B \equiv B_1$.

For nondegenerate continuous diffusions, it is well known that if some bounded domain D is positive recurrent with respect to some point $x \in \overline{D}^c$, then the process is positive recurrent and its generator satisfies the Lyapunov stability hypothesis in (5.1) [4, Lemma 3.3.4]). In Theorem 5.4, we show that the same property holds for the class of operators $\mathfrak{I}_\alpha(\beta,\theta,\lambda)$.

Theorem 5.4. *Let $\mathcal{I} \in \mathfrak{I}_\alpha(\beta,\theta,\lambda)$. We assume that $\mathcal{I}$ satisfies the growth condition in (2.5). Moreover, we assume that $\mathbb{E}_x[\tau(B^c)] < \infty$ for some x in $\overline{B}^c$. Then $u(x) := \mathbb{E}_x[\tau(B^c)]$ is a viscosity solution to*

$$
\begin{aligned}
\mathcal{I}u &= -1 \quad \text{in } \overline{B}^c,\\
u &= 0 \quad \text{in } \overline{B}.
\end{aligned}
$$

To prove Theorem 5.4, we need the following two lemmas.

Lemma 5.2. *Let $\mathcal{I} \in \mathfrak{I}_\alpha(\beta,\theta,\lambda)$ and G a bounded open set containing $\overline{B}$. Then there exist positive constants r_0 and M_0 depending only on G such that*

$$
\int_{\overline{B}^c(x)} \mathbb{E}_z[\tau(B^c)]\,\frac{1}{|z|^{d+\alpha}}\,\mathrm{d}z < \frac{M_0}{r^\alpha}\,\mathbb{E}_x[\tau(B^c)]
$$

for every $r < \operatorname{dist}(x,B)\wedge r_0$ and for all $x \in G\setminus\overline{B}$, such that $\mathbb{E}_x[\tau(B^c)] < \infty$.

Proof. Let $\check{\tau} := \tau(B^c)$ and $\hat{\tau}_r := \tau\big(B_r(x)\big)$. We select r_0 as in Lemma 4.2, and without loss of generality, we assume $r_0 \le 1$. We have

$$\mathbb{E}_x\Big[\mathbf{1}_{\{\hat{\tau}_r<\check{\tau}\}}\mathbb{E}_{X_{\hat{\tau}_r}}[\check{\tau}]\Big] \le \mathbb{E}_x[\check{\tau}]. \tag{5.24}$$

By Definition 3.1, we have

$$k(y,z) \ge \lambda_G^{-1} > 0 \quad \forall y \in B_{r_0}(x).$$

Let $A \subset \overline{B}_r^c(x) \cap \overline{B}^c$ be any Borel set. Using Proposition 5.2, we have

$$\begin{aligned}
\mathbb{P}_x(X_{\hat{\tau}_r\wedge t} \in A) &= \mathbb{E}_x\left[\sum_{s\le\hat{\tau}_r\wedge t} \mathbf{1}_{\{X_{s-}\in B_r(x),\, X_s\in A\}}\right]\\
&= \mathbb{E}_x\bigg[\int_0^{\hat{\tau}_r\wedge t} \mathbf{1}_{\{X_s\in B_r(x)\}} \int_A \pi(X_s, z - X_s)\,\mathrm{d}z\,\mathrm{d}s\bigg]\\
&\ge \lambda_G^{-1}\,\mathbb{E}_x\bigg[\int_0^{\hat{\tau}_r\wedge t}\int_A \frac{1}{|z|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}s\bigg]\\
&\ge \lambda_G^{-1}\,\mathbb{E}_x[\hat{\tau}_r\wedge t]\int_A \frac{1}{|z|^{d+\alpha}}\,\mathrm{d}z.
\end{aligned}$$

Letting $t \to \infty$, we obtain

$$\mathbb{P}_x(X_{\hat{\tau}_r} \in A) \ge \lambda_G^{-1}\,\mathbb{E}_x[\hat{\tau}_r]\int_A \frac{1}{|z|^{d+\alpha}}\,\mathrm{d}z. \tag{5.25}$$

By Lemma 4.2, it holds that $\mathbb{E}_x[\hat{\tau}_r] > \kappa_1\, r^\alpha$ for some positive constant κ_1 which depends on G. Hence combining (5.24) and (5.25), we obtain

$$\begin{aligned}
\lambda_G^{-1}\,\kappa_1\, r^\alpha \int_{\overline{B}^c(x)} \mathbb{E}_z[\check{\tau}]\,\frac{1}{|z|^{d+\alpha}}\,\mathrm{d}z &\le \mathbb{E}_x\big[\mathbf{1}_{\{X_{\hat{\tau}_r}\in\overline{B}^c\}}\mathbb{E}_{X_{\hat{\tau}_r}}[\check{\tau}]\big]\\
&\le \mathbb{E}_x[\check{\tau}],
\end{aligned}$$

where the first inequality follows by the standard approximation technique using step functions. This completes the proof. □

Lemma 5.2 of course implies that if $\mathbb{E}_x[\tau(B^c)] < \infty$ at some point $x \in \overline{B}^c$, then $\mathbb{E}_x[\tau(B^c)]$ is finite a.e.-x. We can express the bound in Lemma 5.2 without reference to Lemma 4.2 as

$$\int_{\overline{B}^c(x)} \mathbb{E}_z[\tau(B^c)]\,\frac{1}{|z|^{d+\alpha}}\,\mathrm{d}z \le \lambda_G\,\frac{\mathbb{E}_x[\tau(B^c)]}{\mathbb{E}_x[\overline{B}^c]}.$$

Now let x' be any point such that $\mathrm{dist}(x',x) \wedge \mathrm{dist}(x',B) = 2r$. We obtain

$$\frac{\omega(r)}{|2r|^{d+\alpha}}\,\inf_{z\in B_r(x')}\,\mathbb{E}_z[\tau(B^c)] \le \frac{M_0}{r^\alpha}\,\mathbb{E}_x[\tau(B^c)].$$

Therefore for some $y \in B_r(x')$, we have $\mathbb{E}_y[\tau(B^c)] < C_1\,\mathbb{E}_x[\tau(B^c)]$. Applying Lemma 5.2 once more, we obtain

$$\int_{\mathbb{R}^d} \mathbb{E}_{x+z}[\tau(B^c)]\,\frac{1}{(1+|z|)^{d+\alpha}}\,\mathrm{d}z \le C_0\mathbb{E}_x[\tau(B^c)],$$

with the constant C_0 depending only on $\mathrm{dist}(x,B)$ and the parameter λ, i.e., the local bounds on k. We introduce the following notation.

Definition 5.2. We say that $\nu \in L^1(\mathbb{R}^d, s)$ if

$$\int_{\mathbb{R}^d} \frac{|\nu(z)|}{(1+|z|)^{d+\alpha}}\,\mathrm{d}z < \infty.$$

Thus we have the following.

Corollary 5.1. *If $\mathbb{E}_{x_0}[\tau(B^c)] < \infty$ for some $x_0 \in \overline{B}^c$, then the function $u(x) := \mathbb{E}_x[\tau(B^c)]$ is in $L^1(\mathbb{R}^d, s)$.*

In what follows, without loss of generality, we assume that $\beta < s$. Then, by Theorem 3.2, $u_n(x) := \mathbb{E}_x\big[\tau(B_n \cap \overline{B}^c)\big]$ is the unique solution in $C^{\alpha+\beta}(B_n \setminus \overline{B}) \cap C(\overline{B}_n \setminus B)$ of

$$\begin{aligned} \mathcal{I}u_n &= -1 \quad \text{in} \quad B_n \cap \overline{B}^c, \\ u_n &= 0 \quad \text{in} \quad B_n^c \cup B. \end{aligned} \tag{5.26}$$

The following lemma provides a uniform barrier on the solutions u_n near B.

Lemma 5.3. *Let $\mathcal{I} \in \mathfrak{I}_\alpha(\beta, \theta, \lambda)$, and*

$$\widetilde{\tau}_n := \tau(B_n \cap \overline{B}^c), \quad n \in \mathbb{N}.$$

Then, provided that $\sup_{x\in F} \mathbb{E}_x[\tau(B^c)] < \infty$ for all compact sets $F \subset \overline{B}^c$, there exists a continuous, non-negative radial function φ that vanishes on B, and satisfies, for some $\eta > 0$,

$$\mathbb{E}_x[\widetilde{\tau}_n] \le \varphi(x) \quad \forall\, x \in B_{1+\eta} \setminus B,\ \forall\, n > 1.$$

Proof. The proof relies on the construction of barrier. Let $\hat{k}(x,z) = k(x,z) - k(x,0)$. By Lemma 3.2, for $q \in (\alpha - 1/2, \alpha/2)$, there exists a constant $c_0 > 0$ such that for $\varphi_q(x) := [(1-|x|)^+]^q$, we have

$$(-\Delta)^{\alpha/2}\varphi_q(x) > c_0\,(1-|x|)^{q-\alpha} \quad \forall\, x \in B.$$

We recall the Kelvin transform from [34]. Define $\hat{\varphi}(x) = |x|^{\alpha-d}\varphi_q(x^*)$ where $x^* := \frac{x}{|x|^2}$. Then by [34, Proposition A.1], there exists a positive constant c_1 such that

$$(-\Delta)^{\alpha/2}\hat{\varphi}(x) > c_1\,(|x|-1)^{q-\alpha} \quad \forall\, x \in B_2 \setminus \overline{B}.$$

We restrict $\hat{\varphi}$ outside a large compact set, so that it is bounded on $\mathbb{R}^d$. By $\widehat{\mathcal{I}}$, we denote the operator

$$\widehat{\mathcal{I}}u(x) = b(x)\cdot\nabla u(x) + \int_{\mathbb{R}^d} \mathfrak{d}u(x;z)\,\frac{\hat{k}(x,z)}{|z|^{d+\alpha}}\,\mathrm{d}z.$$

It is clear that $|\nabla\hat{\varphi}(x)| \le c_2(|x|-1)^{q-1}$ for all $|x| \in (1,2)$, for some constant c_2. Also, using the fact that $\hat{\varphi}$ is Hölder continuous of exponent q and (3.1), we obtain

$$\left|\int_{\mathbb{R}^d} \mathfrak{d}\hat{\varphi}(x;z)\,\frac{\hat{k}(x,z)}{|z|^{d+\alpha}}\,\mathrm{d}z\right| \le c_3(|x|-1)^{q+\theta-\alpha} \quad \forall\, x \in B_2 \setminus \overline{B},$$

for some constant c_3. Hence

$$\big|\widehat{\mathcal{I}}\hat{\varphi}(x)\big| \le c_4\,\big(|x|-1\big)^{(q-1)\wedge(q+\theta-\alpha)}, \quad \text{for } x \in B_2 \setminus \overline{B},$$

for some constant c_4. Since $\theta > 0, \alpha > 1$, and $\mathcal{I} = \widehat{\mathcal{I}} - k(x,0)(-\Delta)^{\alpha/2}$, it follows that we can find η small enough such that

$$\mathcal{I}\hat{\varphi}(x) < -4, \quad \text{for } x \in B_{1+\eta} \setminus \overline{B}.$$

Let K be a compact set containing $B_{1+\eta}$. We define

$$\tilde{\varphi}(x) = \hat{\varphi}(x)\,\mathbf{1}_K(x) + \mathbb{E}_x[\tau(B^c)]\,\mathbf{1}_{K^c}(x).$$

Since the hypotheses of Lemma 5.2 are met, we conclude that $\mathbf{1}_{K^c}(x)\mathbb{E}_x[\tau(B^c)]$ is integrable with respect to the kernel π. For $x \in B_{1+\eta} \setminus \overline{B}$, we obtain

$$\begin{aligned}\mathcal{I}\tilde{\varphi}(x) &< -4 + \int_{\mathbb{R}^d} \big(\mathbb{E}_{x+z}[\tau(B^c)] - \hat{\varphi}(x+z)\big)\,\mathbf{1}_{K^c}(x+z)\,\pi(x,z)\,\mathrm{d}z \\ &= -4 + \int_{K^c} \mathbb{E}_z[\tau(B^c)]\,\frac{\pi(x,z-x)}{\pi(x,z)}\pi(x,z)\,\mathrm{d}z - \int_{\mathbb{R}^d} \hat{\varphi}(x+z)\,\mathbf{1}_{K^c}(x+z)\,\pi(x,z)\,\mathrm{d}z.\end{aligned}$$

Since the kernel is comparable to $|z|^{-d-\alpha}$ on any compact set, we may choose K large enough and use Lemma 5.2 to obtain

$$\mathcal{I}\tilde{\varphi}(x) < -2 \quad \forall\, x \in B_{1+\eta} \setminus \overline{B}.$$

Let

$$\psi(x) := \Big(1 \vee \sup_{z \in K\setminus B_{1+\eta}} \mathbb{E}_z[\tau(B^c)]\Big)\Big(1 \vee \sup_{z \in K\setminus B_{1+\eta}} \frac{1}{\tilde{\varphi}(z)}\Big)\,\tilde{\varphi}(x).$$

Then, $\mathcal{I}\psi < -2$ on $B_{1+\eta} \setminus \overline{B}$, while $\psi \geq u_n$ on $B^c_{1+\eta} \cup B$. Therefore, by the comparison principle, $u_n \leq \psi$ on $B_{1+\eta} \setminus \overline{B}$ for all $n \in \mathbb{N}$ and the proof is complete. □

Proof of Theorem 5.4. Consider the sequence of solutions $\{u_n\}$ defined in (5.26). First we note that $u_n(x) \leq \mathbb{E}_x[\tau(B^c)]$ for all x. Clearly $u_{n+1} - u_n$ is bounded, non-negative, and harmonic in $B_n \setminus \overline{B}$. By Theorem 4.1, the operator $\mathcal{I}$ has the Harnack property. Therefore

$$\sup_{x\in F} \sum_{n\geq 1} \big(u_{n+1}(x) - u_n(x)\big) < \infty$$

for any compact subset F in $\overline{B}^c$. Hence Lemma 2.3 combined with Fatou's lemma implies that $\sup_{x\in F} \mathbb{E}_x[\tau(B^c)] < \infty$ for any compact set $F \subset \overline{B}^c$.

We write

$$u_n = u_1 + \sum_{m=1}^{n-1} \big(u_{m+1}(x) - u_m(x)\big),$$

and use the Harnack property once more to conclude that $u_n \nearrow u$ uniformly over compact subsets of $\overline{B}^c$. Since $u \leq \varphi$ in a neighborhood of ∂B by Lemma 5.3, and φ vanishes on ∂B, it follows that $u \in C(\mathbb{R}^d)$. That u is a viscosity solution follows from the fact that $u_n \to u$ uniformly over compacta as $n \to \infty$ and Lemma 5.2. □

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