SUPPLEMENT TO "CONTROLLED EQUILIBRIUM SELECTION IN STOCHASTICALLY PERTURBED DYNAMICS"

BY ARI ARAPOSTATHIS [*], ANUP BISWAS [†], AND VIVEK S. BORKAR [‡]

The University of Texas at Austin[*],
Indian Institute of Science Education and Research, Pune[†],
and Indian Institute of Technology Bombay[‡]

This document contains the proofs of Lemma 1.3 and Theorem 1.4 in Section 1.3, and Lemma 1.17 and Theorem 1.19 in Section 1.5.

1. Proofs of Lemma 1.3 and Theorem 1.4. We start with the proof of Lemma 1.3.

PROOF OF LEMMA 1.3. The proof is standard. Let U be given, and define $M_t := \mathbb{E}\bigl[\int_0^t |U_s|^2\,\mathrm{d}s\bigr]$, $t\in\mathbb{R}_+$. For $T>0$, let $\mathcal{H}^2_T$ denote the space of $\{\mathfrak{F}_t\}$-adapted processes Y defined on $[0,T]$, having continuous sample paths, and satisfying $\mathbb{E}\bigl[\sup_{0\le t\le T}|Y_t|^2\bigr]<\infty$. The space $\mathcal{H}^2_T$ (more precisely the set of equivalence classes in $\mathcal{H}^2_T$) is a Banach space under the norm

$$\|Y\|_{\mathcal{H}^2_T} \;:=\; \biggl(\mathbb{E}\biggl[\sup_{0\le t\le T}|Y_t|^2\biggr]\biggr)^{1/2}.$$

It is standard to show, for example following the proof of (Arapostathis, Borkar and Ghosh, 2012, Theorem 2.2.2), that any solution X of

$$X_t \;=\; X_0 + \int_0^t \bigl(m(X_s)+\varepsilon\,U_s\bigr)\,\mathrm{d}s + \varepsilon^{\nu}W_t\,,\quad t\;\ge\;0\,, \tag{1}$$

satisfies

$$\|X-X_0\|^2_{\mathcal{H}^2_t} \;\le\; \kappa_0 t(1+t)\bigl(1+M_t+\mathbb{E}\bigl[|X_0|^2\bigr]\bigr)\,\mathrm{e}^{\kappa_1 t}\qquad \forall\,t\ge 0\,, \tag{2}$$

for some constants κ_0 and κ_1 that depend only on m. The existence of a pathwise unique solution then follows by applying the contraction mapping theorem as in (Arapostathis, Borkar and Ghosh, 2012, Theorem 2.2.4). $\Box$

The rest of this section is devoted to the proof of Theorem 1.4. Without loss of generality we fix $\varepsilon=1$, and suppress the dependence on ε in all the variables. Also, throughout the rest of this section, without loss of generality we assume that $\ell \ge 0$.

We proceed by establishing two key lemmas, followed by the proof of Theorem 1.4. Recall that the running cost $\mathcal{R}$ is defined by

$$\mathcal{R}(x,u) \;:=\; \ell(x) + \frac{1}{2}\,|u|^2\,,$$

and that since ℓ is Lipschitz, there exists a constant $\bar{c}_\ell$ independent of ε such that

$$\int \ell\,\mathrm{d}\eta_0^\varepsilon \;\le\; \bar{c}_\ell\,. \tag{3}$$

For $x \in \mathbb{R}^d$, and $\alpha > 0$, we define the subset $\mathfrak{U}_x^\alpha$ of admissible controls by

$$\mathfrak{U}_x^\alpha \;:=\; \biggl\{U \in \mathfrak{U}:\; \mathbb{E}_x^U\biggl[\int_0^\infty \mathrm{e}^{-\alpha s}\,\mathcal{R}(X_s,U_s)\,\mathrm{d}s\biggr] \;<\; \infty\biggr\}\,, \tag{4}$$

where $\mathbb{E}_x^U$ denotes the expectation under the law of (X,U) which solves the Itô equation

$$X_t \;=\; x + \int_0^t m(X_s)\,\mathrm{d}s + \int_0^t U_s\,\mathrm{d}s + W_t\,, \quad t \ge 0\,, \tag{5}$$

with $X_0 = x$.

Lemma 1. *The equation*

$$\tfrac{1}{2}\,\Delta V_\alpha + \langle m, \nabla V_\alpha\rangle - \tfrac{1}{2}\,|\nabla V_\alpha|^2 + \ell \;=\; \alpha V_\alpha \tag{6}$$

has a solution in $\mathcal{C}^2(\mathbb{R}^d)$ for all $\alpha \in (0,1)$. Moreover, for all $\alpha \in (0,1)$, we have the following.

(*i*) *For some constant $c_0 > 0$, not depending on α, it holds that*

$$|\nabla V_\alpha(x)| \;\le\; c_0\sqrt{1+|x|}\,, \quad \text{and} \;\; |\alpha V_\alpha(x)| \;\le\; \ell(x) + \tfrac{c_0}{\alpha} \tag{7}$$

for all $x \in \mathbb{R}^d$.

(*ii*) *The function V_α satisfies*

$$V_\alpha(x) \;\le\; \inf_{U \in \mathfrak{U}_x^\alpha}\; \mathbb{E}_x^U\biggl[\int_0^\infty \mathrm{e}^{-\alpha s}\,\mathcal{R}(X_s,U_s)\,\mathrm{d}s\biggr]\,, \qquad \forall\, x \in \mathbb{R}^d\,. \tag{8}$$

(*iii*) *With $\bar{c}_\ell$ the constant in (3), we have*

$$\inf_{\{x\,:\,\ell(x)\,\le\,\bar{c}_\ell\}} \alpha V_\alpha \;=\; \inf_{\mathbb{R}^d} \alpha V_\alpha \;\le\; \bar{c}_\ell\,.$$

PROOF. In (Bensoussan and Frehse, 2002, Theorem 4.18, p. 177) it is proved that (6) has a solution in $\mathcal{C}^2(\mathbb{R}^d)$, and it is also shown in the proof of this theorem that there exists a constant $\kappa_0 > 0$ which does not depend on α such that

$$\alpha V_\alpha(x) \;\geq\; -\kappa_0 \qquad \forall\, x \in \mathbb{R}^d\,. \tag{9}$$

By (Ichihara, 2012, Theorem B.1) there exists a constant C not depending on $R > 0$ such that

$$\sup_{B_R} |\nabla V_\alpha| \;\leq\; C\Bigl(1 + \sup_{B_{R+1}} \sqrt{(\alpha V_\alpha)^-} + \sup_{B_{R+1}} \sqrt{\ell^+} + \sup_{B_{R+1}} |\nabla \ell|^{1/3}\Bigr)\,, \tag{10}$$

from which gradient estimate in (7) follows. The following structural hypothesis on the Hamiltonian $h(x,p)$ is assumed in (Ichihara, 2012, Theorem B.1): $p \mapsto h(x,p)$ is strictly convex for all $x \in \mathbb{R}^d$, and there exists some constant $k_0 > 0$ such that

$$k_0\,|p|^2 \;\leq\; h(x,p) \;\leq\; k_0^{-1}\,|p|^2\,, \qquad |\nabla_x h(x,p)| \;\leq\; k_0^{-1}\bigl(1+|p|^2\bigr)\,, \tag{11}$$

for $(x,p) \in \mathbb{R}^{2d}$. This Hamiltonian corresponds to $h(x,p) = \frac{1}{2}|p|^2 - \langle m, p\rangle$ for the equation in (6), and the first bound in (11) is not satisfied. However, replacing this bound with

$$k_0\,\bigl(|p|^2 - k_1\bigr) \;\leq\; h(x,p) \;\leq\; k_0^{-1}\,\bigl(|p|^2 + k_1\bigr)\,,$$

for some constant $k_1 \geq 0$, the proof of (Ichihara, 2012, Theorem B.1) goes through unmodified.

Recall that the class of controls $\widehat{\mathfrak{U}}$ is defined by

$$\widehat{\mathfrak{U}} \;:=\; \biggl\{U \in \mathfrak{U}:\; \mathbb{E}\biggl[\int_0^t |U_s|^2\,\mathrm{d}s\biggr] < \infty \;\text{ for all }\; t \geq 0\biggr\}\,.$$

Writing (6) in HJB form, and applying Itô's formula, we obtain

$$V_\alpha(x) - \mathrm{e}^{-\alpha t}\,\mathbb{E}_x^U\bigl[V_\alpha(X_t)\bigr] \;\leq\; \mathbb{E}_x^U\biggl[\int_0^t \mathrm{e}^{-\alpha s}\,\mathcal{R}(X_s,U_s)\,\mathrm{d}s\biggr] \qquad \forall\, t > 0\,, \tag{12}$$

and all $U \in \widehat{\mathfrak{U}}$. Since m is bounded, it is standard to show using (5) that

$$\mathbb{E}_x^U\biggl[\sup_{0\leq s\leq t} |X(s) - x|\biggr] \;\leq\; \|m\|_\infty\, t + \sqrt{t} + \mathbb{E}_x^U\biggl[\int_0^t |U_s|\,\mathrm{d}s\biggr] \;<\; \infty \tag{13}$$

for all $U \in \widehat{\mathfrak{U}}$ and $t > 0$. Also, if $\mathbb{E}_x^0$ denotes the expectation $\mathbb{E}_x^U$ with $U = 0$, then by (5), we have the estimate

$$\mathbb{E}_x^0\bigl[|X_t|^2\bigr] \;\leq\; \kappa_2\bigl(1 + t^2 + |x|^2\bigr) \;<\; \infty \qquad \forall\, t > 0\,, \tag{14}$$

for some constant κ_2. As shown in the proof of (Bensoussan and Frehse, 2002, Theorem 4.18, p. 177), $\alpha \mapsto \alpha V_\alpha(0)$ is bounded on $(0,1)$, which together with the gradient estimate in (7) we have already established, provides us with a liberal bound of V_α of the form $|V_\alpha(x)| \le C\bigl(1+|x|^2\bigr)$ for some constant C. This combined with (14) implies that $\mathrm{e}^{-\alpha t}\,\mathbb{E}^0_x\bigl[V_\alpha(X_t)\bigr] \to 0$ as $t\to\infty$. Therefore, using (13), and the Lipschitz constant C_ℓ of ℓ, we obtain by (12) that

$$\begin{aligned}\alpha V_\alpha(x) \;&\le\; \mathbb{E}^0_x\biggl[\int_0^\infty \alpha\,\mathrm{e}^{-\alpha s}\,\ell(X_s)\,\mathrm{d}s\biggr]\\ &\le\; \ell(x) + C_\ell\int_0^\infty \alpha\,\mathrm{e}^{-\alpha s}\bigl(\|m\|_\infty\, s + 2\sqrt{s}\bigr)\,\mathrm{d}s \qquad \forall\, x\in\mathbb{R}^d\,,\end{aligned}$$

which results in the estimate given in (7), where without loss of generality we use a common constant c_0 for the two inequalities. This completes the proof of part (i).

Let $g(x,t) := |x| + \|m\|_\infty\, t + 2\sqrt{t}$. Multiplying both sides of (13) by $\mathrm{e}^{-\alpha t}$, then strengthening the inequality, and applying the Hölder inequality, we obtain

$$\begin{aligned}\mathrm{e}^{-\alpha t}\,\mathbb{E}^U_x\bigl[|X_t|\bigr] \;&\le\; g(x,t)\,\mathrm{e}^{-\alpha t} + \mathrm{e}^{-\frac{\alpha}{2}t}\,\mathbb{E}^U_x\biggl[\int_0^t \mathrm{e}^{-\frac{\alpha}{2}s}\,|U_s|\,\mathrm{d}s\biggr]\\ &\le\; g(x,t)\,\mathrm{e}^{-\alpha t} + \sqrt{t}\mathrm{e}^{-\frac{\alpha}{4}t}\biggl(\mathbb{E}^U_x\biggl[\int_0^t \mathrm{e}^{-\alpha s}\,|U_s|^2\,\mathrm{d}s\biggr]\biggr)^{1/2}\\ &\xrightarrow[t\to\infty]{}\; 0 \qquad \forall\, U\in\mathfrak{U}^\alpha_x\,,\end{aligned} \tag{15}$$

with $\mathfrak{U}^\alpha_x$ as defined in (4). Taking limits as $t\to\infty$ in (12), and using (15), and the bound of V_α in (7) together with the inequality $|\ell(x)| \le C_l|x| + |\ell(0)|$, we obtain (8).

We now turn to part (iii). Let

$$\chi(x) \;:=\; \frac{1}{\sqrt{3}}\biggl(\min_{y\,\in\, B_1(x)}\Bigl[\ell(y) - (d+1+2\sqrt{d}\,\|m\|_\infty)^2\Bigr]\biggr)^{1/2},$$

and

$$\psi(x) \;:=\; V_\alpha(x) + \tfrac{2\kappa_0}{\alpha} - \chi(x_0)\bigl(1-|x-x_0|^2\bigr)\,, \qquad x\in B_1(x_0)\,,$$

where $\kappa_0 > 0$ is the constant in (9). With $\phi(x) := |x - x_0|^2$, we have

$$
\begin{aligned}
-\tfrac{1}{2}\,\Delta\psi - \langle m - \nabla V_\alpha, \nabla\psi\rangle + \alpha\psi
&= \Bigl(-\tfrac{1}{2}\,\Delta V_\alpha - \langle m, \nabla V_\alpha\rangle + \tfrac{1}{2}\,|\nabla V_\alpha|^2 + \alpha V_\alpha\Bigr) \\
&\quad + \tfrac{1}{2}\,\bigl|\nabla V_\alpha - \chi(x_0)\nabla\phi\bigr|^2 - 2\chi^2(x_0)\phi + 2\kappa_0 \\
&\quad - \chi(x_0)\Bigl(\tfrac{1}{2}\,\Delta\phi + \langle m, \nabla\phi\rangle + \alpha(1-\phi)\Bigr) \\
&\ge\ \ell - 2\chi^2(x_0) + 2\kappa_0 - \bigl(d + 2\sqrt{d}\,\|m\|_\infty + 1\bigr)\chi(x_0) \\
&\ge\ \ell - 3\chi^2(x_0) - \bigl(d + 1 + 2\sqrt{d}\,\|m\|_\infty\bigr)^2 \quad \text{in} \ \ B_1(x_0)\,,
\end{aligned}
$$

for all $\alpha \in (0,1)$, where we use (6) and the fact that $\kappa_0 \ge 0$. Since $\psi > 0$ on $\partial B_1(x_0)$ by (9), an application of the strong maximum principle shows that $\psi \ge 0$ in $B_1(x_0)$, which implies that

$$
\alpha V_\alpha(x) \ \ge\ \alpha\chi(x) + \kappa_0 \qquad \forall\, x \in \mathbb{R}^d\,.
$$

Since ℓ is inf-compact, and the same is true for χ by its definition, this shows that αV_α is inf-compact. In particular, it attains its infimum in $\mathbb{R}^d$. With η_0 denoting the invariant probability measure of the diffusion in (5) under the control $U = 0$, using (8), we obtain

$$
\inf_{\mathbb{R}^d}\ V_\alpha \ \le\ \int V_\alpha\,\mathrm{d}\eta_0 \ \le\ \int_{\mathbb{R}^d} \mathbb{E}_x\biggl[\int_0^\infty \mathrm{e}^{-\alpha s}\ell(X_s)\,\mathrm{d}s\biggr]\,\eta_0(\mathrm{d}x) \ \le\ \frac{\bar{c}_\ell}{\alpha}\,, \tag{16}
$$

where the last inequality follows by (3). One more application of the maximum principle implies that if V_α attains its infimum at $\hat{x} \in \mathbb{R}^d$ then $\ell(\hat{x}) \le \alpha V_\alpha(\hat{x})$. This together with (16) implies part (iii). □

REMARK 2. We should mention, even though we do not need it for the proof of the main theorem, that (8) holds with equality, and thus V_α is indeed the value of the infinite horizon discounted control problem. The proof of this assertion goes as follows. Since ∇V_α has at most linear growth, the diffusion in (5) under the Markov control $v_\alpha = -\nabla V_\alpha$ has a unique strong solution. It is also clear by (7) that for any $\alpha > 0$ we can select a constant $\kappa_1(\alpha)$ such that $|\nabla V_\alpha(x)| \le \kappa_1(\alpha) + \frac{\alpha}{16}x$. Thus using a standard estimate (Arapostathis, Borkar and Ghosh, 2012, Theorem 2.2.2) we obtain

$$
\mathbb{E}_x^{v_\alpha}\biggl[\sup_{0\le s\le t}\,|X(s)|^2\biggr] \ \le\ \kappa_2(\alpha)(1+t^2)(1+|x|^2)\mathrm{e}^{\frac{\alpha}{2}t} \tag{17}
$$

for some constant $\kappa_2(\alpha) > 0$. With τ_R denoting the first exit time from B_R, applying Dynkin's formula we obtain

$$V_\alpha(x) \;=\; \mathbb{E}_x^{v_\alpha}\left[\int_0^{t\wedge\tau_R} \mathrm{e}^{-\alpha s}\,\mathcal{R}(X_s, v_\alpha(X_s))\,\mathrm{d}s\right] + \mathbb{E}_x^{v_\alpha}\left[\mathrm{e}^{-\alpha(t\wedge\tau_R)}\,V_\alpha(X_{t\wedge\tau_R})\right].$$

We write

$$\mathbb{E}_x^{v_\alpha}\left[\mathrm{e}^{-\alpha(t\wedge\tau_R)}\,V_\alpha(X_{t\wedge\tau_R})\right] \;=\; A_1(t,R) + A_2(t,R)\,,$$

with

$$\begin{aligned} A_1(t,R) \;&:=\; \mathbb{E}_x^{v_\alpha}\left[\mathrm{e}^{-\alpha t}\,V_\alpha(X_{t\wedge\tau_R})\,\mathbb{1}_{\{t\le\tau_R\}}\right],\\ A_2(t,R) \;&:=\; \mathbb{E}_x^{v_\alpha}\left[\mathrm{e}^{-\alpha\tau_R}\,V_\alpha(X_{t\wedge\tau_R})\,\mathbb{1}_{\{\tau_R<t\}}\right]. \end{aligned}$$

Since V_α has at most linear growth in x by (7), it follows by (17) that

$$\lim_{t\to\infty}\;\limsup_{R\to\infty}\;|A_1(t,R)| \;=\; 0\,.$$

We also have $\limsup_{R\to\infty}|A_2(t,R)| = 0$ by dominated convergence, since $\mathbb{P}_x^{v_\alpha}\bigl(\tau_R < t\bigr) \to 0$ as $R\to\infty$. Thus, taking limits first as $R\to\infty$, and then as $t\to\infty$ in (17), we obtain

$$V_\alpha(x) \;\ge\; \mathbb{E}_x^{v_\alpha}\left[\int_0^\infty \mathrm{e}^{-\alpha s}\,\mathcal{R}(X_s, v_\alpha(X_s))\,\mathrm{d}s\right].$$

Thus the converse inequality to (8) also holds.

Define the class of controls $\overline{\mathfrak{U}}_x$ by

$$\overline{\mathfrak{U}}_x \;:=\; \left\{U\in\mathfrak{U}:\; \limsup_{T\to\infty}\;\frac{1}{T}\;\mathbb{E}_x^U\left[\int_0^T \mathcal{R}(X_s,U_s)\,\mathrm{d}s\right] \;<\; \infty\right\}.$$

Lemma 3. *There exists an inf-compact $V\in\mathcal{C}^2(\mathbb{R}^d)$ which satisfies*

$$\mathcal{A}[V](x) \;:=\; \tfrac{1}{2}\,\Delta V + \langle m,\nabla V\rangle - \tfrac{1}{2}\,|\nabla V|^2 + \ell \;=\; \beta\,, \tag{18}$$

with

$$\beta \;=\; \beta_* \;:=\; \inf_{U\in\overline{\mathfrak{U}}_x}\;\limsup_{T\to\infty}\;\frac{1}{T}\;\mathbb{E}_x^U\left[\int_0^T \mathcal{R}(X_s,U_s)\,\mathrm{d}s\right],$$

and

$$|\nabla V(x)| \;\le\; c_0\sqrt{1+|x|} \qquad \forall\,x\in\mathbb{R}^d\,, \tag{19}$$

and for some positive constant c_0. In addition, under the Markov control $U_t = v_*(X_t)$, with $v_* = -\nabla V$, the diffusion in (5) is positive recurrent, and $\beta_* = \int_{\mathbb{R}^d}\mathcal{R}[v_*](x)\,\mathrm{d}\eta_*$, where η_* is the invariant probability measure corresponding to the control v_*.

Proof. The existence of a solution to (18) is established as a limit of $V_\alpha(\cdot) - V_\alpha(0)$, along some sequence $\alpha_n \searrow 0$, where V_α is the solution of (6) in Lemma 1 (Bensoussan and Frehse, 2002, p. 175). That V is inf-compact follows by (Bensoussan and Frehse, 2002, Theorem 4.21). It also follows from the proof of this convergence result that $\beta \le \limsup_{\alpha\searrow 0} \alpha V_\alpha(x)$ for all $x \in \mathbb{R}^d$.

We first show that $\beta \le \beta_*$. For this, we employ the following assertion which is a special case of the Hardy–Littlewood theorem Sznajder and Filar (1992). For any sequence $\{a_n\}$ of non-negative real numbers, it holds that

$$\limsup_{\theta\nearrow 1}\,(1-\theta)\sum_{n=1}^{\infty}\theta^n a_n \;\le\; \limsup_{N\to\infty}\,\frac{1}{N}\sum_{n=1}^{N} a_n\,. \tag{20}$$

Concerning this assertion, note that if the right hand side of (20) is finite, then the set $\{\frac{a_n}{n}\}$ is bounded. Therefore, $\sum_{n=1}^{\infty}\theta^n a_n$ is finite for every $\theta < 1$. Hence, we can apply (Sznajder and Filar, 1992, Theorem 2.2) to obtain (20).

Fix $x \in \mathbb{R}^d$, and $U \in \overline{\mathfrak{U}}_x$. Define

$$a_n \;:=\; \mathbb{E}^U_x\biggl[\int_{n-1}^{n} \mathcal{R}(X_s, U_s)\,\mathrm{d}s\biggr]\,,\quad n \ge 1\,,$$

and let $\theta = \mathrm{e}^{-\alpha}$. Applying (20), with N running over the set of natural numbers, we obtain

$$\begin{aligned}\limsup_{N\to\infty}\,\frac{1}{N}\,\mathbb{E}^U_x\biggl[\int_0^N \mathcal{R}(X_s, U_s)\,\mathrm{d}s\biggr]\\ &\ge\; \limsup_{\theta\nearrow 1}\,(1-\theta)\sum_{n=1}^{\infty}\theta^n a_n\\ &\ge\; \limsup_{\alpha\searrow 0}\,(1-\mathrm{e}^{-\alpha})\sum_{n=1}^{\infty}\mathbb{E}^U_x\biggl[\int_{n-1}^{n}\mathrm{e}^{-\alpha s}\,\mathcal{R}(X_s, U_s)\Bigr)\,\mathrm{d}s\biggr]\\ &\ge\; \limsup_{\alpha\searrow 0}\,(1-\mathrm{e}^{-\alpha})\,\mathbb{E}^U_x\biggl[\int_0^{\infty}\mathrm{e}^{-\alpha s}\,\mathcal{R}(X_s, U_s)\,\mathrm{d}s\biggr] \;\ge\; \beta\,.\end{aligned} \tag{21}$$

For the last inequality in (21), we use the fact that $\limsup_{\alpha\searrow 0} \alpha V_\alpha(x) \ge \beta$. Since $U \in \overline{\mathfrak{U}}_x$ is arbitrary, (21) together with the definition of β_* imply that $\beta \le \beta_*$. Note also that (21) implies that $\mathfrak{U}^\alpha_x \subset \overline{\mathfrak{U}}_x$ for all $\alpha \in (0,1)$.

Next, we prove the converse inequality. It is clear that (19) follows from (7). Therefore, since the Markov control $v_* := -\nabla V(x)$ has at most linear growth, there exists a unique strong solution to (7) under the control v_*. Applying Itô's formula to (18), and using the notation

$$\mathcal{R}[v](x) \;:=\; \mathcal{R}\bigl(x, v(x)\bigr) \;=\; \ell(x) + \frac{1}{2}\,|v(x)|^2\,,$$

we obtain

$$\mathbb{E}_x^{v_*}\bigl[V(X_{T\wedge\tau_R})\bigr] - V(x) + \mathbb{E}_x\biggl[\int_0^{T\wedge\tau_R} \mathcal{R}[v_*](X_s)\,\mathrm{d}s\biggr] \;=\; \beta\,\mathbb{E}_x\bigl[T\wedge\tau_R\bigr]\,,$$

where τ_R denotes the exit time from the ball of radius $R>0$ centered at 0. Since V is bounded from below and $\tau_R\to\infty$ a.s., as $R\to\infty$, by first using Fatou's lemma for the integral term in the above display, and then dividing by T and taking limits as $T\to\infty$, we obtain

$$\limsup_{T\to\infty}\,\frac{1}{T}\,\mathbb{E}_x^{v_*}\biggl[\int_0^T \mathcal{R}[v_*](X_s)\,\mathrm{d}s\biggr] \;\le\; \beta\,.$$

Thus $\beta=\beta_*$. Since ℓ is inf-compact, this also implies that the diffusion under the control v_* is positive recurrent, and by Birkhoff's ergodic theorem we obtain $\beta_*=\int_{\mathbb{R}^d}\mathcal{R}[v_*](x)\,\mathrm{d}\eta_*$. This completes the proof. □

Let $\mathcal{L}\colon\mathcal{C}^2(\mathbb{R}^d)\to\mathcal{C}(\mathbb{R}^d\times\mathbb{R}^d)$, and $\mathcal{L}_v$ denote the operators defined by

$$\mathcal{L}[f](x,u) \;:=\; \frac{1}{2}\Delta f(x) + \bigl\langle m(x)+u, \nabla f(x)\bigr\rangle\,,$$

and

$$\mathcal{L}_v f(x) \;:=\; \frac{1}{2}\,\Delta f(x) + \bigl\langle m(x)+v(x), \nabla f(x)\bigr\rangle\,, \qquad f\in\mathcal{C}^2(\mathbb{R}^d)\,,$$

respectively. Also, let $\mathfrak{P}$ denote the set of infinitesimal ergodic occupation measures, i.e., the set of probability measures $\pi\in\mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d)$ which satisfy

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathcal{L}[f](x,u)\,\pi(\mathrm{d}x,\mathrm{d}u) \;=\; 0 \qquad \forall\, f\in\mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^d)\,, \tag{22}$$

where $\mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^d)$ denotes the class of real-valued smooth functions with compact support. Note that if $\pi=\eta_v\circledast v\in\mathfrak{P}$ then (22) can be written as $\int_{\mathbb{R}^d}\mathcal{L}_v f(x)\,\eta_v(\mathrm{d}x)=0$.

Let $\hat{v}(x)=\int v(\mathrm{d}u\,|\,x)$. Since

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}|u|^2\eta_v(\mathrm{d}x)\,v(\mathrm{d}u\,|\,x) \;\ge\; \int_{\mathbb{R}^d}|\hat{v}(x)|^2\eta_v(\mathrm{d}x)\,, \tag{23}$$

and since $\eta_v \circledast \hat{v}$ is also an infinitesimal ergodic occupation measure, it is evident that as far as the proof of strong duality is concerned we may restrict our attention to the subset of $\mathfrak{P}$ that corresponds to precise controls, which we denote as $\mathfrak{P}_\circ$.

We have the following lemma.

Lemma 4. *If $\pi = \eta_v \circledast v \in \mathfrak{P}_\circ$ is such that*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{R}[v](x)\,\eta_v(\mathrm{d}x) \;<\; \infty\,, \tag{24}$$

then

$$\int_{\mathbb{R}^d} \mathcal{R}[v](x)\,\eta_v(\mathrm{d}x) \;=\; \beta_* + \frac{1}{2}\int_{\mathbb{R}^d} \bigl|v(x)+\nabla V(x)\bigr|^2\,\eta_v(\mathrm{d}x)\,. \tag{25}$$

In addition the measure η_v has a density $\varrho_v \in L^{d/(d-1)}(\mathbb{R}^d)$.

Proof. Let χ be a concave $\mathcal{C}^2(\mathbb{R}^d)$ function such that $\chi(x) = x$ for $x \le 0$, and $\chi(x) = 1$ for $x \ge 1$. Then χ' and $-\chi''$ are nonnegative on $(0,1)$. Define $\chi_R(x) := R + \chi(x-R)$ for $R > 0$. Using (18), and completing the square, we obtain

$$\mathcal{L}_v V - \tfrac{1}{2}|v+\nabla V|^2 + \mathcal{R}[v] - \beta_* \;=\; 0\,.$$

Applying $\mathcal{L}_v$ to the function $\chi_R(V)$, results in

$$\begin{aligned}\mathcal{L}_v\chi_R(V) - \tfrac{1}{2}\chi_R''(V)\,|\nabla V|^2 - \tfrac{1}{2}\chi_R'(V)\,|v+\nabla V|^2&\\ + \chi_R'(V)\mathcal{R}[v] - \chi_R'(V)\beta_* \;=\; 0\,.&\end{aligned} \tag{26}$$

Observe that $\chi_R(V)-R-1$ is compactly supported by construction. Thus $\int_{\mathbb{R}^d} \mathcal{L}_v\chi_R(V(x))\eta_v(\mathrm{d}x) = 0$ for all $R > 0$. Since $\int_{\mathbb{R}^d} \ell(x)\eta_v(\mathrm{d}x) < \infty$ by (24), the bound in (10) shows that

$$\int_{\mathbb{R}^d} |\nabla V(x)|^2\,\eta_v(\mathrm{d}x) \;<\; \infty\,. \tag{27}$$

Integrating (26) with respect to η_v, using (27), and passing to the limit as $R \to \infty$, we obtain (25). We have thus shown that

$$\int_{\mathbb{R}^d} |m(x)+v(x)|^2\,\eta_v(\mathrm{d}x) \;<\; \infty\,.$$

By Theorem 1.1 in Bogachev, Krylov and Röckner (1996), this implies that the measure η_v has a density in $L^{d/(d-1)}(\mathbb{R}^d)$. This completes the proof. $\square$

Proof of Theorem 1.4. Without loss of generality we assume $\varepsilon = 1$, and we suppress the explicit dependence on ε in the notation used in the theorem. The statement concerning existence of solutions and the behavior above and below a critical value for β follows by the results in Ichihara (2011). For this, we need to first verify a Foster–Lyapunov type hypothesis, which is part of the assumptions. Note that the operator F in Ichihara (2011) has a negative sign in the Laplacian so that $\mathcal{A}[\varphi] = -F[\varphi]$, where $\mathcal{A}$ is the operator defined in (18). So, given that ℓ is inf-compact, $\varphi_0 = 0$ is an obvious choice to satisfy (A4) in Ichihara (2011). Then of course $-\mathcal{A}[\varphi_0] \to -\infty$ as $|x| \to \infty$. Note that Theorem 2.2 in Ichihara (2011) then asserts that V is bounded below in $\mathbb{R}^d$.

Next, consider $\varphi_1 = -a_1\sqrt{\mathcal{V}}$ with $a_1 := \inf_{\mathcal{K}^c} \frac{|\langle m, \nabla\mathcal{V}\rangle|\sqrt{\mathcal{V}}}{|\nabla\mathcal{V}|^2}$, where $\mathcal{K}$ is as in Hypothesis 1.1 (3), and $\mathcal{V}$ is as in Lemma 2.3. Since $\mathcal{V}$ agrees with $\bar{\mathcal{V}}$ outside some compact set by Lemma 2.3, it follows by Hypothesis 1.1 (3) that $a_1 > 0$. Then we obtain

$$\begin{aligned}
\tfrac{1}{2}\Delta\varphi_1 + \langle m, \nabla\varphi_1\rangle - \tfrac{1}{2}|\nabla\varphi_1|^2 \;&=\; \tfrac{a_1}{4\sqrt{\mathcal{V}}}\,\Delta\mathcal{V} - \tfrac{a_1}{2\sqrt{\mathcal{V}}}\Big(\langle m, \nabla\mathcal{V}\rangle + \tfrac{a_1\sqrt{\mathcal{V}}-1}{4\mathcal{V}}\,|\nabla\mathcal{V}|^2\Big)\\
&\ge\; \tfrac{a_1}{4\sqrt{\mathcal{V}}}\big(\Delta\mathcal{V} - \langle m, \nabla\mathcal{V}\rangle\big) \qquad \text{on } \ \mathcal{K}^c\,.
\end{aligned}$$

Thus, since $\Delta\bar{\mathcal{V}}$ is bounded by Hypothesis 1.1 (3b), we obtain $-\mathcal{A}[\varphi_1] \to -\infty$ as $|x| \to \infty$. It is also clear that $\phi_0(x) - \phi_1(x) \to \infty$ as $|x| \to \infty$. Thus, Hypothesis (A.4)$'$ in Ichihara (2011) is also satisfied. Therefore, as shown in (Ichihara, 2011, Theorem 2.1), there exists some critical value λ_* such that the HJB equation

$$\frac{1}{2}\Delta V + \min_{u\in\mathbb{R}^d}\Big[\langle m+u, \nabla V\rangle + \ell + \tfrac{1}{2}\,|u|^2\Big] \;=\; \beta$$

has no solution for $\beta > \lambda^*$. Also by Theorem 2.2 and Corollary 2.3 in Ichihara (2011), if V is a solution for $\beta < \lambda^*$, then under the control $v = -\nabla V$, the diffusion is transient. Moreover, for $\beta = \lambda^*$ there exists a unique solution $V = V_*$ (up to an additive constant), and under the control $v_* = -\nabla V_*$ the diffusion

$$X_t \;=\; X_0 + \int_0^t \big(m(X_s) + v(X_s)\big)\,\mathrm{d}s + W_t\,, \quad t \;\ge\; 0\,,$$

is positive recurrent. It is clear then that Lemma 3 implies that $\lambda^* = \beta_*$.

We next turn to the proof of items (a)–(e). Part (a) follows directly by (Metafune, Pallara and Rhandi, 2005, Lemma 5.1). Note that a sharper estimate was established in the proof of Lemma 3 when $\beta = \beta_*$. The uniqueness of the solution for $\beta = \beta_*$ follows by the results in Ichihara (2011, 2012) discussed above, while the rest of the assertions in part (b) follow by Lemma 3. Part (c) follows by Lemma 4.

We now turn to part (d). It is enough to show that for any sequence $\{U^n\} \subset \widehat{\mathfrak{U}}$ and a sequence of times $\{t_n\}$ diverging to ∞,

$$\liminf_{n\to\infty} \frac{1}{t_n}\, \mathbb{E}_x\biggl[\int_0^{t_n} \mathcal{R}(X^n_s, U^n_s)\,\mathrm{d}s\biggr] \;\ge\; \beta^\varepsilon_*\,, \tag{28}$$

where X^n denotes the process controlled by U^n. All the terms in (28) are finite, since $\int_0^T \mathbb{E}^U_x[\mathcal{R}(X_s, U_s]\,\mathrm{d}s < \infty$ for any $U \in \widehat{\mathfrak{U}}$, a fact which clearly follows by (2). We include the dependence on the initial condition $X^n_0 = x$ explicitly in the notation, and thus we denote the corresponding sequence of mean empirical measures by $\Phi^{U^n}_{x,t_n}$. Recall that this is defined by

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,u)\,\Phi^{U^n}_{x,t_n}(\mathrm{d}x,\mathrm{d}u) \;=\; \mathbb{E}\biggl[\int_0^{t_n} f(X^n_s, U^n_s)\,\mathrm{d}s \;\Bigm|\; X^n_0 = x\biggr]$$

for all $f \in \mathcal{C}_b(\mathbb{R}^d \times \mathbb{U})$.

Extract a subsequence of $\{t_n\}$ over which the terms on the left hand side of (28) converge to the '$\liminf$' and suppose without loss of generality that this limit is finite. Then the corresponding subsequence of mean empirical measures is tight. Let $\uppi \in \mathfrak{P}$ be any limit point of this subsequence. It follows that the left hand side of (28) is lower bounded by $\uppi(\mathcal{R})$. However, $\uppi(\mathcal{R}) \ge \beta_*$ by (23) and Lemma 4. This completes the proof of part (d).

It remains to prove part (e). Let $\uppi = \eta_v \circledast v \in \mathfrak{P}_\circ$ be any optimal ergodic occupation measure, and $\uppi_* := \eta_* \circledast v_*$, with $v_* = -\nabla V$. By Lemma 4, η_v has density, which we denote by ρ_v. Let

$$\xi_v \;:=\; \frac{\rho_v}{\rho_v + \rho_*}\,, \qquad \text{and} \quad \xi_* \;:=\; \frac{\rho_*}{\rho_v + \rho_*}\,,$$

and also define $\bar{v} := \xi_v v + \xi_* v_*$ and $\bar{\eta} := \frac{1}{2}(\eta_v + \eta_*)$. Using the property that the drift of (1) is an affine function of the control, together with (Arapostathis, Borkar and Ghosh, 2012, Lemma 3.2.3), it is straightforward to verify that $\bar{\eta} \circledast \bar{v} \in \mathfrak{P}_\circ$.

By optimality, we have

$$
\begin{aligned}
0 \;&\le\; 2\int_{\mathbb{R}^d} \mathcal{R}[\bar v]\,\mathrm{d}\bar\eta - \int_{\mathbb{R}^d} \mathcal{R}[v]\,\mathrm{d}\eta_v - \int_{\mathbb{R}^d} \mathcal{R}[v_*]\,\mathrm{d}\eta_* \\
&=\; \int_{\mathbb{R}^d} |\xi_v\, v + \xi_*\, v_*|^2\,\mathrm{d}\bar\eta - \tfrac{1}{2}\int_{\mathbb{R}^d} |v|^2\,\mathrm{d}\eta_v - \tfrac{1}{2}\int_{\mathbb{R}^d} |v_*|^2\,\mathrm{d}\eta_* \\
&=\; \int_{\mathbb{R}^d} \Bigl(|\xi_v\, v + \xi_*\, v_*|^2 - \xi_v\,|v|^2 - \xi_*\,|v_*|^2\Bigr)\,\mathrm{d}\bar\eta \\
&=\; -\frac{1}{2}\int_{\mathbb{R}^d} \frac{\rho_v(x)\,\rho_*(x)}{\rho_v(x)+\rho_*(x)}\,|v(x)-v_*(x)|^2\,\mathrm{d}x\,.
\end{aligned} \tag{29}
$$

Since ρ_* is strictly positive, (29) implies that $\rho_v\,|v-v_*|=0$ a.e. in $\mathbb{R}^d$, and thus $v=v_*$ on the support of η_v. It is clear that if v is modified outside the support of η_v, then the modified $\eta_v \circledast v$ is also an infinitesimal ergodic occupation measure. Therefore $\eta_v \circledast v_* \in \mathfrak{P}_\circ$. The uniqueness of the invariant measure of the diffusion with generator $\mathcal{L}_{v_*}$ then implies that $\eta_v=\eta_*$, which in turn implies (since $v=v_*$ on the support of η_v) that $v=-\nabla V$ a.e. in $\mathbb{R}^d$. This completes the proof of part (e), and also of the theorem. □

2. Proofs of Lemma 1.17 and Theorem 1.19. We start with the proof of Lemma 1.17.

Proof of Lemma 1.17. Suppose that M has a number q of eigenvalues on the open right half complex plane. Using a similarity transformation we can transform M to a matrix of the form $\operatorname{diag}(M_1,-M_2)$ where $M_1\in\mathbb{R}^{(d-q)\times(d-q)}$ and $M_2\in\mathbb{R}^{q\times q}$ are Hurwitz matrices. So without loss of generality, we assume M has this form. Let S_1 and S_2 be the unique symmetric positive definite matrices solving the Lyapunov equations

$$
S_1M_1 + M_1^{\mathsf{T}}S_1 \;=\; -I\,,\qquad \text{and}\quad S_2M_2 + M_2^{\mathsf{T}}S_2 \;=\; -I\,,
$$

respectively. Extend these to symmetric matrices in $\mathbb{R}^{d\times d}$ by defining $\tilde S_1=\operatorname{diag}(S_1,0)$ and $\tilde S_2=\operatorname{diag}(0,S_2)$, and also define, for $\alpha>0$,

$$
\varphi_1(x) \;:=\; \mathrm{e}^{-\alpha\langle x,\tilde S_1x\rangle}\,,\quad \varphi_2(x) \;:=\; \mathrm{e}^{-\alpha\langle x,\tilde S_2x\rangle}\,,\quad \text{and}\;\; \varphi \;:=\; 1+\varphi_1-\varphi_2\,.
$$

Let $T_1=\operatorname{diag}(I_{(d-q)\times(d-q)},0_{q\times q})$, and $T_2=\operatorname{diag}(0_{(d-q)\times(d-q)},I_{q\times q})$. Then, with $\mathcal{L}_v f(x) := \frac{1}{2}\Delta f(x) + \big\langle Mx + v(x), \nabla f(x)\big\rangle$, we obtain

$$
\begin{aligned}
\mathcal{L}_v\big(1-\varphi_2(x)\big) &= \alpha\varphi_2(x)\Big(\operatorname{trace}(\tilde{S}_2) - 2\alpha\big\langle x, \tilde{S}_2^2 x\big\rangle + |T_2 x|^2 \\
&\qquad\qquad + 2\langle v(x), \tilde{S}_2 x\rangle\Big) \\
&\geq \alpha\varphi_2(x)\Big(\operatorname{trace}(\tilde{S}_2) + |T_2 x|^2 - \tfrac{1}{2}|T_2 x|^2 \\
&\qquad\qquad - 2\alpha\|\tilde{S}_2\|^2|T_2 x|^2 - 2\|\tilde{S}_2\|^2|v(x)|^2\Big) \\
&= \alpha\varphi_2(x)\Big(\operatorname{trace}(S_2) + \big(\tfrac{1}{2} - 2\alpha\|\tilde{S}_2\|^2\big)|T_2 x|^2 \\
&\qquad\qquad - 2\|\tilde{S}_2\|^2|v(x)|^2\Big)\,.
\end{aligned}
\tag{30}
$$

For the inequality in (30) we use

$$
\begin{aligned}
2\big\langle v(x), \tilde{S}_2 x\big\rangle = 2\big\langle \tilde{S}_2 v(x), T_2 x\big\rangle &\geq -\big|\tfrac{T_2 x}{\sqrt{2}}\big|^2 - |\sqrt{2}\tilde{S}_2 v(x)|^2 \\
&\geq -\tfrac{1}{2}|T_2 x|^2 - 2\|\tilde{S}_2\|^2|v(x)|^2\,.
\end{aligned}
$$

Using the analogous inequality for $\mathcal{L}_v\varphi_1(x)$ and combining the equations we obtain

$$
\begin{aligned}
\mathcal{L}_v\varphi(x) &\geq \alpha\mathrm{e}^{-\alpha\langle x, \tilde{S}_1 x\rangle}\Big(-\operatorname{trace}(S_1) + \big(\tfrac{1}{2} + 2\alpha\|\tilde{S}_1\|^2\big)|T_1 x|^2 \\
&\qquad\qquad - 2\|\tilde{S}_1\|^2|T_1 v(x)|^2\Big) \\
&\quad + \alpha\mathrm{e}^{-\alpha\langle x, \tilde{S}_2 x\rangle}\Big(\operatorname{trace}(S_2) + \big(\tfrac{1}{2} - 2\alpha\|\tilde{S}_2\|^2\big)|T_2 x|^2 \\
&\qquad\qquad - 2\|\tilde{S}_2\|^2|T_2 v(x)|^2\Big) \\
&\geq \alpha\Big(-\operatorname{trace}(S_1) + \mathrm{e}^{-\alpha\langle x, Sx\rangle}\big(\tfrac{1}{2} - 2\alpha\|S\|^2\big)|x|^2 \\
&\qquad\qquad - 2\|S\|^2|v(x)|^2\Big)\,,
\end{aligned}
\tag{31}
$$

with $S := \operatorname{diag}(S_1, S_2)$.

Applying Itô's formula to (31), dividing by α, and also using the fact that $\varphi \geq 0$ and $\|\varphi\|_\infty = 2$, we obtain

$$
\begin{aligned}
\mathbb{E}_x\bigg[\int_0^T &\Big(-\operatorname{trace}(S_1) + \mathrm{e}^{-\alpha\langle X_t, SX_t\rangle}\big(\tfrac{1}{2} - 2\alpha\|S\|^2\big)|X_t|^2 \\
&\qquad - 2\|S\|^2|v(X_t)|^2\Big)\,\mathrm{d}t\bigg] \leq \frac{2}{\alpha}\,.
\end{aligned}
$$

Dividing by T, letting $T \nearrow \infty$ and rearranging terms, we conclude that $\mathrm{e}^{-\alpha\langle x,Sx\rangle}|x|^2$ is integrable with respect to invariant probability measure μ_v under the control v for any $\alpha < \frac{1}{4\|S\|^2}$, and the following bound holds

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\alpha\langle x,Sx\rangle}|x|^2\,\mu_v(\mathrm{d}x) \;\le\; \frac{\operatorname{trace}(S_1)}{\frac{1}{2}-2\alpha\|S\|^2} + \frac{2\|S\|^2}{\frac{1}{2}-2\alpha\|S\|^2}\int_{\mathbb{R}^d}|v(x)|^2\,\mu_v(\mathrm{d}x)\,.$$

Taking limits as $\alpha \searrow 0$, using monotone convergence, we obtain

$$\int_{\mathbb{R}^d}|x|^2\,\mu_v(\mathrm{d}x) \;\le\; 2\operatorname{trace}(S_1) + 4\|S\|^2\int_{\mathbb{R}^d}|v(x)|^2\,\mu_v(\mathrm{d}x)\,.$$

The proof is complete. □

Proof Theorem 1.19. It is well known (Brockett, 1970, Theorem 3, p. 150) that there exists at most one symmetric matrix Q satisfying

$$M^{\mathsf{T}}Q + QM \;=\; Q^2 \tag{32}$$

and

$$(M-Q)\Sigma + \Sigma(M-Q)^{\mathsf{T}} \;=\; -I\,. \tag{33}$$

For $\kappa > 0$, consider the ergodic control problem of minimizing

$$J_\kappa(v) \;:=\; \limsup_{T\to\infty}\,\frac{1}{T}\,\mathbb{E}\biggl[\int_0^T\Bigl(\kappa\,|X_s|^2 + \tfrac{1}{2}\,|v(X_s)|^2\Bigr)\,\mathrm{d}s\biggr], \tag{34}$$

over $v \in \overline{\mathfrak{U}}_{\mathrm{SSM}}$, subject to the linear controlled diffusion

$$X_t \;=\; X_0 + \int_0^t \bigl(MX_s + v(X_s)\bigr)\,\mathrm{d}s + W_t\,, \quad t \;\ge\; 0\,. \tag{35}$$

As is also well known, an optimal stationary Markov control for this problem takes the form $v(x) = -Q_\kappa x$, where Q_κ is the unique positive definite symmetric solution to the matrix Riccati equation

$$Q_\kappa^2 - M^{\mathsf{T}}Q_\kappa - Q_\kappa M \;=\; 2\,\kappa I\,. \tag{36}$$

Moreover, Q_κ has the following property. Consider a deterministic linear control system $\dot{x}(t) = Mx(t) + u(t)$, with $x, u \in \mathbb{R}^d$, and initial condition $x(0) = x_0$. Let $\mathcal{U}$ denote the space of controls u satisfying $\int_0^T |u(t)|^2\,\mathrm{d}t < \infty$ for all $T > 0$, and $\phi_t^u(x_0)$ denote the solution of the differential equation under a control $u \in \mathcal{U}$. Then

$$\bigl\langle x_0, Q_\kappa x_0\bigr\rangle \;=\; \min_{u\in\mathcal{U}}\int_0^\infty\Bigl(|u(t)|^2 + 2\kappa|\phi_t^u(x_0)|^2\Bigr)\,\mathrm{d}t\,. \tag{37}$$

For these assertions, see (Brockett, 1970, Theorem 1, p. 147, and Theorem 5, p. 151).

On the other hand, $\Psi_\kappa(x) = \frac{1}{2}\langle x, Q_\kappa x\rangle$ is a solution of the associated HJB equation

$$\frac{1}{2}\Delta\Psi_\kappa(x) + \min_{u\in\mathbb{R}^d}\Big[\big\langle Mx+u, \nabla\Psi_\kappa(x)\big\rangle + \tfrac{1}{2}|u|^2\Big] + \kappa\,|x|^2 \;=\; \tfrac{1}{2}\operatorname{trace}(Q_\kappa)\,. \tag{38}$$

The HJB equation (38) characterizes the optimal cost, i.e.,

$$\inf_{v\in\overline{\mathfrak{U}}_{\mathrm{SSM}}} J_\kappa(v) \;=\; \tfrac{1}{2}\operatorname{trace}(Q_\kappa)\,.$$

Let $\mathcal{G}(M)$, $M\in\mathbb{R}^{d\times d}$, denote the collection of all matrices $G\in\mathbb{R}^{d\times d}$ such that $M-G$ is Hurwitz. Also, for $G\in\mathcal{G}(M)$, let Σ_G denote the (unique) symmetric solution of the Lyapunov equation

$$(M-G)\,\Sigma_G + \Sigma_G\,(M-G)^{\mathsf{T}} \;=\; -I\,, \tag{39}$$

and define

$$\begin{aligned} \mathcal{J}_G(M) \;&:=\; \frac{1}{2}\operatorname{trace}\big(G\,\Sigma_G\,G^{\mathsf{T}}\big)\,,\\ \mathcal{J}_*(M) \;&:=\; \inf_{G\in\mathcal{G}(M)} \mathcal{J}_G(M)\,. \end{aligned} \tag{40}$$

Since the stationary probability distribution of (35) under the control $v(x) = -Q_\kappa x$ is Gaussian, it follows by (34) that $G = Q_\kappa$ minimizes

$$\tilde{\mathcal{J}}_{G;\kappa}(M) \;:=\; \kappa\,\operatorname{trace}\big(\Sigma_G\big) + \tfrac{1}{2}\operatorname{trace}\big(G\,\Sigma_G\,G^{\mathsf{T}}\big)$$

over all matrices $G\in\mathcal{G}(M)$, where Σ_G is as in (39) (note that $\tilde{\mathcal{J}}_{G;0}(M) = \mathcal{J}_G(M)$ which is the right hand side of (40)). Combining this with (38), we obtain

$$\inf_{G\in\mathcal{G}(M)} \tilde{\mathcal{J}}_{G;\kappa}(M) \;=\; \tilde{\mathcal{J}}_{Q_\kappa;\kappa}(M) \;=\; \tfrac{1}{2}\operatorname{trace}(Q_\kappa)\,. \tag{41}$$

By Lemma 1.17, we have

$$\begin{aligned} \operatorname{trace}(\Sigma_{Q_\kappa}) \;&\le\; \widetilde{C}_0\big(1 + \tilde{\mathcal{J}}_{Q_\kappa;\kappa}(M)\big)\\ &=\; \widetilde{C}_0\big(1 + \tfrac{1}{2}\operatorname{trace}(Q_\kappa)\big)\,. \end{aligned} \tag{42}$$

It also follows by (37) that $Q_{\kappa'} - Q_\kappa$ is nonnegative definite if $\kappa' \ge \kappa$. Therefore Q_κ has a unique limit Q as $\kappa \searrow 0$. It is evident that Q is nonnegative semidefinite, and (36) shows that it satisfies (32). Since $\operatorname{trace}(\Sigma_{Q_\kappa})$

is bounded by (42), it follows that Σ_{Q_κ} converges along some subsequence $\kappa_n \searrow 0$ to a symmetric positive semidefinite matrix Σ. Thus (33) holds. However, (33) implies that Σ is invertible, and therefore, it is positive definite. In turn, (33) implies that $M - Q$ is Hurwitz.

Since $v_G(x) = -Gx$, $G \in \mathcal{G}(M)$, is in general suboptimal for the criterion $J_\kappa(v)$, applying Lemma 1.17 once more, we obtain

$$\mathcal{J}_{Q_\kappa}(M) \;\le\; \tilde{\mathcal{J}}_{Q_\kappa;\kappa}(M) \;\le\; \kappa\,\widetilde{C}_0\bigl(1 + \mathcal{J}_G(M)\bigr) + \mathcal{J}_G(M) \qquad \forall\, G \in \mathcal{G}(M)\,.$$

Therefore, we have

$$\mathcal{J}_*(M) \;\le\; \tilde{\mathcal{J}}_{Q_\kappa;\kappa}(M) \;\le\; \kappa\,\widetilde{C}_0\bigl(1 + \mathcal{J}_*(M)\bigr) + \mathcal{J}_*(M)\,,$$

and taking limits as $\kappa \searrow 0$, this implies by (41) that $\mathcal{J}_*(M) \;=\; \frac{1}{2}\,\mathrm{trace}(Q)$.

It remains to show that $\mathit{\Lambda}^+(M) \;=\; \frac{1}{2}\,\mathrm{trace}(Q)$. Let T be a unitary matrix such that $\tilde{Q} := TQT^{\mathsf{T}}$ takes the form $\tilde{Q} = \mathrm{diag}(0, \tilde{Q}_2)$, with $\tilde{Q}_2 \in \mathbb{R}^{q\times q}$ a positive definite matrix, for some $0 \le q \le d$. Write the corresponding block structure of TMT^{T} as

$$\tilde{M} \;:=\; TMT^{\mathsf{T}} \;=\; \begin{pmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{pmatrix}\,,$$

with $\tilde{M}_{22} \in \mathbb{R}^{q\times q}$. Since $M^{\mathsf{T}}Q + QM = Q^2$, we obtain $\tilde{M}^{\mathsf{T}}\tilde{Q} + \tilde{Q}\tilde{M} = \tilde{Q}^2$, and block multiplication shows that $\tilde{Q}_2\tilde{M}_{21} = 0$, which implies that $\tilde{M}_{21} = 0$. Since $M - Q$ is similar to $\tilde{M} - \tilde{Q}$, the latter must be Hurwitz, which implies that $\tilde{M}_{11}$ is Hurwitz. By block multiplication we have

$$\tilde{M}_{22}^{\mathsf{T}}\tilde{Q}_2 + \tilde{Q}_2\tilde{M}_{22} = \tilde{Q}_2^2\,. \tag{43}$$

Since $\tilde{Q}_2$ is positive definite, the matrix $-\tilde{M}_{22}$ is Hurwitz by the Lyapunov theorem. Thus $\mathit{\Lambda}^+(M) = \mathrm{trace}(\tilde{M}_{22})$, since $\tilde{M}_{11}$ is Hurwitz. Therefore, since $\tilde{Q}_2$ is invertible, and $\mathrm{trace}(Q) = \mathrm{trace}(\tilde{Q})$, we obtain by (43) that

$$\begin{aligned} \mathrm{trace}(Q) \;&=\; \mathrm{trace}(\tilde{Q}_2) \;=\; \mathrm{trace}(\tilde{M}_{22}^{\mathsf{T}} + \tilde{M}_{22}) \\ &=\; 2\,\mathrm{trace}(\tilde{M}_{22}) \;=\; 2\mathit{\Lambda}^+(M)\,. \end{aligned}$$

This proves part (a).

Now let $\hat{v} \in \overline{\mathfrak{U}}_{\mathrm{SSM}}$ be any control. Let $\bar{V}(x) = \frac{1}{2}\langle x, Qx\rangle$. Then $\bar{V}$ satisfies (38) with $\kappa = 0$. Since

$$\begin{aligned} \min_{u\in\mathbb{R}^d} \Bigl[\bigl\langle Mx + u, \nabla\bar{V}(x)\bigr\rangle + \tfrac{1}{2}|u|^2\Bigr] \;&=\; \bigl\langle Mx, \nabla\bar{V}(x)\bigr\rangle - \tfrac{1}{2}\bigl|Qx\bigr|^2 \\ &=\; \bigl\langle Mx + \hat{v}(x), \nabla\bar{V}(x)\bigr\rangle + \tfrac{1}{2}|\hat{v}(x)|^2 \\ &\qquad - \tfrac{1}{2}\bigl|Qx + \hat{v}(x)\bigr|^2\,, \end{aligned}$$

we obtain

$$\tfrac{1}{2}\Delta\bar{V}(x) + \bigl\langle Mx + \hat{v}(x), \nabla\bar{V}(x)\bigr\rangle + \tfrac{1}{2}|\hat{v}(x)|^2 = \tfrac{1}{2}\,\mathrm{trace}(Q) + \tfrac{1}{2}\bigl|Qx + \hat{v}(x)\bigr|^2\,. \tag{44}$$

Applying Itô's formula to (44), and using the fact that $\mu_{\hat{v}}$ has finite second moments as shown in Lemma 1.17, and $\bar{V}$ is quadratic, a standard argument gives

$$\int_{\mathbb{R}^d}\Bigl(\tfrac{1}{2}|\hat{v}(x)|^2 - \tfrac{1}{2}|Qx + \hat{v}(x)|^2\Bigr)\,\mu_{\hat{v}}(\mathrm{d}x) \;=\; \tfrac{1}{2}\,\mathrm{trace}(Q)\,. \tag{45}$$

Thus $\int_{\mathbb{R}^d}\frac{1}{2}|\hat{v}(x)|^2\,\mu_{\hat{v}}(\mathrm{d}x) \ge \frac{1}{2}\,\mathrm{trace}(Q) = \mathcal{J}_*(M)$. Therefore, we have

$$\inf_{v\in\bar{\mathfrak{U}}_{\mathrm{SSM}}}\int_{\mathbb{R}^d}\tfrac{1}{2}|v(x)|^2\,\mu_v(\mathrm{d}x) \;=\; \mathit{\Lambda}^+(M)\,. \tag{46}$$

Suppose $\hat{v}$ is optimal, i.e., attains the infimum in (46). By (45), we obtain

$$\int_{\mathbb{R}^d}|Qx + \hat{v}(x)|^2\,\mu_{\hat{v}}(\mathrm{d}x) \;=\; 0\,.$$

Therefore, since $\mu_{\hat{v}}$ has a positive density, it holds that $\hat{v}(x) = -Qx$ a.e. in $\mathbb{R}^d$. This completes the proof of part (b).

We have already shown that $\bar{V}(x) = \frac{1}{2}\langle x, Qx\rangle$ satisfies

$$\frac{1}{2}\,\Delta\bar{V}(x) + \bigl\langle Mx, \nabla\bar{V}(x)\bigr\rangle - \frac{|\nabla\bar{V}(x)|^2}{2} \;=\; \mathit{\Lambda}^+(M)\,,$$

and that the associated process is positive recurrent. Therefore, as in the proof of Theorem 1.4 for a bounded m, part (c) follows by Theorems 2.1–2.2 and Corollary 2.3 in Ichihara (2011). Note that Hypothesis (A4) in Ichihara (2011) is easily satisfied for the linear problem. Indeed, since M is exponentially dichotomous, then as seen in the proof of Theorem 2.2, there exist symmetric matrices S and $\hat{S}$, with $\hat{S}$ positive definite such that $M^{\mathsf{T}}S + SM = \hat{S}$. Consider the function $\varphi_0(x) := a\,\langle x, Sx\rangle$, with $a := \frac{1}{4}\bigl(\|\hat{S}^{-1}\|\|S\|^2\bigr)^{-1}$. Since

$$\|\hat{S}^{-1}\|\,\langle x, \hat{S}x\rangle \;\ge\; |x|^2 \;\ge\; \|S\|^{-2}\,|Sx|^2\,,$$

we obtain

$$\begin{aligned}
\bar{\mathcal{A}}[\varphi_0](x) \;&:=\; \tfrac{1}{2}\,\Delta\varphi_0(x) + \langle Mx, \nabla\varphi_0(x)\rangle - \tfrac{1}{2}\,|\nabla\varphi_0(x)|^2\\
&=\; a\;\mathrm{trace}\,S + a\,\langle x, \hat{S}x\rangle - 2a^2\,|Sx|^2\\
&>\; \tfrac{a}{2}\,\bigl(2\,\mathrm{trace}\,S + \langle x, \hat{S}x\rangle\bigr)\,.
\end{aligned}$$

Thus $\bar{\mathcal{A}}[\varphi_0](x) \to \infty$ as $|x| \to \infty$. This completes the proof. □

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Dept. of Electrical and Computer Eng.
The University of Texas at Austin
2501 Speedway, EER 7.824
Austin, TX 78712
E-mail: ari@ece.utexas.edu

Indian Institute of Science Education and Research
Dr. Homi Bhabha Road
Pune 411008, India
E-mail: anup@iiserpune.ac.in

Department of Electrical Engineering
Indian Institute of Technology
Powai, Mumbai, India
E-mail: borkar@ee.iitb.ac.in