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A CORRECTION TO "A RELATIVE VALUE ITERATION ALGORITHM FOR NONDEGENERATE CONTROLLED DIFFUSIONS"*

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Abstract. In A Relative Value Iteration Algorithm for Nondegenerate Controlled Diffusions, [*SIAM J. Control Optim.*, 50 (2012), pp. 1886–1902], convergence of the relative value iteration for the ergodic control problem for a nondegenerate diffusion controlled through its drift was established, under the assumption of geometric ergodicity, using two methods: (a) the theory of monotone dynamical systems and (b) the theory of reverse martingales. However, in the proof using (a) it is wrongly claimed that the semiflow is strong order preserving. In this note, we provide a simple generic proof and also comment on how to relax the uniform geometric ergodicity hypothesis.

Key words. controlled diffusions, ergodic control, Hamilton–Jacobi–Bellman equation, relative value iteration, monotone dynamical systems

AMS subject classifications. Primary, 93E15, 93E20; Secondary, 60J25, 60J60, 90C40

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1. Introduction.

The study in [1] concerns the value iteration (VI), and relative VI, for a controlled diffusion process $X = \{X_t,\ t \geq 0\}$ in $\mathbb{R}^d$, governed by the Itô stochastic differential equation

$$\mathrm{d}X_t \;=\; b(X_t, U_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\,. \tag{1}$$

All random processes in (1) live in a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. The process W is a d-dimensional standard Wiener process independent of the initial condition X_0. The control process U takes values in a compact, metrizable set $\mathbb{U}$, and $U_t(\omega)$ is jointly measurable in $(t,\omega) \in [0,\infty) \times \Omega$. Moreover, it is *nonanticipative*: for $s < t$, $W_t - W_s$ is independent of

$$\mathfrak{F}_s \;\triangleq\; \text{the completion of } \sigma\{X_0, U_r, W_r,\ r \leq s\} \text{ relative to } (\mathfrak{F}, \mathbb{P})\,.$$

As is customary, such a process U is called an *admissible control*, and we let $\mathfrak{U}$ denote the set of all admissible controls. Standard assumptions are imposed on b and σ to guarantee existence and uniqueness of solutions to (1); see (A1)–(A3) in [1].

For $u \in \mathbb{U}$, we define $L^u\colon \mathcal{C}^2(\mathbb{R}^d) \mapsto \mathcal{C}(\mathbb{R}^d)$ by

$$L^u f(x) \;\triangleq\; \sum_{i,j} a^{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_i b^i(x,u) \frac{\partial f}{\partial x_i}(x)\,, \quad u \in \mathbb{U}\,,$$

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and extend its definition to admissible controls or stationary Markov controls, denoted as $\mathfrak{U}_{\mathrm{SM}}$, as in [1].

The running cost function $r\colon \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}_+$ is continuous and locally Lipschitz in its first argument uniformly in $u \in \mathbb{U}$.

The following uniform geometric ergodicity assumption is considered in [1].

Assumption 1.1. There exists a nonnegative, inf-compact $\mathcal{V}\colon \mathbb{R}^d \to \mathbb{R}$ and positive constants c_0, c_1, and c_2 satisfying

$$
\begin{aligned}
L^u \mathcal{V}(x) \;&\le\; c_0 - c_1 \mathcal{V}(x) \qquad \forall\, u \in \mathbb{U}\,, \\
\sup_{u\in\mathbb{U}}\, r(x,u) \;&\le\; c_2 \mathcal{V}(x)
\end{aligned} \tag{2}
$$

for all $x \in \mathbb{R}^d$. Without loss of generality we assume $\mathcal{V} \ge 1$.

We let μ_v denote the unique invariant probability measure on $\mathbb{R}^d$ for the diffusion under the control $v \in \mathfrak{U}_{\mathrm{SM}}$. We let $\mathcal{C}_{\mathcal{V}}(\mathbb{R}^d)$ denote the Banach space of functions in $\mathcal{C}(\mathbb{R}^d)$ with norm $\|f\|_{\mathcal{V}} \triangleq \sup_{x\in\mathbb{R}^d} \bigl|\frac{f(x)}{\mathcal{V}(x)}\bigr|$. It is well known (see [2, 3]) that (2) implies that

$$
\mathbb{E}_x^U \bigl[\mathcal{V}(X_t)\bigr] \;\le\; \frac{c_0}{c_1} + \mathcal{V}(x)\mathrm{e}^{-c_1 t} \qquad \forall\, x \in \mathbb{R}^d\,, \;\; \forall\, U \in \mathfrak{U}\,. \tag{3}
$$

Also, there exist constants C_0 and γ such that

$$
\left| \mathbb{E}_x^v \bigl[h(X_t)\bigr] - \int_{\mathbb{R}^d} h(x)\,\mu_v(\mathrm{d}x) \right| \;\le\; C_0 \mathrm{e}^{-\gamma t} \bigl\|h\bigr\|_{\mathcal{V}} \mathcal{V}(x)\,, \quad t \ge 0\,,\; x \in \mathbb{R}^d\,, \tag{4}
$$

for all $h \in \mathcal{C}_{\mathcal{V}}(\mathbb{R}^d)$.

1.1. The VI. Under Assumption 1.1 there exists a unique solution $V^* \in \mathcal{C}_{\mathcal{V}}(\mathbb{R}^d) \cap \mathcal{C}^2(\mathbb{R}^d)$, satisfying $V^*(0) = 0$, of the ergodic Hamilton–Jacobi–Bellman equation

$$
\min_{u\in\mathbb{U}} \bigl[L^u V^*(x) + r(x,u)\bigr] \;=\; \beta\,, \tag{5}
$$

where β is the optimal ergodic value (see equation (2.8) in [1]).

Let $\mathcal{H} \triangleq \mathcal{C}_{\mathcal{V}}(\mathbb{R}^d) \cap \mathcal{C}^2(\mathbb{R}^d)$. The VI equation introduced in [1] takes the form of the Cauchy problem

$$
\partial_t \Phi_t[h](x) \;=\; \min_{u\in\mathbb{U}} \bigl[L^u \Phi_t[h](x) + r(x,u)\bigr] - \beta\,, \qquad \Phi_0(h)(x) = h(x)\,, \tag{6}
$$

with $h \in \mathcal{H}$.

As shown in [1, Lemma 4.1], $\Phi_t[h] \in \mathcal{H}$ for all $t \ge 0$, and by (4.9)–(4.10) in [1] it satisfies

$$
\mathbb{E}_x^{\bar v}\bigl[h(X_t) - V^*(X_t)\bigr] \;\le\; \Phi_t[h](x) - V^*(x) \;\le\; \mathbb{E}_x^{v^*}\bigl[h(X_t) - V^*(X_t)\bigr]\,, \tag{7}
$$

where v^* is any measurable selector from the minimizer in (5), i.e., an optimal stationary Markov control, and $\bar v$ is any measurable selector from the minimizer in (6). It follows by (7) that the orbit of h under the semiflow Φ_t, defined by $O(h) \triangleq \{\Phi_t[h]\colon t \ge 0\}$, is bounded in $\mathcal{C}_{\mathcal{V}}(\mathbb{R}^d)$, and as argued in the proof of [1, Theorem 4.5] it is relatively compact in $\mathcal{H}$. It follows that the *ω-limit set* of h, which is denoted by $\omega(h)$ and defined as $\omega(h) \triangleq \cap_{t>0} \overline{\cup_{s\ge t}\, \Phi_t[h]}$, is nonempty, compact, connected, and invariant and satisfies $\mathrm{dist}\big(\omega(h), \Phi_t[h]\big) \to 0$ as $t \to \infty$ (see [5]), where dist is a metric for $\mathcal{H}$, for example, as given in the proof of [1, Theorem 4.5].

For $h, h' \in \mathcal{H}$ we write $h \preceq h'$ if $h'(x) - h(x) \geq 0$ for all $x \in \mathbb{R}^d$, and we use $\prec$ for $\preceq$ but $\neq$. We also write $h \prec\!\!\prec h'$ if $h' - h$ lies in the interior of the positive cone of $\mathcal{C}_{\mathcal{V}}(\mathbb{R}^d)$.

If $h \prec h'$, then by (6) we obtain

$$\mathbb{E}_x^{v'}\big[h'(X_t) - h(X_t)\big] \;\leq\; \Phi_t[h'](x) - \Phi_t[h](x) \qquad \forall\, t > 0\,, \quad \forall\, x \in \mathbb{R}^d\,, \tag{8}$$

where v' is a Markov control associated with a measurable selector from the minimizer in (6) corresponding to the solution starting at h'. Equation (8) has been used in the proof of [1, Theorem 4.5] to erroneously argue that Φ_t is *strongly monotone*, which means that $h \prec h'$ implies that $\Phi_t[h] \prec\!\!\prec \Phi_t[h']$ for all $t > 0$. This is incorrect. In the next section we provide a simple proof of Theorem 4.5 in [1].

2. A simple proof of convergence of the VI. Convergence of the VI is asserted in Theorem 4.5 in [1], which we quote as follows.

THEOREM 2.1. *For each $h \in \mathcal{H}$, we have $\Phi_t[h](x) \to V^*(x) + c$ as $t \to \infty$, for some $c \in \mathbb{R}$ which depends on h.*

Proof. By (3) and (7) we have

$$\big|V^*(x) - \Phi_t[h](x)\big| \;\leq\; \big\|V^* - h\big\|_{\mathcal{V}} \left(\frac{c_0}{c_1} + \mathcal{V}(x)\mathrm{e}^{-c_1 t}\right) \qquad \forall\, x \in \mathbb{R}^d\,, \quad \forall\, t \geq 0\,. \tag{9}$$

Hence every $g \in \omega(h)$ satisfies

$$\big\|V^* - g\big\|_\infty \;\leq\; \frac{c_0}{c_1}\,\big\|V^* - h\big\|_{\mathcal{V}}\,. \tag{10}$$

Applying Itô's formula to (6) we obtain

$$\begin{aligned}
\Phi_t[h](x) &\leq \mathbb{E}_x^{v^*}\left[\int_0^{t-\tau} r\big(X_s, v^*(X_s)\big)\,\mathrm{d}s - \beta(t-\tau) + \Phi_\tau[h](X_{t-\tau})\right] \\
&= \mathbb{E}_x^{v^*}\left[\int_0^{t-\tau} r\big(X_s, v^*(X_s)\big)\,\mathrm{d}s - \beta(t-\tau) + V^*(X_{t-\tau})\right] \\
&\quad + \mathbb{E}_x^{v^*}\big[\Phi_\tau[h](X_{t-\tau}) - V^*(X_{t-\tau})\big] \\
&= V^*(x) + \mathbb{E}_x^{v^*}\big[\Phi_\tau[h](X_{t-\tau}) - V^*(X_{t-\tau})\big] \qquad \forall\, \tau \in [0, t]
\end{aligned}$$

and all $x \in \mathbb{R}^d$. Therefore, we have

$$\Phi_t[h](x) - V^*(x) \;\leq\; \mathbb{E}_x^{v^*}\big[\Phi_\tau[h](X_{t-\tau}) - V^*(X_{t-\tau})\big] \qquad \forall\, \tau \in [0, t]\,, \tag{11}$$

and since $|\Phi_t[h] - V^*|$ is integrable with respect to μ_{v^*} by (9), it follows by integrating (11) with respect to μ_{v^*} that the map

$$t \;\mapsto\; \int_{\mathbb{R}^d} \big(\Phi_t[h](x) - V^*(x)\big)\,\mu_{v^*}(\mathrm{d}x)$$

is nonincreasing. Since it is also bounded by (9), it follows that the map

$$\mathcal{G}(g) \;\triangleq\; \int_{\mathbb{R}^d} \big(g(x) - V^*(x)\big)\,\mu_{v^*}(\mathrm{d}x)\,, \qquad g \in \mathcal{H}\,,$$

must be constant on $\omega(h)$. Let Γ_h denote this constant. By the invariance of $\omega(h)$ we have

$$\int_{\mathbb{R}^d} \bigl(\Phi_t[g](x) - V^*(x)\bigr)\, \mu_{v^*}(\mathrm{d}x) \;=\; \Gamma_h \qquad \forall\, g \in \omega(h)\,, \quad \forall\, t \geq 0\,. \tag{12}$$

For $g \in \omega(h)$ define

$$C_g \;\triangleq\; \sup_{\mathbb{R}^d}\, (g - V^*)\,.$$

It follows by (10) that C_g is finite. By the definition of C_g we have $V^* + C_g - g \geq 0$ and $\inf_{\mathbb{R}^d}\, (V^* + C_g - g) = 0$.

We claim that $V^* + C_g - g = 0$ for all $g \in \omega(h)$. If the claim is true, then $C_g = \Gamma_h$ by (12), and thus $\Phi_t[h](x) \to V^*(x) + \Gamma_h$, which proves the theorem.

We prove the claim by contradiction. Suppose that for some $\hat{g} \in \omega(h)$ we have $V^* + C_{\hat{g}} - \hat{g} \neq 0$. Let t_n be an increasing sequence such that $\Phi_{t_n}[h] \to \hat{g}$ and $t_{n+1} - t_n \to \infty$ as $n \to \infty$. By (7), and the semigroup property of Φ_t, we have

$$\mathbb{E}_x^{v^*}\bigl[V^*(X_{t_{n+1}-t_n}) + C_{\hat{g}} - \Phi_{t_n}[h](X_{t_{n+1}-t_n})\bigr] \;\leq\; V^*(x) + C_{\hat{g}} - \Phi_{t_{n+1}}[h](x)\,. \tag{13}$$

Since $V^* + C_{\hat{g}} - \hat{g} \gneqq 0$ and $V^* + C_{\hat{g}} - \hat{g}$ is bounded, then

$$\mathbb{E}_x^{v^*}\bigl[V^*(X_{t_{n+1}-t_n}) + C_{\hat{g}} - \hat{g}(X_{t_{n+1}-t_n})\bigr]$$

converges to some positive constant κ as $n \to \infty$ by (4). In addition, since $\bigl\|\Phi_{t_n}[h] - \hat{g}\bigr\|_{\mathcal{V}} \to 0$ as $n \to \infty$ by (9), it follows that the left-hand side of (13) converges to the same constant κ. Thus by (13) we obtain $V^* + C_{\hat{g}} - \hat{g} \geq \kappa > 0$, which contradicts the definition of $C_{\hat{g}}$. This proves the claim and completes the proof of the theorem. $\square$

3. Convergence in the absence of geometric ergodicity. The key properties used in the proof of Theorem 2.1 is a bound of the form

$$\sup_{x \in \mathbb{R}^d}\; \limsup_{t \to \infty}\; \bigl|V^*(x) - \Phi_t[h](x)\bigr| \;\leq\; M \qquad \forall\, t \geq 0 \tag{14}$$

for some constant M, which follows by (9), and the integrability of V^* under μ_{v^*}.

We replace geometric ergodicity in Assumption 1.1 by the following stability hypothesis.

Assumption 3.1. There exist nonnegative, inf-compact functions $\mathcal{V}_k \colon \mathbb{R}^d \to \mathbb{R}$, $k = 0, 1$, and positive constants κ_0, κ_1 satisfying

$$L^u \mathcal{V}_1(x) \;\leq\; \kappa_1 - \kappa_0 \mathcal{V}_0(x) \qquad \forall\, u \in \mathbb{U}\,, \quad \forall\, x \in \mathbb{R}^d\,,$$

and

$$\frac{1}{\mathcal{V}_0(x)}\, \sup_{u \in \mathbb{U}}\, r(x,u) \;\xrightarrow[|x| \to \infty]{}\; 0\,.$$

It is well known that under Assumption 3.1 there exists a unique solution $V^* \in \mathcal{C}_{\mathcal{V}_1}(\mathbb{R}^d) \cap \mathcal{C}^2(\mathbb{R}^d)$ of (5) satisfying $V^*(0) = 0$ [2, Theorem 3.7.11].

We have the following convergence result.

COROLLARY 3.2. *Let Assumption 3.1 hold, and suppose that the function $\mathcal{V}_1$ is integrable under μ_{v^*} for some optimal control $v^* \in \mathfrak{U}_{\mathrm{SM}}$. Then for any $h \in \mathcal{C}_{\mathcal{V}_1}(\mathbb{R}^d) \cap \mathcal{C}^2(\mathbb{R}^d)$, such that $\inf_{\mathbb{R}^d}\, (h - V^*) > -\infty$, it holds that $\Phi_t[h](x) \to V^*(x) + c$ as $t \to \infty$, for some $c \in \mathbb{R}$ which depends on h.*

Proof. Since $V^* \in \mathcal{C}_{\mathcal{V}_1}(\mathbb{R}^d)$, the right-hand side of (7) converges to a constant as $t \to \infty$ by [4, Theorem 4.12]. Also by the hypothesis of the corollary, we have

$$\inf_{x \in \mathbb{R}^d} \liminf_{t \to \infty} \mathbb{E}_x^U \big[h(X_t) - V^*(X_t)\big] > -\infty \qquad \forall\, U \in \mathfrak{U}\,.$$

Therefore, (14) follows by (7), and the proof follows by the argument used in the proof of Theorem 2.1. ∎

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