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On solutions of mean field games with ergodic cost

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ABSTRACT

A general class of mean field games are considered where the governing dynamics are controlled diffusions in $\mathbb{R}^d$. The optimization criterion is the long time average of a running cost function. Under various sets of hypotheses, we establish the existence of mean field game solutions. We also study the long time behavior of the mean field game solutions associated with the finite horizon problem, and under the assumption of geometric ergodicity for the dynamics, we show that these converge to the ergodic mean field game solution as the horizon tends to infinity. Lastly, we study the associated N-player games, show existence of Nash equilibria, and establish the convergence of the solutions associated to Nash equilibria of the game to a mean field game solution as $N \to \infty$.

RÉSUMÉ

Nous considérons une classe générale de jeux en champ moyen où la dynamique est gouvernrnée par des diffusions contrôlées dans $\mathbb{R}^d$. Le critère d'optimisation est la moyenne à long terme d'une fonction de coût de fonctionnement. Sous divers ensembles d'hypothèses, on établit l'existence des solutions des jeux à champ moyen. On étudie aussi le comportement en temps long des solutions des jeux à champ moyen associées au problème d'horizon fini, et si l'on suppose l'ergodicité géométrique des dynamiques, on montre que ces solutions convergent vers la solution du jeu ergodique à champ moyen, lorsque l'horizon tend vers l'infini. Enfin, on étudie les jeux correspondants avec N-joueurs, on montre l'existence d'équilibres de Nash, et l'on établit la convergence des solutions associées aux équilibres de Nash du jeu vers une solution du jeu à champ moyen, lorsque $N \to \infty$.

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1. Introduction

Mean field games (MFG) were introduced by J.M. Lasry and P.L. Lions [38–40], and independently, by Huang, Malhamé and Caines [31]. Mean field games are the limiting models for symmetric, non-zero sum, non-cooperative N-player games with the interaction between the players being of mean field type. In view of the theory of McKean–Vlasov limits and propagation of chaos for uncontrolled weakly interacting particle systems [44], one may expect to obtain convergence result for N-player game Nash equilibria, at least under some symmetry conditions. With this heuristic in mind, Lasry and Lions introduced the field of mean field games. Recently, rigorous results have been established for finite horizon control problems [20,36], for mean field games with ergodic cost [18], and for discrete time Markov processes with ergodic cost [8]. On the other hand, it is also known that one can construct ε-Nash equilibria for N-player games from mean field game solutions. See for example [15,16,31,34,35]. Mean field games have seen a wide variety of applications, and have been studied extensively during the last decade using both analytic and probabilistic techniques. We refer to the surveys in [6,11,23,28] for recent developments in the area of mean field games.

In this paper, we model the controlled dynamics of the ith player, $i = 1, \dots, N$, by the Itô equation

$$dX_t^i \;=\; b(X_t^i, U_t^i)\,\mathrm{d}t + \sigma(X_t^i)\,\mathrm{d}W_t^i\,,$$

where $\{W^i\}_{\{1\le i\le N\}}$ is a collection of independent Wiener processes in $\mathbb{R}^d$ and U^i is an *admissible* control, taking values in a compact metric space $\mathbb{U}$, adapted to the filtration generated by W^i. Thus the players do not have access to the full state vector for purposes of control. Such strategies are referred to as *narrow strategies* [20]. The *running cost* is given by a continuous function $r\colon \mathbb{R}^d \times \mathbb{U} \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}_+$. The goal of the i-th player is to minimize the (ergodic) criterion

$$J^i(\boldsymbol{U}) \;:=\; \limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}\biggl[\int_0^T r(X_t^i, U_t^i, \mu_t^N)\,\mathrm{d}t\biggr], \qquad \text{with} \quad \mu_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}\,,$$

and $\boldsymbol{U} = (U^1, \dots, U^N)$. We note that the running cost function r may depend upon the empirical distribution μ^N of the private states of the players. Since each player's objective depend on the action of others we naturally look for Nash equilibria.

If the number of players N is very large, the contribution of the i-th player in the empirical distribution μ^N is negligible, and therefore μ^N may as well be treated fixed for player i. This heuristic argument leads to the mean field game formulation which can be described as follows:

(a) For a fixed element $\eta \in \mathcal{C}([0,\infty), \mathcal{P}(\mathbb{R}^d))$ solve the optimal control problem,

$$\begin{aligned} &\text{minimize} \quad \limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}\biggl[\int_0^T r(X_t, U_t, \eta_t)\,\mathrm{d}t\biggr], \\ &\text{subject to} \quad \mathrm{d}X_t \;=\; b(X_t, U_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\,, \quad \text{law of } X_0 = \eta(0)\,. \end{aligned} \tag{1.1}$$

(b) Find an optimal control U^* for the above control problem, and let η^* denote the law of the state dynamics under the optimal control U^*.

(c) Find a fixed point of the map $\eta \mapsto \eta^*$.

The above model can be interpreted as follows: there is a single representative agent whose reward function is effected by an environment distribution (coming from the large number of agents), and the state process of

the representative can not influence the environment while solving its own optimization problem. Moreover, since all agents have identical dynamics and the same objective function, the distribution of the state process of the representative agent should agree with the environment distribution. We observe that the above problem is not a *typical* optimal control problem. The cost function here is not being optimized over all possible pairs (X, η) where X_t has distribution η_t and X satisfies the dynamics in (1.1). This later class of problems are known as mean field type control problems [6].

There are three major issues of interest in mean field games, (1) existence and uniqueness of solutions of MFG, (2) long time behavior of the finite horizon MFG, and (3) establishing rigorous connection of N-player games with MFG. The topic in (3) can also be divided in two parts: (3a) convergence of a N-player game Nash equilibria to a MFG solution, and (3b) construction of ε-Nash equilibria for the N-player game from a MFG solution. The main goal of this paper is to answer the questions in (1), (2) and (3a) for the class of models we consider.

During the last decade many papers have been devoted to the study of the topics above. Existence of mean field game solutions with ergodic cost for a compact state space is studied in [18,40]. For existence of mean field game solutions for finite horizon control problems we refer the reader to [7,14–16,36]. These papers also allow the drift to depend on the environment distribution η. The existence problem for finite state processes is addressed in [24,25,27], and a more general class of discrete time Markov processes to study the existence result when the cost is ergodic is considered in [8]. Analytical results on the existence of solution for the stationary mean field games can found in [26,42]. Linear-Quadratic mean field games with ergodic costs are considered in [5], and existence results are established. However there is not much improvement as far as uniqueness in concerned. A L^2 type monotonicity condition (or a variant of it) is generally used to claim uniqueness of the mean field game solution (see [11,40]).

In Section 2 we study the existence of MFG solutions. We show that existence of MFG solutions is related to the existence of $(V, \tilde{\varrho}, \mu) \in \mathcal{C}^2(\mathbb{R}^d) \times \mathbb{R}_+ \times \mathcal{P}(\mathbb{R}^d)$ satisfying the following coupled equations (see Theorems 2.1 and 3.2)

$$\min_{u \in \mathbb{U}} \left[L^u V(x) + r(x,u,\mu) \right] \;=\; L^v V + r\big(x, v(x), \mu\big) \;=\; \tilde{\varrho} \quad \text{a.e. } x \in \mathbb{R}^d\,, \tag{1.2}$$

$$\int_{\mathbb{R}^d} L^v f(x)\, \mu(\mathrm{d}x) \;=\; 0 \qquad \forall\, f \in \mathcal{C}_c^2(\mathbb{R}^d). \tag{1.3}$$

Here L^u (see (2.3)) denotes the controlled extended generator of the controlled diffusion in (1.1). As well known, (1.2) is the Hamilton–Jacobi–Bellman (HJB) equation for an optimal ergodic control problem with running cost function $(x,u) \mapsto r(x,u,\mu)$, whereas (1.3) characterizes μ as the invariant probability measure corresponding to an optimal (stationary) Markov control v of (1.2). We use convex analytic tools (see Section 3) and the Kakutani–Fan–Glicksberg fixed point theorem to establish existence of a solution to (1.2)–(1.3).

One may also consider the finite horizon problem (say, with time horizon T) for the mean field model. In this situation the solution is again determined by two coupled equations, where one equation depicts the evolution of transition density (or transition probability) and the other one is the HJB for the finite horizon optimal control problem. For a model with a compact state space, it is shown in [12,13] that, as $T \to \infty$, the solution of the finite horizon control problem tends to the solution of (1.2)–(1.3) under suitable normalization. Similar long time behavior are also studied in [19,24,25] for finite state space processes. In Section 4 we study the analogous problem for our model. We compensate for the non-compactness of the state space by imposing a Lyapunov stability hypothesis to control behavior at infinity. We show that as the horizon $T \to \infty$, the law of the process for the finite horizon MFG tends to a stationary law with marginals μ (see (1.3)), and the value function of the finite horizon problem, suitably normalized, tends to V in (1.2), uniformly over compact sets (see Theorems 4.3 and 4.4 for details).

Next we discuss topic (3). As stated earlier there are several papers in which construction of approximate Nash equilibria is done using a MFG solution. In fact, a similar construction is also possible in our set up as well. However, the opposite direction is probably more natural and interesting [11, Remark 3.9]. Existence of Nash equilibria for N-player games with ergodic cost, and convergence to them is studied in [18], for a model with compact state space and a running cost function that has a special separable structure. Recently, [14,20,35] have addressed the same question (assuming existence of approximate Nash equilibria) for a general class of finite horizon control problems where the drift b and the diffusion matrix σ may also depend on μ^N. The approach in these papers uses the martingale formulation, and the method of weak convergence. Under suitable conditions, and for finite horizon control problems, it is established in [20,35] that a certain type of *averages* of approximate Nash equilibria are tight and their subsequential limits are a solution for the MFG problem. The results in Section 5 are quite similar to that of [18] (compare Theorem 5.2 with [18, Theorem 2]). Since the state space is not compact, we work under the assumption of geometric stability. Also we impose fairly general hypotheses on the running cost function, which are satisfied by a large class of functions (Assumption 5.3). For the analysis, we have borrowed several results from [3]. The representation formula of the ergodic value function is shown to be quite useful in proving Theorem 5.2. Let us also mention that the convergence results for Nash equilibria we present are somewhat stronger than those obtained in [20,35]. In fact, we show that the maximum distance between the invariant measures in the Nash equilibrium tuple tends to 0 as the number of players increases to infinity (see Theorem 5.2(b)).

Summarizing our contributions in this paper, we

- establish the existence of MFG solutions for a large class of mean field games;
- prove the convergence of the finite horizon MFG solution to the stationary one, under the assumption of geometric stability;
- study the existence of Nash equilibria for N-player games and prove that they converge to a MFG solution.

The organization of the paper is as follows: In Section 2 we introduce the model and the basic assumptions, and state the main result (Theorem 2.1) on the existence of MFG solutions. Various convex analytic results are in Section 3, where we also prove the main results. In Section 4 we study the long time behavior of the finite horizon problem. Finally, in Section 5 we show existence of Nash equilibria for the N-player games and study their convergence to MFG solutions.

1.1. Notation

The standard Euclidean norm in $\mathbb{R}^d$ is denoted by $|\cdot|$. The set of nonnegative real numbers is denoted by $\mathbb{R}_+$, $\mathbb{N}$ stands for the set of natural numbers, and $\mathbb{1}$ denotes the indicator function. The interior, closure, the boundary and the complement of a set $A \subset \mathbb{R}^d$ are denoted by A^o, $\overline{A}$, ∂A and A^c, respectively. The open ball of radius R around 0 is denoted by B_R. Given two real numbers a and b, the minimum (maximum) is denoted by $a \wedge b$ ($a \vee b$), respectively. By δ_x we denote the Dirac mass at x.

For a continuous function $g\colon \mathbb{R}^d \to [1,\infty)$ we let $\mathcal{O}(g)$ denote the space of Borel measurable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ satisfying $\operatorname{ess\,sup}_{x\in\mathbb{R}^d} \frac{|f(x)|}{g(x)} < \infty$, and by $\mathfrak{o}(g)$ those functions satisfying

$$\limsup_{R\to\infty} \operatorname*{ess\,sup}_{x\in B_R^c} \frac{|f(x)|}{g(x)} \;=\; 0\,.$$

We also let $\mathcal{C}_g(\mathbb{R}^d)$ denote the Banach space of continuous functions under the norm

$$\|f\|_g \;:=\; \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{g(x)}\,.$$

For two nonnegative functions f and g, we use the notation $f \sim g$ to indicate that $f \in \mathcal{O}(1+g)$ and $g \in \mathcal{O}(1+f)$.

We denote by $L^p_{\rm loc}(\mathbb{R}^d)$, $p \geq 1$, the set of real-valued functions that are locally p-integrable and by $\mathcal{W}^{k,p}_{\rm loc}(\mathbb{R}^d)$ the set of functions in $L^p_{\rm loc}(\mathbb{R}^d)$ whose i-th weak derivatives, $i = 1, \dots, k$, are in $L^p_{\rm loc}(\mathbb{R}^d)$. The set of all bounded continuous functions is denoted by $\mathcal{C}_b(\mathbb{R}^d)$. By $\mathcal{C}^{k,\alpha}_{\rm loc}(\mathbb{R}^d)$ we denote the set of functions that are k-times continuously differentiable and whose k-th derivatives are locally Hölder continuous with exponent α. We define $\mathcal{C}^k_b(\mathbb{R}^d)$, $k \geq 0$, as the set of functions whose i-th derivatives, $i = 1, \dots, k$, are continuous and bounded in $\mathbb{R}^d$ and denote by $\mathcal{C}^k_c(\mathbb{R}^d)$ the subset of $\mathcal{C}^k_b(\mathbb{R}^d)$ with compact support.

Given any Polish space $\mathcal{X}$, we denote by $\mathcal{B}(\mathcal{X})$ its Borel σ-field, by $\mathcal{P}(\mathcal{X})$ the set of probability measures on $\mathcal{B}(\mathcal{X})$, and by $\mathcal{M}(\mathcal{X})$ the set of all bounded signed measures on $\mathcal{B}(\mathcal{X})$. For $\nu \in \mathcal{P}(\mathcal{X})$ and a Borel measurable map $f\colon \mathcal{X} \to \mathbb{R}$, we often use the abbreviated notation $\nu(f) := \int_{\mathcal{X}} f \,\mathrm{d}\nu$. The space of all continuous maps from $[0,\infty)$ to $\mathcal{X}$ is denoted by $\mathcal{C}([0,\infty),\mathcal{X})$. The law of a random variable X is denoted by $\mathcal{L}(X)$. For presentation purposes the time variable appears as a subscript for the diffusion process X. Also $\kappa_1, \kappa_2, \dots$ and $C_1, C_2, \dots$ are used as generic constants whose values might vary from place to place.

2. Existence of solutions to MFG

2.1. Controlled diffusions

The dynamics are modeled by a controlled diffusion process $X = \{X_t,\ t \geq 0\}$ taking values in the d-dimensional Euclidean space $\mathbb{R}^d$, and governed by the Itô stochastic differential equation

$$\mathrm{d}X_t \ = \ b(X_t, U_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\,. \tag{2.1}$$

All random processes in (2.1) live in a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. The process W is a d-dimensional standard Wiener process independent of the initial condition X_0. The control process U takes values in a compact metric space $(\mathbb{U}, d_{\mathbb{U}})$, and $U_t(\omega)$ is jointly measurable in $(t,\omega) \in [0,\infty) \times \Omega$. Moreover, it is *non-anticipative*: for $s < t$, $W_t - W_s$ is independent of

$$\mathfrak{F}_s \ := \ \text{the completion of } \sigma\{X_0, U_r, W_r,\ r \leq s\} \text{ relative to } (\mathfrak{F}, \mathbb{P})\,.$$

Such a process U is called an *admissible control*, and we let $\mathfrak{U}$ denote the set of all admissible controls.

We impose the following standard assumptions on the drift b and the diffusion matrix σ to guarantee existence and uniqueness of solutions to (2.1).

(A1) *Local Lipschitz continuity:* The functions

$$b \ = \ \bigl[b^1, \dots, b^d\bigr]^{\mathsf{T}}\colon \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}^d \quad \text{and} \quad \sigma \ = \ \bigl[\sigma^{ij}\bigr]\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$$

are locally Lipschitz in x with a Lipschitz constant $C_R > 0$ depending on $R > 0$. In other words, for all $x, y \in B_R$ and $u \in \mathbb{U}$,

$$|b(x,u) - b(y,u)| + \|\sigma(x) - \sigma(y)\| \ \leq \ C_R\,|x-y| \qquad \forall\, x, y \in B_R\,.$$

We also assume that b is continuous in (x,u).

(A2) *Affine growth condition:* b and σ satisfy a global growth condition of the form

$$|b(x,u)|^2 + \|\sigma(x)\|^2 \ \leq \ C_1\bigl(1 + |x|^2\bigr) \qquad \forall\,(x,u) \in \mathbb{R}^d \times \mathbb{U}\,,$$

where $\|\sigma\|^2 \ := \ \operatorname{trace}\bigl(\sigma\sigma^{\mathsf{T}}\bigr)$.

(A3) *Local nondegeneracy:* For each $R > 0$, it holds that

$$\sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \;\geq\; C_R^{-1}|\xi|^2 \qquad \forall\, x \in B_R\,,$$

for all $\xi = (\xi_1, \dots, \xi_d)^{\mathsf{T}} \in \mathbb{R}^d$, where $a := \frac{1}{2}\sigma\sigma^{\mathsf{T}}$.

In integral form, (2.1) is written as

$$X_t \;=\; X_0 + \int_0^t b(X_s, U_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s\,. \tag{2.2}$$

The third term on the right hand side of (2.2) is an Itô stochastic integral. We say that a process $X = \{X_t(\omega)\}$ is a solution of (2.1), if it is $\mathfrak{F}_t$-adapted, continuous in t, defined for all $\omega \in \Omega$ and $t \in [0, \infty)$, and satisfies (2.2) for all $t \in [0, \infty)$ a.s. It is well known that under (A1)–(A3), for any admissible control there exists a unique solution of (2.1) [3, Theorem 2.2.4].

We define the family of operators $L^u \colon \mathcal{C}^2(\mathbb{R}^d) \to \mathcal{C}(\mathbb{R}^d)$, where $u \in \mathbb{U}$ plays the role of a parameter, by

$$L^u f(x) \;:=\; a^{ij}(x)\,\partial_{ij} f(x) + b(x,u)\cdot\nabla f(x)\,, \quad (x,u) \in \mathbb{R}^d \times \mathbb{U}\,. \tag{2.3}$$

We refer to L^u as the *controlled extended generator* of the diffusion. In (2.3) and elsewhere in this paper we have adopted the notation $\partial_t := \frac{\partial}{\partial t}$, $\partial_i := \frac{\partial}{\partial x_i}$, and $\partial_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}$. We also use the standard summation rule that repeated subscripts and superscripts are summed from 1 through d. In other words, the right hand side of (2.3) stands for

$$\sum_{i,j=1}^{d} a^{ij}(x)\,\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b^i(x,u)\,\frac{\partial f}{\partial x_i}(x)\,.$$

Of fundamental importance in the study of functionals of X is Itô's formula. For $f \in \mathcal{C}^2(\mathbb{R}^d)$ and with L^u as defined in (2.3), it holds that

$$f(X_t) \;=\; f(X_0) + \int_0^t L^{U_s} f(X_s)\,\mathrm{d}s + M_t\,, \quad \text{a.s.},$$

where

$$M_t \;:=\; \int_0^t \big\langle \nabla f(X_s), \sigma(X_s)\,\mathrm{d}W_s \big\rangle$$

is a local martingale.

Recall that a control is called *Markov* if $U_t = v(t, X_t)$ for a measurable map $v \colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{U}$, and it is called *stationary Markov* if v does not depend on t, i.e., $v \colon \mathbb{R}^d \to \mathbb{U}$. Correspondingly (2.1) is said to have a *strong solution* if given a Wiener process $(W_t, \mathfrak{F}_t)$ on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, there exists a process X on $(\Omega, \mathfrak{F}, \mathbb{P})$, with X_0 as specified by the initial condition, which is continuous, $\mathfrak{F}_t$-adapted, and satisfies (2.2) for all t a.s. A strong solution is called *unique*, if any two such solutions X and X' agree $\mathbb{P}$-a.s., when viewed as elements of $\mathcal{C}\big([0,\infty), \mathbb{R}^d\big)$. It is well known that under Assumptions (A1)–(A3), for any Markov control v, (2.1) has a unique strong solution [29].

Let $\mathfrak{U}_{\rm SM}$ denote the set of stationary Markov controls. Under $v \in \mathfrak{U}_{\rm SM}$, the process X is strong Markov, and we denote its transition function by $P_t^v(x,\cdot)$. It also follows from the work of [9,43] that under $v \in \mathfrak{U}_{\rm SM}$, the transition probabilities of X have densities which are locally Hölder continuous. Thus L^v defined by

$$L^v f(x) \;:=\; a^{ij}(x)\,\partial_{ij} f(x) + b^i\big(x, v(x)\big)\,\partial_i f(x)\,, \quad v \in \mathfrak{U}_{\rm SM}\,,$$

for $f \in \mathcal{C}^2(\mathbb{R}^d)$, is the generator of a strongly-continuous semigroup on $\mathcal{C}_b(\mathbb{R}^d)$, which is strong Feller. We let $\mathbb{P}_x^v$ denote the probability measure and $\mathbb{E}_x^v$ the expectation operator on the canonical space of the process under the control $v \in \mathfrak{U}_{\rm SM}$, conditioned on the process X starting from $x \in \mathbb{R}^d$ at $t = 0$. The expectation operator $\mathbb{E}_x^U$ is of course also well defined for $U \in \mathfrak{U}$.

Recall that a control $v \in \mathfrak{U}_{\rm SM}$ is called *stable* if the associated diffusion is positive recurrent. We denote the set of such controls by $\mathfrak{U}_{\rm SSM}$, and let μ_v denote the unique invariant probability measure on $\mathbb{R}^d$ for the diffusion under the control $v \in \mathfrak{U}_{\rm SSM}$. It is well known that $v \in \mathfrak{U}_{\rm SSM}$ if and only if there exists an inf-compact function $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$, a bounded domain $D \subset \mathbb{R}^d$, and a constant $\varepsilon > 0$ satisfying

$$L^v \mathcal{V}(x) \;\le\; -\varepsilon \qquad \forall\, x \in D^c\,.$$

We denote by $\uptau(A)$ the *first exit time* of a process $\{X_t\,,\ t \in \mathbb{R}_+\}$ from a set $A \subset \mathbb{R}^d$, defined by

$$\uptau(A) \;:=\; \inf\{t > 0 :\ X_t \notin A\}\,.$$

The open ball of radius R in $\mathbb{R}^d$, centered at the origin, is denoted by B_R, and we let $\uptau_R \;:=\; \uptau(B_R)$, and $\breve{\uptau}_R := \uptau(B_R^c)$.

2.2. Topologies on $\mathcal{P}(\mathbb{R}^d)$

We endow the space $\mathcal{P}(\mathbb{R}^d)$ with the Prokhorov metric $d_{\rm P}$ that renders $\mathcal{P}(\mathbb{R}^d)$ the topology of weak convergence. As is well known this is defined by

$$d_{\rm P}(\mu_1,\mu_2) \;:=\; \inf\big\{\varepsilon > 0 :\ \mu_1(F) \le \mu_2(F^\varepsilon) + \varepsilon \text{ for all Borel } F \subset \mathbb{R}^d\big\}\,. \tag{2.4}$$

It is well known that $(\mathcal{P}(\mathbb{R}^d), d_{\rm P})$ is a Polish space and $d_{\rm P}(\mu_n,\mu) \to 0$ as $n \to \infty$ if and only if, for every $f \in \mathcal{C}_b(\mathbb{R}^d)$, we have $\mu_n(f) \to \mu(f)$ as $n \to \infty$. By $\mathcal{P}_p(\mathbb{R}^d)$, $p \ge 1$, we denote the subset of $\mathcal{P}(\mathbb{R}^d)$ containing all probability measures μ with the property that $\int_{\mathbb{R}^d} |x|^p \mu(\mathrm{d}x) < \infty$. The Wasserstein metric on $\mathcal{P}_p(\mathbb{R}^d)$ is defined as follows:

$$\mathcal{W}_p(\mu_1,\mu_2) \;:=\; \inf\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \nu(\mathrm{d}x,\mathrm{d}y)\right)^{1/p} :\ \nu \in \mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d) \text{ has marginals } \mu_1,\,\mu_2\right\}. \tag{2.5}$$

It is well known that $(\mathcal{P}_p(\mathbb{R}^d), \mathcal{W}_p)$, $p \ge 1$, is a Polish space. The topology generated by $\mathcal{W}_p$ on $\mathcal{P}_p(\mathbb{R}^d)$ is finer than the one induced by $d_{\rm P}$. In fact, we have the following assertion [45, Theorem 7.12].

Proposition 2.1. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures in $\mathcal{P}_p(\mathbb{R}^d)$, and let $\mu \in \mathcal{P}(\mathbb{R}^d)$. Then, the following statements are equivalent:*

1. $\mathcal{W}_p(\mu_n,\mu) \to 0$, *as* $n \to \infty$.
2. $d_{\rm P}(\mu_n,\mu) \to 0$ *as* $n \to \infty$, *and*

$$\int_{\mathbb{R}^d} |x|^p \mu_n(\mathrm{d}x) \;\xrightarrow[n\to\infty]{}\; \int_{\mathbb{R}^d} |x|^p \mu(\mathrm{d}x)\,.$$

3. $d_{\mathrm{P}}(\mu_n,\mu)\to 0$ *as* $n\to\infty$, *and* $\{\mu_n\}$ *satisfies the following condition:*

$$\lim_{R\to\infty}\ \limsup_{n\to\infty}\ \int_{B_R^c(0)} |x|^p\,\mathrm{d}\mu_n \;=\; 0\,.$$

Therefore, a set $\mathcal{K}$ which is compact in $(\mathcal{P}_p(\mathbb{R}^d),\mathcal{W}_p)$ is also compact in $(\mathcal{P}_p(\mathbb{R}^d),d_{\mathrm{P}})$. In the rest of the paper $\mathcal{P}_p(\mathbb{R}^d)$ and $\mathcal{P}(\mathbb{R}^d)$ are always meant to be metric spaces endowed with the metrics $\mathcal{W}_p$ and d_{P}, respectively, unless mentioned otherwise.

2.3. The ergodic control problem

In this paper we consider dynamics as in (2.1) and associated running cost functions belonging to one of the three classes described in Assumption 2.1 below. We use the notation $r_\mu(x,u):=r(x,u,\mu)$. Also we write $r_\mu\in\mathfrak{o}(h)$ for $h\colon\mathbb{R}^d\times\mathbb{U}\to\mathbb{R}_+$, provided that

$$\limsup_{|x|\to\infty}\ \sup_{u\in\mathbb{U}}\ \frac{|r_\mu(x,u)|}{1+h(x,u)}\;=\;0\,.$$

Assumption 2.1. One of the following conditions holds:

(C1) The running cost $r\colon\mathbb{R}^d\times\mathbb{U}\times\mathcal{P}(\mathbb{R}^d)\to\mathbb{R}_+$ is continuous, and for each $\mu\in\mathcal{P}(\mathbb{R}^d)$, $r_\mu(\cdot\,,\cdot)$ is locally Lipschitz in its first argument uniformly with respect to the second. Moreover, for any compact subset $\mathcal{K}$ of $\mathcal{P}(\mathbb{R}^d)$ there exists $\theta>0$ such that

$$\liminf_{|x|\to\infty}\ \inf_{u\in\mathbb{U}}\ \frac{r(x,u,\mu)}{r(x,u,\mu')}\;>\;\theta\qquad\forall\,\mu,\mu'\in\mathcal{K}\,, \tag{2.6}$$

and

$$\inf_{(u,\mu)\,\in\,\mathbb{U}\times\mathcal{K}} r(x,u,\mu)\;\xrightarrow[|x|\to\infty]{}\;\infty\,. \tag{2.7}$$

(C2) The running cost takes the form $r_\mu(x,u)=\mathring{r}(x,u)+F(x,\mu)$, where $F\colon\mathbb{R}^d\times\mathcal{P}_p(\mathbb{R}^d)\to\mathbb{R}_+$ is a continuous function, satisfying

$$F(x,\mu)\;\le\;\kappa_0\left(1+|x|^p+\int_{\mathbb{R}^d}|x|^p\,\mu(\mathrm{d}x)\right)\qquad\forall\,(x,\mu)\in\mathbb{R}^d\times\mathcal{P}_p(\mathbb{R}^d)\,,$$

for some constant κ_0 and $p\ge 1$. Also, $\mathring{r}\colon\mathbb{R}^d\times\mathbb{U}\to\mathbb{R}_+$ is continuous and locally Lipschitz in x uniformly in $u\in\mathbb{U}$, and satisfies

$$\min_{u\in\mathbb{U}}\ \frac{\mathring{r}(x,u)}{1+|x|^p}\;\xrightarrow[|x|\to\infty]{}\;\infty\,.$$

(C3) The running cost $r\colon\mathbb{R}^d\times\mathbb{U}\times\mathcal{P}(\mathbb{R}^d)\to\mathbb{R}_+$ is continuous, and $x\mapsto r(x,u,\mu)$ is locally Lipschitz uniformly in $u\in\mathbb{U}$ and μ in compact subsets of $\mathcal{P}(\mathbb{R}^d)$. Also

(C3a) There exist inf-compact functions $\mathcal{V}\in\mathcal{C}^2(\mathbb{R}^d)$ and $h\in\mathcal{C}(\mathbb{R}^d\times\mathbb{U})$ such that for some positive constants c_1 and c_2 we have

$$L^u\mathcal{V}(x)\;\le\;c_1-c_2h(x,u)\qquad\forall\,(x,u)\in\mathbb{R}^d\times\mathbb{U}\,, \tag{2.8}$$

(C3b) For any compact $\mathcal{K} \subset \mathcal{P}(\mathbb{R}^d)$ it holds that

$$\sup_{\mu\in\mathcal{K}} r_\mu \in \mathfrak{o}(h)\,.$$

A typical example of F in (C2) is $F(x,\mu) = \int |x-y|^p\,\mu(\mathrm{d}y)$. Also $r(x,u,\mu) = r_1(x) + r_2(x,u,\mu)$, with $r_2 \in \mathcal{C}_b\big(\mathbb{R}^d \times \mathbb{U} \times \mathcal{P}(\mathbb{R}^d)\big)$, $r_1 \in \mathcal{C}^{0,1}_{\mathrm{loc}}(\mathbb{R}^d)$, and $\lim_{|x|\to\infty} r_1(x) = +\infty$ is an example of running cost satisfying (C1).

The running costs in (C1) and (C2) satisfy the condition of *near monotonicity* [3], while (2.8) implies that the controlled diffusion is uniformly stable. In [18,40] cost functions satisfying (C2) on a compact state space are considered. But in the current scenario the state space is $\mathbb{R}^d$ which is not compact. The cost functions in (C3) are allowed to take more general forms. Since $\mathcal{V}$ and h are bounded from below (being inf-compact), without loss of generality we assume that $\mathcal{V} \ge 1$, and $h \ge 0$.

In general, $\mathbb{U}$ may not be a convex set. It is therefore often useful to enlarge the control set to $\mathcal{P}(\mathbb{U})$. To do so, for $v \in \mathcal{P}(\mathbb{U})$ we replace the drift and the running cost with

$$\bar b(x,v) \;:=\; \int_{\mathbb{U}} b(x,u)\,v(\mathrm{d}u)\,, \quad \text{and} \quad \bar r(x,v,\mu) \;:=\; \int_{\mathbb{U}} r(x,u,\mu)\,v(\mathrm{d}u)\,. \tag{2.9}$$

It is easy to see that $\bar b$ satisfies (A1)–(A2), while and running cost $\bar r$ inherits the properties in Assumption 2.1 from r. In what follows we assume that all the controls take values in $\mathcal{P}(\mathbb{U})$. These controls are generally referred to as *relaxed* controls. We endow the set of relaxed stationary Markov controls with the following topology: $v_n \to v$ in $\mathfrak{U}_{\mathrm{SM}}$ if and only if

$$\int_{\mathbb{R}^d} f(x) \int_{\mathbb{U}} g(x,u) v_n(\mathrm{d}u\,|\,x)\,\mathrm{d}x \;\xrightarrow[n\to\infty]{}\; \int_{\mathbb{R}^d} f(x) \int_{\mathbb{U}} g(x,u) v(\mathrm{d}u\,|\,x)\,\mathrm{d}x$$

for all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $g \in \mathcal{C}_b(\mathbb{R}^d \times \mathbb{U})$. Then $\mathfrak{U}_{\mathrm{SM}}$ is a compact metric space under this topology [3, Section 2.4]. We refer to this topology as the *topology of Markov controls*. A control is said to be *precise/strict* if it takes values in $\mathbb{U}$. It is easy to see that any precise control U_t can also be understood as a relaxed control by $U_t(\mathrm{d}u) = \delta_{U_t}$. Abusing the notation we denote the drift and running cost by b and r, respectively, and the action of a relaxed control on them is understood as in (2.9).

Now we introduce the control problem. Let $\eta \in \mathcal{C}\big([0,\infty), \mathcal{P}(\mathbb{R}^d)\big)$. We define the ergodic cost as follows:

$$J_x(U,\eta) \;:=\; \limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}^U_x\left[\int_0^T r(X_t,U_t,\eta_t)\,\mathrm{d}t\right], \quad U \in \mathfrak{U}\,, \quad x \in \mathbb{R}^d\,. \tag{2.10}$$

Let

$$\varrho_\eta(x) \;:=\; \inf_{U\in\mathfrak{U}} J_x(U,\eta)\,.$$

Definition 2.1. $\eta \in \mathcal{C}([0,\infty), \mathcal{P}(\mathbb{R}^d))$ is said to be a Mean Field Game (MFG) solution starting at $x \in \mathbb{R}^d$ if there exists an admissible control v such that

$$\begin{aligned} \mathrm{d}X_t \;&=\; b(X_t,v_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\,,\\ \text{with} \quad \mathcal{L}(X_t) \;&=\; \eta_t\,, \quad X_0 = x\,, \end{aligned}$$

and $J_x(U,\eta) \ge J_x(v,\eta)$ for all admissible U. We say the MFG solution is relaxed (strict) if the control v is a stationary Markov control taking values in $\mathcal{P}(\mathbb{U})$ ($\mathbb{U}$, respectively).

One of our main goals in this paper is to establish the existence of MFG solutions. First we review some basic facts about ergodic occupation measures and invariant probability measures for a controlled diffusion as in (2.1). The set of all ergodic occupation measures is defined as

$$\mathcal{G} \;:=\; \Big\{ \pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{U}) \;:\; \int_{\mathbb{R}^d \times \mathbb{U}} L^u f(x)\, \pi(\mathrm{d}x, \mathrm{d}u) = 0 \quad \forall\, f \in \mathcal{C}_c^2(\mathbb{R}^d) \Big\}. \tag{2.11}$$

By [3, Lemma 3.2.3] $\mathcal{G}$ is a closed and convex subset on $\mathcal{P}(\mathbb{R}^d \times \mathbb{U})$. Disintegrating an ergodic occupation measure π we write $\pi(\mathrm{d}x, \mathrm{d}u) = \mu_v(\mathrm{d}x) v(\mathrm{d}u \,|\, x)$ for some $\mu_v \in \mathcal{P}(\mathbb{R}^d)$ and some measurable kernel $v \colon \mathbb{R}^d \to \mathcal{P}(\mathbb{U})$. We use the notation $\pi = \mu_v \circledast v$ to denote this disintegration. It straightforward to verify that μ_v satisfies

$$\int_{\mathbb{R}^d} L^v f(x)\, \mu_v(\mathrm{d}x) \;=\; 0 \quad \text{for all } f \in \mathcal{C}_c^2(\mathbb{R}^d),$$

and is therefore an invariant probability measure for the diffusion controlled by v. It follows that $v \in \mathfrak{U}_{\mathrm{SSM}}$. Conversely, if $v \in \mathfrak{U}_{\mathrm{SSM}}$, then there exists a unique invariant probability measure for the diffusion under the control $v \in \mathfrak{U}_{\mathrm{SSM}}$, and $\pi_v := \mu_v \circledast v$ is an ergodic occupation measure.

Thus, the set of all invariant probability measures may be defined as

$$\mathcal{H} \;:=\; \big\{ \nu \in \mathcal{P}(\mathbb{R}^d) \;:\; \nu \circledast v \in \mathcal{G} \text{ for some } v \in \mathfrak{U}_{\mathrm{SM}} \big\}. \tag{2.12}$$

This is a convex subset of $\mathcal{P}(\mathbb{R}^d)$. We refer to π_v (μ_v) as the ergodic occupation measure (invariant probability measure) associated with $v \in \mathfrak{U}_{\mathrm{SSM}}$.

The sets $\mathcal{G}$ and $\mathcal{H}$ play a key role in the analysis of the ergodic control problem. In fact, we are going to exhibit MFG solutions associated with $v \in \mathfrak{U}_{\mathrm{SSM}}$ and $\pi \in \mathcal{G}$ that satisfy the following

$$\pi \;=\; \mu_v \circledast v, \qquad \min_{u \in \mathbb{U}} \big[L^u V_{\mu_v}(x) + r_{\mu_v}(x,u) \big] \;=\; \varrho_{\mu_v}, \tag{2.13}$$

for some function $V_{\mu_v} \in \mathcal{C}^2(\mathbb{R}^d)$. Existence results of solutions to (2.13) are in Section 3. Such existence results are generally shown using fixed point arguments [40]. Compactness of invariant measures plays an important role in this. When the state space is compact, then of course $\mathcal{H}$ is also compact. But this is not true in general for non-compact state spaces. We adopt the following notation. For any $G \subset \mathcal{G}$ we let $\mathcal{H}[G]$ denote the corresponding set of invariant measures, i.e.,

$$\mathcal{H}[G] \;:=\; \{ \mu \in \mathcal{H} :\; \mu \circledast v \in G \text{ for some } v \in \mathfrak{U}_{\mathrm{SSM}} \}.$$

Consider the following assumption.

Assumption 2.2. The following hold:

(i) There exist $\mu_0 \in \mathcal{H}$ and $\pi_0 \in \mathcal{G}$ such that $\pi_0(r_{\mu_0}) < \infty$.
(ii) For models satisfying Assumption 2.1(C1), there exists a nonempty compact set $\mathcal{K} \subset \mathcal{G}$ such that

$$\pi(r_\mu) \;>\; \tilde{\varrho}_\mu \qquad \forall\, \pi \in \mathcal{G} \cap \mathcal{K}^c,$$

and for all $\mu \in \mathcal{H}$ where $\tilde{\varrho}_\mu = \inf_{\pi \in \mathcal{G}} \pi(r_\mu)$.

Remark 2.1. Assumption 2.2(i) is rather standard in ergodic control—if it is violated, the problem is vacuous. Note that Assumption 2.2(i) always holds for the model in (C3) of Assumption 2.1. Also, Assumption 2.2(i) implies that for running costs satisfying (C1)–(C2) of Assumption 2.1 it holds that $\pi_0(r_\mu) < \infty$ for all $\mu \in \mathcal{H}$.

Define $h(p,x,u,\mu) := p \cdot b(x,u) + r(x,u,\mu)$, $p \in \mathbb{R}^d$. Our main result of this section is the following.

Theorem 2.1. *Let Assumptions 2.1 and 2.2 hold. Then, for any $x \in \mathbb{R}^d$, there exists a relaxed MFG solution starting at x in the sense of Definition 2.1. Moreover, if $\mathbb{U}$ is convex and $u \mapsto h(p,x,u,\mu)$ is strictly convex for all $x,p \in \mathbb{R}^d$, and $\mu \in \mathcal{P}(\mathbb{R}^d)$, then there exists a strict MFG solution.*

3. MFG solutions for HJB

In this section we investigate the existence of MFG solutions for the associated Hamilton–Jacobi–Bellman (HJB) equation given by (2.13).

Recall the notation $r_\mu(x,u) = r(x,u,\mu)$. Consider the ergodic control problem

$$\varrho^*_\mu := \inf_{U \in \mathfrak{U}} \limsup_{T\to\infty} \frac{1}{T}\, \mathbb{E}^U_x \left[\int_0^T r_\mu(X_t,U_t)\,\mathrm{d}t \right]$$

for fixed $\mu \in \mathcal{P}(\mathbb{R}^d)$. Also recall the abbreviated notation $\pi(r) = \int_{\mathbb{R}^d\times\mathbb{U}} r\,\mathrm{d}\pi$. We need the following definition:

Definition 3.1. Let $f\colon \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}_+$. We say that $\bar\pi \in \mathcal{G}$ is *optimal relative to f* (for the ergodic cost criterion) if $\bar\pi(f) = \inf_{\pi\in\mathcal{G}} \pi(f)$. For $\mu \in \mathcal{H}$, we let $\mathcal{A}(\mu) \subset \mathcal{G}$ denote the set of optimal ergodic occupation measures relative to r_μ, and $\mathcal{A}^*(\mu) \subset \mathcal{H}$ denote the corresponding set of invariant probability measures. We also let $\tilde\varrho_\mu := \inf_{\pi\in\mathcal{G}} \pi(r_\mu)$.

There are two general models for which there exists an optimal ergodic occupation relative to r_μ for $\mu \in \mathcal{P}(\mathbb{R}^d)$, and optimality can be characterized by the HJB equation:

(H1) The running cost r_μ satisfies $\liminf_{|x|\to\infty} \inf_{u\in\mathbb{U}} r_\mu(x,u) > \tilde\varrho_\mu$, and $\tilde\varrho_\mu < \infty$.
(H2) The set $\mathcal{H}$ is compact, and r_μ is uniformly integrable with respect to $\mathcal{H}$.

For models in (H1)–(H2) we assume that $r\colon \mathbb{R}^d \times \mathbb{U} \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}_+$ is continuous. Hypothesis (H2) is equivalent to (C3a) of Assumption 2.1, with h satisfying $r_\mu \in \mathfrak{o}(h)$ by [3, Theorem 3.7.2].

We quote the following result which is contained in Theorems 3.6.10 and Theorem 3.7.12 of [3].

Theorem 3.1. *If (H1) holds, then there exists a unique $V_\mu \in \mathcal{C}^2(\mathbb{R}^d)$ which is bounded below in $\mathbb{R}^d$ and satisfies*

$$\min_{u\in\mathbb{U}} \left[L^u V_\mu(x) + r_\mu(x,u) \right] = \tilde\varrho_\mu\,, \quad V_\mu(0) = 0\,. \tag{3.1}$$

Under (H2), there exists a unique $V_\mu \in \mathcal{C}^2(\mathbb{R}^d) \cap \mathfrak{o}(\mathcal{V})$ satisfying (3.1) (see [3, Theorem 3.7.12]). In either case, $\tilde\varrho_\mu = \varrho^*_\mu$, and $v \in \mathfrak{U}_{\mathrm{SM}}$ is optimal for the ergodic control problem if and only if it satisfies

$$\min_{u\in\mathbb{U}} \left[L^u V_\mu(x) + r_\mu(x,u) \right] = L^v V_\mu(x) + r_\mu\big(x,v(x)\big) \quad \text{almost everywhere in } \mathbb{R}^d\,. \tag{3.2}$$

It follows by Theorem 3.1 that if (H1) or (H2) hold, then the set valued maps $\mathcal{A}$ and $\mathcal{A}^*$ can be characterized by

$$\mathcal{A}(\mu) \;=\; \bigl\{\pi \in \mathcal{G} \,:\, \pi = \pi_v\,,\ \text{where } v \in \mathfrak{U}_{\rm SSM} \text{ satisfy (3.2)}\bigr\}\,,$$

$$\mathcal{A}^*(\mu) \;=\; \{\nu \in \mathcal{H} \,:\, \nu \circledast v \in \mathcal{A}(\mu) \text{ for some } v \in \mathfrak{U}_{\rm SSM}\}\,.$$

This motivates the definition of the following notion of an MFG solution.

Definition 3.2. An invariant probability measure $\mu \in \mathcal{H}$ is said to be a MFG solution if $\mu \in \mathcal{A}^*(\mu)$ and there exists $V_\mu \in \mathcal{C}^2(\mathbb{R}^d)$ and $v \in \mathfrak{U}_{\rm SSM}$ such that

$$\min_{u\in\mathbb{U}} \bigl[L^u V_\mu(x) + r_\mu(x,u)\bigr] \;=\; L^v\, V_\mu + r_\mu\bigl(x, v(x)\bigr) \;=\; \tilde\varrho_\mu \quad \text{a.e. } x \in \mathbb{R}^d\,, \tag{3.3}$$

$$\int_{\mathbb{R}^d} L^v f(x)\, \mu(\mathrm{d}x) \;=\; 0 \qquad \forall\, f \in \mathcal{C}_c^2(\mathbb{R}^d)\,. \tag{3.4}$$

We retain the notion of a relaxed, or strict solution form Definition 2.1.

Equation (3.3) is the HJB equation corresponding to the ergodic control problem with running cost r_μ, while (3.4) asserts that $\mu = \mu_v$ is the invariant probability measure associated with the optimal Markov control v.

Remark 3.1. The reader should have noticed the relation between Definitions 2.1 and 3.2. It should observed that the initial distribution in Definition 2.1 is a Dirac mass at x. In fact, one may consider any *nice* distribution as initial condition in Definition 2.1. For example, if we fix the initial condition to be μ satisfying (3.4), then a solution $\mu \in \mathcal{P}$ according to Definition 3.2 gives rise to a solution according to Definition 2.1 due to stationarity.

We start with the following lemma:

Lemma 3.1. *Suppose that either (H1) or (H2) hold. Then the set $\mathcal{A}^*(\mu)$ is non-empty, convex and compact in $\mathcal{P}(\mathbb{R}^d)$ under the total variation norm topology.*

Proof. It is well known that $\mathcal{G}$ is convex (see [3, Lemma 3.2.3]). The convexity of $\mathcal{A}(\mu)$ follows by the linearity of the map $\pi \to \int_{\mathbb{R}^d\times\mathbb{U}} r_\mu(x,u)\pi(\mathrm{d}x,\mathrm{d}u)$. It then follows that $\mathcal{A}^*(\mu)$ is convex by the linearity of the projection.

Compactness of $\mathcal{A}^*(\mu)$ is obvious under (H2). So suppose (H1) holds. To prove compactness, let $\{\nu_n\}$ be a sequence in $\mathcal{A}^*(\mu)$, and $\{\pi_n\}$ be a corresponding sequence in $\mathcal{A}(\mu)$ i.e., $\pi_n = \nu_n \circledast v_n$ for some $v_n \in \mathfrak{U}_{\rm SSM}$ that satisfies (3.3). Let $(\mathbb{R}^d \times \mathbb{U}) \cup \{\infty\}$ be the one point compactification of $(\mathbb{R}^d \times \mathbb{U})$. If $\{\pi_n\}$ is not tight in $\mathcal{P}(\mathbb{R}^d \times \mathbb{U})$ then there exist a constant $\varepsilon > 0$, and a subsequence, also denoted by $\{\pi_n\}$, such that π_n converges to a probability measure of the form π' on $(\mathbb{R}^d \times \mathbb{U}) \cup \{\infty\}$ such that $\pi'(\infty) \ge \varepsilon$. It is evident from the near monotone condition in (H1) that $\pi'(\mathbb{R}^d \times \mathbb{U}) > 0$. It is also standard to show that $\frac{\pi'}{1-\pi'(\infty)}$ is an ergodic occupation measure on $\mathbb{R}^d \times \mathbb{U}$ which implies by optimality that

$$\frac{\pi'(r_\mu)}{1-\pi'(\infty)} \;\ge\; \tilde\varrho_\mu\,. \tag{3.5}$$

However, the lower semicontinuity of the map $\pi \mapsto \pi(r_\mu)$ and (H1) imply that $\pi'(r_\mu) < \bigl(1-\pi'(\infty)\bigr)\tilde\varrho_\mu$, which contradicts (3.5). Therefore $\{\pi_n\}$ must be tight in $\mathcal{P}(\mathbb{R}^d\times\mathbb{U})$ which implies that $\{\nu_n\}$ is tight. On the

other hand, under (H2), $\{\nu_n\}$ is trivially tight. Consider any subsequence such that $\nu_n \to \nu$ in $\mathcal{P}(\mathbb{R}^d)$ and $v_n \to v$ in $\mathfrak{U}_{\rm SM}$ under the topology of Markov controls, and let $\pi_n := \nu_n \circledast v_n$. It follows by [3, Lemma 3.2.6] that $\pi_n \to \pi := \nu \circledast v \in \mathcal{G}$ as $n \to \infty$. By the lower semicontinuity of the map $\pi \mapsto \pi(r_\mu)$ we have

$$\tilde{\varrho}_\mu \;=\; \liminf_{n\to\infty} \pi_n(r_\mu) \;\ge\; \pi(r_\mu) \;\ge\; \tilde{\varrho}_\mu\,,$$

which implies that $\mathcal{A}(\mu)$ is closed and therefore compact. It then follows that $\mathcal{A}^*(\mu)$ is compact in $\mathcal{P}(\mathbb{R}^d)$ under the total variation norm topology [3, Lemma 3.2.5]. □

Remark 3.2. It follows by Proposition 2.1, that for a running cost satisfying (C2), $\mathcal{A}^*(\mu)$ is compact in $\mathcal{P}_p(\mathbb{R}^d)$ for all $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ such that $\tilde{\varrho}_\mu < \infty$.

The following theorem asserts the existence of MFG solutions in the sense of Definition 3.2.

Theorem 3.2. *Suppose that Assumptions 2.1–2.2 hold. Then there exists a relaxed MFG solution in the sense of Definition 3.2. Moreover, if $\mathbb{U}$ is convex and $u \mapsto h(p,x,u,\mu)$ is strictly convex for all $x,p \in \mathbb{R}^d$, and $\mu \in \mathcal{P}(\mathbb{R}^d)$, then there exists a strict MFG solution.*

The rest of this section is devoted in proving the above result. The proof is an application of the Kakutani–Fan–Glicksberg fixed point theorem. A similar fixed point theorem has been applied in [36] to obtain MFG solutions for finite horizon control problems. Readers may consult [1, Chapter 17] for some basic properties of set-valued maps used in the proofs below.

We recall the definition of *hemicontinuity* [1, Section 17.3].

Definition 3.3. The map $\mu \mapsto \mathcal{A}^*(\mu)$ is said to be *upper hemicontinuous* if whenever $\mu_n \to \mu$ as $n \to \infty$, and $\nu_n \in \mathcal{A}^*(\mu_n)$ for all n, then the sequence $\{\nu_n\}$ has a limit point in $\mathcal{A}^*(\mu)$. The map $\mu \mapsto \mathcal{A}^*(\mu)$ is said to be *lower hemicontinuous* if whenever $\mu_n \to \mu$ as $n \to \infty$ and $\nu \in \mathcal{A}^*(\mu)$, then there exists a subsequence $\{\nu_{n_k}\}$ such that $\nu_{n_k} \in \mathcal{A}^*(\mu_{n_k})$ and $\nu_{n_k} \to \nu$ as $n_k \to \infty$. The map $\mu \mapsto \mathcal{A}^*(\mu)$ is said to be *continuous* if it is both upper and lower hemicontinuous.

We have the following general lemma:

Lemma 3.2. *Suppose that*

(a) $r\colon \mathbb{R}^d \times \mathbb{U} \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}_+$ *is continuous;*
(b) $\mu \mapsto \tilde{\varrho}_\mu$ *is upper semicontinuous;*
(c) *whenever $\mu_n \to \mu$, then $\mathcal{A}^*(\mu_n)$ is tight along some subsequence.*

Then $\mu \mapsto \mathcal{A}^*(\mu)$ is upper hemicontinuous, and $\mu \mapsto \tilde{\varrho}_\mu$ is continuous.

Proof. Since $\mathfrak{U}_{\rm SM}$ is compact under the topology of Markov controls, and (c) holds, it is enough to show that $\mu \mapsto \mathcal{A}(\mu)$ is upper hemicontinuous. So suppose $\mu_n \to \mu$ as $n \to \infty$ and $\pi_n \in \mathcal{A}(\mu_n)$. Let $\hat{\pi}$ be the limit of π_n along some subsequence also denoted as $\{\pi_n\}$. Then,

$$\begin{aligned}\liminf_{n\to\infty} \tilde{\varrho}_{\mu_n} \;&=\; \liminf_{n\to\infty} \pi_n(r_{\mu_n}) \\ &\ge\; \hat{\pi}(r_\mu) \\ &\ge\; \tilde{\varrho}_\mu\,. \end{aligned} \tag{3.6}$$

Since by hypothesis $\limsup_{n\to\infty} \tilde{\varrho}_{\mu_n} \le \tilde{\varrho}_\mu$, equality follows in (3.6). Since $\hat{\pi} \in \mathcal{G}$ and $\hat{\pi}(r_\mu) = \tilde{\varrho}_\mu$, we have that $\hat{\pi} \in \mathcal{A}(\mu)$, and upper hemicontinuity of $\mu \mapsto \mathcal{A}(\mu)$ follows. Moreover, it follows by (3.6) and (b) that $\mu \mapsto \tilde{\varrho}_\mu$ is necessarily continuous. □

Consider the model in (H1). Note that Assumption 2.1(C2) implies (2.6) and (2.7).

Lemma 3.3. *Suppose that Assumptions 2.1(C1) and 2.2(i) hold. Then $\mu \mapsto \mathcal{A}^*(\mu)$ is upper hemicontinuous on $\mathcal{H}$.*

Proof. It is evident that since $\tilde{\varrho}_\mu$ is finite for some $\mu \in \mathcal{H}$, then (2.6) implies that it is finite for all $\mu \in \mathcal{H}$. It then follows by (2.6) that $\cup_{\mu\in\mathcal{K}}\mathcal{A}^*(\mu)$ is tight, and it is routine to show that this together with (2.6) imply that $\mu \mapsto \tilde{\varrho}_\mu$ is continuous on $\mathcal{H}$. The result then follows by Lemma 3.2. □

Next we turn to the model in (H2). By [3, Theorem 3.7.2], Assumption 2.1(C3) is equivalent to

$$\sup_{\pi\in\mathcal{G}} \int_{B_R^c\times\mathbb{U}} \sup_{\mu\in\mathcal{H}} r_\mu(x,u)\,\pi(\mathrm{d}x,\mathrm{d}u) \xrightarrow[R\to\infty]{} 0\,.$$

We work under a weaker hypothesis.

Lemma 3.4. *Let (H2) hold and suppose that*

$$\sup_{\pi\in\mathcal{G}} \sup_{\mu\in\mathcal{H}} \int_{B_R^c\times\mathbb{U}} r_\mu(x,u)\,\pi(\mathrm{d}x,\mathrm{d}u) \xrightarrow[R\to\infty]{} 0\,.$$

Then $\mu \mapsto \mathcal{A}^*(\mu)$ is upper hemicontinuous on $\mathcal{H}$.

Proof. If $\mu_n \to \mu$ and $\bar{\pi}^\mu \in \mathcal{G}$ is optimal relative to r_μ, then by uniform integrability we have

$$\limsup_{n\to\infty} \tilde{\varrho}_{\mu_n} \;\le\; \limsup_{n\to\infty} \bar{\pi}^\mu(r_{\mu_n}) \;=\; \tilde{\varrho}_\mu\,,$$

which implies that $\mu \mapsto \tilde{\varrho}_\mu$ is continuous. The result then follows by Lemma 3.2. □

Lemmas 3.3 and 3.4 imply the following.

Corollary 3.1. *Let Assumptions 2.1–2.2 hold. Then $\mu \mapsto \mathcal{A}^*(\mu)$ is upper hemicontinuous on $\mathcal{H}$.*

In order to apply the fixed point theorem it remains to show that there exists some nonempty, convex and compact set $\mathcal{K} \subset \mathcal{H}$ such that $\mathcal{A}^*(\mu) \subset \mathcal{K}$ for all $\mu \in \mathcal{K}$. For the models satisfying (H2) we can select $\mathcal{K} \equiv \mathcal{H}$.

We have the following lemma:

Lemma 3.5. *Let Assumptions 2.1 and 2.2 hold. There exists a non-empty, convex and compact set $\mathcal{K} \subset \mathcal{P}(\mathbb{R}^d)$ such that for $\mu \in \mathcal{K}$ we have $\mathcal{A}^*(\mu) \subset \mathcal{K}$.*

Proof. Under Assumption 2.1(C3) we choose $\mathcal{K} = \mathcal{H}$.

Suppose that Assumption 2.1(C1) holds and let $\mathcal{K}$ be as in Assumption 2.2. Then $\mathcal{H}\big[\mathcal{K}\big]$ is compact under the total variation norm topology [3, Lemma 3.2.5]. Therefore it follows that $\overline{\operatorname{conv}}\,\mathcal{H}\big[\mathcal{K}\big]$ is also compact in

the total variation norm topology [1, Theorem 5.35]. Defining $\mathcal{K} = \overline{\operatorname{conv} \mathcal{H}[\mathcal{K}]}$ we see that $\mathcal{K}$ is a convex, compact subset of $\mathcal{P}(\mathbb{R}^d)$. Again by Assumption 2.2(ii) it easy to see that $\mathcal{K}$ has required property.

Next, consider Assumption 2.1(C2), and let $\pi_0 = \nu_0 \circledast v_0$ be as in Assumption 2.2(i). For $R > 0$, let $M_R, N_R \subset \mathcal{H}$ be defined by

$$M_R := \left\{ \mu \in \mathcal{H} : \kappa_0 \left(1 + \int_{\mathbb{R}^d} |x|^p \, \nu_0(\mathrm{d}x) + \int_{\mathbb{R}^d} |x|^p \, \mu(\mathrm{d}x) \right) \le R \right\},$$

$$N_R := \left\{ \mu \in \mathcal{H} : \int_{\mathbb{R}^d} \left(\min_{u \in \mathbb{U}} \mathring{r}(x,u) \right) \nu(\mathrm{d}x) \le \pi_0(\mathring{r}) + R \right\}.$$

By Assumption 2.1(C2), there exists $R_0 > 0$ such that $M_{R_0} \supset N_{R_0}$. It is evident that N_{R_0} is convex and compact in $\mathcal{P}_p(\mathbb{R}^d)$. Let $\mu \in N_{R_0} \subset M_{R_0}$. If $\pi = \nu \circledast v \in \mathcal{G}$, and $\nu \in N_{R_0}^c$, then

$$\begin{aligned} \pi(\mathring{r}) + \int_{\mathbb{R}^d} F(x,\mu) \, \nu(\mathrm{d}x) &> \pi_0(\mathring{r}) + R_0 \\ &\ge \pi_0(\mathring{r}) + \kappa_0 \left(1 + \int_{\mathbb{R}^d} |x|^p \, \nu_0(\mathrm{d}x) + \int_{\mathbb{R}^d} |x|^p \, \mu(\mathrm{d}x) \right) \\ &\ge \pi_0(\mathring{r}) + F(\nu_0, \mu), \end{aligned}$$

where the first inequality follows since $\nu \in N_{R_0}^c$, while the second follows from the hypothesis that $\mu \in N_{R_0} \subset M_{R_0}$. This of course implies that $\pi \notin \mathcal{A}(\mu)$. Therefore $\mathcal{A}^*(\mu) \in N_{R_0}$ for all $\mu \in N_{R_0}$. This completes the proof. □

Next we prove Theorem 3.2.

Proof of Theorem 3.2. Consider the map $\mu \in \mathcal{K} \mapsto \mathcal{A}^*(\mu) \in 2^{\mathcal{K}}$ where $\mathcal{K}$ is chosen from Lemma 3.5. We note that $\mathcal{K}$ is a non-empty, convex and compact subset of $\mathcal{M}(\mathbb{R}^d)$ which is a locally convex Hausdorff space under the weak topology. By Lemma 3.1 $\mathcal{A}^*(\mu)$ is non-empty, convex and compact. From Lemma 3.1, Corollary 3.1 and [1, Theorem 17.10] we conclude that the map $\mu \mapsto A^*(\mu)$ has closed graph. Therefore applying the Kakutani–Fan–Glicksberg fixed point theorem (see [1, Corollary 17.55]) there exists $\mu \in \mathcal{K}$ satisfying $\mu \in \mathcal{A}^*(\mu)$. This proves the existence of a relaxed MFG solution in the sense of Definition 3.2.

Suppose now that $\mathbb{U}$ is convex and $u \mapsto h(p,x,u,\mu)$ is strictly convex for all $x, p \in \mathbb{R}^d$, $\mu \in \mathcal{P}(\mathbb{R}^d)$. Then we can find a unique continuous, strict Markov control $v \colon \mathbb{R}^d \to \mathbb{U}$ such that

$$\min_{u \in \mathbb{U}} \left[L^u V_\mu(x) + r_\mu(x,u) \right] = L^v V_\mu(x) + r_\mu\big(x, v(x)\big) \qquad \forall \, x \in \mathbb{R}^d .$$

Note that in this case $\mathcal{A}^*(\mu)$ is a singleton, and $\mu \mapsto \mathcal{A}^*(\mu)$ is continuous in $\mathcal{P}(\mathbb{R}^d)$. Hence, an application of the Schauder–Tychonoff fixed point theorem suffices to assert existence of a strict MFG solution. □

Remark 3.3. It is possible to allow the drift b to depend on the measure μ. In case (C3) we can even consider a continuous $b \colon \mathbb{R}^d \times \mathbb{U} \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ such that $b(\cdot, u, \mu)$ is locally Lipschitz uniformly with respect to $u \in \mathbb{U}$ for all $\mu \in \mathcal{P}(\mathbb{R}^d)$. The argument in the proof of Theorem 3.2 holds in this case if (2.8) is satisfied. In particular, consider $b(x,u,\mu) \equiv b(x,u) + e(\mu)$ for some bounded continuous vector valued map $e \colon \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ where b satisfies the following: there exists $\mathcal{V}, h$ satisfying (2.8) when the operator L is defined using the drift $b(x,u)$ and $|\nabla V| \in \mathfrak{o}(h)$. Then it is easy to see that (2.8) holds for the original drift $b(x,u,\mu)$ with the same functions $\mathcal{V}$ and h.

Proof of Theorem 2.1. Consider a MFG solution μ in the sense of Definition 3.2 and take a relaxed/strict control $v \in \mathfrak{U}_{\rm SSM}$ associated to it in $\mathcal{A}(\mu)$. The existence of such a v is assured by Theorem 3.2. We know that there exists a unique strong Markov process corresponding to v satisfying (2.1) i.e.,

$$\mathrm{d}X_t \;=\; b(X_t, v(X_t))\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\,, \quad X_0 = x\,.$$

By definition, μ is the unique invariant probability measure of the process X under the control v. Let $\eta \in \mathcal{C}([0,\infty), \mathcal{P}(\mathbb{R}^d))$ be the path of transition probabilities of this process. It suffices to show that $J_x(v,\eta) = \tilde{\varrho}_\mu$ for all $x \in \mathbb{R}^d$, and that for any admissible control $U \in \mathfrak{U}$ we have

$$J_x(U,\eta) \;\geq\; \tilde{\varrho}_\mu \qquad \forall\, U \in \mathfrak{U}\,, \qquad \forall\, x \in \mathbb{R}^d\,, \tag{3.7}$$

where J is defined by (2.10). We divide the proof in three cases.

Case 1. Consider models satisfying (C3). Applying [32, Proposition 2.6] it follows that there exists a compact set $G \in \mathcal{P}(\mathbb{R}^d)$ such that $\eta_t \in G$ for all $t \geq 0$. Therefore

$$r_{\eta_t} \;\leq\; \sup_{\nu \in G} r_\nu \;\in\; \mathfrak{o}(h) \qquad \forall\, t \geq 0\,. \tag{3.8}$$

Also by (2.8), it follows that

$$\limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}^U_x\biggl[\int_0^T h(X_t,U_t)\,\mathrm{d}t\biggr] \leq \frac{1}{c_2}\bigl(c_1 - \mathcal{V}(x)\bigr) \qquad \forall\, U \in \mathfrak{U}\,.$$

This shows that

$$\limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}^U_x\biggl[\int_0^T \mathbb{1}_{B_R^c}(X_t) \sup_{\nu\in G} r_\nu(X_t,U_t)\,\mathrm{d}t\biggr] \xrightarrow[R\to\infty]{} 0 \qquad \forall\, U \in \mathfrak{U}\,. \tag{3.9}$$

Therefore since r is continuous and $\eta_t \to \mu$ in $\mathcal{P}(\mathbb{R}^d)$ as $t \to \infty$, we obtain by (3.8) and (3.9) that

$$\limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}^v_x\biggl[\int_0^T r(X_t,v(X_t),\eta_t)\,\mathrm{d}t\biggr] \;=\; \limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}^v_x\biggl[\int_0^T r(X_t,v(X_t),\mu)\,\mathrm{d}t\biggr] \;=\; \tilde{\varrho}_\mu\,. \tag{3.10}$$

It remains to show that

$$\limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}^U_x\biggl[\int_0^T r(X_t,U_t,\eta_t)\,\mathrm{d}t\biggr] \;\geq\; \tilde{\varrho}_\mu \qquad \forall\, U \in \mathfrak{U}\,. \tag{3.11}$$

For this purpose, we consider a smooth cut-off function ϕ_R that equals 1 on B_R and vanishes outside B_{R+1}. By ω we denote the local modulus of continuity of r, defined by

$$\omega(R,G,\varepsilon) \;:=\; \sup\bigl\{|r(x,u,\mu - r(\bar{x},\bar{u},\bar{\mu})| :\; |x-\bar{x}| + d_{\mathbb{U}}(u,\bar{u}) + d_{\mathrm{P}}(\mu,\bar{\mu}) \leq \varepsilon,$$
$$x,\bar{x} \in \bar{B}_R,\; \mu,\bar{\mu} \in G\,,\; u,\bar{u} \in \mathbb{U}\bigr\}.$$

Since the mean empirical measures of the process (X_t,U_t) are tight, applying Theorem 3.4.7 in [3], we obtain

$$\liminf_{T\to\infty} \frac{1}{T}\,\mathbb{E}_x^U\left[\int_0^T \phi_R(X_t)\, r(X_t,U_t,\mu)\,\mathrm{d}t\right] \;\geq\; \inf_{\pi\in\mathfrak{S}} \int_{\mathbb{R}^d\times\mathbb{U}} \phi_R(x) r_\mu(x,u)\,\pi(\mathrm{d}x,\mathrm{d}u)\,.$$

Thus, using the inequality

$$\begin{aligned} r(X_t,U_t,\eta_t) \;&\geq\; \phi_R(X_t)\, r(X_t,U_t,\eta_t) \\ &\geq\; \phi_R(X_t)\, r(X_t,U_t,\mu) - \phi_R(X_t)\,\omega\big(R+1,G,d_{\mathrm{P}}(\eta_t,\mu)\big)\,, \end{aligned}$$

and the fact that $d_{\mathrm{P}}(\eta_t,\mu)\to 0$ as $t\to\infty$, we obtain

$$\liminf_{T\to\infty} \frac{1}{T}\,\mathbb{E}_x^U\left[\int_0^T r(X_t,U_t,\eta_t)\,\mathrm{d}t\right] \;\geq\; \inf_{\pi\in\mathfrak{S}} \int_{\mathbb{R}^d\times\mathbb{U}} \phi_R(x) r_\mu(x,u)\,\pi(\mathrm{d}x,\mathrm{d}u)\,.$$

Letting $R\to\infty$, and using the fact $\tilde\varrho_\mu$ is the optimal value, we obtain (3.11).

Case 2. We consider running costs satisfying (C1). From the HJB equation we have

$$L^v V_\mu(x) + r_\mu(x,v(x)) \;=\; \tilde\varrho_\mu\,,$$

with $\mu\in\mathcal{A}^*(\mu)$. Hence, by [3, Lemma 3.7.2], there exist nonnegative, inf-compact functions $\mathcal{V}\in\mathcal{C}^2(\mathbb{R}^d)$ and $h\in\mathcal{C}(\mathbb{R}^d)$ such that $r_\mu(\cdot,v(\cdot))\in\mathfrak{o}(h)$, and satisfy

$$L^v\mathcal{V}(x) \;\leq\; c_0 - h(x) \tag{3.12}$$

for some constant c_0. It follows by (3.12) that $\eta_t\in G$ for all $t\geq 0$, where G is a compact subset of $\mathcal{P}(\mathbb{R}^d)$. By (2.6) we have

$$\sup_{\nu\in G} r_\nu(\cdot,v(\cdot)) \;\in\; \mathfrak{o}(h)\,.$$

Repeating the argument used in Case 1, we obtain (3.10). Also (3.11) follows as in Case 1 by using the near-monotone property of r_μ.

Case 3. We consider (C2). To show (3.7) in this case, it is enough to show that $F(x,\eta_t)\to F(x,\mu)$ as $t\to\infty$ uniformly in x on compact subsets of $\mathbb{R}^d$. Since F is continuous in $\mathbb{R}^d\times\mathcal{P}_p(\mathbb{R}^d)$, we need to show that $\mathcal{W}_p(\eta_t,\mu)\to 0$ as $t\to\infty$. Since $\int_{\mathbb{R}^d}\mathring{r}(x,v(x))\mu(\mathrm{d}x)\leq\tilde\varrho_\mu$, it follows that for any continuous ϕ with $\phi\in\mathcal{O}(\mathring{r})$ we have $\int\phi\,\mathrm{d}\mu<\infty$. Then by [32, Proposition 2.6] we have $\int\phi\,\mathrm{d}\eta_t\to\int\phi\,\mathrm{d}\mu$ as $t\to\infty$ for every initial condition x. Combining this fact with Proposition 2.1 we obtain that $\mathcal{W}_p(\eta_t,\mu)\to 0$ as $t\to\infty$. This shows (3.7). It is also easy to see that $\varrho_\eta=\tilde\varrho_\mu$. □

Remark 3.4. We note that for models satisfying (C3) we can strengthen the assumption on r depending on the growth rate of h. For example, if $h\sim|x|^p$ for $p\geq 1$, then one may a consider continuous r defined on $\mathbb{R}^d\times\mathbb{U}\times\mathcal{P}_p(\mathbb{R}^d)$ that is locally Lipschitz in first and third arguments uniformly in $u\in\mathbb{U}$, and with the property that $\sup_{\mu\in\mathcal{K}} r_\mu\in\mathfrak{o}(h)$ for any compact $\mathcal{K}\subset\mathcal{P}_p(\mathbb{R}^d)$. The results of Theorem 2.1 continue to hold in this case.

4. Long time behavior and the relative value iteration

In this section we study the long time behavior of the finite horizon mean field game equations. The problem is as follows. We are given a running cost function $r(x,u,\mu)$, a horizon $T>0$, a 'terminal cost function' $\varphi_0\in\mathcal{C}^2(\mathbb{R}^d)$, and an initial distribution $\eta\in\mathcal{P}(\mathbb{R}^d)$. For $U\in\mathfrak{U}$ and $\{\mu_t\in\mathcal{P}(\mathbb{R}^d)\,,\ t\in[0,T]\}$ we define

$$\mathcal{J}(U,\mu;\eta) \;:=\; \mathbb{E}^U_\eta\left[\int_0^T r(X_t,U_t,\mu_t)\,\mathrm{d}t + V_T(X_T)\right],$$

where X_t is governed by (2.1) with $\mathcal{L}(X_0)=\eta$. Let $\mathcal{L}^U_\eta(X_t)$ denote the law of the process X_t governed by (2.1) under a control U with $\mathcal{L}(X_0)=\eta$. Then $\{\mu^*_t\in\mathcal{P}(\mathbb{R}^d)\,,\ t\in[0,T]\}$ is called an MFG solution for the problem on $[0,T]$ if there exists an admissible control U^* such that $\mathcal{L}^{U^*}_\eta(X_t)=\mu^*_t$ for all $t\in[0,T]$

$$\mathcal{J}(U,\mu^*;\eta) \;\ge\; \mathcal{J}(U^*,\mu^*;\eta) \qquad \forall\, U\in\mathfrak{U}\,.$$

We assume that $r(x,u,\mu)$ has the separable form $\mathring{r}(x,u)+F(x,\mu)$, so that the Hamiltonian $H(x,p)$ is given by

$$H(x,p) \;=\; \min_u\,\{b(x,u)\cdot p+\mathring{r}(x,u)\}\,. \tag{4.1}$$

Denoting by $\chi(t,\cdot\,)$ the density of $\mu_{T-t}=\mathcal{L}(X_{T-t})$, the dynamic programming formulation amounts to solving

$$\begin{aligned} \partial_t V \;&=\; a^{ij}\partial_{ij}V+H(x,\nabla V)+F(x,\mu_{T-t})\,,\\ &\qquad V(0,x)=\varphi_0(x)\quad \forall\, x\in\mathbb{R}^d\,, \end{aligned} \tag{4.2a}$$

$$\begin{aligned} -\partial_t\chi \;&=\; \partial_i\big(a^{ij}\partial_j\chi+(\partial_j a^{ij})\chi\big)-\operatorname{div}\left(\tfrac{\partial H}{\partial p}(x,\nabla V)\chi\right),\\ &\qquad \chi(T,\cdot\,)\text{ is the density of }\eta\,. \end{aligned} \tag{4.2b}$$

Equation (4.2b) is the Kolmogorov equation for the density $\chi(t,\cdot\,)$, running in backward time. Therefore, if (V,χ) is a solution of (4.2a)–(4.2b), then $\mu^*_t(\mathrm{d}x)=\chi(T-t,x)\,\mathrm{d}x$ for $t\in[0,T]$ is a MFG solution in the sense of the above definition. It also follows by the dynamic programming principle that the solution $V_T(t,x)$, where the T in the subscript denotes the dependence of the solution on the horizon $[0,T]$, has the stochastic representation

$$V_T(t,x) \;=\; \inf_{U\in\mathfrak{U}}\,\mathbb{E}^U_x\left[\int_0^{T-t} r\big(X_s,U_s,\mu^*_{T-t+s}\big)\,\mathrm{d}t+h(X_{T-t})\right] \qquad \forall\,(t,x)\in[0,T]\times\mathbb{R}^d\,,$$

where the process X is governed by (2.1).

Lasry and Lions have examined thoroughly the case where $b(x,u)=-u$, σ is the identity matrix, $r(x,u)=1/2|u|^2$, and $F(x,\mu)$ takes the form $F(x,\chi(x))$, where χ is the density of μ. Moreover, they assume that $F(x,t)\in\mathbb{R}^d\times\mathbb{R}$ is $\mathcal{C}^1$, is strictly increasing in t, and is $\mathbb{Z}^d$-periodic in x. As a result the state space is a d-dimensional torus $\mathbb{T}$. Under the assumption that the density η is Hölder continuous and has finite second moments, they have shown the existence and uniqueness of a solution to (4.2a)–(4.2b) for this problem [40]. They also consider a general class of F that includes non-local interactions, and present a detailed analysis of the stationary ergodic problem, proving the existence and uniqueness of a stationary solution. The behavior over a long horizon for this model has been studied, with both local and non-local interactions in [12,13]. In the case of non-local interactions, they establish convergence in the average sense, i.e., $\lim_{T\to\infty}\frac{1}{T}V(\gamma T)=(1-\gamma)\bar\varrho$, for $\gamma\in(0,1)$, where $\bar\varrho$ is the value of the associated ergodic problem (see (4.8) below), and also convergence in $L^2(\mathbb{T})$ uniformly over compact intervals of time. Also, they show that the density χ_T converges to the density of the stationary solution in $L^2(\mathbb{T})$ uniformly over compact intervals of time. Under stronger assumptions on F they show that convergence is exponential in T.

For the problem on $\mathbb{R}^d$ we are dealing with, in order to avoid restrictive assumptions on F, we have to compensate for the non-compactness of the state space by imposing a uniform stability hypothesis on the dynamics in (2.1). We describe these assumptions in the next section.

4.1. Assumptions and basic properties

Existence and uniqueness of solutions to (4.2a)–(4.2b) under general vector fields requires strong regularity of the data. We refer the reader to [31,33,34]. We note here that the results in this paper can be extended to include a drift b, a diffusion matrix σ and running cost r that all depend on μ, albeit necessitating various assumptions on the smoothness of the data.

In this paper we are not interested in the regularity of the Fokker–Planck equation (4.2b). If a MFG solution μ_t is provided, then $F(x,\mu_t)$ is a given function of $(t,x)\in[0,T]\times\mathbb{R}^d$, and the Hamiltonian does not depend on μ_t. If $F(x,\mu_t)$ is Hölder in x and continuous in t, and φ_0 is smooth enough, then (4.2b) has almost classical solutions. Therefore we concentrate on a set of assumptions that guarantee the existence and uniqueness of a MFG solution, and at the same time maintain sufficient regularity for the solutions of (4.2b).

For $\eta\in\mathcal{P}(\mathbb{R}^d)$ we let $\mathcal{M}_\eta([0,T])$ denote the set of all trajectories $\{\mu_t=\mathcal{L}^U_\eta(X_t)\,,\ U\in\mathfrak{U}\,,t\in[0,T]\}$, and we define

$$\mathcal{P}(\eta)\ :=\ \left\{\mathcal{L}^U_\eta(X_t)\in\mathcal{P}(\mathbb{R}^d)\,:\,U\in\mathfrak{U}\,,\ t\ge 0\right\}.$$

Assumption 4.1. The following hold:

(i) Assumptions (A1)–(A3) on the data hold, and $\mathring{r}\colon\mathbb{R}^d\times\mathbb{U}\to\mathbb{R}_+$ is continuous and locally Lipschitz in x uniformly in $u\in\mathbb{U}$.

(ii) The function F is defined on $\mathbb{R}^d\times\tilde{\mathcal{P}}$, where $\tilde{\mathcal{P}}$ is some subset of $\mathcal{P}(\mathbb{R}^d)$ which contains $\mathcal{P}(\eta)$, and satisfies

$$\mathcal{F}(\mu,\mu')\ :=\ \int_{\mathbb{R}^d}\big(F(x,\mu)-F(x,\mu')\big)\big(\mu(\mathrm{d}x)-\mu'(\mathrm{d}x)\big)\ \ge\ 0\qquad\forall\mu,\mu'\in\tilde{\mathcal{P}}\,. \tag{4.3}$$

Moreover, $F(x,\mu)$ is locally Lipschitz in x uniformly on compact subsets of $\tilde{\mathcal{P}}$, and $\mu\mapsto F(\cdot\,,\mu)$ is a continuous map from $\tilde{\mathcal{P}}$ to $\mathcal{C}(\mathbb{R}^d)$ under the topology of uniform convergence on compact sets.

(iii) The terminal cost φ_0 is in $\mathcal{C}^2(\mathbb{R}^d)$ and the density of the initial distribution η is Hölder continuous, and has a finite second moment.

(iv) There exists a unique u^* that minimizes the Hamiltonian in (4.1).

Under Assumption 4.1, existence of an MFG solution is asserted in [36, Theorem 2.1]. In fact, this can be obtained applying a fixed point argument same as above. The (non-strict) monotonicity hypothesis (4.3) together with the fact that $\mathcal{A}^*(\mu)$ is a singleton implied by Assumption 4.1(iv), is enough to guarantee uniqueness of the MFG solution for the ergodic problem. The monotonicity hypothesis has become a standard assumption in the literature [13,40].

Recall the definition of the weighted Banach space $\mathcal{C}_{\mathcal{V}}(\mathbb{R}^d)$ from Section 1.1. The following assumption is a strengthening of the stability hypothesis in (C3).

Assumption 4.2. A number $p\ge 1$ is specified as a parameter. There exists a nonnegative, inf-compact $\mathcal{V}\in\mathcal{C}^2(\mathbb{R}^d)$, and positive constants c_0 and c_1 satisfying

$$L^u\mathcal{V}(x)\ \le\ c_0-c_1\mathcal{V}(x)\qquad\forall\,(x,u)\in\mathbb{R}^d\times\mathbb{U}\,. \tag{4.4}$$

Without loss of generality we assume $\mathcal{V} \geq 1$. Also

(i) It holds that

$$\frac{\|\sigma(x)\|^2 + |x|^p + \sup_{u\in\mathbb{U}} \mathring{r}(x,u)}{\mathcal{V}(x)} \xrightarrow[|x|\to\infty]{} 0\,.$$

(ii) For any compact $\mathcal{K} \subset \mathcal{P}_p(\mathbb{R}^d)$ and $R > 0$, there exists a constant $M_p(R) > 0$ such that such that

$$\bigl|F(x,\mu) - F(x',\mu)\bigr| \;\leq\; M_p(R)|x - x'| \qquad \forall\, x, x' \in B_R\,,\ \forall\, \mu \in \mathcal{K}\,.$$

(iii) The map $\mu \to F(\cdot\,,\mu)$ from $\mathcal{P}_p(\mathbb{R}^d)$ to $\mathcal{C}_{\mathcal{V}}(\mathbb{R}^d)$ is continuous.

It is well known (see [3,22]) that (4.4) implies that

$$\mathbb{E}_x^U\bigl[\mathcal{V}(X_t)\bigr] \;\leq\; \frac{c_0}{c_1} + \mathcal{V}(x)\mathrm{e}^{-c_1 t} \qquad \forall\, x \in \mathbb{R}^d\,,\ \forall\, U \in \mathfrak{U}\,. \tag{4.5}$$

It follows by (4.5) that all stationary Markov controls $\mathfrak{U}_{\mathrm{SM}}$ are stable and that

$$\int_{\mathbb{R}^d} \mathcal{V}(x)\,\mu_v(\mathrm{d}x) \;\leq\; \frac{c_0}{c_1}\,,$$

where, as usual, μ_v denotes the unique invariant probability measure of the diffusion controlled under v. Therefore, $\mu_v \in \mathcal{P}_p(\mathbb{R}^d)$ for all $v \in \mathfrak{U}_{\mathrm{SM}}$.

It also follows that for any $v \in \mathfrak{U}_{\mathrm{SM}}$ the controlled process under v is $\mathcal{V}$-geometrically ergodic (see [17,21]), or in other words, that there exist constants M_0 and $\gamma > 0$ such that, if $h\colon \mathbb{R}^d \to \mathbb{R}$ is Borel measurable and $h \in \mathcal{O}(\mathcal{V})$, then

$$\left|\mathbb{E}_x^v\bigl[h(X_t)\bigr] - \int_{\mathbb{R}^d} h(x)\,\mu_v(\mathrm{d}x)\right| \;\leq\; M_0\mathrm{e}^{-\gamma t}\bigl\|h\bigr\|_{\mathcal{V}}\bigl(1 + \mathcal{V}(x)\bigr) \tag{4.6}$$

for all $t \geq 0$ and $x \in \mathbb{R}^d$.

We have the following simple assertion.

Lemma 4.1. *Under Assumption 4.2, here exists a constant $\tilde{M}_p$ which depends only on $p \geq 1$ and $\mathcal{L}(X_0) \in \mathcal{P}_p(\mathbb{R}^d)$ such that*

$$\mathcal{W}_p\bigl(\mathcal{L}(X_t), \mathcal{L}(X_s)\bigr) \;\leq\; \tilde{M}_p\sqrt{|t-s|} \qquad \forall\, s,t \in \mathbb{R}_+\,,\ |s-t| < 1\,, \qquad \textit{under any } U \in \mathfrak{U}\,.$$

Proof. By the Burkholder–Davis–Gundy inequality, for some constant $\kappa_p > 0$, we obtain

$$\mathbb{E}\left[\sup_{s\leq r\leq t} |X_r - X_s|^p\right] \;\leq\; 2^{p-1}(t-s)^{p-1}\mathbb{E}\left[\int_s^t |b(X_r,U_r)|^p\,\mathrm{d}r\right] + 2^{p-1}\kappa_p\mathbb{E}\left[\left(\int_s^t \|\sigma(X_r)\|^2\,\mathrm{d}r\right)^{p/2}\right]. \tag{4.7}$$

The expectation in the second term on the right hand side of (4.7) is bounded by $\left(\int_s^t \mathbb{E}\left[\|\sigma(X_r)\|^2\right]\mathrm{d}r\right)^{p/2}$ when $p \in [1,2]$. On the other hand, if $p > 2$, using the Hölder inequality, we bound this term by

$$(t-s)^{\frac{p-2}{2}} \int_s^t \mathbb{E}\left[\|\sigma(X_r)\|^p\right] \mathrm{d}r\,.$$

Therefore, since $\sup_{u\in\mathbb{U}} |b(\cdot,u)|^p \in \mathcal{O}(\mathcal{V})$ and $\|\sigma(\cdot)\|^{2\vee p} \in \mathcal{O}(\mathcal{V})$ by (A2) and Assumption 4.2(i), the result follows by (4.5) and (4.7). □

Let

$$\mathcal{M}_0 \;:=\; \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) \,:\, \int_{\mathbb{R}^d} \mathcal{V}(x)\,\mu(\mathrm{d}x) \le \frac{c_0}{c_1} \right\}.$$

The set $\mathcal{M}_0$ is compact in $\mathcal{P}_p(\mathbb{R}^d)$ and $\mathcal{H} \subset \mathcal{M}_0$ by Assumption 4.2. We have not assumed that F is nonnegative. Nevertheless, Assumption 4.2 implies that $\inf_{\mu,\mu'\in\mathcal{H}} \int F(x,\mu)\,\mu'(\mathrm{d}x) < -\infty$, and therefore, the ergodic cost problem is well posed. Combining the preceding discussion with the results in Section 2, we have the following:

Theorem 4.1. *Let Assumptions 4.1–4.2 hold. Then there exists a unique MFG solution $\bar\mu \in \mathcal{H}$ to the ergodic control problem. Associated with that, we obtain a unique $\bar V \in \mathcal{C}^2(\mathbb{R}^d) \cap \mathcal{C}_{\mathcal{V}}(\mathbb{R}^d)$, satisfying $\bar V \in \mathfrak{o}(\mathcal{V})$ and $\bar V(0) = 0$, which solves*

$$a^{ij}(x)\partial_{ij}\bar V(x) + H(x,\nabla\bar V) + F(x,\bar\mu) \;=\; \bar\varrho \tag{4.8}$$

with $\bar\varrho = \varrho_{\bar\mu}$.

For the rest of this section we let $\bar v$ denote some Markov control associated with the stationary solution in (4.8), i.e., a measurable selector from the minimizer of the Hamiltonian $H(x,\nabla\bar V)$. By uniqueness of the solutions we have $\mu_{\bar v} = \bar\mu$.

4.2. The relative value iteration

Note that the Markov control associated with (4.2a) is computed 'backward' in time. We need the following definition:

Definition 4.1. Let $\hat v = \{\hat v_t\,,\ t\in[0,T]\}$ denote a measurable selector from the minimizer of the Hamiltonian in (4.2a). For each $T>0$ we define the (nonstationary) Markov control

$$\hat v^T \;:=\; \left\{\hat v^T_s \;=\; \hat v_{T-s}\,,\ s\in[0,T]\right\}.$$

We also let $\hat\eta^T_s$ denote the law of X_s, $s\in[0,T]$ under the control $\hat v^T$. As remarked earlier $\hat\eta^T_s(\cdot) = \mathcal{L}^{\hat v}_\eta(X_s)$ for $s\in[0,T]$, and thus, $\hat\eta^T_0$ agrees with the initial law η, which we also denote by $\hat\eta_0$.

We modify (4.2a) by normalizing it as follows:

$$\partial_t\varphi(t,x) \;=\; a^{ij}(x)\partial_{ij}\varphi(t,x) + H(x,\nabla\varphi(t,x)) + F(x,\hat\eta^T_{T-t}) - \bar\varrho\,, \quad \varphi(0,x) = \varphi_0(x)\,, \tag{4.9}$$

where $\varphi_0 \in \mathcal{C}^2(\mathbb{R}^d)\cap\mathfrak{o}(\mathcal{V})$ denotes the terminal cost. It is evident the solution φ depends also on the horizon $[0,T]$ and to distinguish among these solutions we adopt the notation $\varphi^T(t,x)$, or $\varphi^T_t(x)$.

For existence and uniqueness of solutions to (4.9) in cylinders we refer the reader to [37, Theorem 6.1, p. 452] and to p. 492 of the same reference for the Cauchy problem. See also [2,4] for the Cauchy problem

in (4.9) as well as (4.11) below. We need to mention though that Theorem 6.1 in [37] concerns solutions in Hölder spaces, and in order to satisfy the assumptions of this theorem $t \mapsto F(x, \hat\eta^T_{T-t})$ has to be Hölder continuous. However under our assumptions it is only continuous, which means that the time derivative of the solution $\varphi(t,x)$ is not necessarily Hölder continuous. In general then, (4.9) has to be solved in the parabolic Sobolev space $\mathcal{W}^{1,2,q}_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^d)$ (see [37, Section IV.9]). We don't require more regularity than that in this paper.

We are concerned here only with the solution φ^T which agrees with the stochastic representation

$$\begin{aligned}\varphi^T_t(x) &= \inf_U \mathbb{E}^U_x\left[\int_0^t r_{\hat\eta^T_{T-t+s}}(X_s,U_s)\,\mathrm{d}s + \varphi_0(X_t)\right] - \bar\varrho\, t \\ &= \mathbb{E}^{\hat v^T}_x\left[\int_0^t r_{\hat\eta^T_{T-t+s}}\big(X_s, \hat v^T_{T-t+s}(X_s)\big)\,\mathrm{d}s + \varphi_0(X_t)\right] - \bar\varrho\, t \qquad \forall\, t\in[0,T]\,, \end{aligned} \tag{4.10}$$

and in general, for any $[t_1,t_2]\in[0,T]$,

$$\varphi^T_{t_2}(x) = \mathbb{E}^{\hat v^T}_x\left[\int_0^{t_2-t_1} r_{\hat\eta^T_{T-t_2+s}}\big(X_s, \hat v^T_{T-t_2+s}(X_s)\big)\,\mathrm{d}s + \varphi^T_{t_1}(X_{t_2-t_1})\right] - \bar\varrho\,(t_2-t_1)\,.$$

We also consider the following variation of (4.9):

$$\partial_t\psi^T_t(x) = a^{ij}(x)\partial_{ij}\psi^T_t(x) + H(x,\nabla\psi^T_t(x)) + F(x,\hat\eta^T_{T-t}) - \psi^T_t(0)\,, \quad \psi^T_0(x)=\varphi_0(x)\,. \tag{4.11}$$

It is straightforward to show that $\hat v_t$ is also a measurable selector from the minimizer of the Hamiltonian in (4.11), and that φ^T and ψ^T are related by

$$\varphi^T_t(x) = \psi^T_t(x) - \bar\varrho\, t + \int_0^t \psi^T_s(0)\,\mathrm{d}s\,, \qquad (t,x)\in[0,T]\times\mathbb{R}^d\,.$$

We have in particular that

$$\varphi^T_t(x) - \varphi^T_t(0) = \psi^T_t(x) - \psi^T_t(0)\,, \qquad (t,x)\in[0,T]\times\mathbb{R}^d\,. \tag{4.12}$$

Conversely, if φ^T is a solution of (4.9), then one obtains a corresponding solution of (4.11) that takes the form [2, Lemma 4.4]:

$$\psi^T_t(x) = \varphi^T_t(x) - \int_0^t \mathrm{e}^{s-t}\,\varphi^T_s(0)\,\mathrm{d}s + \bar\varrho\,(1-\mathrm{e}^{-t})\,, \qquad (t,x)\in[0,T]\times\mathbb{R}^d\,. \tag{4.13}$$

We refer to (4.9), and (4.11) as the *value iteration* (VI), and *relative value iteration* (RVI) equations, respectively.

The following technique is rather standard. For $\eta\in\mathcal{P}(\mathbb{R}^d)$ and $v\in\mathfrak{U}_{\mathrm{SM}}$ we define

$$\bar F(\eta,\mu) := \int F(x,\mu)\,\eta(\mathrm{d}x)\,, \quad \text{and} \quad \bar{\mathring r}(\eta,v) := \int \mathring r\big(x,v(x)\big)\,\eta(\mathrm{d}x)$$

for $\eta \in \mathcal{P}(\mathbb{R}^d)$. We consider X in (2.1) under the following Markov controls: $\hat{v}_t$ which a measurable selector from the minimizer in (4.9), and the stationary control $\bar{v}$ which corresponds to (4.8). Applying (4.10) and integrating with respect to $\hat{\eta}_0^T$ and $\bar{\mu}$, respectively, we obtain

$$\hat{\eta}_0^T(\varphi_T^T) \;=\; \int_0^T \big(\bar{\mathring{r}}(\hat{\eta}_t^T, \hat{v}_t) + \bar{F}(\hat{\eta}_t^T, \hat{\eta}_t^T) - \bar{\varrho}\big)\,\mathrm{d}t + \hat{\eta}_T^T(\varphi_0)\,, \tag{4.14}$$

$$\bar{\mu}(\varphi_T^T) \;\le\; \int_0^T \big(\bar{\mathring{r}}(\bar{\mu}, \bar{v}) + \bar{F}(\bar{\mu}, \hat{\eta}_t^T) - \bar{\varrho}\big)\,\mathrm{d}t + \bar{\mu}(\varphi_0)\,. \tag{4.15}$$

Repeating this with terminal cost $\bar{V}$, and using (4.8), we obtain

$$\bar{\mu}(\bar{V}) \;=\; \int_0^T \big(\bar{\mathring{r}}(\bar{\mu}, \bar{v}) + \bar{F}(\bar{\mu}, \bar{\mu}) - \bar{\varrho}\big)\,\mathrm{d}t + \bar{\mu}(\bar{V})\,, \tag{4.16}$$

$$\hat{\eta}_0^T(\bar{V}) \;\le\; \int_0^T \big(\bar{\mathring{r}}(\hat{\eta}_t^T, \hat{v}_t) + \bar{F}(\hat{\eta}_t^T, \bar{\mu}) - \bar{\varrho}\big)\,\mathrm{d}t + \hat{\eta}_T^T(\bar{V})\,. \tag{4.17}$$

Adding together (4.14)–(4.16) and subtracting (4.15)–(4.17) we obtain

$$\int_0^T \mathcal{F}(\hat{\eta}_t^T, \bar{\mu})\,\mathrm{d}t \;\le\; (\hat{\eta}_0^T - \bar{\mu})(\varphi_T^T - \bar{V}) - (\hat{\eta}_T^T - \bar{\mu})(\varphi_0 - \bar{V})\,. \tag{4.18}$$

Define

$$\mathit{\Gamma}_T(t) \;:=\; \big(\hat{\eta}_{T-t}^T - \bar{\mu}\big)\big(\varphi_t^T - \bar{V}\big)\,, \quad t \in [0,T]\,.$$

In complete analogy to (4.18) we have

$$\int_{t_1}^{t_2} \mathcal{F}(\hat{\eta}_{T-s}^T, \bar{\mu})\,\mathrm{d}s \;\le\; \mathit{\Gamma}_T(t_2) - \Gamma_T(t_1)\,, \qquad t_1 \le t_2\,. \tag{4.19}$$

Remark 4.1. We often use in the proofs the following fact: if f_t, $h_t\colon \mathbb{R}^d \to \mathbb{R}$ and $g\colon \mathbb{R}^d \to \mathbb{R}$ are such that $\sup_{t\ge 0} \|f_t\|_{\mathcal{V}} < \infty$, $\|g\|_{\mathcal{V}} < \infty$, and $h_t(x) = \mathbb{E}_x^{\bar{v}}\big[\int_0^t f_s(X_s)\,\mathrm{d}s + g(X_t)\big]$, then it holds that $\sup_{t>0} \|h_t(x) - h_t(0)\|_{\mathcal{V}} < \infty$. Indeed, by (4.6) we have

$$\left| h_t(x) - \int_0^t \bar{\mu}(f_s)\,\mathrm{d}s - \bar{\mu}(g) \right| \;\le\; \big(1 + \mathcal{V}(x)\big)\left(\int_0^t M_0\,\mathrm{e}^{-\gamma s} \|f_s\|_{\mathcal{V}}\,\mathrm{d}s + \|g\|_{\mathcal{V}} \right),$$

so that

$$\begin{aligned} |h_t(x) - h_t(0)| \;&\le\; \big(2 + \mathcal{V}(x) + \mathcal{V}(0)\big)\left(\int_0^t M_0\,\mathrm{e}^{-\gamma s} \|f_s\|_{\mathcal{V}}\,\mathrm{d}s + \mathrm{e}^{-\gamma t}\,\|g\|_{\mathcal{V}} \right) \\ &\le\; \big(2 + \mathcal{V}(x) + \mathcal{V}(0)\big)\left(\frac{M_0}{\gamma} \sup_{s\ge 0} \|f_s\|_{\mathcal{V}} + \mathrm{e}^{-\gamma t}\,\|g\|_{\mathcal{V}} \right). \end{aligned}$$

We start with the following result:

Theorem 4.2. *Let Assumptions 4.1–4.2 hold. Then, for all $\lambda \in [0,1)$, we have*

$$\frac{1}{(1-\lambda)T} \int_{\lambda T}^{T} \hat\eta_t^T \,\mathrm{d}t \xrightarrow[T\to\infty]{} \bar\mu \quad \textit{in } \mathcal{P}(\mathbb{R}^d)\,.$$

Moreover,

$$\sup_{T>0} \int_0^T \mathcal{F}(\hat\eta_t^T, \bar\mu)\,\mathrm{d}t \;<\; \infty\,.$$

Proof. Let

$$\hat g_t^T(x) \;:=\; \mathbb{E}_x^{\hat v^T}\biggl[\int_0^t r_{\bar\mu}\bigl(X_s, \hat v_{T-t+s}^T(X_s)\bigr)\,\mathrm{d}s + \bar V(X_t)\biggr] - \bar\varrho\, t\,,$$

$$\bar g_t^T(x) \;:=\; \mathbb{E}_x^{\bar v}\biggl[\int_0^t r_{\hat\eta_{T-t+s}^T}\bigl(X_s, \bar v(X_s)\bigr)\,\mathrm{d}s + \varphi_0(X_t)\biggr] - \bar\varrho\, t\,.$$

Since

$$\bar g_t^T(x) - \bar V(x) \;=\; \mathbb{E}_x^{\bar v}\biggl[\int_0^t \bigl(F(X_s, \hat\eta_{T-t+s}^T) - F(X_s,\bar\mu)\bigr)\,\mathrm{d}s + \varphi_0(X_t) - \bar V(X_t)\biggr]\,,$$

it follows by (4.6) and Remark 4.1 that

$$\sup_{T>0}\sup_{t\,\in\,[0,T]} \bigl\|\bar g_t^T(x) - \bar V(x) - \bar g_t^T(0) - \bar V(0)\bigr\|_{\mathcal{V}} \;<\; \infty\,.$$

Therefore, for some constant C, we have $|(\hat\eta_{T-t}^T - \bar\mu)(\bar g_t^T - \bar V)| < C$ for all $t \in [0,T]$ and $T > 0$. Hence, by (4.3), we have

$$\begin{aligned}\hat\eta_{T-t}^T(\varphi_t^T) \;&\ge\; \hat\eta_{T-t}^T(\hat g_t^T) + \bar\mu(\bar g_t^T - \bar V) + (\hat\eta_T^T - \bar\mu)(\varphi_0 - \bar V)\\ &\ge\; \hat\eta_{T-t}^T(\hat g_t^T) + \hat\eta_{T-t}^T(\bar g_t^T - \bar V) - C + (\hat\eta_T^T - \bar\mu)(\varphi_0 - \bar V)\end{aligned}$$

for all $t \in [0,T]$ and $T > 0$. By suboptimality $\hat g_t^T \ge \bar V$ and $\bar g_t^T \ge \varphi_t^T$. Also

$$|(\hat\eta_T^T - \bar\mu)(\varphi_0 - \bar V)| \;\le\; 2\frac{c_0}{c_1} + \|\varphi_0 - \bar V\|_{\mathcal{V}}\bigl(\hat\eta_0(\mathcal{V}) + \bar\mu(\mathcal{V})\bigr)\,.$$

Hence, for some constant C' we obtain

$$\begin{aligned}0 \;&\le\; \hat\eta_{T-t}^T(\hat g_t^T - \bar V) \;\le\; C'\,,\\ 0 \;&\le\; \hat\eta_{T-t}^T(\bar g_t^T - \varphi_t^T) \;\le\; C'\end{aligned} \tag{4.20}$$

for all $t \in [0,T]$ and $T > 0$. By the first equation in (4.20), and since $\hat\eta^T_T(\bar V)$, $\hat\eta^T_T(\varphi_0)$, and $\int_0^T \big(\bar F(\mathcal{L}^{\bar v}_{\hat\eta_0}(X_t),\bar\mu) - \bar F(\bar\mu,\bar\mu)\big)\,\mathrm{d}t$ are bounded uniformly in $T>0$, using a triangle inequality we obtain

$$\sup_{T>0}\left|\int_0^T \big(\bar F(\hat\eta^T_t,\bar\mu) - \bar F(\bar\mu,\bar\mu)\big)\,\mathrm{d}t\right| \;<\; C''$$

for some constant C''. Similarly from the second equation, using the same constant C'', without loss of generality, we have

$$\sup_{T>0}\left|\int_0^T \big(\bar F(\bar\mu,\hat\eta^T_t) - \bar F(\hat\eta^T_t,\hat\eta^T_t)\big)\,\mathrm{d}t\right| \;<\; C''\,.$$

The second assertion of the theorem follows from these bounds.

From the first inequality in (4.20) we obtain $0 \le \hat\eta^T_{(1-\lambda)T}(\hat g^T_{\lambda T} - \bar V) \le C$ for all $\lambda \in [0,1]$. Therefore, we have

$$\frac{1}{\lambda T}\,\hat\eta^T_{(1-\lambda)T}(\hat g^T_{\lambda T}) \;\xrightarrow[T\to\infty]{}\; 0 \qquad \forall\,\lambda\in(0,1]\,. \tag{4.21}$$

Let

$$\breve\pi^T_\lambda(\mathrm{d}x,\mathrm{d}u) \;:=\; \frac{1}{\lambda T}\int_0^{\lambda T} \hat\eta^T_{(1-\lambda)T+s}(\mathrm{d}x)\circledast \hat v^T_{(1-\lambda)T+s}(\mathrm{d}u\mid x)\,\mathrm{d}s\,,\quad \lambda\in(0,1]\,,$$

and $\bar\pi := \bar\mu\circledast\bar v$. Since $\bar\pi(r_{\bar\mu}) = \bar\varrho$, and write (4.21) as

$$0 \;\ge\; \int_{\mathbb{R}^d\times\mathbb{U}} \Big(\breve\pi^T_\lambda(\mathrm{d}x,\mathrm{d}u) - \bar\pi(\mathrm{d}x,\mathrm{d}u)\Big)\, r_{\bar\mu}(x,u) \;\xrightarrow[T\to\infty]{}\; 0\,.$$

Since $\{\breve\pi^T_\lambda\,,\ T>0\}$ is tight, any limit point of $\breve\pi^T_\lambda$ as $T\to\infty$ is an element of $\mathcal{G}$ [3, Lemma 3.4.6]. Let $\{T_n\}$ be any sequence, and select a subsequence also denoted as $\{T_n\}$ along which $\breve\pi^{T_n}_\lambda \to \breve\pi^* \in \mathcal{G}$. Then $\breve\pi^*(r_{\bar\mu}) = \bar\pi(r_{\bar\mu})$, and since by Assumption 4.1(iv) the set $\mathcal{A}(\bar\mu)$ is a singleton it follows that $\breve\pi^* = \bar\pi$ which, in turn, implies the first assertion in the theorem. □

We also have the following simple lemma concerning the growth of $\varphi^T_t(0)$ in t.

Lemma 4.2. *Let Assumptions 4.1–4.2 hold. Then there exists a constant $\bar C_0 > 0$ which depends only on φ_0 and $\mathcal{L}(X_0)$, such that*

$$\big|\varphi^T_t(0) - \varphi^T_{t-\tau}(0)\big| \;\le\; \bar C_0(1+\tau) \qquad \textit{for all } t\in[0,T]\,,\ \tau\in[0,t]\,,\ \textit{and } T>0\,.$$

Proof. By Assumptions 4.1 and 4.2, there is a constant $\bar C$ depending only on φ_0 and $\mathcal{L}(X_0)$, such that

$$\mathbb{E}^U_0\left|r_{\hat\eta^T_{T-t}}(X_s,U_s)\right| \;\le\; \bar C$$

for all $t\in[0,T]$, $s\ge 0$, $U\in\mathfrak{U}$, and $T>0$. Therefore, we have

$$
\begin{aligned}
\bigl|\varphi^T_{t-\tau}(0)-\varphi^T_t(0)\bigr| \;&=\; \Bigl|\inf_U \mathbb{E}^U_0\biggl[\int_0^{t-\tau} r_{\hat\eta^T_{T-t+\tau+s}}(X_s,U_s)\,\mathrm{d}s+\varphi_0(X_{t-\tau})\biggr]\\
&\qquad -\inf_U \mathbb{E}^U_0\biggl[\int_0^{t} r_{\hat\eta^T_{T-t+s}}(X_s,U_s)\,\mathrm{d}s+\varphi_0(X_t)\biggr]\Bigr|\\
&\le\; \sup_U \mathbb{E}^U_0\biggl[\int_{t-\tau}^{t} \bigl|r_{\hat\eta^T_{T-t+s}}(X_s,U_s)\bigr|\,\mathrm{d}s+\bigl|\varphi_0(X_{t-\tau})-\varphi_0(X_t)\bigr|\biggr]\\
&\le\; \bar C\tau+2\|\varphi_0\|_{\mathcal{V}}\bigl(\tfrac{c_0}{c_1}+\mathrm{e}^{-c_1 t}\mathcal{V}(0)\bigr)\,. \qquad \square
\end{aligned}
$$

4.3. Convergence of the RVI

Theorem 4.2 shows that $\hat\eta^T$ converges to $\bar\mu$ in a time average sense. We wish to show that $\psi^T_{\lambda T}$ converges to $\bar V-\bar\varrho$ as $T\to\infty$. It is evident by (4.12) that this cannot happen unless $\varphi^T_{\lambda T}-\varphi^T_{\lambda T}(0)$ is at least locally bounded, uniformly in $T>0$. We state this necessary condition for convergence as a property.

Property 4.1. *Define* $\overline{\varphi}^T_t := \varphi^T_t-\varphi^T_t(0)$. *Suppose that* $\hat\eta_0(\mathcal{V})\le\kappa_1$ *and* $\|\varphi_0\|_{\mathcal{V}}\le\kappa_2$, *where* $\mathcal{V}$ *is as in Assumption 4.2. Then there exists a constant* $\tilde C_1=\tilde C_1(\kappa_1,\kappa_2)$ *such that*

$$
\sup_{T>0}\sup_{t\,\in\,[0,T]}\bigl\|\overline{\varphi}^T_t\bigr\|_{\mathcal{V}} \;<\; \tilde C_1\,.
$$

It is unclear if Assumption 4.2 suffices to establish Property 4.1. Instead of imposing additional assumptions on F, we choose instead to show that this property is satisfied for a large class of controlled diffusions, and then assume only Property 4.1 in the statement of the main results.

We introduce the following notation: for x, z in $\mathbb{R}^d$ define

$$
\begin{aligned}
\Delta_z b(x,u) \;&:=\; b(x+z,u)-b(x,u)\,,\\
\Delta_z \sigma(x) \;&:=\; \sigma(x+z)-\sigma(x)\,,\\
\tilde a(x;z) \;&:=\; \Delta_z\sigma(x)\Delta_z\sigma^{\mathsf{T}}(x)\,.
\end{aligned}
$$

Definition 4.2. We say that the controlled diffusion in (2.1) is *asymptotically flat* if the following hold:

(a) The diffusion matrix σ is Lipschitz continuous.
(b) There exist a symmetric positive definite matrix Q and a constant $r>0$ such that for $x,z\in\mathbb{R}^d$, with $z\neq 0$, and $u\in\mathbb{U}$, it holds that

$$
2\Delta_z b^{\mathsf{T}}(x,u)Qz-\frac{|\Delta_z\sigma^{\mathsf{T}}(x)Qz|^2}{z^{\mathsf{T}}Qz}+\operatorname{trace}\bigl(\tilde a(x;z)Q\bigr) \;\le\; -r|z|^2\,.
$$

A standard model of asymptotically flat diffusions is given by $\mathbb{U}=[0,1]^d$, $b(x,u)=Bx+Du$, where B, D are constant $d\times d$ matrices and B is Hurwitz (i.e., its eigenvalues have negative real parts). Note also that if σ is constant, then asymptotic flatness amounts to the requirement that $\langle b(x+z,u)-b(x,u),Qz\rangle\le -r|z|^2$. Nevertheless, the class of asymptotically flat diffusions is significantly richer than models with stable linear drifts. Asymptotically flat diffusions satisfy an "incremental stability" property. For recent work along similar directions see [10,41].

We quote the following result [3, Lemmas 7.3.4 and 7.3.6].

Lemma 4.3. *Suppose that the diffusion in (2.1) is asymptotically flat, and let X_t^x be the solution with initial condition $X_0 = x$, corresponding to an admissible relaxed control U. Then there exist constants $\hat{c}_0 > 0$ and $\hat{c}_1 > 0$, which do not depend on U, such that*

$$\mathbb{E}^U \left| X_t^x - X_t^y \right| \;\le\; \hat{c}_0 \, \mathrm{e}^{-\hat{c}_1 t} |x - y| \qquad \forall\, x,\, y \in \mathbb{R}^d \,. \tag{4.22}$$

Moreover there exists a nonnegative, inf-compact $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$ satisfying (4.4), and such that

$$\frac{\mathcal{V}(x)}{1 + |x|^{p_0}} \;\xrightarrow[|x| \to \infty]{}\; \infty \tag{4.23}$$

for some $p_0 > 1$.

Let $\mathrm{Lip}(f)$ denote a Lipschitz constant for a function f, which is assumed Lipschitz. We often use the fact that for an asymptotically flat diffusion, if $g_t(x) = \mathbb{E}_x^U[f(X_t]$, then $\mathrm{Lip}(g_t) \le \hat{c}_0 \mathrm{e}^{-\hat{c}_1 t} \, \mathrm{Lip}(f)$ for all $U \in \mathfrak{U}$.

We have the following lemma:

Lemma 4.4. *Let Assumptions 4.1 and 4.2 hold, and suppose that the diffusion is asymptotically flat, $\hat{\eta}_0(\mathcal{V}) < \infty$, and $\varphi_0 \in \mathcal{C}^2(\mathbb{R}^d)$ is Lipschitz. We also assume that $\mathring{r}(\cdot\,, u)$ is Lipschitz uniformly in $u \in \mathbb{U}$, and that Assumption 4.2(ii) holds for a constant M_1 which is independent of R. Then there exists a constant $\tilde{C}_1'$ which depends only on $\hat{\eta}_0(\mathcal{V})$ and $\mathrm{Lip}(\varphi_0)$ such that*

$$\mathrm{Lip}\big(\varphi_t^T\big) \;\le\; \tilde{C}_1' \qquad \forall\, t \in [0, T] \,,\; \forall\, T > 0 \,.$$

In particular,

$$\sup_{T>0} \sup_{t \in [0,T]} \big\| \overline{\varphi}_t^T \big\|_{\mathcal{V}} \;\le\; \tilde{C}_1' \,,$$

and thus Property 4.1 holds.

Proof. We fix some compact set $\mathcal{K}_0 \subset \mathcal{P}_1(\mathbb{R}^d)$ of initial distributions that contains $\bar{\mu}$, and satisfies $\sup_{\mu \in \mathcal{K}_0} \mu(\mathcal{V}) < \infty$. The initial distribution $\hat{\eta}_0$ is assumed to lie in the set $\mathcal{K}_0$. The corresponding collection $\mathcal{P}(\mathcal{K}_0) := \{\mathcal{L}_\mu^U(X_t) : \mu \in \mathcal{K}_0 \,,\; U \in \mathfrak{U} \,,\; t > 0\}$ is compact in $\mathcal{P}_1(\mathbb{R}^d)$ by (4.4) and (4.23). Therefore, for some constant $\tilde{C}_0$, it holds that $\mathrm{Lip}(F(\cdot, \mu)) \le \tilde{C}_0$ and $|F(x, \mu)| \le \tilde{C}_0(1 + |x|)$ for all $\mu \in \mathcal{K}_0$. It is straightforward to show that, under asymptotic flatness, $\bar{V}$ is Lipschitz. Without loss of generality, we let $\tilde{C}_0$ be also a Lipschitz constant for φ_0, $\bar{V}$, and $\mathring{r}(\cdot\,, u)$ as well.

By Lemma 4.3, we have

$$\begin{aligned}
\big|\varphi_t^T(x) - \varphi_t^T(y)\big| \;&=\; \Bigg| \inf_U \; \mathbb{E}_x^U \bigg[\int_0^t r_{\hat{\eta}_{T-t+s}^T}(X_s, U_s) \, \mathrm{d}s + \varphi_0(X_t) \bigg] \\
&\qquad - \inf_U \; \mathbb{E}_y^U \bigg[\int_0^t r_{\hat{\eta}_{T-t+s}^T}(X_s, U_s) \, \mathrm{d}s + \varphi_0(X_t) \bigg] \Bigg| \\
&\le \sup_U \; \mathbb{E}^U \bigg[\int_0^t \big| r_{\hat{\eta}_{T-t+s}^T}(X_s^x, U_s) - r_{\hat{\eta}_{T-t+s}^T}(X_s^y, U_s) \big| \, \mathrm{d}s \bigg] \\
&\qquad + \sup_U \; \mathbb{E}^U \Big[\big| \varphi_0(X_t^x) - \varphi_0(X_t^y) \big| \Big]
\end{aligned}$$

$$\le\ 2\tilde{C}_0\frac{\hat{c}_0}{\hat{c}_1}\,|x-y|+\hat{c}_0|x-y| \qquad \text{for all } x,y\in\mathbb{R}^d\,,\ t\in[0,T]\,,\ \text{and } T>0\,.$$

This completes the proof. □

Remark 4.2. Even though $\mathring{r}$ and F have been assumed Lipschitz in x, running costs with higher growth in x can be treated, depending on the diffusion matrix. In particular, if the diffusion matrix is constant, then (4.22) can be replaced by

$$\mathbb{E}^U\left|X_t^x-X_t^y\right|^2\ \le\ \hat{c}_0\mathrm{e}^{-\hat{c}_1 t}|x-y|^2 \qquad \forall\, x,\,y\in\mathbb{R}^d\,.$$

Thus, in this case, the results of Lemma 4.4 can be extended to include running costs with up to quadratic growth.

As mentioned earlier, Property 4.1, which is implied by asymptotic flatness, together with Assumption 4.2 are sufficient to prove convergence. So in the statement of the main results we use Property 4.1 in lieu of asymptotic flatness.

Since $(\hat{\eta}_0^T-\bar{\mu})(\varphi_T^T-\bar{V})=(\hat{\eta}_0^T-\bar{\mu})(\overline{\varphi}_T^T-\bar{V})$, it is evident that if Property 4.1 holds, then the right hand side of (4.18) is bounded uniformly in $T>0$. Therefore, by (4.19), we have the following.

Corollary 4.1. *Let Assumptions 4.1–4.2 and Property 4.1 hold, and suppose that $\varphi_0\in\mathcal{C}^2(\mathbb{R}^d)\cap\mathcal{C}_{\mathcal{V}}(\mathbb{R}^d)$ and $\hat{\eta}_0(\mathcal{V})<\infty$. Then there exists a constant C_0 such that*

$$\int_{t_1}^{t_2}\mathcal{F}(\hat{\eta}_{T-s}^T,\bar{\mu})\,\mathrm{d}s\ \le\ \mathit{\Gamma}_T(t_2)-\Gamma_T(t_1)\ \le\ C_0\,,\qquad t_1\le t_2\,.$$

In particular, $t\mapsto\mathit{\Gamma}_T(t)$ is nondecreasing and bounded on $t\in[0,T]$ uniformly in $T>0$.

We are now ready to state the main results.

Theorem 4.3. *Let Assumptions 4.1–4.2 and Property 4.1 hold. Also suppose that $\hat{\eta}_0(\mathcal{V})<\infty$ and $\varphi_0\in\mathcal{C}^2(\mathbb{R}^d)\cap\mathcal{C}_{\mathcal{V}}(\mathbb{R}^d)$. Then for any $\lambda\in(0,1)$, and $t_0>0$ we have*

$$\sup_{t\in[0,t_0]}\ \mathcal{W}_1(\hat{\eta}_{\lambda T-t}^T,\bar{\mu})\ \xrightarrow[T\to\infty]{}\ 0\,. \tag{4.24}$$

Moreover,

$$\sup_{t\in[0,t_0]}\left\|\varphi_{\lambda T-t}^T-\varphi_{\lambda T-t}^T(0)-\bar{V}\right\|_{\mathcal{V}}\ \xrightarrow[T\to\infty]{}\ 0\,, \tag{4.25}$$

and

$$\sup_{t\in[0,t_0]}\left|\varphi_{\lambda T}^T(0)-\varphi_{\lambda T-t}^T(0)\right|\ \xrightarrow[T\to\infty]{}\ 0\,. \tag{4.26}$$

Proof. Let $\varepsilon\in\left(0,\frac{1}{2}\min(\lambda,1-\lambda)\right)$, and $\tau>0$. Consider the interval $[(1-\varepsilon)T-\tau,T]$ and let $\mathcal{I}_T$ be the collection of consecutive closed intervals $[(1-\varepsilon)T,(1-\varepsilon)T+2\tau]$, $[(1-\varepsilon)T+2\tau,(1-\varepsilon)T+4\tau],\dots$ contained in it. Let $T_n\to\infty$ be any sequence. By Corollary 4.1 there exists a sequence $[t_n-\tau,t_n+\tau]\in\mathcal{I}_{T_n}$ such that $\mathit{\Gamma}_{T_n}(t_n+\tau)-\mathit{\Gamma}_{T_n}(t_n-\tau)\to 0$ as $n\to\infty$.

Property 4.1 together with Lemma 4.2 imply that $(s,x)\mapsto\varphi_{t+s}^T(x)-\varphi_t^T(0)$ is bounded on compact sets of $\mathbb{R}_+\times\mathbb{R}^d$ uniformly in $T>0$ and $t\in[0,T]$. By well known interior estimates of parabolic equations,

this means that the maps $(s,x) \mapsto \varphi^T_{t+s}(x) - \varphi^T_t(0)$ are locally Hölder equicontinuous on compact sets of $(1,\infty)\times\mathbb{R}^d$. Therefore, $(t,x) \mapsto \varphi^{T_n}_t(x) - \varphi^{T_n}_{t_n}(0)$ is equicontinuous on $[t_n-\tau, t_n+\tau]$. At the same time, by Lemma 4.1, the laws $\{\hat\eta^{T_n}_{T_n-t_n+s}\,,\ s\in[-\tau,\tau]\}$ are precompact in $[-\tau,\tau]\times\mathcal{P}_1(\mathbb{R}^d)$. Passing to the limit along a subsequence, also denoted as $\{T_n\}$, we define

$$\varphi^*_t \;:=\; \lim_{n\to\infty}\big(\varphi^{T_n}_{t_n-\tau+t} - \varphi^{T_n}_{t_n}(0)\big)\,, \quad \text{and} \quad \hat\eta^*_t \;:=\; \lim_{n\to\infty}\hat\eta^{T_n}_{T_n-t_n-\tau+t}\,, \qquad t\in[0,2\tau]\,. \tag{4.27}$$

Let $\overline{\varphi}^*_t := \varphi^*_t - \varphi^*_t(0)$. By Property 4.1 and Lemma 4.2 we have

$$\sup_{t\,\in\,[0,2\tau]} \|\varphi^*_t\|_{\mathcal{V}} \;<\; \tilde C_1 + \bar C_0(1+\tau)\,, \quad \text{and} \quad \sup_{t\,\in\,[0,2\tau]} \|\overline{\varphi}^*_t\|_{\mathcal{V}} \;<\; \tilde C_1\,. \tag{4.28}$$

It is evident that $\{\hat\eta^*_t\,,\ t\in[0,2\tau]\}$ is a MFG solution for the finite horizon problem on $[0,2\tau]$ with initial law $\hat\eta^*_0$ and terminal cost φ^*_0. Therefore,

$$\varphi^*_t(x) \;=\; \inf_{U\in\mathfrak{U}} \mathbb{E}^U_x\left[\int_0^t r_{\hat\eta^*_s}\big(X_s,U_s\big)\,\mathrm{d}s + \varphi^*_0(X_t)\right] - \bar\varrho\, t\,.$$

Let $\hat v^*$ be a Markov control that realizes this infimum, i.e., $\hat v^*$ is the a.e. unique minimizer from the Hamiltonian of the associated HJB. By suboptimality we have

$$\varphi^*_t(x) \;\le\; \mathbb{E}^{\bar v}_x\left[\int_0^t r_{\hat\eta^*_s}\big(X_s,\bar v(X_s)\big)\,\mathrm{d}s + \varphi^*_0(X_t)\right] - \bar\varrho\, t\,, \tag{4.29a}$$

$$\bar V(x) \;\le\; \mathbb{E}^{\hat v^*}_x\left[\int_0^t r_{\bar\mu}\big(X_s,\hat v^*_s(X_s)\big)\,\mathrm{d}s + \bar V(X_t)\right] - \bar\varrho\, t \tag{4.29b}$$

for $t\in[0,\tau]$. Since $\mathit{\Gamma}_{T_n}(t_n+\tau) - \mathit{\Gamma}_{T_n}(t_n-\tau) \to 0$ as $n\to\infty$ along the subsequence, taking limits we have

$$(\hat\eta^*_0 - \bar\mu)(\varphi^*_{2\tau} - \bar V) - (\hat\eta^*_{2\tau} - \bar\mu)(\varphi^*_0 - \bar V) \;=\; 0\,. \tag{4.30}$$

Thus

$$\int_0^{2\tau} \mathcal{F}(\hat\eta^*_s,\bar\mu)\,\mathrm{d}s \;=\; 0 \tag{4.31}$$

by (4.18). However, since $\bar\mu$ and $\hat\eta^*_0$ have strictly positive density, then (4.30)–(4.31) imply that (4.29a) and (4.29b) must hold with equality. By a.e. uniqueness of the minimizer in the Hamiltonian, we must have $\hat v^* = \bar v$ a.e. in $[0,2\tau]\times\mathbb{R}^d$. Recall that $P^{\bar v}_t(x,\cdot\,)$ denotes the transition probability of the process X in (2.1) under the control $\bar v$. Thus, by (4.6) we have

$$\hat\eta^*_t(\cdot) \;=\; \int_{\mathbb{R}^d} \hat\eta^*_0(\mathrm{d}y)\, P^{\bar v}_t(y,\cdot\,)\,, \quad t\in[0,2\tau]\,,$$

and using (4.6) and (4.28) we obtain

$$\begin{aligned}\big|(\hat\eta^*_t - \bar\mu)(\varphi^*_0 - \bar V)\big| \;&\le\; M_0\,\mathrm{e}^{-\gamma t}\big\|\overline{\varphi}^*_0 - \bar V\big\|_{\mathcal{V}}\big(1+\hat\eta^*_0(\mathcal{V})\big)\\ &\le\; M_0\big(\tilde C_1 + \|\bar V\|_{\mathcal{V}}\big)\,\big(1+\tfrac{c_0}{c_1}+\hat\eta_0(\mathcal{V})\big)\,\mathrm{e}^{-\gamma t}\,, \quad t\in[0,2\tau]\,.\end{aligned} \tag{4.32}$$

Note also that by the Kantorovich duality theorem, we have the estimate

$$\mathcal{W}_1(\hat\eta_t^*,\bar\mu) \;\le\; M_0\,\mathrm{e}^{-\gamma t}\bigl(1+\hat\eta_0^*(\mathcal{V})\bigr)\sup_{x\in\mathbb{R}^d}\tfrac{|x|}{\mathcal{V}(x)}\,,\quad t\in[0,2\tau]\,. \tag{4.33}$$

We claim that $\hat v^*=\bar v$ a.e. in $[0,2\tau]\times\mathbb{R}^d$ also implies that

$$\sup_{t\,\in\,[\frac{\tau}{4},\frac{3\tau}{2}]}\bigl\|\overline{\varphi}_t^*-\bar V\bigr\|_{\mathcal{V}}\;\xrightarrow[\tau\to\infty]{}\;0\,, \tag{4.34}$$

and

$$\sup_{t\,\in\,[\frac{\tau}{2},\frac{3\tau}{2}]}\bigl|\varphi_t^*(0)-\varphi_\tau^*(0)\bigr|\;\xrightarrow[\tau\to\infty]{}\;0\,. \tag{4.35}$$

To prove the claim, we estimate φ_t^* by

$$\begin{aligned}\varphi_t^*(x)-\varphi_0^*(0) \;&=\; \mathbb{E}_x^{\bar v}\biggl[\int_0^t r_{\hat\eta^*_{2\tau-t+s}}\bigl(X_s,\bar v(X_s)\bigr)\,\mathrm{d}s+\overline{\varphi}_0^*(X_t)\biggr]-\bar\varrho\,t\\ &=\; \mathbb{E}_x^{\bar v}\biggl[\int_0^t r_{\bar\mu}\bigl(X_s,\bar v(X_s)\bigr)\,\mathrm{d}s+\bar V(X_t)-\bar\varrho\,t\biggr]\\ &\quad+\mathbb{E}_x^{\bar v}\biggl[\int_0^t\Bigl(F\bigl(X_s,\hat\eta^*_{2\tau-t+s}\bigr)-F\bigl(X_s,\bar\mu\bigr)\Bigr)\,\mathrm{d}s+\overline{\varphi}_0^*(X_t)-\bar V(X_t)\biggr]\,.\end{aligned} \tag{4.36}$$

The first term in (4.36) equals $\bar V(x)$. We use the estimate

$$\bigl|\mathbb{E}_x^{\bar v}\bigl[\overline{\varphi}_0^*(X_t)-\bar V(X_t)\bigr]-\bar\mu\bigl(\overline{\varphi}_0^*-\bar V\bigr)\bigr|\;\le\;M_0\mathrm{e}^{-\gamma t}\bigl\|\overline{\varphi}_0^*-\bar V\bigr\|_{\mathcal{V}}\bigl(1+\mathcal{V}(x)\bigr)\,, \tag{4.37}$$

which holds by (4.6). Similarly, with $\tilde F_\mu(x):=F(x,\mu)-F(x,\bar\mu)$, we have

$$\biggl|\mathbb{E}_x^{\bar v}\biggl[\int_0^t\tilde F_{\hat\eta^*_{2\tau-t+s}}(X_s)\,\mathrm{d}s\biggr]-\int_0^t\bar\mu\bigl(\tilde F_{\hat\eta^*_{2\tau-t+s}}\bigr)\,\mathrm{d}s\biggr|\;\le\;M_0\bigl(1+\mathcal{V}(x)\bigr)\int_0^t\mathrm{e}^{-\gamma s}\bigl\|\tilde F_{\hat\eta^*_{2\tau-t+s}}\bigr\|_{\mathcal{V}}\,\mathrm{d}s\,. \tag{4.38}$$

Let

$$\zeta(t)\;:=\;\bar\mu\bigl(\overline{\varphi}_0^*-\bar V\bigr)+\int_0^t\bar\mu\bigl(\tilde F_{\hat\eta^*_{2\tau-t+s}}\bigr)\,\mathrm{d}s\,.$$

We evaluate $\varphi_t^*(x)-\varphi_0^*(0)$ in (4.36) first at x and then at $x=0$, using also (4.37)–(4.38) to form a triangle inequality, as well as the fact that $\bar V(0)=0$, to obtain

$$\begin{aligned}\bigl|\overline{\varphi}_t^*(x)-\bar V(x)\bigr|\;&\le\;\bigl|\varphi_t^*(x)-\varphi_0^*(0)-\bar V(x)-\zeta(t)\bigr|+\bigl|\varphi_t^*(0)-\varphi_0^*(0)-\bar V(0)-\zeta(t)\bigr|\\ &\le\;M_0\bigl(2+\mathcal{V}(x)+\mathcal{V}(0)\bigr)\biggl(\mathrm{e}^{-\gamma t}\bigl\|\overline{\varphi}_0^*-\bar V\bigr\|_{\mathcal{V}}+\int_0^t\mathrm{e}^{-\gamma s}\bigl\|\tilde F_{\hat\eta^*_{2\tau-t+s}}\bigr\|_{\mathcal{V}}\,\mathrm{d}s\biggr)\,.\end{aligned} \tag{4.39}$$

By Assumption 4.2(iii), which holds with $p=1$, and (4.33) we have

$$\sup_{t\,\in\,[\tau/2,2\tau]} \big\|\tilde{F}_{\hat{\eta}^*_t}\big\|_{\mathcal{V}} \xrightarrow[\tau\to\infty]{} 0\,. \tag{4.40}$$

Therefore (4.34) follows by (4.39)–(4.40). Using (4.36) once more, we obtain

$$\begin{aligned}\big|\varphi^*_t(0)-\varphi^*_{\tau/4}(0)\big| \;\le\; \bar{\mu}\big(\overline{\varphi}^*_{\tau/4}-\bar{V}\big) + \int_0^{t-\tau/4} \bar{\mu}\big(\tilde{F}_{\hat{\eta}^*_{2\tau-t+s}}\big)\,\mathrm{d}s \\ + M_0\big(1+\mathcal{V}(0)\big)\left(\mathrm{e}^{-\gamma(t-\tau/4)}\big\|\overline{\varphi}^*_{\tau/4}-\bar{V}\big\|_{\mathcal{V}} + \int_0^{t-\tau/4} \mathrm{e}^{-\gamma s}\big\|\tilde{F}_{\hat{\eta}^*_{2\tau-t+s}}\big\|_{\mathcal{V}}\,\mathrm{d}s\right)\end{aligned} \tag{4.41}$$

for $t \ge \tau/4$. The first term in (4.41) vanishes as $\tau \to \infty$ by (4.34), and the same holds for the integrals by (4.40). This proves (4.35).

Repeating the same argument on the interval $[0, \varepsilon T]$, we obtain the analogous to (4.32). Combining the two, and using the fact that $\mathit{\Gamma}_T((1-\varepsilon)T) - \mathit{\Gamma}_T(\varepsilon T) \to 0$ as $T \to \infty$, and $\mathit{\Gamma}_T(t) - \mathit{\Gamma}_T(t') \ge 0$ for $t \ge t'$, we deduce that

$$\sup_{\lambda\,\in\,[\varepsilon,1-\varepsilon]} \mathit{\Gamma}_T(\lambda T) \xrightarrow[T\to\infty]{} 0\,. \tag{4.42}$$

Let $\tilde{\varepsilon} > 0$ be given. By (4.32) and (4.39)–(4.40) we can select τ_0 such that, if $\tau > \tau_0$ then any limits φ^* and $\hat{\eta}^*$ as defined in (4.27) satisfy

$$\sup_{t\,\in\,[\frac{\tau}{2},\frac{3\tau}{2}]} \max\Big(\mathcal{W}_1(\hat{\eta}^*_t,\bar{\mu}), \big|\varphi^*_t(0)-\varphi^*_\tau(0)\big|, \big\|\overline{\varphi}^*_t-\bar{V}\big\|_{\mathcal{V}}\Big) \;\le\; \frac{\tilde{\varepsilon}}{4}\,. \tag{4.43}$$

Next, we select any interval of the form $[\lambda T-\tau, \lambda T+\tau] \subset [\varepsilon T, (1-\varepsilon)T]$, $\tau > \tau_0$. Given any sequence $T_n \to 0$, we can take limits along some subsequence $T_n \to \infty$ by (4.42), as done earlier, and define

$$\varphi^*_t \;:=\; \lim_{n\to\infty}\big(\varphi^{T_n}_{\lambda T_n-\tau+t} - \varphi^{T_n}_{\lambda T_n}(0)\big)\,, \qquad \hat{\eta}^*_t \;=\; \lim_{n\to\infty} \hat{\eta}^{T_n}_{(1-\lambda)T_n-\tau+t}\,, \qquad t\in[0,2\tau]\,.$$

Therefore, $\overline{\varphi}^*_t = \lim_{n\to\infty} \overline{\varphi}^{T_n}_{\lambda T_n-\tau+t}$. Since convergence is uniform on $[0,2\tau]$, there exists $n_0 \in \mathbb{N}$, such that

$$\sup_{t\,\in\,[0,2\tau]} \max\Big(\mathcal{W}_1\big(\hat{\eta}^*_t, \hat{\eta}^{T_n}_{(1-\lambda)T_n-\tau+t}\big), \big\|\overline{\varphi}^*_t - \overline{\varphi}^{T_n}_{\lambda T_n-\tau+t}\big\|_{\mathcal{V}}\Big) \;\le\; \frac{\tilde{\varepsilon}}{2} \qquad \forall\, n \ge n_0\,. \tag{4.44}$$

Since

$$\varphi^*_t(0) - \varphi^*_\tau(0) \;=\; \lim_{n\to\infty}\big(\varphi^{T_n}_{\lambda T_n-\tau+t}(0) - \varphi^{T_n}_{\lambda T_n}(0)\big)\,,$$

the result clearly follows by (4.43)–(4.44), and a standard triangle inequality. □

Convergence of the (RVI) is asserted in the following theorem:

Theorem 4.4. *Under the assumptions of Theorem 4.3, it holds that*

$$\big\|\psi^T_{\lambda T}(x) - \bar{V}(x) + \bar{\varrho}\big\|_{\mathcal{V}} \xrightarrow[T\to\infty]{} 0 \tag{4.45}$$

for all $\lambda \in (0,1)$.

Proof. We write (4.13) as

$$\psi^T_{\lambda T}(x) \;=\; \overline{\varphi}^T_{\lambda T}(x) - \int_0^{\lambda T} \mathrm{e}^{s-\lambda T}\left(\varphi^T_s(0) - \varphi^T_{\lambda T}(0)\right) \mathrm{d}s + \mathrm{e}^{-\lambda T}\varphi^T_{\lambda T}(0) + \bar{\varrho}\,(1-\mathrm{e}^{-\lambda T}) \tag{4.46}$$

for $(t,x) \in [0,T] \times \mathbb{R}^d$. We expand the integral as

$$\int_0^{\lambda T - t_0} \mathrm{e}^{s-\lambda T}\left(\varphi^T_s(0) - \varphi^T_{\lambda T}(0)\right) \mathrm{d}s + \int_{\lambda T - t_0}^{\lambda T} \mathrm{e}^{s-\lambda T}\left(\varphi^T_s(0) - \varphi^T_{\lambda T}(0)\right) \mathrm{d}s$$

for $t_0 > 0$. The first integral has a bound $\kappa \mathrm{e}^{-t_0}$ for some constant κ by Lemma 4.2, while second integral vanishes as $T \to \infty$ by (4.26). Since $t_0 > 0$ is arbitrary, (4.45) follows by (4.25) and (4.46). □

Remark 4.3. The result of Theorem 4.4 can be improved to assert that convergence is uniform on $[\varepsilon T, (1-\varepsilon)T]$, or in other words that for any $\varepsilon \in (0, 1/2)$ we have

$$\sup_{\lambda \in [\varepsilon,(1-\varepsilon)]} \left\| \psi^T_{\lambda T}(x) - \bar{V}(x) + \bar{\varrho} \right\|_{\mathcal{V}} \xrightarrow[T\to\infty]{} 0\,.$$

The same applies to the convergence in (4.24)–(4.26). To establish this one may follow the argument in the proof of Theorem 4.3. First, under the hypotheses, the map

$$\overline{\mathcal{T}} : \mathcal{P}_p(\mathbb{R}^d) \times \left(\mathcal{C}^2(\mathbb{R}^d) \cap \mathcal{C}_{\mathcal{V}}(\mathbb{R}^d)\right) \to \mathcal{C}\left([0,2\tau], \mathcal{P}_p(\mathbb{R}^d)\right)$$

which determines the MFG solution for the finite horizon problem on an interval $[0,2\tau]$ from an initial distribution $\eta \in \mathcal{P}_p(\mathbb{R}^d)$ and a terminal cost $g \in \mathcal{C}^2(\mathbb{R}^d) \cap \mathcal{C}_{\mathcal{V}}(\mathbb{R}^d)$ is continuous in η and $g - g(0)$. Since $\hat{\eta}^T_t$ lives in some compact set $\mathcal{K}$ of $\mathcal{P}_p(\mathbb{R}^d)$ and $\overline{\varphi}^T_t$ lives in some compact set $\mathcal{G}$ of $\mathcal{C}^2(\mathbb{R}^d) \cap \mathcal{C}_{\mathcal{V}}(\mathbb{R}^d)$ for all $t \in [0,T]$ and $T > 0$, then by the uniform continuity of the map $\overline{\mathcal{T}}$ on $\mathcal{K} \times \mathcal{G}$, it is evident from the proof of Theorem 4.3 that the convergence in (4.24)–(4.26) is uniform in $\lambda \in [\varepsilon T, (1-\varepsilon)T]$.

Remark 4.4. As shown in [13], under the hypothesis $\|F(x,\mu) - F(x,\mu')\|_{\mathcal{C}^{1+\alpha}} \le C\|\mu - \mu'\|_{H_0^{-1}}$, convergence is exponential in T for the problem on the d-dimensional torus $\mathbb{T}$. For the model addressed in this paper, it would be interesting to investigate whether strengthening Assumption 4.2(ii) and (iii) to $|F(x,\mu) - F(x',\mu')| \le C\big(|x-x'| + \mathcal{W}_p(\mu,\mu')\big)$ is sufficient to guarantee exponential convergence.

5. Limits of N-player games

In this section we consider certain classes of N player games, and show that as $N \to \infty$, the limiting value function and invariant probability measure solve mean field games. As earlier, we consider a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ on which we are given N independent d-dimensional standard Brownian motions $\{W^1, \dots, W^N\}$ with respect to a complete filtration $\{\mathfrak{F}^N_t\}$. The initial conditions $\{X^i_0\}$ are assumed to be independent of these Brownian motions. The control for the ith player lives in a compact, metrizable control set $\mathbb{U}^i$. The set of all admissible controls is denoted by $\mathfrak{U}^N$ and contains paths $(U^1, \dots, U^N)$, satisfying the following: $\{U^i_t(\omega),\ 1 \le i \le N\}$, is jointly measurable in $(t,\omega) \in [0,\infty) \times \Omega$, takes values in $\mathbb{U}^1 \times \cdots \times \mathbb{U}^N$, and U^i is adapted to the Brownian motion W^i for $1 \le i \le N$. Therefore the game under consideration is non-cooperative. We note that the controls in $\mathfrak{U}^N$ satisfy the non-anticipativity condition. We consider the collection of controlled diffusions

$$\mathrm{d}X_t^i \;=\; b^i(X_t^i,U_t^i)\,\mathrm{d}t+\sigma^i(X_t^i)\,\mathrm{d}W_t^i\,,\quad 1\le i\le N. \tag{5.1}$$

We assume that b^i, σ^i, $1\le i\le N$, satisfy conditions (A1)–(A3) possibly for different constants C_1, C_R. Therefore, for any admissible control $\boldsymbol{U}^N=(U^1,\dots,U^N)\in\mathfrak{U}^N$, (5.1) has a unique strong solution for every deterministic initial condition. It might be convenient to think of this system of diffusions as a single controlled diffusion with state space $\mathbb{R}^{Nd}$. The cost functions

$$r^i\colon \mathbb{R}^d\times\mathbb{U}\times\mathcal{P}(\mathbb{R}^d)\to\mathbb{R}_+\,,$$

are assumed to be continuous and for all μ, r is locally Lipschitz in the variable x uniformly in $u\in\mathbb{U}$. We extend the action space to the relaxed control framework, and assume that the admissible control takes values in $\mathcal{P}(\mathbb{U}^1)\times\cdots\times\mathcal{P}(\mathbb{U}^N)$. Let $\mathfrak{U}_{\mathrm{SM}}=\mathfrak{U}_{\mathrm{SM}}^1\times\cdots\times\mathfrak{U}_{\mathrm{SM}}^N$, where $\mathfrak{U}_{\mathrm{SM}}^i$ denotes the set of measurable maps $v^i\colon\mathbb{R}^d\to\mathcal{P}(\mathbb{U}^i)$. We endow $\mathfrak{U}_{\mathrm{SM}}$ with the product topology; therefore $\mathfrak{U}_{\mathrm{SM}}$ forms a compact space. By $\mathfrak{U}_{\mathrm{SSM}}$ we denote the set of all stable stationary Markov controls in $\mathfrak{U}_{\mathrm{SM}}$. The cost function for the i-th player is given by

$$J^i(\boldsymbol{U}^N)\;:=\;\limsup_{T\to\infty}\frac{1}{T}\,\mathbb{E}_x\biggl[\int_0^T r^i\biggl(X_t^i,U_t^i,\frac{1}{N-1}\sum_{j\ne i}\delta_{X_t^j}\biggr)\,\mathrm{d}t\biggr]. \tag{5.2}$$

From (2.4) it is easy to see that for all $x^j,y^j\in\mathbb{R}^d$, $1\le j\le N-1$, we have

$$d_{\mathrm{P}}\biggl(\frac{1}{N-1}\sum_{j=1}^{N-1}\delta_{x^j},\frac{1}{N-1}\sum_{j=1}^{N-1}\delta_{y^j}\biggr)\;\le\;\sum_{j=1}^{N-1}|x^j-y^j|\,.$$

Therefore, defining $\breve r^i\colon\mathbb{R}^{Nd}\times\mathbb{U}^i\to\mathbb{R}_+$ by

$$\breve r^i(x^1,\dots,x^N,u^i)\;:=\;r^i\biggl(x^i,u^i,\frac{1}{N-1}\sum_{j\ne i}\delta_{x^j}\biggr),$$

it follows that $\breve r^i$ is continuous in $\mathbb{R}^{Nd}$ uniformly in $u^i\in\mathbb{U}^i$. Hence we can redefine the ergodic criterion in (5.2) as

$$J^i(\boldsymbol{U}^N)\;:=\;\limsup_{T\to\infty}\frac{1}{T}\,\mathbb{E}_x\biggl[\int_0^T\breve r^i(X_t^1,\dots,X_t^N,U_t^i)\,\mathrm{d}t\biggr].$$

By $\mathfrak{U}^i$, $1\le i\le N$, we denote the set of all jointly measurable functions $U^i\colon[0,\infty)\times\Omega\to\mathbb{U}^i$ that are adapted to W^i.

5.1. Existence of Nash equilibria

Definition 5.1. A strategy $\boldsymbol{U}=(U^1,\dots,U^N)\in\mathfrak{U}^N$ is called a *Nash equilibrium* for the N-player game if for every $i\in\{1,\dots,N\}$ and $\tilde U^i\in\mathfrak{U}^i$, we have

$$J^i(\boldsymbol{U})\;\le\;J^i(U^1,\dots,U^{i-1},\tilde U^i,U^{i+1},\dots,U^N)\quad\text{for almost for all initial points } x\,.$$

Remark 5.1. The above definition of Nash equilibrium is the one used in [5,18,40]. In [20] such equilibria are referred to as *local* Nash equilibria. In the terminology of [20], $\mathfrak{U}^N$ is the set of all *narrow* strategies.

Let $a^i(x) := \frac{1}{2}\sigma^i(x)\sigma^i(x)^{\mathsf{T}}$. We define the family of operators $L_i^u : \mathcal{C}^2(\mathbb{R}^d) \to \mathcal{C}(\mathbb{R}^d)$, where $u \in \mathbb{U}^i$ plays the role of a parameter, by

$$L_i^u f(x) \;:=\; \operatorname{trace}\bigl(a^i(x)\,\nabla^2 f(x)\bigr) + b^i(x,u)\cdot\nabla f(x)\,, \quad u \in \mathbb{U}^i\,.$$

Therefore, L_i^u is the controlled extended generator of the i-th process in (5.1). We define $\mathcal{G}^i$ and $\mathcal{H}^i$ similar to (2.11) and (2.12) relative to the operator L_i^u. We assume the following.

Assumption 5.1.

(i) For $1 \le i \le N$, there exist an inf-compact $\mathcal{V}^i \in \mathcal{C}^2(\mathbb{R}^d)$, an inf-compact, locally Lipschitz h^i, such that for some positive constants γ_3^i, and γ_4^i we have

$$L_i^u \mathcal{V}^i(x) \;\le\; \gamma_4^i - \gamma_3^i h^i(x) \qquad \text{for all } u \in \mathbb{U}^i. \tag{5.3}$$

Moreover, for any compact $\mathcal{K} \subset \mathcal{P}(\mathbb{R}^d)$ with respect to the metric d_{P}, we have

$$\sup_{u\in\mathbb{U}^i,\,\nu\in\mathcal{K}} r^i(\cdot,u,\nu) \;\in\; \mathfrak{o}(h^i)\,.$$

(ii) There exist non-negative locally Lipschitz functions $g^i \in \mathfrak{o}(h^i)$, $1 \le i \le N$, and $g^0 \in \mathfrak{o}(\min_i h^i)$, satisfying

$$\breve{r}^i(x^1,\dots,x^N,u^i) \;\le\; g^0(x^i) + \frac{1}{N-1}\sum_{j\neq i} g^j(x^j) \qquad \text{for all } u^i \in \mathbb{U}^i\,.$$

There are quite a few cost functions considered in the literature that satisfy Assumption 5.1(ii).

Example 5.1. Consider $g^0 : \mathbb{R}^d \times \mathbb{U}^i \to \mathbb{R}_+$, $g^1 : \mathbb{R}^d \to \mathbb{R}_+$, such that $\sup_{u\in\mathbb{U}^i} g^0(\cdot,u)$ and g^1 are in $\mathfrak{o}(\min_i h^i)$, and define

$$r^i(x,u,\mu) \;:=\; g^0(x,u) + \int g^1 \,\mathrm{d}\mu\,.$$

Note that these running cost functions satisfy Assumption 5.1(ii).

We first show that, under Assumption 5.1, there exists a Nash equilibrium in the sense of Definition (5.1). First, we need to introduce some additional notation. Let

$$\boldsymbol{\mathcal{G}}^N \;:=\; \mathcal{G}^1 \times \cdots \times \mathcal{G}^N, \quad \text{and} \quad \boldsymbol{\mathcal{H}}^N \;:=\; \mathcal{H}^1 \times \cdots \times \mathcal{H}^N\,.$$

By (5.3) the sets $\mathcal{G}^i$, $\mathcal{H}^i$, $1 \le i \le N$, are compact, and as a result, $\boldsymbol{\mathcal{G}}^N$ and $\boldsymbol{\mathcal{H}}^N$ are convex and compact. For $\boldsymbol{\mu} = (\mu^1,\dots,\mu^N) \in \boldsymbol{\mathcal{H}}^N$ we define

$$\breve{r}^i_{\boldsymbol{\mu}}(x,u) \;:=\; \int_{\mathbb{R}^d\times\cdots\times\mathbb{R}^d} \breve{r}^i(x^1,\dots,x^{i-1},x,x^{i+1},\dots,x^N,u) \prod_{j\neq i} \mu^j(\mathrm{d}x^j)\,, \quad 1 \le i \le N. \tag{5.4}$$

Using Assumption 5.1 and the dominated convergence theorem, we deduce that $\breve{r}^i_{\boldsymbol{\mu}} : \mathbb{R}^d \times \mathbb{U}^i \to \mathbb{R}_+$, is a continuous function.

Assumption 5.2. For all $i \in \{1,\dots,N\}$ and $\boldsymbol{\mu} \in \mathcal{H}^N$, the function $\breve{r}^i_{\boldsymbol{\mu}}$ is locally Lipschitz in the variable x uniformly with respect to $u \in \mathbb{U}^i$ and $\boldsymbol{\mu} \in \mathcal{H}^N$.

Example 5.2. Let $F\colon \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}_+$ be a bounded, locally Lipschitz function (with respect to the metric d_{P}). Consider maps $r_i\colon \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}$, $i = 1, 2$, having the property that r_i is locally Lipschitz in first variable uniformly with respect to the second. Define

$$r(x,u,\mu) \;:=\; r_1(x,u) + r_2(x,u)\, F(\mu)\,.$$

It is easy to see that for this running cost, Assumption 5.2 is met.

Example 5.3. Let $\varphi, \varphi_1\colon \mathbb{R}^d \to \mathbb{R}$ be symmetric, locally Lipschitz functions with the property that

$$|\varphi(x) - \varphi(y)| \;\le\; |\varphi_1(x) + \varphi_1(y)|\,|x-y| \qquad \text{for all } x, y \in \mathbb{R}^d\,,$$

and $\varphi, \varphi_1 \in \mathfrak{o}(h)$. Define

$$r(x,u,\mu) \;=\; \int_{\mathbb{R}^d} \varphi(x-y)\,\mu(\mathrm{d}y)\,.$$

Assumption 5.2 is met for this running cost.

By Assumption 5.1(ii), we have $\sup_{u^i \in \mathbb{U}^i} \breve{r}^i_{\boldsymbol{\mu}}(\cdot,\mu) \in \mathfrak{o}(h^i)$ for all $i \in \{1,\dots,N\}$, and all $\boldsymbol{\mu} \in \mathcal{H}^N$. Since $\sup_{\mu \in \mathcal{H}^i} \int h^i \,\mathrm{d}\mu^i < \infty$ for all i by (5.3), we obtain that

$$\sup_{\boldsymbol{\mu} \in \mathcal{H}^N} \sup_{u^i \in \mathbb{U}^i} \breve{r}^i_{\boldsymbol{\mu}}(\cdot,\mu) \;\in\; \mathfrak{o}(h^i) \qquad \text{for all } i \in \{1,\dots,N\}\,. \tag{5.5}$$

Next we treat $\breve{r}^i_{\boldsymbol{\mu}}$ as a running cost, and define the ergodic control problem for $\boldsymbol{\mu} \in \mathcal{H}^N$ as

$$\tilde{\varrho}^i_{\boldsymbol{\mu}} \;:=\; \inf_{U^i \in \mathfrak{U}^i} \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}\biggl[\int_0^T \breve{r}^i_{\boldsymbol{\mu}}(X^i_t, U^i_t)\,\mathrm{d}t\biggr], \quad 1 \le i \le N. \tag{5.6}$$

For every $\boldsymbol{\mu} \in \mathcal{H}^N$, there exists a unique $V^i_{\boldsymbol{\mu}} \in \mathcal{C}^2(\mathbb{R}^d)$, $1 \le i \le N$, satisfying (see [3, Theorem 3.7.12])

$$\min_{u \in \mathbb{U}^i} \bigl[L^u_i V^i_{\boldsymbol{\mu}}(x) + \breve{r}^i_{\boldsymbol{\mu}}(x,u)\bigr] \;=\; \tilde{\varrho}^i_{\boldsymbol{\mu}}\,, \qquad V^i_{\boldsymbol{\mu}}(0) \;=\; 0\,, \qquad V^i_{\boldsymbol{\mu}} \;\in\; \mathfrak{o}(\mathcal{V}^i)\,. \tag{5.7}$$

As we have discussed earlier in (3.2), any measurable selector of (5.7) is an optimal Markov control for (5.6) and vice-versa. We define

$$\begin{aligned}
\boldsymbol{\mathcal{A}}(\boldsymbol{\mu}) \;&:=\; \bigl\{\boldsymbol{\pi} = (\pi^1,\dots,\pi^N) \in \mathfrak{G}^N \;:\; \pi^i = \mu^i_{v^i} \circledast v^i\,,\ \text{and } v^i \in \mathfrak{U}^i_{\mathrm{SSM}} \\
&\qquad \text{is a measurable selector satisfying (5.7) for all } i\bigr\}\,, \\
\boldsymbol{\mathcal{A}}^*(\boldsymbol{\mu}) \;&:=\; \bigl\{\boldsymbol{\nu} = (\nu^1,\dots,\nu^N) \in \mathcal{H}^N \;:\; (\nu^1 \circledast v^1,\dots,\nu^N \circledast v^N) \in \boldsymbol{\mathcal{A}}(\boldsymbol{\mu}) \\
&\qquad \text{for some } \boldsymbol{v} = (v^1,\dots,v^N) \in \mathfrak{U}_{\mathrm{SSM}}\bigr\}\,.
\end{aligned}$$

It is easy to find the analogy of the above maps with $\mathcal{A}$ and $\mathcal{A}^*$ defined in Section 3. The following theorem establishes the existence of a Nash equilibrium for the N-person game.

Theorem 5.1. *Let Assumptions 5.1 and 5.2 hold. Then there exists $\boldsymbol{\mu} \in \mathfrak{H}^N$ satisfying*

$$\boldsymbol{\mu} \;\in\; \boldsymbol{\mathcal{A}}^*(\boldsymbol{\mu})\,. \tag{5.8}$$

In particular, there exists a stable Markov control $\boldsymbol{v} = (v^1, \dots, v^N) \in \mathfrak{U}_{\mathrm{SSM}}$ such that, for all $i \in \{1, \dots, N\}$, we have

$$\tilde{\varrho}^i_{\boldsymbol{\mu}} \;=\; J^i(\boldsymbol{v}) \;\le\; J^i(v^1, \dots, v^{i-1}, \tilde{U}^i, v^{i+1}, \dots, v^N) \qquad \text{for all } \tilde{U}^i \in \mathfrak{U}^i\,. \tag{5.9}$$

Proof. The main idea of the proof is to use the Kakutani–Fan–Glicksberg fixed point theorem as we have done in the proof of Theorem 3.2. Consider the convex, compact set $\mathfrak{H}^N$. Since the product of Hausdorff locally convex spaces is again a Hausdorff locally convex space, it follows that $\mathfrak{H}^N$ is a non-empty, convex, compact subset of $\mathcal{M}(\mathbb{R}^d) \times \cdots \times \mathcal{M}(\mathbb{R}^d)$. Following an argument similar to Lemma 3.1 we deduce that $\boldsymbol{\mathcal{A}}^*(\boldsymbol{\mu})$ is non-empty, convex and compact for all $\boldsymbol{\mu} \in \mathfrak{H}^N$. Let $\boldsymbol{\mu}_n \to \boldsymbol{\mu}$ as $n \to \infty$. Using the non-degeneracy of a^i, $1 \le i \le N$, we can improve this to convergence under the total variation norm ([3, Lemma 3.2.5]). Therefore using (5.5) together with an argument similar to the proofs of Lemmas 3.2–3.4 we obtain that $\boldsymbol{\mu} \mapsto \boldsymbol{\mathcal{A}}^*(\boldsymbol{\mu})$ is upper-hemicontinuous. Hence we can apply the Kakutani–Fan–Glicksberg fixed point theorem [3, Corollary 17.55] to obtain a $\boldsymbol{\mu} \in \mathfrak{H}^N$ satisfying $\boldsymbol{\mu} \in \boldsymbol{\mathcal{A}}^*(\boldsymbol{\mu})$. This proves (5.8).

By the definition of an ergodic occupation measure, we can find $\boldsymbol{v} = (v^1, \dots, v^N) \in \mathfrak{U}_{\mathrm{SSM}}$ such that for $\boldsymbol{\mu} = (\mu^1, \dots, \mu^N)$ we have

$$(\mu^1 \circledast v^1, \dots, \mu^N \circledast v^N) \;\in\; \boldsymbol{\mathcal{A}}(\mu)\,. \tag{5.10}$$

In particular, μ^i the unique invariant probability measure of (5.1) associated to the stationary Markov control v^i. Without loss of generality we fix $i = 1$. To show (5.9) we consider $\tilde{U}^1 \in \mathfrak{U}^i$. Define the occupation measure ξ_T on $\mathbb{R}^d \times \mathbb{U}^1 \times \mathbb{R}^{(N-1)d}$ as follows:

$$\xi_T(A \times B \times C) \;:=\; \frac{1}{T}\mathbb{E}\biggl[\int_0^T \mathbb{1}_{A\times B\times C}(X^1_t, U^1_t, \hat{X}^2_t, \dots, \hat{X}^N_t)\,\mathrm{d}t\biggr], \quad T > 0\,, \tag{5.11}$$

for $A \in \mathcal{B}(\mathbb{R}^d)$, $B \in \mathcal{B}(\mathbb{U}^1)$, $C \in \mathcal{B}(\mathbb{R}^{(N-1)d})$, where (X^1, U^1) solves (5.1) for $i = 1$, and X^j, $j > 1$, are the solutions to (5.1) under the Markov control v^j. Using Assumption 5.1, we deduce that $\{\xi_T, T > 0\}$ is a tight family of probability measures. By weak convergence, we have $\mathbb{E}[f(\hat{X}^i(T))] \to \mu^i(f)$, $i \ge 2$, as $T \to \infty$, for any bounded continuous function $f\colon \mathbb{R}^d \to \mathbb{R}$. Hence using the independence property of $\bigl((X^1, U^1), \hat{X}^2, \dots, \hat{X}^N\bigr)$ and the definition in (5.11), we can easily show that as $T \to \infty$, the limit points of ξ_T as $T \to \infty$ belong to the set

$$\{\pi^1 \times \mu^2 \times \cdots \times \mu^N \;:\; \pi \in \mathcal{G}^1\}\,.$$

For above to hold we also use the fact that the collection

$$\bigl\{g(x^1, u) \cdot f^2(x^2) \cdots f^N(x^N) \;:\; g \in \mathcal{C}_b(\mathbb{R}^d \times \mathbb{U}),\; f^i \in \mathcal{C}_b(\mathbb{R}^d)\bigr\}$$

determines the probability measures on $\mathbb{R}^d \times \mathbb{U} \times (\mathbb{R}^d)^{N-1}$. By lower-semicontinuity we have

$$J^1(U^1, v^2, \dots, v^N) \;\ge\; \int_{\mathbb{R}^d\times\mathbb{U}^1} \breve{r}^1_{\boldsymbol{\mu}}(x, u)\,\mathrm{d}\pi^1$$

for some $\pi^1 \in \mathcal{G}^1$. Hence using (5.10) and [3, Theorem 3.7.12] we obtain

$$J^1(U^1, v^2, \dots, v^N) \;\geq\; \int_{\mathbb{R}^d \times \mathbb{U}^1} \breve{r}^1_{\boldsymbol{\mu}}(x,u)\, \mathrm{d}\pi^1 \;\geq\; \tilde{\varrho}^1_{\boldsymbol{\mu}}\,.$$

To complete the proof we observe that $(\hat{X}^1, \dots, \hat{X}^N)$ is a strong Markov process with invariant probability measure $\mu^1 \times \cdots \times \mu^N$. Therefore, by Birkhoff's ergodic theorem, we have

$$J^1(v^1, \dots, v^N) \;=\; \tilde{\varrho}^1_{\boldsymbol{\mu}} \;=\; \limsup_{T\to\infty} \frac{1}{T}\, \mathbb{E}\biggl[\int_0^T \breve{r}^1\bigl(\hat{X}^1_t, \dots, \hat{X}^N_t, v^1(\hat{X}^1)\bigr)\, \mathrm{d}t\biggr]\,. \qquad \square$$

5.2. Limits of symmetric Nash equilibria

We now let the number of players N tend to infinity, assuming that all the players are identical. Hence, for the rest of this section we assume that

$$\mathbb{U}^i = \mathbb{U}, \quad b^i = b, \quad \sigma^i = \sigma, \quad r^i = r, \quad \mathcal{V}^i = \mathcal{V}, \quad h^i = h \qquad \text{for all } i \in \{1, \dots, N\}\,.$$

A similar argument as in the proof of Theorem 5.1 provides us with a Nash equilibrium of the form $\boldsymbol{v} = (v, \dots, v)$ and $\boldsymbol{\mu} = (\mu, \dots, \mu)$, where μ is the unique invariant probability measure corresponding to the Markov control v, and v is a measurable selector of (5.7) (compare this with [40, Theorem 2.2]). These equilibria are known as the symmetric Nash equilibria.

Remark 5.2. Any (Markovian) Nash equilibrium for the N-player game is related to a fixed point of the map $\boldsymbol{\mathcal{A}}^*(\cdot)$. To elaborate consider any tuple $(v^1, \dots, v^N) \in \mathfrak{U}^N_{\mathbf{SSM}}$ that corresponds to a Nash equilibrium. Let $\boldsymbol{\mu} = (\mu^1, \dots, \mu^N)$ be the corresponding invariant measures. Solve equations (5.6)–(5.7) with above choice of $\boldsymbol{\mu}$. Since $(v^1, \dots, v^N)$ is a Nash equilibrium, it follows that v^i is an optimal Markov control in (5.6). Thus $\boldsymbol{\mu} \in \boldsymbol{\mathcal{A}}^*(\boldsymbol{\mu})$.

Remark 5.3. Assumption 5.2 is not very crucial for Theorem 5.1. If $\breve{r}^i_{\boldsymbol{\mu}}$ is only continuous, then the value function $V^i_{\boldsymbol{\mu}}$ is in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p \geq 1$ instead of $\mathcal{C}^2(\mathbb{R}^d)$, but the conclusion of Theorem 5.1 still holds.

In the rest of this section we discuss the convergence of the N-person game as N tends to infinity. In what follows we work with Wasserstein metric instead of the metric of weak convergence. We also need some additional regularity assumptions on r, which are as follows.

Assumption 5.3. The following hold:

(i) There exists an inf-compact $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$, such that for some positive constants γ_3, γ_4,

$$L^u \mathcal{V}(x) \;\leq\; \gamma_4 - \gamma_3\, |x|^q, \qquad \text{for all } u \in \mathbb{U}, \text{ and } q > 1, \tag{5.12}$$

and for any compact $\mathcal{K} \subset \mathcal{P}(\mathbb{R}^d)$, with respect to the metric $\mathcal{W}_{\bar{q}}$, $\bar{q} \in [1, q)$, we have

$$\sup_{u\in\mathbb{U}, \nu\in\mathcal{K}} r(\cdot, u, \nu) \;\in\; \mathfrak{o}(|x|^q)\,.$$

Moreover, there exists non-negative locally Lipschitz functions $g^0 \in \mathfrak{o}(|x|^q)$ and $g^1 \in \mathfrak{o}(|x|^q)$ satisfying

$$r\biggl(x, u, \frac{1}{N}\sum_{j=1}^N \delta_{y^j}\biggr) \;\leq\; g^0(x) + \frac{1}{N}\sum_{j=1}^N g^1(y^j) \qquad \text{for all } u \in \mathbb{U}, \text{ and } N \geq 1\,. \tag{5.13}$$

Also

$$(x,u) \;\mapsto\; \hat{r}^N_{\boldsymbol{\mu}}(x,u) \;:=\; \int_{\mathbb{R}^{Nd}} r\Big(x,u,\frac{1}{N}\sum_{j=1}^N \delta_{y^j}\Big)\prod_{j=1}^N \mu^j(\mathrm{d}y^j) \tag{5.14}$$

is continuous, and locally Lipschitz in x (the Lipschitz constant might depend on N) uniformly in u, for all $\boldsymbol{\mu}=(\mu^1,\dots,\mu^N)\in\mathfrak{H}^N$.

(ii) For some $\bar{q}\in[1,q)$ we have

$$|r(x,u,\nu)-r(x,u,\tilde{\nu})| \;\le\; \kappa_R\left(1+\int_{\mathbb{R}^d}|y|^{\bar{q}}\nu(\mathrm{d}y)+\int_{\mathbb{R}^d}|y|^{\bar{q}}\tilde{\nu}(\mathrm{d}y)\right)^{1-1/\bar{q}}\mathcal{W}_{\bar{q}}(\nu,\tilde{\nu}) \tag{5.15}$$

for all $|x|\le R$, $u\in\mathbb{U}$, and $R>0$. For every $(x,u)\in\mathbb{R}^d\times\mathbb{U}$ and $R>0$, there exists a constant κ', depending on x, u, and R, such that

$$\left|r\Big(x,u,\frac{1}{N}\sum_{j=1}^N\delta_{y^j}\Big)-r\Big(x,u,\frac{1}{N-1}\sum_{j=1}^{N-1}\delta_{y^j}\Big)\right| \;\le\; \kappa'\,\frac{1}{N}\qquad \forall\, y^j\in B_R\,. \tag{5.16}$$

(iii) $\mathbb{U}$ is a convex set and for all $R>0$ the following holds: for any $\theta\in(0,1)$ there exists $\kappa_{\theta,R}>0$, such that

$$\begin{aligned} &b(x,\theta u+(1-\theta)u')\cdot p+r(x,\theta u+(1-\theta)u',\mu)\\ &\qquad\le\; \theta[b(x,u)\cdot p+r(x,u,\mu)]+(1-\theta)[b(x,u')\cdot p+r(x,u',\mu)]-\kappa_{\theta,R}\end{aligned}$$

for all $u,u'\in\mathbb{U}$, $\mu\in\mathcal{P}(\mathbb{R}^d)$, and $|x|,|p|\le R$.

We note that (5.12) is the uniform stability condition we have used before, and (5.15) is a Lipschitz property of the function r in the variable μ. Note also that for $\bar{q}\in[1,q)$, $\mu\mapsto r(x,u,\mu)$ is locally Lipschitz uniformly with respect to (x,u) in compact subsets of $\mathbb{R}^d\times\mathbb{U}$. Assumption 5.3(iii) is a strict convexity condition that we need in order to resolve the issue of non-uniqueness of the optimal control. Running costs considered in [5,18,40] do satisfy this condition.

Example 5.4. Let $r(x,u,\mu)=R(x,u,\zeta(x,\mu))$ where $R\colon\mathbb{R}^d\times\mathbb{U}\times\mathbb{R}$ is a continuous function and for every compact $K\subset\mathbb{R}^d$ there exists constant γ_K satisfying

$$|R(x,u,z)-R(y,u,z_1)| \;\le\; \gamma_K\,(|x-y|+|z-z_1|)\qquad \text{for all } z,\, z_1\in\mathbb{R}\,,\ \text{and } (x,y)\in K\times K. \tag{5.17}$$

Also suppose that for some $g_0\in\mathfrak{o}(|x|^q)$ we have

$$R(x,u,z) \;\le\; g_0(x)+\kappa|z|\qquad \forall\, x\in\mathbb{R}^d,\ z\in\mathbb{R}\,,$$

for some constant $\kappa>0$, and that for some $\bar{q}\in[1,q)$ we have

$$\zeta(x,\mu) \;=\; \int_{\mathbb{R}^d}|x-y|^{\bar{q}}\,\mathrm{d}\mu\,.$$

Since $a^{\bar{q}}-b^{\bar{q}}\le\bar{q}\,(a^{\bar{q}-1}+b^{\bar{q}-1})|a-b|$ for all $a,b\ge0$ and $\bar{q}\ge1$, then, for any $\gamma\in\mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d)$ with marginals $\nu,\tilde{\nu}$, it holds that

$$\begin{aligned}\zeta(x,\nu)-\zeta(x,\tilde\nu) \;&\le\; \bar q \int_{\mathbb{R}^d\times\mathbb{R}^d} (|x|^{\bar q-1}+|y|^{\bar q-1})|x-y|\gamma(\mathrm{d}x,\mathrm{d}y)\\ &\le\; \kappa\biggl(\int_{\mathbb{R}^d\times\mathbb{R}^d}(1+|x|^{\bar q}+|y|^{\bar q})\gamma(\mathrm{d}x,\mathrm{d}y)\biggr)^{\frac{\bar q-1}{\bar q}}\biggl(\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^{\bar q}\gamma(\mathrm{d}x,\mathrm{d}y)\biggr)^{\frac{1}{\bar q}}.\end{aligned}$$

Since γ is arbitrary, using (5.17), we deduce that r satisfies (5.15). One can also show that (5.14) and (5.16) are also satisfied.

Similar to (5.4) we define for $\boldsymbol{\mu}=(\mu_N^1,\dots,\mu_N^N)\in\mathfrak{H}^N$,

$$\breve r_{\boldsymbol{\mu}}^{i,N}(x,u) \;:=\; \int_{\mathbb{R}^d\times\cdots\times\mathbb{R}^d} \breve r^i(x^1,\dots,x^{i-1},x,x^{i+1},\dots,x^N)\prod_{j\ne i}\mu_N^j(\mathrm{d}x^j)\,,\quad 1\le i\le N.$$

Theorem 5.2. *Let Assumption 5.3 hold. Let $\boldsymbol{\mu}=(\mu_N^1,\dots,\mu_N^N)$ be such that for $\tilde\varrho_N^i\in\mathbb{R}$, $V_N^i\in\mathcal{C}^2(\mathbb{R}^d)$, $1\le i\le N$, we have*

$$\min_{u\in\mathbb{U}}\bigl[L^uV_N^i(x)+\breve r_{\boldsymbol{\mu}}^{i,N}(x,u)\bigr] \;=\; L^{v_N^i}V_N^i(x)+\breve r_{\boldsymbol{\mu}}^{i,N}(x,v_N^i) \;=\; \tilde\varrho_N^i\,,\quad V_N^i(0)=0\,,\quad V_N^i\in\mathfrak{o}(\mathcal{V})\,, \tag{5.18}$$

$$\int_{\mathbb{R}^d} L^{v_N^i}f(x)\,\mu_N^i(\mathrm{d}x) \;=\; 0\quad \textit{for all } f\in\mathcal{C}_c^2(\mathbb{R}^d)\,,\quad 1\le i\le N. \tag{5.19}$$

Then the following hold:

(a) $\{(\tilde\varrho_N^i,V_N^i,\mu_N^i)\}_{i,N}$ *is relatively compact in* $\mathbb{R}\times\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)\times(\mathcal{P}_{\bar q}(\mathbb{R}^d),\mathcal{W}_{\bar q})$ *for any* $1\le p<\infty$;
(b) $\sup_{i,j}\bigl(|\tilde\varrho_N^i-\tilde\varrho_N^j|+\|V_N^i-V_N^j\|_{\mathcal{W}^{2,p}(K)}+\mathcal{W}_{\bar q}(\mu_N^i,\mu_N^j)\bigr)\to 0$ *as* $N\to\infty$, *for all compact subsets* $K\subset\mathbb{R}^d$;
(c) *any limit point* $(\tilde\varrho,V,\mu)$ *of* $\{(\tilde\varrho_N^i,V_N^i,\mu_N^i)\}_{i,N}$ *solves*

$$\min_{u\in\mathbb{U}}\bigl[L^uV(x)+r(x,u,\mu)\bigr] \;=\; L^vV(x)+r(x,v,\mu) \;=\; \tilde\varrho\,,\quad V(0)=0\,,\quad V\in\mathfrak{o}(\mathcal{V})\,, \tag{5.20}$$

$$\int_{\mathbb{R}^d} L^vf(x)\,\mu(\mathrm{d}x) \;=\; 0\quad \textit{for all } f\in\mathcal{C}_c^2(\mathbb{R}^d)\,. \tag{5.21}$$

We see that (5.20)–(5.21) defines a MFG solution in the sense of Definition 3.2. Theorem 5.2 asserts that the limits of N-player games are solutions to mean field games. Similar results are also obtained in [18,40] in the case of a compact state space. One of the key ideas to prove Theorem 5.2 is to use (5.12) to show that one can consider compact subsets of $\mathbb{R}^d$ to approximate integrals. This is done following the method in [11, Corollary 5.13]. To accomplish this we introduce the projection map

$$\mathfrak{P}(x) \;=\; \mathfrak{P}_R(x) \;:=\; \begin{cases} x & \text{if } x\in\bar B_R(0)\,,\\ 0 & \text{otherwise.}\end{cases}$$

Also define $\tilde\mu_N^i(B) \;=\; \mu_N^i(\mathfrak{P}^{-1}(B))$ for $B\in\mathcal{B}(\mathbb{R}^d)$. Then we have the following result.

Lemma 5.1. *Let $\varepsilon>0$ be given. Then for any compact set $C\subset\mathbb{R}^d$, there exists $R>0$, such that*

$$\sup_{(x,u)\in C\times\mathbb{U}}\bigl|\hat r_{\boldsymbol{\mu}}^N(x,u)-\hat r_{\tilde{\boldsymbol{\mu}}}^N(x,u)\bigr| \;\le\; \varepsilon\qquad\forall\, N\ge 1\,,$$

where $\hat r_{\boldsymbol{\mu}}^N$ is given by (5.14).

Proof. We claim that for any $x^j, y^j \in \mathbb{R}^d$, $1 \le j \le N$, we have

$$\mathcal{W}_{\bar q}\Big(\tfrac{1}{N}\textstyle\sum_{j=1}^N \delta_{x^j}, \tfrac{1}{N}\sum_{j=1}^N \delta_{y^j}\Big) \;\le\; \Big(\tfrac{1}{N}\textstyle\sum_{i=1}^N |x^j - y^j|^{\bar q}\Big)^{1/\bar q}. \tag{5.22}$$

Indeed, this can be obtained by choosing $\nu(\mathrm{d}x,\mathrm{d}y) := \frac{1}{N}\sum_{j=1}^N \delta_{(x^j,y^j)}$ in (2.5). Using (5.12) we can find a constant κ_1 such that

$$\sup_{\nu\in\mathcal{H}} \int_{\mathbb{R}^d} |x|^q \,\nu(\mathrm{d}x) \;\le\; \kappa_1\,. \tag{5.23}$$

Then

$$\begin{aligned}
&\big|\hat r^N_{\boldsymbol{\mu}}(x,u) - \hat r^N_{\tilde{\boldsymbol{\mu}}}(x,u)\big| \\
&\quad= \Big|\int_{\mathbb{R}^{Nd}} \Big[r\Big(x,u,\frac{1}{N}\sum_{j=1}^N \delta_{y^j}\Big) - r\Big(x,u,\frac{1}{N}\sum_{j=1}^N \delta_{\mathfrak{P}(y^j)}\Big)\Big]\prod_{j=1}^N \mu_N^j(\mathrm{d}y^j)\Big| \\
&\quad\le \kappa_C \int_{\mathbb{R}^{Nd}} \Big(1+\frac{1}{N}\sum_{j=1}^N\big(|y^j|^{\bar q}+|\mathfrak{P}(y^j)|^{\bar q}\big)\Big)^{\frac{\bar q-1}{\bar q}} \mathcal{W}_{\bar q}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{y^j}, \frac{1}{N}\sum_{j=1}^N \delta_{\mathfrak{P}(y^j)}\Big)\prod_{j=1}^N \mu_N^j(\mathrm{d}y^j) \\
&\quad\le \kappa_C\Big(\int_{\mathbb{R}^{Nd}} \Big(1+\frac{1}{N}\sum_{j=1}^N\big(|y^j|^{\bar q}+|\mathfrak{P}(y^j)|^{\bar q}\big)\Big)\prod_{j=1}^N \mu_N^j(\mathrm{d}y^j)\Big)^{\frac{\bar q-1}{\bar q}} \\
&\qquad\times\Big(\int_{\mathbb{R}^{Nd}}\Big[\mathcal{W}_{\bar q}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{y^j}, \frac{1}{N}\sum_{j=1}^N \delta_{\mathfrak{P}(y^j)}\Big)\Big]^{\bar q}\prod_{j=1}^N \mu_N^j(\mathrm{d}y^j)\Big)^{\frac{1}{\bar q}} \\
&\quad\le \kappa_2\Big(\int_{\mathbb{R}^{Nd}} \frac{1}{N}\sum_{j=1}^N |y^j-\mathfrak{P}(y^j)|^{\bar q}\prod_{j=1}^N \mu_N^j(\mathrm{d}y^j)\Big)^{\frac{1}{\bar q}} \\
&\quad\le \kappa_2\Big(\frac{1}{N}\sum_{i=1}^N \int_{B_R^c(0)} |y^j|^{\bar q}\mu_N^j(\mathrm{d}y^j)\Big)^{\frac{1}{\bar q}} \\
&\quad\le \kappa_3\frac{1}{R^{q-\bar q}} \qquad \forall\,(x,u)\in C\times\mathbb{U}\,,
\end{aligned}$$

for some constants κ_2, κ_3, where in the third line we use (5.15), in the fourth line we use the Hölder inequality, (5.22) is used in the fifth line, and in the last line we use (5.23). Choosing R large enough completes the proof. □

Proof of Theorem 5.2. Since $(g^0 \vee g^1)(x) \le \kappa_2(1+|x|^q)$ we obtain from (5.13) that

$$\check r^{i,N}_{\boldsymbol{\mu}}(x,u) \;\le\; g^0(x) + \frac{\kappa_2}{N-1}\sum_{j=1}^{N-1}\int_{\mathbb{R}^d}(1+|x|^q)\,\mathrm{d}\mu_N^j \;\le\; g^0(x)+\kappa_2(1+\kappa_1)\,, \tag{5.24}$$

where we also use (5.23). Recall that $\check\tau_r$ denotes the hitting time to the ball $B_r(0)$ and τ_R is the exit time from the ball $B_R(0)$. From [3, Lemma 3.3.4] and (5.12) we know that $\sup_{v\in\mathfrak{U}_{\mathrm{SSM}}} \mathbb{E}_x[\check\tau_r] < \infty$ for $r > 0$. Therefore using Itô's formula in (5.12) we obtain that for $r > 0$,

$$\limsup_{R\to\infty}\sup_{v\in\mathfrak{U}_{\rm SSM}}\mathbb{E}_x^v\biggl[\mathcal{V}(X_{\check{\tau}_r\wedge\tau_R})+\gamma_3\int_0^{\check{\tau}_r\wedge\tau_R}|X_t|^q\mathrm{d}t\biggr]\;\le\;\mathcal{V}(x)+\kappa_3 \tag{5.25}$$

for some constant κ_3. Since $V_N^i\in\mathfrak{o}(\mathcal{V})$, for every $\varepsilon>0$, there exists κ_ε satisfying $V_N^i(x)\le\kappa_\varepsilon+\varepsilon\mathcal{V}(x)$. Therefore, using (5.25) we obtain

$$\begin{aligned}\limsup_{R\to\infty}\sup_{v\in\mathfrak{U}_{\rm SSM}}\mathbb{E}_x\bigl[\mathbb{1}_{\{\tau_R<\check{\tau}_r\}}V_N^i(X_{\tau_R})\bigr]\;&\le\;\varepsilon\limsup_{R\to\infty}\sup_{v\in\mathfrak{U}_{\rm SSM}}\mathbb{E}_x\bigl[\mathbb{1}_{\{\tau_R<\check{\tau}_r\}}\mathcal{V}(X_{\tau_R})\bigr]\\&\le\;\varepsilon(\mathcal{V}(x)+\gamma_4)\,.\end{aligned}$$

Since ε is arbitrary, we have $\limsup_{R\to\infty}\sup_{v\in\mathfrak{U}_{\rm SSM}}\mathbb{E}_x\bigl[\mathbb{1}_{\{\tau_R<\check{\tau}_r\}}V_N^i(X_{\tau_R})\bigr]=0$. Thus applying Itô's lemma to (5.18) we have

$$V_N^i(x)\;=\;\inf_{v\in\mathfrak{U}_{\rm SSM}}\mathbb{E}_x^v\biggl[\int_0^{\check{\tau}_r}\bigl(\check{r}_{\boldsymbol{\mu}}^{i,N}(X_t,v(X_t))-\tilde{\varrho}_N^i\bigr)\,\mathrm{d}t+V_N^i(X_{\check{\tau}_r})\biggr],\quad x\in B_r^c(0)\,,\;r>0\,. \tag{5.26}$$

A similar argument as in (5.25) gives us

$$\limsup_{T\to\infty}\sup_{v\in\mathfrak{U}_{\rm SSM}}\mathbb{E}_x^v\biggl[\frac{1}{T}\mathcal{V}(X_T)+\frac{\gamma_3}{T}\int_0^T|X_t|^q\mathrm{d}t\biggr]\;\le\;\gamma_4\,. \tag{5.27}$$

Therefore using (5.27) and the fact that $V_N^i\in\mathfrak{o}(\mathcal{V})$ we have $\lim_{T\to\infty}\frac{1}{T}\sup_{v\in\mathfrak{U}_{\rm SSM}}\mathbb{E}_x[V_N^i(X_T)]=0$. Applying Dynkin's theorem to (5.18), and using (5.24) and (5.27), we obtain

$$\begin{aligned}\tilde{\varrho}_N^i\;&=\;\lim_{T\to\infty}\frac{1}{T}\mathbb{E}_x^{v_N^i}\biggl[\int_0^T\check{r}_{\boldsymbol{\mu}}^{i,N}(X_t,v_N^i(X_t))\,\mathrm{d}t\biggr]\\&\le\;\lim_{T\to\infty}\frac{1}{T}\mathbb{E}_x^{v_N^i}\biggl[\int_0^T g^0(X_t)\,\mathrm{d}t\biggr]+\kappa_2(1+\kappa_1)\\&\le\;\kappa_4\end{aligned}$$

for some constant κ_4, independent of i and N. This shows that $\{\tilde{\varrho}_N^i\}_{i,N}$ is relatively compact. By (5.27) and Proposition 2.1 we see that $\{\mu_N^i\}_{i,N}$ is relatively compact in $(\mathcal{P}_{\bar{q}}(\mathbb{R}^d),\mathcal{W}_{\bar{q}})$. Next we prove the compactness of $\{V_N^i\}_{i,N}$. Define

$$V_{i,N}^\alpha(x)\;:=\;\inf_{U\in\mathfrak{U}}\mathbb{E}_x^U\biggl[\int_0^\infty e^{-\alpha t}\,\check{r}_{\boldsymbol{\mu}}^{i,N}(X_t,U_t)\,\mathrm{d}t\biggr].$$

It is shown in [3, Theorem 3.7.12] that $V_{i,N}^\alpha(x)-V_{i,N}^\alpha(0)$ are locally bounded in $\mathcal{W}^{2,p}_{\rm loc}(\mathbb{R}^d)$ and converge to V_N^i, as $\alpha\to0$, in $\mathcal{W}^{2,p}_{\rm loc}(\mathbb{R}^d)$, $p\ge1$. By (5.24) we have that $\check{r}_{\boldsymbol{\mu}}^{i,N}$ are locally bounded uniformly in N and $i\in\{1,\dots,N\}$. Thus applying [3, Lemma 3.6.3] we obtain that for any $R>0$ there exists a constant ϖ_R, independent of i,N, such that

$$\bigl\|V_{i,N}^\alpha(\cdot)-V_{i,N}^\alpha(0)\bigr\|_{\mathcal{W}^{2,p}(B_R(0))}\;\le\;\varpi_R\Bigl(1+\alpha\sup_{B_R}V_\alpha^{i,N}\Bigr)\,. \tag{5.28}$$

Since $g^0\in\mathfrak{o}(h)$, using (5.27) and (5.12) it is easy to see that

$$\sup_{x\in B_R(0)} \alpha V_\alpha^{i,N}(x) \;\le\; \widehat{\varpi}_R$$

for some constant $\widehat{\varpi}_R$. Thus using (5.28) we have $\|V_{i,N}^\alpha(\cdot) - V_{i,N}^\alpha(0)\|_{\mathrm{W}^{2,p}(B_R(0))} \;\le\; \varpi_R\big(1+\widehat{\varpi}_R\big)$, which gives

$$\|V_N^i\|_{\mathrm{W}^{2,p}(B_R(0))} \;\le\; \varpi_R\big(1+\widehat{\varpi}_R\big)\,, \tag{5.29}$$

with ϖ_R, $\widehat{\varpi}_R$ do not depending on i, N. Since $V_N^i(0) = 0$, we have

$$\sup_{B_1(0)} |V_N^i(x)| \;\le \kappa_5$$

for some constant κ_5. By [3, Lemma 3.7.2] we have

$$x \mapsto \sup_{v\in\mathfrak{U}_{\mathrm{SSM}}} \mathbb{E}_x\left[\int_0^{\check{\tau}_1} (1+g^0(X_t))\,\mathrm{d}t\right] \in \mathfrak{o}(\mathcal{V})\,.$$

Thus from (5.24) and (5.26) we obtain that $\sup_{x\in B_R(0)}|V_N^i(x)| \;\le \kappa_R$ where κ_R is independent of i, N. Therefore using standard elliptic regularity theory in (5.18) we deduce that $\{V_N^i\}$ is bounded in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$. This completes the proof of part (a).

Next we prove part (b). Recall the definition of $\hat{r}^N$ from (5.14). Consider the unique solution $W_N \in \mathcal{C}^2(\mathbb{R}^d)$ of the equation

$$\min_{u\in\mathbb{U}} \big[L^u W_N(x) + \hat{r}_{\boldsymbol{\mu}}^N(x,u)\big] \;=\; \lambda_N, \quad W_N(0) = 0, \quad W_N \in \mathfrak{o}(\mathcal{V}). \tag{5.30}$$

For existence and uniqueness of W_N we refer the reader to [3, Theorem 3.7.12]. From (5.18) and (5.30) we have the following characterizations:

$$\begin{aligned}
\tilde{\varrho}_N^i \;&=\; \min_{\pi\in\mathcal{G}} \int_{\mathbb{R}^d\times\mathbb{U}} \check{r}_{\boldsymbol{\mu}}^{i,N}(x,u)\,\pi(\mathrm{d}x,\mathrm{d}u)\,,\\
\lambda_N \;&=\; \min_{\pi\in\mathcal{G}} \int_{\mathbb{R}^d\times\mathbb{U}} \hat{r}_{\boldsymbol{\mu}}^N(x,u)\,\pi(\mathrm{d}x,\mathrm{d}u)\,.
\end{aligned}$$

It is easy to see that $\hat{r}^N$ satisfies a similar estimate as (5.24) for all x and u. Using (5.23)–(5.24) we see that for any $\varepsilon > 0$, we can find $R > 0$ large enough satisfying

$$\sup_{\pi\in\mathcal{G}} \int_{B_R^c(0)\times\mathbb{U}} \check{r}_{\boldsymbol{\mu}}^{i,N}(x,u)\,\pi(\mathrm{d}x,\mathrm{d}u) + \sup_{\pi\in\mathcal{G}} \int_{B_R^c(0)\times\mathbb{U}} \hat{r}_{\boldsymbol{\mu}}^N(x,u)\,\pi(\mathrm{d}x,\mathrm{d}u) \;\le\; \varepsilon\,. \tag{5.31}$$

Recall the projection map $\mathfrak{P} = \mathfrak{P}_R$ and $\tilde{\mu}_N^i = \mu_N^i \circ \mathfrak{P}^{-1}$. By Lemma 5.1, for each $\varepsilon > 0$, there exists $R_1 > 0$ such that

$$\begin{aligned}
\sup_{(x,u)\in B_R\times\mathbb{U}} \big|\hat{r}_{\boldsymbol{\mu}}^N(x,u) - \hat{r}_{\tilde{\boldsymbol{\mu}}}^N(x,u)\big| \;&\le\; \varepsilon\,,\\
\sup_{(x,u)\in B_R\times\mathbb{U}} \big|\check{r}_{\boldsymbol{\mu}}^{i,N}(x,u) - \check{r}_{\tilde{\boldsymbol{\mu}}}^{i,N}(x,u)\big| \;&\le\; \varepsilon\,.
\end{aligned} \tag{5.32}$$

It is also easy to see that for any $\{y^j\}_{j\ge1} \subset \bar{B}_{R_1}(0)$ we have

$$\mathcal{W}_{\bar q}\Big(\frac{1}{N-1}\sum_{j\neq i}\delta_{y^j},\frac{1}{N}\sum_{j=1}^{N}\delta_{y^j}\Big) \;\le\; \Big[\mathcal{W}_1\Big(\frac{1}{N-1}\sum_{j\neq i}\delta_{y^j},\frac{1}{N}\sum_{j=1}^{N}\delta_{y^j}\Big)\Big]^{\frac{1}{\bar q}}(2R)^{\frac{\bar q-1}{\bar q}} \le \frac{4R}{N^{1/\bar q}},$$

which gives, by (5.15), that

$$\begin{aligned}\sup_{x\in B_R,u\in\mathbb{U}}&\big|\breve r^{i,N}_{\tilde{\boldsymbol\mu}}(x,u)-\hat r^{N}_{\tilde{\boldsymbol\mu}}(x,u)\big|\\ &=\sup_{x\in B_R,u\in\mathbb{U}}\Big|\int_{\mathbb{R}^{Nd}}\Big[r\Big(x,u,\frac{1}{N-1}\sum_{j\neq i}\delta_{y^j}\Big)-r\Big(x,u,\frac{1}{N}\sum_{j=1}^{N}\delta_{y^j}\Big)\Big]\prod_{j=1}^{N}\tilde\mu^j(\mathrm{d}y^j)\Big|\\ &\le\frac{\kappa_1}{N^{1/\bar q}}\end{aligned}\tag{5.33}$$

for some constant κ_1, which depends on R_1 but not on N. Thus combining (5.33) with (5.31) and (5.32) we obtain $\sup_{i,j}|\tilde\varrho^i_N-\lambda_N|\to0$ as $N\to\infty$. An argument similar to (5.26) and (5.29) also gives

$$\begin{gathered}W_N(x)=\inf_{v\in\mathfrak{U}_{\mathrm{SSM}}}\mathbb{E}^v_x\Big[\int_0^{\breve\uptau_r}\big(\hat r^N_{\boldsymbol\mu}(X_t,v(X_t))-\lambda_N\big)\,\mathrm{d}t+W_N(X_{\breve\uptau_r})\Big],\quad r>0,\\ \|W_N\|_{\mathcal{W}^{2,p}(B_R(0))}\le\varpi_1,\quad p\in[1,\infty),\end{gathered}\tag{5.34}$$

for some constant ϖ_1 independent of N but might depend on p. Therefore, by (5.26), (5.29) and (5.34), for every $\varepsilon>0$ we can find $r>0$ small enough such that

$$|V^i_N(x)-W_N(x)|\;\le\sup_{v\in\mathfrak{U}_{\mathrm{SSM}}}\mathbb{E}^v_x\Big|\int_0^{\breve\uptau_r}\Big(\breve r^{i,N}_{\boldsymbol\mu}(X_t,v(X_t))-\hat r^N_{\boldsymbol\mu}(X_t,v(X_t))+\lambda_N-\tilde\varrho^i_N\Big)\,\mathrm{d}t\Big|+\varepsilon.\tag{5.35}$$

Equations (5.32)–(5.33) imply that $|\breve r^{i,N}_{\boldsymbol\mu}(x,u)-\hat r^N_{\boldsymbol\mu}(x,u)|\to0$ as $N\to\infty$ uniformly on compact subsets of $\mathbb{R}^d\times\mathbb{U}$. Since $g^0\in\mathfrak{o}(|x|^q)$, using (5.25) (together with Fatou's lemma), we obtain

$$\limsup_{R\to\infty}\sup_{v\in\mathfrak{U}_{\mathrm{SSM}}}\mathbb{E}^v_x\Big[\int_0^{\breve\uptau_r}\mathbb{1}_{B_R^c(0)}(X_t)g^0(X_t)\,\mathrm{d}t\Big]\;=\;0$$

uniformly in x belonging to compact subsets of $\mathbb{R}^d$. It follows by (5.35) that

$$\sup_i\|V^i_N-W_N\|_{L^\infty(B_R)}\;\xrightarrow[N\to\infty]{}\;0.$$

As earlier, using (5.34) we can show that $\{W_N\}$ is bounded in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p\ge1$. Thus $\{V^i_N-W_N\}$ is bounded in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p\ge1$, which implies the local compactness of $\{V^i_N-V^j_N\}$ in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p\ge1$, and clearly $V^i_N-V^j_N\to0$ in $\mathcal{W}^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$ as $N\to\infty$. Therefore from (5.18) and (5.30) we obtain

$$\frac12\,\mathrm{trace}\big(a(x)\nabla^2(V^i_N-W_N)(x)\big)\;=\;f^i_N(x),$$

where $f^i_N\to0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ uniformly in $1\le i\le N$. Thus using standard results of elliptic pde we get that $\{V^i_N-W_N\}$ converges to 0 in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p\ge1$, uniformly in $1\le i\le N$. Hence $\{V^i_N-V^j_N\}$ converges to 0 in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p\ge1$, uniformly in $1\le i,j\le N$.

Next we show that $\sup_{i,j} \mathcal{W}_p(\mu_N^i, \mu_N^j) \to 0$ as $N \to \infty$. Due to Proposition 2.1 and (5.23) it is enough to show that $\sup_{i,j} \ d_{\mathrm{P}}(\mu_N^i, \mu_N^j) \to 0$ as $N \to \infty$. Let w_N be the continuous selector from the minimizer of (5.30), and μ_{w_N} be the corresponding invariant measure. We show that

$$\sup_{i\in\{1,\dots,N\}} d_{\mathrm{P}}(\mu_N^i, \mu_{w_N}) \ \xrightarrow[N\to\infty]{} \ 0\,. \tag{5.36}$$

By Assumption 5.3 it follows that $u \mapsto L^u V_N^i(x) + \breve{r}_{\boldsymbol{\mu}}^{i,N}(x,u)$ is a strictly convex function. Therefore there exists a unique continuous measurable selector $v_N^i\colon \mathbb{R}^d \to \mathbb{U}$ from the minimizer in (5.18). Also, μ_N^i is the unique invariant probability measure corresponding to v_N^i by (5.19). We claim that

$$(x,u) \ \mapsto \ \breve{r}_{\boldsymbol{\mu}}^{i,N}(x,u) \quad \text{is equicontinuous on compact subsets of } \ \mathbb{R}^d \times \mathbb{U}\,. \tag{5.37}$$

To show (5.37) we use continuity of r on $\mathbb{R}^d \times \mathbb{U} \times \mathcal{P}_{\bar{q}}(\mathbb{R}^d)$. We consider the set $C \times \mathbb{U}$ where C is a compact subset of $\mathbb{R}^d$. Let $\varepsilon > 0$ be given. Then using Lemma 5.1 we can find $R > 0$ and the projected measures $\tilde{\mu}_N^i$ such that

$$\sup_{(x,u)\in C\times\mathbb{U}} \left|\hat{r}_{\boldsymbol{\mu}}^N(x,u) - \hat{r}_{\tilde{\boldsymbol{\mu}}}^N(x,u)\right| \ \le \ \frac{\varepsilon}{4} \qquad \forall\, N \ge 1\,. \tag{5.38}$$

Since $\mathcal{P}(\bar{B}_R(0))$ is a compact set, using the continuity of r, we can find $\delta > 0$ such that

$$|r(x,u,\mu) - r(\bar{x},\bar{u},\mu)| \ \le \ \frac{\varepsilon}{4}\,, \quad \text{whenever } |x-\bar{x}| + d_{\mathbb{U}}(u,\bar{u}) \le \delta, \text{ and } x,\bar{x} \in C\,. \tag{5.39}$$

Thus using (5.39) we obtain

$$\begin{aligned} \left|\hat{r}_{\tilde{\boldsymbol{\mu}}}^N(x,u) - \hat{r}_{\tilde{\boldsymbol{\mu}}}^N(\bar{x},\bar{u})\right| \ &= \ \left| \int_{\bar{B}_R^{Nd}} r\Big(x,u,\frac{1}{N}\sum_{j=1}^N \delta_{y^j}\Big) \prod_{j=1}^n \tilde{\mu}_N^j(\mathrm{d}y^j) - \int_{\bar{B}_R^{Nd}} r\Big(\bar{x},\bar{u},\frac{1}{N}\sum_{j=1}^N \delta_{y^j}\Big) \prod_{j=1}^n \tilde{\mu}_N^j(\mathrm{d}y^j) \right| \\ &\le \ \frac{\varepsilon}{4}\,, \end{aligned}$$

whenever $|x-\bar{x}| + d_{\mathbb{U}}(u,\bar{u}) \le \delta$ and $x,\bar{x} \in C$. Combining this with (5.38) we establish (5.37). This also shows that

$$(x,u) \ \mapsto \ \hat{r}_{\boldsymbol{\mu}}^N(x,u) \quad \text{is equicontinuous on compact subsets of } \ \mathbb{R}^d \times \mathbb{U}\,.$$

Suppose that (5.36) is not true. Then for $\varepsilon > 0$ we can find a subsequence N_k and $i_k \in 1,\dots,N_k$ such that

$$d_{\mathrm{P}}(\mu_{N_k}^{i_k}, \mu_{w_{N_k}}) \ \ge \ \varepsilon \ > \ 0 \qquad \text{for all } N_k\,. \tag{5.40}$$

We can further chose a subsequence of $\{N_k, i_k\}$, relabel it with the same indices, such that the following hold:

$$\begin{aligned} V_{N_k}^{i_k} \to V\,, \quad \text{and} \quad W_{N_k} \to V \quad \text{in } \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)\,, \\ \breve{r}_{\boldsymbol{\mu}}^{i_k N_k} \to \vartheta\,, \quad \text{and} \quad \hat{r}_{\boldsymbol{\mu}}^{N_k} \to \vartheta \quad \text{in } \mathcal{C}_{\mathrm{loc}}(\mathbb{R}^d)\,, \\ \tilde{\varrho}_{N_k}^{i_k} \to \varrho\,, \quad \text{and} \quad \lambda_{N_k} \to \varrho\,, \\ v_{N_k}^{i_k} \to v\,, \quad \text{and} \quad w_{N_k} \to w \quad \text{in } \mathfrak{U}_{\mathrm{SSM}}\,. \end{aligned} \tag{5.41}$$

The convergence in the first and third lines are justified by the compactness property and the uniqueness of the limit which we established earlier. For the second line we use the Arzelà–Ascoli theorem and

(5.32)–(5.33), while the fourth line is a consequence of the compactness property of $\mathfrak{U}_{\rm SM}$ [3, Section 2.4]. We first show that $v = w$. From (5.18), (5.30) and (5.41) we obtain

$$\min_{u\in\mathbb{U}} \left[L^u V(x) + \vartheta(x,u)\right] \;=\; \varrho, \quad V(0) \;=\; 0, \quad V \;\in\; \mathfrak{o}(\mathcal{V})\,. \tag{5.42}$$

Using Assumption 5.3(iii) it is also easy to see that $v^{i_k}_{N_k} \to v$ and $w_{N_k} \to w$ pointwise, as $N_k \to \infty$ and

$$\min_{u\in\mathbb{U}} \left[L^u V(x) + \vartheta(x,u)\right] \;=\; L^v V(x) + \vartheta(x,v(x)) \;=\; L^w V(x) + \vartheta(x,w(x))\,.$$

Thus using the strict convexity of the Hamiltonian we obtain $v(x) = w(x)$ for all x. By [3, Lemma 3.2.6], there exists $\mu \in \mathcal{H}$, corresponding to v, such that

$$d_{\rm P}(\mu^{i_k}_{N_k}, \mu) + d_{\rm P}(\mu_{w_{N_k}}, \mu) \;\xrightarrow[N_k\to\infty]{}\; 0\,.$$

But this contradicts (5.40) and thus (5.36) holds.

Next we prove part (c). In view of (5.42) we only need to show that $\vartheta(x,u) = r(x,u,\mu)$, where μ is the invariant probability measure corresponding to the minimizing selector v. Without loss of generality, we assume that $\hat{r}^N_{\boldsymbol{\mu}}(x,u) \to \varpi(x,u)$ as $N \to \infty$. Fix $(x,u) \in \mathbb{R}^d \times \mathbb{U}$. Then $\nu \mapsto r(x,u,\nu)$ is a continuous map. From part (b) we also have $\sup_{1\le j\le N} d_{\rm P}(\mu^j_N, \mu) \to 0$ as $N \to \infty$. Let $\tilde{\nu}_R := \nu \circ \mathfrak{P}_R^{-1}$. Then it easy to see that

$$\tilde{\nu}_R \;=\; \nu|_{\bar{B}_R(0)} + \nu(B_R^c(0))\,\delta_0\,.$$

Since $\mu^i_N \to \mu$ in total variation (by [3, Lemma 3.2.5]), we deduce that $(\tilde{\mu}_R)^i_N \to \tilde{\mu}_R$ in total variation as well. Therefore using (5.16) and mimicking the arguments in [18, pp. 530] we can show that

$$\left| \int_{\mathbb{R}^{Nd}} r\Big(x,u,\frac{1}{N}\sum_{i=1}^N \delta_{y^j}\Big) \prod_{j=1}^N (\tilde{\mu}_R)^j_N(\mathrm{d}y^j) - \int_{\mathbb{R}^{Nd}} r\Big(x,u,\frac{1}{N}\sum_{i=1}^N \delta_{y^j}\Big) \prod_{j=1}^N \tilde{\mu}_R(\mathrm{d}y^j) \right| \;\xrightarrow[N\to\infty]{}\; 0\,.$$

On the other hand (see [11,30])

$$\int_{\mathbb{R}^{Nd}} r\Big(x,u,\frac{1}{N}\sum_{i=1}^N \delta_{y^j}\Big) \prod_{j=1}^N \tilde{\mu}_R(\mathrm{d}y^j) \;\xrightarrow[N\to\infty]{}\; r(x,u,\tilde{\mu}_R)\,.$$

To complete the proof we use Lemma 5.1 and the fact that $r(x,u,\tilde{\mu}_R) \to r(x,u,\mu)$ as $R \to \infty$. □

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