ERGODICITY OF A LÉVY-DRIVEN SDE ARISING FROM MULTICLASS MANY-SERVER QUEUES[1]

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We study the ergodic properties of a class of multidimensional piecewise Ornstein–Uhlenbeck processes with jumps, which contains the limit of the queueing processes arising in multiclass many-server queues with heavy-tailed arrivals and/or asymptotically negligible service interruptions in the Halfin–Whitt regime as special cases. In these queueing models, the Itô equations have a piecewise linear drift, and are driven by either (1) a Brownian motion and a pure-jump Lévy process, or (2) an anisotropic Lévy process with independent one-dimensional symmetric α-stable components or (3) an anisotropic Lévy process as in (2) and a pure-jump Lévy process. We also study the class of models driven by a subordinate Brownian motion, which contains an isotropic (or rotationally invariant) α-stable Lévy process as a special case. We identify conditions on the parameters in the drift, the Lévy measure and/or covariance function which result in subexponential and/or exponential ergodicity. We show that these assumptions are sharp, and we identify some key necessary conditions for the process to be ergodic. In addition, we show that for the queueing models described above with no abandonment, the rate of convergence is polynomial, and we provide a sharp quantitative characterization of the rate via matching upper and lower bounds.

CONTENTS

1. Introduction.

We consider a d-dimensional stochastic differential equation (SDE) of the form

$$(1.1) \qquad \mathrm{d}X(t) = b\big(X(t)\big)\,\mathrm{d}t + \sigma\big(X(t)\big)\,\mathrm{d}W(t) + \mathrm{d}L(t), \qquad X(0) = x \in \mathbb{R}^d,$$

where

(A1) the function $b\colon \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$b(x) = \ell - M\big(x - \langle e, x\rangle^+ v\big) - \langle e, x\rangle^+ \Gamma v$$

$$= \begin{cases} \ell - (M + (\Gamma - M)ve')x, & e'x > 0, \\ \ell - Mx, & e'x \le 0, \end{cases}$$

where $\ell \in \mathbb{R}^d$, $v \in \mathbb{R}^d_+$ satisfies $\langle e, v\rangle = e'v = 1$ with $e = (1, \ldots, 1)' \in \mathbb{R}^d$, $M \in \mathbb{R}^{d \times d}$ is a nonsingular M-matrix such that the vector $e'M$ has nonnegative components, and $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_d)$ with $\gamma_i \in \mathbb{R}_+$, $i = 1, \ldots, d$;

(A2) $\{W(t)\}_{t \ge 0}$ is a standard n-dimensional Brownian motion, and the covariance function $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d \times n}$ is locally Lipschitz and satisfies, for some constant $\kappa > 0$,

$$\big\|\sigma(x)\big\|^2 \le \kappa\big(1 + |x|^2\big), \qquad x \in \mathbb{R}^d;$$

(A3) $\{L(t)\}_{t \ge 0}$ is a d-dimensional pure-jump Lévy process specified by a drift $\vartheta \in \mathbb{R}^d$ and Lévy measure $\nu(\mathrm{d}y)$.

In (A1)–(A3), $\|M\| := (\mathrm{Tr}\, MM')^{\frac{1}{2}}$ denotes the Hilbert–Schmidt norm of a $d \times n$ matrix M, and $\langle \cdot, \cdot \rangle$ stands for the inner product on $\mathbb{R}^d$. For a square matrix M, $\mathrm{Tr}\, M$ stands for the trace of M, and for a vector x and a matrix M, x' and M' stand for their transposes, respectively. A $d \times d$ matrix M is called an M-matrix if it can be expressed as $M = s\mathbb{I} - N$ for some $s > 0$ and some nonnegative $d \times d$ matrix N with the property that $\rho(N) \le s$, where $\mathbb{I}$ and $\rho(N)$ denote the $d \times d$ identity matrix and spectral radius of N, respectively. Clearly, the matrix M is nonsingular if $\rho(N) < s$.

Such a SDE is often called a piecewise Ornstein–Uhlenbeck (O–U) process with jumps. Recall that a Lévy measure $\nu(\mathrm{d}y)$ is a σ-finite measure on $\mathbb{R}^d_* := \mathbb{R}^d \setminus \{0\}$

satisfying $\int_{\mathbb{R}^d_*}(1 \wedge |y|^2)\nu(\mathrm{d}y) < \infty$. It is well known that the SDE (1.1) admits a unique nonexplosive strong solution $\{X(t)\}_{t \geq 0}$ which is a strong Markov process and it satisfies the C_b-Feller property (see [1], Theorem 3.1, and Propositions 4.2 and 4.3). In addition, in the same reference, it is shown that the infinitesimal generator $(\mathcal{A}^X, \mathcal{D}_{\mathcal{A}^X})$ of $\{X(t)\}_{t \geq 0}$ (with respect to the Banach space $(\mathcal{B}_b(\mathbb{R}^d), \|\cdot\|_\infty)$) satisfies $C_c^2(\mathbb{R}^d) \subseteq \mathcal{D}_{\mathcal{A}^X}$ and

$$
\begin{aligned}
(1.2) \quad \mathcal{A}^X|_{C_c^2(\mathbb{R}^d)} f(x) &= \frac{1}{2} \operatorname{Tr}\big(a(x)\nabla^2 f(x)\big) + \langle b(x) + \vartheta, \nabla f(x) \rangle \\
&\quad + \int_{\mathbb{R}^d_*} \mathfrak{d}_1 f(x; y)\nu(\mathrm{d}y),
\end{aligned}
$$

with $\nabla^2 f(x)$ denoting the Hessian of $f(x)$. Here, $\mathcal{D}_{\mathcal{A}^X}$, $\mathcal{B}_b(\mathbb{R}^d)$ and $C_c^2(\mathbb{R}^d)$ denote the domain of $\mathcal{A}^X$, the space of bounded Borel measurable functions and the space of twice continuously differentiable functions with compact support, respectively. In (1.2), we use the notation $a(x) = (a^{ij}(x))_{1 \leq i, j \leq d} := \sigma(x)\sigma(x)'$, and

$$
(1.3) \quad \mathfrak{d}_1 f(x; y) := f(x + y) - f(x) - \mathbb{1}_\mathcal{B}(y)\langle y, \nabla f(x) \rangle, \qquad f \in C^1(\mathbb{R}^d),
$$

where $\mathcal{B}$ denotes the unit ball in $\mathbb{R}^d$ centered at 0, and $\mathbb{1}_\mathcal{B}$ its indicator function.

The goal of this paper is to investigate the ergodic properties of $\{X(t)\}_{t \geq 0}$. This process arises as a limit of the suitably scaled queueing processes of multiclass many-server queueing networks with heavy-tailed (bursty) arrivals and/or asymptotically negligible service interruptions. In these models, if the scheduling policy is based on a static priority assignment on the queues, then the vector v in the limiting diffusion (1.1) corresponds a constant control, that is, an element of the set

$$
\Delta := \{v \in \mathbb{R}^d_+ : \langle e, v \rangle = 1\}.
$$

The process $\{X(t)\}_{t \geq 0}$ also arises in many-server queues with phase-type service times, where the constant vector v corresponds to the probability distribution of the phases.

These queueing models are described in detail in Section 4. It is important to note that for a multiclass queueing network with independent heavy-tailed arrivals, the process $\{L(t)\}_{t \geq 0}$ in (1.1) is an anisotropic Lévy process consisting of independent one-dimensional symmetric α-stable components; for a detailed description, see Section 4.1. Such processes have a highly singular Lévy measure and lack the regularity properties of the standard isotropic (or rotationally invariant) α-stable d-dimensional Lévy processes. Notably, as shown in [11], the Harnack inequality, an essential tool in showing regularity of the invariant probability measure for nondegenerate continuous diffusions, fails for SDEs driven by this anisotropic Lévy process. In Theorem 3.1, we establish the open-set irreducibility of solutions of (1.1) driven by an anisotropic α stable process. This is required in the results which follow. Other than the work in [10, 11, 14], they have not been studied much.

Under service interruptions, $\{L(t)\}_{t\geq 0}$ is either a compound Poisson process (under $\sqrt{n}$ scaling), or an anisotropic Lévy process described above together with a compound Poisson component (under $n^{\frac{1}{\alpha}}$ scaling for $\alpha \in (1,2)$). In this paper, however, we study the ergodic properties of (1.1) for a much broader class of Lévy processes $\{L(t)\}_{t\geq 0}$.

If the control (scheduling policy) is a function of the state of the system, then $v(x)$ in the diffusion limit is, in general, a Borel measurable map from $\mathbb{R}^d$ to Δ. We call such a $v(x)$ a stationary Markov control and denote the set of such controls by $\mathfrak{U}_{\mathrm{SM}}$. If (1.1) is driven by a Wiener process only, it follows from the results in [25] that, under any $v \in \mathfrak{U}_{\mathrm{SM}}$, the diffusion has a unique strong solution. On the other hand, as shown in [52], if the Lévy measure is finite, the solution of (1.1) can be constructed in a piecewise fashion, and thus, in such a case we have a unique strong solution under any $v \in \mathfrak{U}_{\mathrm{SM}}$. There are no such sharp results on existence of solutions to (1.1) with a measurable drift, when this is driven by a general Lévy process. However, the well-posedness of the martingale problem for SDEs with measurable drifts driven by an α-stable process has been studied (see [63] and references therein). We are not concerned with this problem in this paper, especially since the results involving Markov controls concern only on necessary conditions, and we clarify that whenever we state a result involving Markov controls it is implied that the martingale problem is well posed. Parenthetically, we mention here that for Markov controls that are locally Lipschitz the problem is always well posed for the model we consider (see the discussion in the beginning of Section 5.4).

1.1. *Summary of the results.* Broadly speaking, the results in this paper have two flavors. On the one hand, we present sufficient conditions under which $\{X(t)\}_{t\geq 0}$ is ergodic under any constant control $v \in \Delta$ (Theorems 3.2, 3.4(a), and 3.5), while on the other, we present necessary conditions for ergodicity under any Markov control (Theorems 3.3, 3.4(b), Lemma 5.7 and Corollary 5.1).

It turns out that these conditions are sharp and they match. We discuss these results in the context of a many-server queueing network with heavy-tailed arrivals and/or service interruptions, even though the results are applicable to a larger class of SDEs. There are two important parameters involved. One concerns the heaviness of the tail of the Lévy measure, and to describe this we define

$$(1.4) \qquad \Theta_c := \left\{\theta > 0 : \int_{\mathcal{B}^c} |y|^\theta \nu(\mathrm{d}y) < \infty\right\} \quad \text{and} \quad \theta_c := \sup\{\theta \in \Theta_c\}.$$

It follows from its definition that, if bounded, Θ_c is an open or left-open interval, that is, the interval $(0, \alpha)$ in the case of an α-stable process (isotropic or not), or it could be an interval of the form $(0, \theta_c]$. When more than one Lévy components are involved, Θ_c refers to the intersection of the individual intervals. The other parameter is the constant term ℓ in the drift which arises as the limit of the spare capacity of the network, when driven only by a Wiener process (see (4.2)). It turns out that this constant should be modified to account for the drift in the Lévy process. Recall

that ℓ is the constant in the drift in (A1) (see (1.1)) and that ϑ and $\nu(\mathrm{d}y)$ are the drift and the Lévy measure of the process $\{L(t)\}_{t\geq 0}$. We define

$$(1.5) \qquad \tilde{\ell} := \begin{cases} \ell + \vartheta + \displaystyle\int_{\mathcal{B}^c} y\,\nu(\mathrm{d}y) & \text{if } \displaystyle\int_{\mathcal{B}^c} |y|\nu(\mathrm{d}y) < \infty, \\ \ell + \vartheta & \text{otherwise,} \end{cases}$$

and

$$(1.6) \qquad \tilde{\varrho} := -\langle e, M^{-1}\tilde{\ell}\rangle.$$

We refer to $\tilde{\varrho}$ as the (effective) spare capacity.

The richest and most interesting set of results concerns networks where the abandonment rate is 0, and this corresponds to $\Gamma = 0$, or more generally, when the control gives lowest priority to queues whose abandonment rate is 0 (this is equivalent to $\Gamma v = 0$). In this scenario, we establish in Theorem 3.3 and Lemma 5.7 that $\tilde{\varrho} > 0$ and $1 \in \Theta_c$ are both necessary conditions for the state process $\{X(t)\}_{t\geq 0}$ to have an invariant probability measure under some Markov control $v \in \mathfrak{U}_{\mathrm{SM}}$ (see also Corollary 4.1). This translates to the requirement that $\alpha > 1$ if the system has heavy-tailed arrivals, and/or $\theta_c \geq 1$ if there are service interruptions. If these conditions are met, we show in Theorem 3.2 that the process is ergodic under any constant control $v \in \Delta$. Moreover, we prove in Theorems 3.2 and 3.4 that convergence to the invariant measure in total variation has a polynomial rate $r(t) \approx t^{\theta_c - 1}$ for any constant control, and by this we mean that the rate is $r(t) = t^{\theta_c - \epsilon - 1}$ for all $\epsilon \in (0, \theta_c - 1)$ for $\theta_c > 1$, and $\epsilon = 0$ for $\theta_c = 1$. This is accomplished by deriving matching upper and lower bounds for convergence (see (3.8)). An interesting related result is that the spare capacity $\tilde{\varrho}$ is equal to the average idleness of the system (idle servers) under any Markov control v satisfying $\Gamma v(x) = 0$ a.e., and such that the state process is ergodic (see Corollary 5.1).

In the context of many-server queueing networks, stability is defined as the finiteness of the average value of the sum of the queue lengths, and this translates into the requirement that the map $x \mapsto \langle e, x\rangle^+$ be integrable under the invariant probability measure of the process. In turn, a necessary and sufficient condition for this is that the invariant probability measure has a finite first absolute moment (see Remark 5.1). We refer to a control attaining this property as stabilizing. Lemma 5.7 shows that if there is no abandonment, then no Markov control is stabilizing unless $2 \in \Theta_c$, while under abandonment it is necessary that $1 \in \Theta_c$ (see Corollary 4.1 and Theorem 3.5). This means that for a system with heavy-tailed arrivals (resulting in an α-stable limit with $\alpha \in (1, 2)$), there are no stabilizing controls, unless some abandonment rate is positive. On the other hand, for a system under service interruptions and no heavy-tailed arrivals, a necessary and sufficient condition for the existence of stabilizing controls, under no abandonment, is that the Lévy measure has a finite second moment. If such is the case, then every constant control is stabilizing by Theorem 3.2.

Another set of results concern the case $\Gamma v \neq 0$. Here, we show that $\{X(t)\}_{t \geq 0}$ has an exponential rate of convergence (Theorem 3.5), and that every constant control is stabilizing, provided $1 \in \Theta_c$.

1.2. *Literature review.* Our work relates to the active research on Lévy-driven (generalized) O–U processes, and the vast literature on SDEs with jumps. In [18, 23, 29, 35, 51, 57, 60], the ergodic properties of a general class of Lévy-driven O–U processes are established using Foster–Lyapunov and coupling methods. In all of these works, the process is governed by a linear drift function. In [47], a one-dimensional piecewise O–U process driven by a spectrally one-sided Lévy process is studied. The authors have shown the existence and characterization of the invariant distribution, and ergodicity of the process. In [17], motivated by the many-server queuing model with phase-type service times, the authors have established ergodicity and exponential ergodicity of a piecewise O–U process driven by Brownian motion only. See also Remark 3.4 on the comparison of the models and contributions.

For general diffusions with jumps, ergodic properties are studied in [31, 36, 37, 44, 49, 59, 61], under suitable conditions on the drift, covariance function and jump component. In this paper, we take advantage of the explicit form of the drift and carry out detailed calculations which yield important insights on the rates of convergence and ergodic properties. Some of the estimates in the proofs may be of independent interest to future work on the subject. Our results also lay important foundations for the study of ergodic control problems for Lévy-driven SDEs (see a recent development in [6]), especially those arising from the multiclass many-server queueing systems; recent studies on Markovian queueing models are in [4, 7–9].

A surprising discovery of this study is a class of models in (1.1) possessing a "polynomial" ergodicity property in the total variation norm. Subexponential ergodicity of Markov processes, including diffusions and SDEs with jumps, has been a very active research area in recent years; see, for example, [2, 13, 15, 18, 19, 21, 23, 24, 26, 34, 54] and references therein. Note that in [18], some interesting diffusion models and an O–U process (linear drift) driven by a compound Poisson process with a heavy-tailed jump is studied as examples for the general theory of subexponential ergodicity. Our work identifies a concrete, yet highly nontrivial, class of SDEs with jumps that satisfy the conditions for polynomial ergodicity in [18] (see also [26]). This may be of great interest to a broad audience on the subject of ergodicity of Markov processes.

The rate of convergence for the limiting queueing process of multiclass many-server networks under heavy-tailed arrivals and/or asymptotically negligible service interruptions has not been studied up to now. Finally, it is worth mentioning that there exist very scarce results on subexponential ergodicity in queueing theory; see, for example, [28].

1.3. *Organization of the paper.* In the next subsection, we summarize some notation used in this paper. In Section 2, we review some background material on the ergodicity of Markov processes that is relevant to our study. The main results are presented in Section 3. In Section 4, we provide some motivating examples of multiclass many-server queues which have queueing process limits as in (1.1), and state the relevant ergodic properties. Section 5 is devoted to the proofs of Theorems 3.2 to 3.5, and contains some additional results. The Appendix contains the proof of Theorem 3.1.

1.4. *Notation.* We summarize some notation used throughout the paper. We use $\mathbb{R}^d$ (and $\mathbb{R}^d_+$), $d \geq 1$, to denote real-valued d-dimensional (nonnegative) vectors, and write $\mathbb{R}$ ($\mathbb{R}_+$) for $d = 1$. For $x, y \in \mathbb{R}$, $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. Let $D^d = D([0, \infty), \mathbb{R}^d)$ denote the $\mathbb{R}^d$-valued function space of all right-continuous functions on $[0, \infty)$ with left limits everywhere in $(0, \infty)$. Let (D^d, M_1) denote the space D^d equipped with the Skorohod M_1 topology. Denote $D \equiv D^1$. Let $(D_d, M_1) = (D, M_1) \times \cdots \times (D, M_1)$ be the d-fold product of (D, M_1) with the product topology [62]. For a set $A \subseteq \mathbb{R}^d$, we use A^c and $\mathbb{1}_A$ to denote the complement and the indicator function of A, respectively. A ball of radius $r > 0$ in $\mathbb{R}^d$ around a point x is denoted by $\mathcal{B}_r(x)$, or simply as $\mathcal{B}_r$ if $x = 0$. We also let $\mathcal{B} \equiv \mathcal{B}_1$. The Euclidean norm on $\mathbb{R}^d$ is denoted by $|\cdot|$. We let $\mathfrak{B}(\mathbb{R}^d)$ stand for the Borel σ-algebra on $\mathbb{R}^d$. For a Borel probability measure $\pi(\mathrm{d}x)$ on $\mathfrak{B}(\mathbb{R}^d)$ and a measurable function $f(x)$, which is integrable under $\pi(\mathrm{d}x)$, we often use the convenient notation $\pi(f) = \int_{\mathbb{R}^d} f(x)\pi(\mathrm{d}x)$.

2. Preliminaries. Let $(\Omega, \mathcal{F}, \mathcal{F}(t), M(t), \theta(t), \{\mathbb{P}^x\}_{x \in \mathbb{R}^d})$, $t \in [0, \infty)$, denoted by $\{M(t)\}_{t \geq 0}$ in the sequel, be a Markov process with càdlàg sample paths and state space $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$ (see [12], page 20). We let $P_t^M(x, \mathrm{d}y) := \mathbb{P}^x(M(t) \in \mathrm{d}y)$, $t \geq 0$ and $x \in \mathbb{R}^d$, denote the transition probability of $\{M(t)\}_{t \geq 0}$. Also, in the sequel we assume that $P_t^M(x, \mathrm{d}y)$, $t \geq 0$ and $x \in \mathbb{R}^d$, is a probability measure, that is, $\{M(t)\}_{t \geq 0}$ does not admit a cemetery point in the sense of [12]. Observe that this is not a restriction since, as we have already commented, $\{X(t)\}_{t \geq 0}$ is nonexplosive. The process $\{M(t)\}_{t \geq 0}$ is called:

(i) φ-irreducible if there exists a σ-finite measure $\varphi(\mathrm{d}y)$ on $\mathfrak{B}(\mathbb{R}^d)$ such that whenever $\varphi(B) > 0$ we have $\int_0^\infty P_t^M(x, B)\,\mathrm{d}t > 0$ for all $x \in \mathbb{R}^d$;

(ii) transient if it is φ-irreducible, and if there exists a countable covering of $\mathbb{R}^d$ with sets $\{B_j\}_{j \in \mathbb{N}} \subseteq \mathfrak{B}(\mathbb{R}^d)$, and for each $j \in \mathbb{N}$ there exists a finite constant $c_j \geq 0$ such that $\int_0^\infty P_t^M(x, B_j)\,\mathrm{d}t \leq c_j$ holds for all $x \in \mathbb{R}^d$;

(iii) recurrent if it is φ-irreducible, and $\varphi(B) > 0$ implies $\int_0^\infty P_t^M(x, B)\,\mathrm{d}t = \infty$ for all $x \in \mathbb{R}^d$.

Let us remark that if $\{M(t)\}_{t \geq 0}$ is a φ-irreducible Markov process, then the irreducibility measure $\varphi(\mathrm{d}y)$ can be maximized. This means that there exists a unique

"maximal" irreducibility measure ψ such that for any measure $\bar{\varphi}(\mathrm{d}y)$, $\{M(t)\}_{t \geq 0}$ is $\bar{\varphi}$-irreducible if, and only if, $\bar{\varphi} \ll \psi$ (see [55], Theorem 2.1). In view to this, when we refer to an irreducibility measure we actually refer to the maximal irreducibility measure. It is also well known that every ψ-irreducible Markov process is either transient or recurrent (see [55], Theorem 2.3).

Recall, a Markov process $\{M(t)\}_{t \geq 0}$ is called:

(1) open-set irreducible if its maximal irreducibility measure $\psi(\mathrm{d}y)$ is fully supported, that is, $\psi(O) > 0$ for every open set $O \subseteq \mathbb{R}^d$;

(2) aperiodic if it admits an irreducible skeleton chain, that is, there exist $t_0 > 0$ and a σ-finite measure $\phi(\mathrm{d}y)$ on $\mathfrak{B}(\mathbb{R}^d)$, such that $\phi(B) > 0$ implies $\sum_{n=0}^{\infty} P_{nt_0}^M(x, B) > 0$ for all $x \in \mathbb{R}^d$.

Let $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{P}(\mathbb{R}^d)$ denote the classes of Borel measurable functions and Borel probability measures on $\mathbb{R}^d$, respectively. We adopt the usual notation

$$\pi P_t^M(\mathrm{d}y) = \int_{\mathbb{R}^d} \pi(\mathrm{d}x) P_t^M(x, \mathrm{d}y)$$

for $\pi \in \mathcal{P}(\mathbb{R}^d)$, $t \geq 0$, and $P_t^M f(x) = \int_{\mathbb{R}^d} P_t^M(x, \mathrm{d}y) f(y)$ for $t \geq 0$, $x \in \mathbb{R}^d$ and $f \in \mathcal{B}(\mathbb{R}^d)$. Therefore, with δ_x denoting the Dirac measure concentrated at $x \in \mathbb{R}^d$, we have $\delta_x P_t^M(\mathrm{d}y) = P_t^M(x, \mathrm{d}y)$, $t \geq 0$.

Recall that a probability measure $\overline{\pi} \in \mathcal{P}(\mathbb{R}^d)$ is called invariant for $\{M(t)\}_{t \geq 0}$ if $\int_{\mathbb{R}^d} P_t^M(x, \mathrm{d}y) \overline{\pi}(\mathrm{d}x) = \overline{\pi}(\mathrm{d}y)$ for all $t > 0$. It is well known that if $\{M(t)\}_{t \geq 0}$ is recurrent, then it possesses a unique (up to constant multiples) invariant measure $\overline{\pi}(\mathrm{d}y)$ (see [55], Theorem 2.6). If the invariant measure is finite, then it may be normalized to a probability measure. If $\{M(t)\}_{t \geq 0}$ is recurrent with finite invariant measure, then $\{M(t)\}_{t \geq 0}$ is called positive recurrent; otherwise, it is called null recurrent. Note that a transient Markov process cannot have a finite invariant measure. Indeed, assume that $\{M(t)\}_{t \geq 0}$ is transient and that it admits a finite invariant measure $\overline{\pi}(\mathrm{d}y)$, and fix some $t > 0$. Then, for each $j \in \mathbb{N}$, with c_j and B_j as in (ii) above, we have

$$t\overline{\pi}(B_j) = \int_0^t \int_{\mathbb{R}^d} P_t^M(x, B_j) \overline{\pi}(\mathrm{d}x) \, \mathrm{d}s \leq c_j \overline{\pi}(\mathbb{R}^d).$$

Now, by letting $t \to \infty$ we obtain $\overline{\pi}(B_j) = 0$ for all $j \in \mathbb{N}$, which is impossible.

A Markov process $\{M(t)\}_{t \geq 0}$ is called ergodic if it possesses an invariant probability measure $\overline{\pi}(\mathrm{d}y)$ and there exists a nondecreasing function $r \colon \mathbb{R}_+ \to [1, \infty)$ such that

$$\lim_{t \to \infty} r(t) \| P_t^M(x, \mathrm{d}y) - \overline{\pi}(\mathrm{d}y) \|_{\mathsf{TV}} = 0, \qquad x \in \mathbb{R}^d.$$

Here, $\|\cdot\|_{\mathsf{TV}}$ denotes the total variation norm on the space of signed measures on $\mathfrak{B}(\mathbb{R}^d)$. For a function $f \colon \mathbb{R}^d \to [1, \infty)$, we define the f-norm of a signed measure μ as

$$\|\mu\|_f := \sup_{g \in \mathcal{B}(\mathbb{R}^d), |g| \leq f} \left| \int_{\mathbb{R}^d} g(y) \mu(\mathrm{d}y) \right|.$$

Observe that $\|\cdot\|_1 = \|\cdot\|_{\mathsf{TV}}$. We say that $\{M(t)\}_{t\geq0}$ is subexponentially ergodic if it is ergodic and $\lim_{t\to\infty} \frac{\ln r(t)}{t} = 0$, and that it is exponentially ergodic if it is ergodic and $r(t) = e^{\kappa t}$ for some $\kappa > 0$. Let us remark that (under the assumptions of open-set irreducibility and aperiodicity) ergodicity is equivalent to positive recurrence (see [38], Theorem 13.0.1, [39], Theorem 6.1, and [55], Theorems 4.1, 4.2 and 7.1).

Since $\{M(t)\}_{t\geq0}$ is a Markov process, $P_t^M f(x) = \int_{\mathbb{R}^d} f(y) P_t^M(x, \mathrm{d}y)$, $x \in \mathbb{R}^d$, defines a semigroup of linear operators $\{P_t^M\}_{t\geq0}$ on the Banach space $(\mathcal{B}_b(\mathbb{R}^d), \|\cdot\|_\infty)$, that is, $P_s^M \circ P_t^M = P_{s+t}^M$ for all $s, t \geq 0$, and $P_0^M f = f$. Here, $\|\cdot\|_\infty$ denotes the supremum norm on the space $\mathcal{B}_b(\mathbb{R}^d)$. The infinitesimal generator $(\mathcal{A}^M, \mathcal{D}_{\mathcal{A}^M})$ of the semigroup $\{P_t^M\}_{t\geq0}$ of a Markov process $\{M(t)\}_{t\geq0}$ is a linear operator $\mathcal{A}^M : \mathcal{D}_{\mathcal{A}^M} \longrightarrow \mathcal{B}_b(\mathbb{R}^d)$ defined by

$$\mathcal{A}^M f := \lim_{t\to 0} \frac{P_t^M f - f}{t},$$

$$f \in \mathcal{D}_{\mathcal{A}^M} := \left\{ f \in \mathcal{B}_b(\mathbb{R}^d) : \lim_{t\to 0} \frac{P_t^M f - f}{t} \text{ exists in } \|\cdot\|_\infty \right\}.$$

Let $C_b(\mathbb{R}^d)$ denote the space of continuous bounded functions. A Markov process $\{M(t)\}_{t\geq0}$ is called a C_b-Feller process if its corresponding semigroup satisfies $P_t^M(C_b(\mathbb{R}^d)) \subseteq C_b(\mathbb{R}^d)$ for all $t \geq 0$, and it is called a strong Feller process if $P_t^M(\mathcal{B}_b(\mathbb{R}^d)) \subseteq C_b(\mathbb{R}^d)$ for all $t > 0$.

Recall that the extended domain of $\{M(t)\}_{t\geq0}$, denoted by $\mathcal{D}_{\bar{\mathcal{A}}^M}$, is defined as the set of all $f \in \mathcal{B}(\mathbb{R}^d)$ such that $f(M(t)) - f(M(0)) - \int_0^t g(M(s)) \,\mathrm{d}s$ is a local $\{\mathbb{P}^x\}_{x\in\mathbb{R}^d}$-martingale for some $g \in \mathcal{B}(\mathbb{R}^d)$. Let us remark that in general the function g does not have to be unique (see [22], page 24). For $f \in \mathcal{D}_{\bar{\mathcal{A}}^M}$, we define

$$\bar{\mathcal{A}}^M f := \left\{ g \in \mathcal{B}(\mathbb{R}^d) : f(M(t)) - f(M(0)) - \int_0^t g(M(s)) \,\mathrm{d}s \right.$$

$$\left. \text{is a local } \{\mathbb{P}^x\}_{x\in\mathbb{R}^d}\text{-martingale} \right\}.$$

We call $\bar{\mathcal{A}}^M$ the extended generator of $\{M(t)\}_{t\geq0}$. A function $g \in \bar{\mathcal{A}}^M f$ is usually abbreviated by $\bar{\mathcal{A}}^M f(x) := g(x)$. A well-known fact is that if $(\mathcal{A}^M, \mathcal{D}_{\mathcal{A}^M})$ is the infinitesimal generator of $\{M(t)\}_{t\geq0}$, then $\mathcal{D}_{\mathcal{A}^M} \subseteq \mathcal{D}_{\bar{\mathcal{A}}^M}$ and for $f \in \mathcal{D}_{\mathcal{A}^M}$ the function $\mathcal{A}^M f$ is contained in $\bar{\mathcal{A}}^M f$ (see [22], Proposition IV.1.7). In the case of the process $\{X(t)\}_{t\geq0}$, it has been shown in [36, 37], Lemma 3.7, that

$$(2.1) \qquad \mathcal{D} := \left\{ f \in C^2(\mathbb{R}^d) : x \longmapsto \left| \int_{\mathcal{B}^c} f(x + y)\nu(\mathrm{d}y) \right| \text{ is locally bounded} \right\}$$

is a subset of $\mathcal{D}_{\bar{\mathcal{A}}^X}$, and on this set, for the function $\bar{\mathcal{A}}^X f(x)$ we can take exactly $\mathcal{A}^X f(x)$, where $\mathcal{A}^X$ is given by (1.2).

3. Ergodic properties. We start by examining the irreducibility and aperiodicity of the process $\{X(t)\}_{t \geq 0}$ in (1.1). This is the topic of the following theorem whose proof can be found in the Appendix.

THEOREM 3.1. *Suppose that one of the following four conditions holds*:

(i) $v(\mathbb{R}^d) < \infty$, *and for every $R > 0$ there exists $c_R > 0$ such that*

$$\langle y, a(x)y \rangle \geq c_R |y|^2, \qquad x, y \in \mathbb{R}^d, |x|, |y| \leq R.$$

(ii) $v(O) > 0$ *for any nonempty open set $O \subseteq \mathcal{B}$, and $\sigma \colon \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is Lipschitz continuous and invertible for any $x \in \mathbb{R}^d$, satisfying*

$$\delta := \sup_{x \in \mathbb{R}^d} \left\| \sigma^{-1}(x) \right\| > 0.$$

(iii) $\sigma(x) \equiv \sigma$ *and $\{L(t)\}_{t \geq 0}$ is of the form $L(t) = L_1(t) + L_2(t)$, $t \geq 0$, where $\{L_1(t)\}_{t \geq 0}$ and $\{L_2(t)\}_{t \geq 0}$ are independent d-dimensional pure-jump Lévy processes, such that $\{L_1(t)\}_{t \geq 0}$ is a subordinate Brownian motion.*

(iv) $\sigma(x) \equiv 0$ *and $\{L(t)\}_{t \geq 0}$ is of the form $L(t) = L_1(t) + L_2(t)$, $t \geq 0$, where $\{L_1(t)\}_{t \geq 0}$ and $\{L_2(t)\}_{t \geq 0}$ are independent d-dimensional pure-jump Lévy processes, such that $\{L_1(t)\}_{t \geq 0}$ is an anisotropic Lévy process with independent symmetric one-dimensional α-stable components for $\alpha \in (0, 2)$, and $\{L_2(t)\}_{t \geq 0}$ is a compound Poisson process.*

Then the process $\{X(t)\}_{t \geq 0}$ is open-set irreducible and aperiodic.

Recall that a Lévy process $\{L(t)\}_{t \geq 0}$ is a d-dimensional subordinate Brownian motion if it is of the form $L(t) = W(S(t))$, $t \geq 0$, where $\{W(t)\}_{t \geq 0}$ is a d-dimensional Brownian motion and $\{S(t)\}_{t \geq 0}$ is a subordinator (a one-dimensional nonnegative increasing Lévy process with $S(0) = 0$) independent of $\{W(t)\}_{t \geq 0}$. Moreover, any isotropic α-stable Lévy process can be obtained as a subordinate Brownian motion with $\frac{\alpha}{2}$-stable subordinator, hence part (iii) of Theorem 3.1 includes a d-dimensional isotropic stable Lévy process as a special case. We also note that in Theorem 3.1(iii), the component $\{L_2(t)\}_{t \geq 0}$ can be any pure-jump Lévy process or vanish, and in addition, we require that $\sigma(x)$ is constant, but it can either be a $d \times n$ or $d \times d$ singular or nonsingular matrix, and it can vanish. In the interest of brevity, we often refer to the process $\{L_1(t)\}_{t \geq 0}$ in Theorem 3.1(iv) as the anisotropic α-stable process. Unless otherwise specified, by an α-stable process we refer to both the isotropic and anisotropic models.

We remark that the hypotheses in Theorem 3.1 include a broader class of processes $\{X(t)\}_{t \geq 0}$ than those encountered in multiclass many-server queues described in Section 4.

We continue with the main results of the paper concerning the ergodicity of the process $\{X(t)\}_{t \geq 0}$ in (1.1). We present four theorems whose proofs can be found in Section 5. In all these theorems, the hypotheses of Theorem 3.1 are granted in order to guarantee that $\{X(t)\}_{t \geq 0}$ is open-set irreducible and aperiodic. This is important

when applying the Foster–Lyapunov drift condition in (3.3), (3.6) and (3.11) in order to conclude (3.4), (3.7) and (3.12), respectively (see [18], Theorem 3.2, and [20], Theorem 5.2).

We start by introducing the following notation.

NOTATION 3.1. For a vector $z \in \mathbb{R}^d$, we write $z \geq 0$ ($z > 0$) to indicate that all the components of z are nonnegative (positive), and analogously for a matrix in $\mathbb{R}^{d \times d}$. The notation $z \lneqq 0$ stands for $-z \geq 0$ and $z \neq 0$. For a symmetric matrix $S \in \mathbb{R}^{d \times d}$, we write $S \succeq 0$ ($S \succ 0$) to indicate that it is positive semidefinite (positive definite), and we let $\mathcal{M}_+$ denote the class of positive definite symmetric matrices in $\mathbb{R}^{d \times d}$. For $Q \in \mathcal{M}_+$, we let $\|x\|_Q := \langle x, Qx \rangle^{\frac{1}{2}}$ for $x \in \mathbb{R}^d$. Let $\hat{\phi}(x)$ be some fixed positive, convex smooth function which agrees with $\|x\|_Q$ on the complement of the unit ball centered at 0 in $\mathbb{R}^d$. For $\delta > 0$, we define $V_{Q,\delta}(x) := (\hat{\phi}(x))^\delta$, and $\widetilde{V}_{Q,\delta}(x) := e^{\delta \hat{\phi}(x)}$. For $r > 0$, we let τ_r denote the first hitting time of $\mathcal{B}_r$, and $\hat{\tau}_r$ the first hitting time of $\mathcal{B}_r^c$. Recall also that a continuous function $V \colon \mathbb{R}^d \to \mathbb{R}$ is called inf-compact if the set $\{x \colon V(x) \leq r\}$ is compact (or empty) for all $r \in \mathbb{R}$. By $\mathcal{P}_p(\mathbb{R}^d)$, $p > 0$, we denote the subset of $\mathcal{P}(\mathbb{R}^d)$ containing all probability measures $\mu(\mathrm{d}x)$ with the property that $\int_{\mathbb{R}^d} |x|^p \mu(\mathrm{d}x) < \infty$. We let $\mathcal{K}_\delta \subset \mathbb{R}^d$, $\delta > 0$, stand for the cone

$$(3.1) \qquad \mathcal{K}_\delta := \{x \in \mathbb{R}^d \colon \langle e, x \rangle > \delta |x|\}.$$

In the multiclass queueing context, the theorem that follows concerns the case where the jobs do not abandon the queue, or more generally when those jobs that abandon the queue are given higher priority in service than those that do not (i.e., when not all γ_i's are positive, then v_i must be equal to 0 if $\gamma_i > 0$). The Lévy process here refers to any, or a combination, of processes in Theorem 3.1. Recall the definitions in (1.4)–(1.6).

THEOREM 3.2. *Assume the hypotheses of Theorem* 3.1, $1 \in \Theta_c$, *and suppose* $\Gamma v = 0$, *with* $v \in \Delta$. *Then, provided that* $\tilde{\varrho} > 0$, *the following hold*:

(i) *Suppose*

$$(3.2) \qquad \limsup_{|x| \to \infty} \frac{\|a(x)\|}{|x|} < \infty.$$

Then there exist $Q \in \mathcal{M}_+$, *depending on* v, *and positive constants* $c_0 = c_0(\theta)$, c_1, *and* δ, *such that for any* $\theta \in \Theta_c$, $\theta \geq 1$, *we have*

$$(3.3) \qquad \mathcal{A}^X V_{Q,\theta}(x) \leq c_0(\theta) - c_1 V_{Q,\theta}(x) \mathbb{1}_{\mathcal{K}_\delta^c}(x) - c_1 V_{Q,(\theta-1)}(x) \mathbb{1}_{\mathcal{K}_\delta}(x)$$

for all $x \in \mathbb{R}^d$. *The process* $\{X(t)\}_{t \geq 0}$ *admits a unique invariant probability measure* $\overline{\pi} \in \mathcal{P}(\mathbb{R}^d)$, *and satisfies*

$$(3.4) \qquad \lim_{t \to \infty} t^{\theta-1} \|\pi P_t^X(\mathrm{d}y) - \overline{\pi}(\mathrm{d}y)\|_{\mathsf{TV}} = 0, \qquad \pi \in \mathcal{P}_\theta(\mathbb{R}^d).$$

In addition, when $\theta = 1$, *then* (3.4) *holds for any* $\pi \in \mathcal{P}(\mathbb{R}^d)$.

(ii) *If $\sigma(x)$ is bounded and*

$$(3.5) \qquad \int_{\mathcal{B}^c} e^{\theta|y|} \nu(dy) < \infty$$

for some $\theta > 0$, then there exist $Q \in \mathcal{M}_+$ and positive constants $\tilde{c}_0, \tilde{c}_1$ such that

$$(3.6) \qquad \mathcal{A}^X \widetilde{V}_{Q,p}(x) \leq \tilde{c}_0 - \tilde{c}_1 \widetilde{V}_{Q,p}(x), \qquad x \in \mathbb{R}^d,$$

where $0 < p < \theta \|Q\|^{-\frac{1}{2}}$. The process $\{X(t)\}_{t \geq 0}$ admits a unique invariant probability measure $\overline{\pi} \in \mathcal{P}(\mathbb{R}^d)$, and for any $\gamma \in (0, c_1)$ there exists a positive constant C_γ such that

$$(3.7) \qquad \|\delta_x P_t^X(dy) - \overline{\pi}(dy)\|_{\widetilde{V}_{Q,p}} \leq C_\gamma \widetilde{V}_{Q,p}(x) e^{-\gamma t}, \qquad x \in \mathbb{R}^d, t \geq 0.$$

REMARK 3.1. Note that $\tilde{\varrho} > 0$ is always satisfied if $\tilde{\ell} \lneqq 0$. This is because M^{-1} is a positive matrix (see page 1307 in [17]). The same is true if M is diagonal matrix with positive diagonal elements.

The assumption $\tilde{\varrho} > 0$ in Theorem 3.2 is rather sharp as the following theorem shows.

THEOREM 3.3. *Suppose that* (1.1) *is driven by any or a combination of* (a)–(c) *below, while conforming to* (i)–(iv) *of Theorem* 3.1.

(a) *A Brownian motion with $\sigma(x)$ bounded.*

(b) *A Lévy process $L(t)$ which is either an anisotropic process with independent symmetric one-dimensional α-stable components, or an α-stable process, with $\alpha \in (1, 2)$.*

(c) *A Lévy process with a finite Lévy measure $\nu(dy)$, supported on a half-line in $\mathbb{R}^d$ of the form $\{tw : t \in [0, \infty)\}$, and with $1 \in \Theta_c$.*

Under these hypotheses, if $\tilde{\varrho} < 0$ ($\tilde{\varrho} = 0$), then the process $\{X(t)\}_{t \geq 0}$ is transient (cannot be positive recurrent) under any Markov control $v(x)$ satisfying $\Gamma v(x) = 0$ a.e.

Theorem 3.3 should be compared to Lemma 5.7 which does not assume that $\sigma(x)$ is bounded. However, Theorem 3.3 establishes a stronger result when $\tilde{\varrho} < 0$.

In general, if $\theta_c < \infty$, then under the assumptions of Theorem 3.2(i) we cannot have exponential ergodicity, as the next theorem shows. This is always true for models where the Lévy process is a subordinate Brownian motion, or an anisotropic process with independent symmetric one-dimensional α-stable components. However, in the case of a compound Poisson process, the asymmetry of the Lévy measure may be beneficial to ergodicity (see Remark 5.3). For the limiting SDEs that arise from stochastic networks under service interruptions, the Lévy measure is supported on a half-line in the direction of some $w \in \mathbb{R}_+^d$. In the theorem that follows, we enforce this as a hypothesis in part (ii).

THEOREM 3.4. *Grant the structural hypotheses of Theorem* 3.1, *assume the growth condition* (3.2), *and either, or both of the following*:

 (i) *The Lévy process is a subordinate Brownian motion such that* $1 \in \Theta_c$ *and* $\theta_c < \infty$, *or an anisotropic process with independent symmetric one-dimensional* α-*stable components, with* $\alpha \in (1, 2)$.

(ii) *The Lévy process has a finite Lévy measure such that* $1 \in \Theta_c$ *and* $\theta_c < \infty$, *and* $\nu(\mathrm{d}y)$ *is supported on a half-line of the form* $\{tw \colon t \in [0, \infty)\}$, *with* $\langle e, M^{-1}w \rangle > 0$.

Then the following hold:

(a) *Suppose* $v \in \Delta$, $\Gamma v = 0$, *and* $\tilde{\varrho} > 0$. *Then* $\{X(t)\}_{t \geq 0}$ *is polynomially ergodic, and its rate of convergence is* $r(t) \approx t^{\theta_c - 1}$. *In particular, in the case of an* α-*stable process* (*isotropic or not*), *we obtain the following quantitative bounds. There exist positive constants* $\tilde{C}_1$, *and* $\tilde{C}_2(\epsilon)$ *such that for all* $\epsilon \in (0, \alpha - 1)$, *we have*

$$(3.8) \qquad \tilde{C}_1 \left(\frac{t \vee 1}{\epsilon} + |x|^{\alpha - \epsilon} \right)^{\frac{1-\alpha}{1-\epsilon}} \leq \big\| \delta_x P_t^X(\mathrm{d}y) - \overline{\pi}(\mathrm{d}y) \big\|_{\mathrm{TV}}$$

$$\leq \tilde{C}_2(\epsilon)(t \vee 1)^{1+\epsilon - \alpha} |x|^{\alpha - \epsilon}$$

for all $t > 0$, *and all* $x \in \mathbb{R}^d$.

On the other hand, in the case of a Lévy process in (ii) *we obtain the following lower bound. There exists a positive constant* $\tilde{C}_3(\epsilon)$ *such that for all* $\epsilon \in (0, \frac{1}{3})$, *and all* $x \in \mathbb{R}^d$, *we have*

$$(3.9) \qquad \big\| \delta_x P_{t_n}^X(\mathrm{d}x) - \overline{\pi}(\mathrm{d}x) \big\|_{\mathrm{TV}} \geq \tilde{C}_3(\epsilon)\big(t_n + |x|^{\theta_c - \epsilon}\big)^{-\frac{\theta_c - 1 + 2\epsilon}{1 - 3\epsilon}}$$

for some sequence $\{t_n\}_{n \in \mathbb{N}} \subset [0, \infty)$, $t_n \to \infty$, *depending on* x. *The upper bound has the same form as the one in* (3.8), *but with* α *replaced by* θ_c.

(b) *If under some control* $v \in \mathfrak{U}_{\mathrm{SM}}$, *such that* $\Gamma v(x) = 0$ *a.e. the process* $\{X(t)\}_{t \geq 0}$ *has an invariant probability measure* $\overline{\pi} \in \mathcal{P}_p(\mathbb{R}^d)$, $p \geq 0$, *then* $p + 1 \in \Theta_c$, *and* $\tilde{\varrho} > 0$. *In addition,* $\tilde{\varrho} = \int_{\mathbb{R}^d} \langle e, x \rangle^{-} \overline{\pi}(\mathrm{d}x)$. *Conversely, if* v *is a constant control such that* $\Gamma v = 0$, $\tilde{\varrho} > 0$, *and* $p \geq 0$ *is such that* $p + 1 \in \Theta_c$, *then* $\{X(t)\}_{t \geq 0}$ *admits a unique invariant probability measure* $\overline{\pi} \in \mathcal{P}_p(\mathbb{R}^d)$.

REMARK 3.2. Roughly speaking, the mechanism that results in polynomial ergodicity can be described as follows. In rough terms, exponential ergodicity is related to the existence of a supersolution $V(x) \geq 1$ of $\mathcal{A}^X V(x) \leq c_0 \mathbb{1}_{\mathcal{B}}(x) - c_1 V(x)$, for some positive constants c_0 and c_1 (see [20]). Consider the model where $\{L(t)\}_{t \geq 0}$ is an isotropic α-stable process with $\alpha > 1$. For $V(x)$ to be integrable under the Lévy measure inherited from the α-stable kernel $\frac{1}{|y|^{d+\alpha}}$, it cannot grow faster than $|x|^{\alpha}$. On the other hand, the nonlocal part of the infinitesimal generator

acts like a derivative of order α (see Lemma 5.3). Thus, since $\alpha > 1$, such a super-solution must satisfy $\langle b(x), \nabla V(x) \rangle \leq -\epsilon V(x)$ for some $\epsilon > 0$, and all $|x|$ large enough. Since $V(x)$ has polynomial growth, this requires the drift $b(x)$ to have at least linear growth in x (see also Corollary 5.3).

REMARK 3.3. Suppose that the Lévy process is an anisotropic process with independent symmetric one-dimensional α-stable components. Then, by Theorem 3.4, the invariant probability measure $\bar{\pi}(dx)$ of $\{X(t)\}_{t \geq 0}$ cannot have a finite first absolute moment. In the context of queueing networks, this means that, under heavy-tailed arrivals with an α-stable limit, $\alpha \in (1, 2)$, a constant control v cannot stabilize the network unless $\Gamma v \neq 0$.

The next theorem asserts exponential ergodicity for models corresponding to queueing problems with reneging (abandonment).

THEOREM 3.5. *Grant the hypotheses of Theorem* 3.1. *Suppose that* $\theta \in \Theta_c$,

$$(3.10) \qquad \limsup_{|x| \to \infty} \frac{\|a(x)\|}{|x|^2} = 0,$$

and that one of the following holds:

  (i) $Mv \geq \Gamma v \gneqq 0$;
  (ii) $M = \mathrm{diag}(m_1, \ldots, m_d)$ *with* $m_i > 0$, $i = 1, \ldots, d$, *and* $\Gamma v \neq 0$.

Then there exists $Q \in \mathcal{M}_+$ *such that*

$$MQ + QM \succ 0, \quad and \quad (M - ev'(M - \Gamma))Q + Q(M - (M - \Gamma)ve') \succ 0,$$

and positive constants $\bar{c}_0$, $\bar{c}_1$ *satisfying*

$$(3.11) \qquad \mathcal{A}^X V_{Q,\theta}(x) \leq \bar{c}_0 - \bar{c}_1 V_{Q,\theta}(x), \qquad x \in \mathbb{R}^d.$$

The process $\{X(t)\}_{t \geq 0}$ *admits a unique invariant probability measure* $\bar{\pi} \in \mathcal{P}(\mathbb{R}^d)$, *and for any* $\gamma \in (0, \bar{c}_1)$ *there exists a positive constant* C_γ *such that for* $p \in (0, \theta]$,

$$(3.12) \qquad \|\delta_x P_t(dy) - \bar{\pi}(dy)\|_{V_{Q,p}} \leq C_\gamma V_{Q,p}(x) e^{-\gamma t}, \qquad x \in \mathbb{R}^d, t \geq 0.$$

In addition, $\bar{\pi} \in \mathcal{P}_q(\mathbb{R}^d)$ *if, and only if,* $q \in \Theta_c$.

REMARK 3.4. Theorems 3.2 and 3.5 generalize [17], Theorems 2 and 3, for the corresponding diffusion models. In [17], Theorem 2, the model in (1.1) is driven by a Brownian motion $\{W(t)\}_{t \geq 0}$, $\ell = -lv$ for some $l > 0$ and $\Gamma = 0$, and it is shown that $\{X(t)\}_{t \geq 0}$ admits a unique invariant probability measure $\bar{\pi}(dx)$ and is ergodic. For the same model, but with $\ell = -lv$ and $\Gamma = c\mathbb{I}$ for some $l \in \mathbb{R}$ and $c > 0$, [17], Theorem 3, establishes exponential ergodicity for $\{X(t)\}_{t \geq 0}$. Theorem 3.2 improves [17], Theorem 2, to exponential ergodicity of the process

$\{X(t)\}_{t \geq 0}$, under a weaker hypothesis on ℓ, which is shown to be also necessary for positive recurrence. Moreover, in the proof of [17], Theorem 3, a sophisticated nonquadratic Lyapunov function is constructed, whereas we employ a quadratic-type Lyapunov function (e.g., $V_{Q,\theta}(x)$ in (3.11)) in the proof of Theorem 3.5.

We note that the hypothesis that M is diagonal in (ii) of Theorem 3.5 can be waived if we assume that $\Gamma v = \gamma v$ for some $\gamma > 0$ (which is a rather restrictive assumption). In such a case a slight modification of the arguments in [17], Theorem 3, and Theorem 3.2, shows that the process $\{X(t)\}_{t \geq 0}$ is exponentially ergodic.

4. Multiclass many-server queueing models. In this section we present some examples of many-server queueing systems for which the class of piecewise O–U processes with jumps in (1.1) arises as a limit in the so-called (modified) Halfin–Whitt (H–W) heavy-traffic regime [27].

In the queueing context, we identify three classes of processes $\{X(t)\}_{t \geq 0}$:

(C1) $\sigma(x) \equiv \sigma$ is a $d \times d$ nonsingular matrix and the process $\{L(t)\}_{t \geq 0}$ is a d-dimensional pure-jump Lévy process, with $\nu(\mathbb{R}^d) < \infty$;

(C2) $\sigma(x) \equiv 0$, and $L(t) = L_1(t)$, with $\{L_1(t)\}_{t \geq 0}$ the anisotropic Lévy process from Theorem 3.1(iv) with $\alpha \in (1, 2)$;

(C3) $\sigma(x) \equiv 0$ and the process $\{L(t)\}_{t \geq 0}$ takes the form in Theorem 3.1(iv) with $\alpha \in (1, 2)$.

Case (C1) corresponds to a multiclass many-server queueing network having service interruptions (with the $\sqrt{n}$ scaling), (C2) to heavy-tailed arrivals, and (C3) to a combination of both (with the $n^{\frac{1}{\alpha}}$ scaling for $\alpha \in (1, 2)$). Case (C1) is covered by (i) in Theorem 3.1, and cases (C2) and (C3) are covered by (iv). We describe how these arise, and summarize the ergodic properties of the limiting processes for these queueing models in Sections 4.1 and 4.2.

4.1. *Multiclass $G/M/n + M$ queues with heavy-tailed arrivals.* In [42], a functional central limit theorem (FCLT) is proved for the queueing process in the $G/M/n + M$ model with first-come-first-served (FCFS) service discipline in a modified H–W regime. Customers waiting in queue can abandon before receiving service (the $+M$ in the notation). The limit process is a one-dimensional SDE with a piecewise-linear drift, driven by a symmetric α-stable Lévy process (a special case of the process $\{X(t)\}_{t \geq 0}$ in (1.1)). This analysis can be easily extended to multiclass $G/M/n + M$ queues under a constant Markov control.

Consider a sequence of $G/M/n + M$ queues with d classes of customers, indexed by n and let $n \to \infty$. Customers of each class form their own queue and are served in the order of their arrival. Let A_i^n, $i = 1, \ldots, d$, be the arrival process of class-i customers with arrival rate λ_i^n. Assume that A_i^n's are mutually independent. The service and patience times are exponentially distributed, with

class-dependent rates, μ_i and γ_i, respectively, for class-i customers. The arrival, service and abandonment processes of each class are mutually independent. Define the FCLT-scaled arrival processes $\hat{A}^n = (\hat{A}_1^n, \dots, \hat{A}_d^n)'$ by $\hat{A}_i^n := n^{-\frac{1}{\alpha}}(A_i^n - \lambda_i^n \varpi)$, $i = 1, \dots, d$, where $\varpi(t) \equiv t$ for each $t \geq 0$, and $\alpha \in (1, 2]$. We assume that

$$(4.1) \qquad \frac{\lambda_i^n}{n} \to \lambda_i > 0 \quad \text{and} \quad \hat{\ell}_i^n := n^{-\frac{1}{\alpha}}(\lambda_i^n - n\lambda_i) \to \hat{\ell}_i \in \mathbb{R},$$

for each $i = 1, \dots, d$, as $n \to \infty$. It follows from (4.1) that

$$(4.2) \qquad n^{1-\frac{1}{\alpha}}(1 - \rho^n) \xrightarrow[n \to \infty]{} \hat{\rho} = -\sum_{i=1}^{d} \frac{\hat{\ell}_i}{\mu_i},$$

where $\rho^n := \sum_{i=1}^{d} \frac{\lambda_i^n}{n\mu_i}$ is the aggregate traffic intensity. Under (4.1) and (4.2), the system is critically loaded, that is, it satisfies $\sum_{i=1}^{d} \frac{\lambda_i}{\mu_i} = 1$. Assume that the arrival processes satisfy a FCLT

$$(4.3) \qquad \hat{A}^n \Rightarrow \hat{A} = (\hat{A}_1, \dots, \hat{A}_d)' \qquad \text{in } (D_d, M_1), \text{ as } n \to \infty,$$

where the limit processes $\hat{A}_i$, $i = 1, \dots, d$, are mutually independent symmetric α-stable processes with $\hat{A}_i(0) \equiv 0$, and $\Rightarrow$ denotes weak convergence. The processes $\hat{A}_i$ have the same stability parameter α, with possibly different "scale" parameters η_i. These determine the characteristic function of $\hat{A}$ which takes the form

$$\varphi_{\hat{A}(t)}(\xi) = e^{-t \sum_{i=1}^{d} \eta_i |\xi_i|^\alpha}, \qquad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d, t \geq 0.$$

Note that if the arrival process of each class is renewal with regularly varying interarrival times of parameter α, then we obtain the above limit process.

Next, we provide a representation of the generator of the process $\hat{A}$. Let $\widetilde{\mathcal{N}}(dt, dy)$ be a martingale measure in $\mathbb{R}_*$, corresponding to a standard Poisson random measure $\mathcal{N}(t, dy)$, and $\widetilde{\mathcal{N}}(t, dy) = \widehat{N}(t, dy) - t\mathcal{N}(dy)$, with $\mathbb{E}\widetilde{\mathcal{N}}(t, dy) = t\mathcal{N}(dy)$, and with $\mathcal{N}$ being a σ-finite measure on $\mathbb{R}_*$ given by $\mathcal{N}(dy) = \frac{dy}{|y|^{1+\alpha}}$. Let $\widetilde{\mathcal{N}}_1, \dots, \widetilde{\mathcal{N}}_d$ be d independent copies of $\widetilde{\mathcal{N}}$. We can then write

$$d\hat{A}_i(t) := \eta_i C(1, \alpha) \int_{\mathbb{R}_*} y \widetilde{\mathcal{N}}_i(dt, dy),$$

where

$$(4.4) \qquad C(d, \alpha) := \frac{\alpha 2^{\alpha-1} \Gamma(\frac{\alpha+d}{2})}{\pi^{\frac{d}{2}} \Gamma(1 - \frac{\alpha}{2})}.$$

Note that for α close to 2 we have $C(d, \alpha) \approx (2 - \alpha)d$.

Thus, the generator $\mathcal{L}$ of the process $\hat{A}$ then takes the form

$$\mathcal{L}f(x) = C(1, \alpha) \sum_{i=1}^{d} \int_{\mathbb{R}_*} \mathfrak{d}_1 f(x; y_i e_i) \frac{\eta_i \, dy_i}{|y_i|^{1+\alpha}} = \int_{\mathbb{R}_*^d} \mathfrak{d}_1 f(x; y) \nu(dy),$$

where $\nu(dy)$ is of the form $\nu(dy) = \sum_{i=1}^{d} \nu_i(dy_i)$ with $\nu_i(dy_i)$ supported on the ith coordinate axis in $\mathbb{R}^d$. Recall that the characteristic function of an isotropic α-stable process $\{L(t)\}_{t\geq 0}$ has the form $\varphi_{L(t)}(\xi) = e^{-t\eta|\xi|^\alpha}$ for some $\eta > 0$. Thus, $\hat{A}$ is not an isotropic α-stable Lévy process. According to [48], Theorem 2.1.5, $\hat{A}$ is a symmetric d-dimensional α-stable Lévy process. Since it is not isotropic, it is not a subordinate Brownian motion with $\frac{\alpha}{2}$-stable subordinator, although each component $\hat{A}_i$ is.

Let $X^n = (X_1^n, \dots, X_d^n)'$, $Q^n = (Q_1^n, \dots, Q_d^n)'$ and $Z^n = (Z_1^n, \dots, Z_d^n)'$ be the processes counting the number of customers of each class in the system, in queue, and in service, respectively. Then it is evident that $X_i^n = Q_i^n + Z_i^n$ for each i and $\sum_{i=1}^{d} Z_i^n \leq n$. We consider work-conserving scheduling policies that are nonanticipative and allow preemption. Namely, no server will idle if there is any customer waiting in a queue, and service of a customer can be interrupted at any time to serve some other class of customers and will be resumed at a later time. Scheduling policies determine the allocation of service capacity, that is, the Z^n process, which must satisfy the condition that $\langle e, Z^n \rangle = \langle e, X^n \rangle \wedge n$ at each time. Define the FCLT-scaled processes $\hat{X}^n = (\hat{X}_1^n, \dots, \hat{X}_d^n)'$, $\hat{Q}^n = (\hat{Q}_1^n, \dots, \hat{Q}_d^n)'$ and $\hat{Z}^n = (\hat{Z}_1^n, \dots, \hat{Z}_d^n)'$ by

$$(4.5) \qquad \hat{X}_i^n := n^{-\frac{1}{\alpha}}(X_i^n - \rho_i n), \qquad \hat{Q}_i^n := n^{-\frac{1}{\alpha}} Q_i^n, \; \hat{Z}_i^n := n^{-\frac{1}{\alpha}}(Z_i^n - \rho_i n).$$

Then under the work-conserving preemptive scheduling policies, given the controls Z^n, the processes $\hat{Q}^n$ and $\hat{Z}^n$ can be parameterized as follows: for adapted $\hat{V}^n \in \Delta$,

$$\hat{Q}_i^n = \langle e, \hat{X}^n \rangle^+ \hat{V}_i^n, \qquad \hat{Z}_i^n = \hat{X}_i^n - \langle e, \hat{X}^n \rangle^+ \hat{V}_i^n.$$

The controls $\hat{V}^n$ represent the fraction of class-i customers in the queue when the total queue size is positive. When $\hat{Q}^n \equiv 0$, we set $\hat{V}^n = (0, \dots, 0, 1)'$. In the limit process, the control takes values in Δ, and will be regarded as a fixed parameter, that is, this falls into the framework of our study when the control is constant. We obtain the following FCLT.

THEOREM 4.1.   *Under a fixed constant scheduling control $V \in \Delta$, provided there exists $\hat{X}(0)$ such that $\hat{X}^n(0) \Rightarrow \hat{X}(0)$ as $n \to \infty$, we have*

$$(4.6) \qquad\qquad \hat{X}^n \Rightarrow \hat{X} \qquad in \; (D^d, M_1) \; as \; n \to \infty,$$

where the limit process $\hat{X}$ is a unique strong solution to the SDE

$$(4.7) \qquad\qquad d\hat{X}(t) = \hat{b}(\hat{X}(t), V) \, dt + d\hat{A}(t) - \sigma_\alpha \, dW(t),$$

with an initial condition $\hat{X}(0)$. Here, the drift $\hat{b}(x, v) \colon \mathbb{R}^d \times \Delta \to \mathbb{R}^d$ takes the form

$$(4.8) \qquad\qquad \hat{b}(x, v) = \hat{\ell} - R(x - \langle e, x \rangle^+ v) - \langle e, x \rangle^+ \Gamma v,$$

with $R = \operatorname{diag}(\mu_1, \ldots, \mu_d)$, $\Gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_d)$, and $\hat{\ell} := (\hat{\ell}_1, \ldots, \hat{\ell}_d)'$ for $\hat{\ell}_i$ in (4.1). In (4.7), $\hat{A}$ is the limit of the arrival process, W is a standard d-dimensional Brownian motion, independent of $\hat{A}$, and the covariance matrix σ_α satisfies $\sigma_\alpha \sigma_\alpha' = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$ if $\alpha = 2$ and $\sigma_\alpha = 0$ if $\alpha \in (1, 2)$.

PROOF. The FCLT-scaled processes $\hat{X}_i^n$, $i = 1, \ldots, d$, can be represented as

$$\hat{X}_i^n(t) = \hat{X}_i^n(0) + \hat{\ell}_i^n t - \mu_i \int_0^t \hat{Z}_i^n(s) \, ds - \gamma_i \int_0^t \hat{Q}_i^n(s) \, ds$$
$$+ \hat{A}_i^n(t) - \hat{M}_{S,i}^n(t) - \hat{M}_{R,i}^n(t),$$

where $\hat{\ell}_i^n$ is defined in (4.1),

$$\hat{M}_{S,i}^n(t) = n^{-\frac{1}{\alpha}} \left(S_i^n \left(\mu_i \int_0^t Z_i^n(s) \, ds \right) - \mu_i \int_0^t Z_i^n(s) \, ds \right),$$

$$\hat{M}_{R,i}^n(t) = n^{-\frac{1}{\alpha}} \left(R_i^n \left(\theta_i \int_0^t Q^n(s) \, ds \right) - \theta_i \int_0^t Q_i^n(s) \, ds \right),$$

and S_i^n, R_i^n, $i = 1, \ldots, d$, are mutually independent rate-one Poisson processes, representing the service and reneging (abandonment), respectively. We can then establish an FCLT for the processes $\hat{X}^n$, by following a similar argument as Theorem 2.1 in [42], if we prove the continuity in the Skorohod M_1 topology of the d-dimensional integral mapping $\phi \colon D^d \to D^d$ defined by

$$y(t) = x(t) + \int_0^t h(y(s)) \, ds, \qquad t \geq 0,$$

where $h \colon \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz function. In Theorem 1.1 of [42], the integral mapping is from D to D, but a slight modification of the argument of that proof can show our claim in the multidimensional setting. Specifically, the parametric representations can be constructed in the same way with the spatial component being multidimensional, and the time component satisfying the conditions in Theorem 1.2 of [42]. □

In analogy to Theorems 3.2 to 3.5, we obtain Corollary 4.1 which follows. For multiclass many-server queues, the model in Theorem 3.2 corresponds to systems without abandonment, that is, $\Gamma = 0$. In such systems, M is a diagonal matrix, so the results in Theorem 3.2 are more general than needed for the queueing models.

Recall the quantity $\hat{\rho}$ defined in (4.2). As mentioned earlier, this quantity is the spare capacity of the network, and when it is positive it amounts to the so-called $\sqrt{n}$ safety staffing for the network. By Theorem 3.3 and Lemma 5.7(a), both $\alpha > 1$ and $\hat{\rho} > 0$ are necessary for the process to be ergodic. However, for the queueing model, the limit process has an interpretation only if $\alpha > 1$, and this is reflected in the statement of the corollary.

COROLLARY 4.1. *For the multiclass many-server queues with heavy-tailed arrivals with $\alpha \in (1, 2)$, the following hold*:

(1) *For the process $\{\hat{X}(t)\}_{t\geq 0}$ in (4.7) to be ergodic under some Markov control $v \in \mathfrak{U}_{SM}$, satisfying $\Gamma v(x) = 0$ a.e., it is necessary and sufficient that $\hat{\rho} > 0$.*
(2) *Suppose that $\hat{\rho} > 0$.*

   (2a) *The process $\{\hat{X}(t)\}_{t\geq 0}$ is polynomially ergodic under any constant control satisfying $\Gamma v = 0$, and its rate of convergence is $r(t) \approx t^{\alpha-1}$. In addition, the conclusions of Theorem 3.2(i) hold for any $\theta \in [1, \alpha)$.*

   (2b) *For any Markov control $v \in \mathfrak{U}_{SM}$ satisfying $\Gamma v(x) = 0$ a.e., which renders the process $\{\hat{X}(t)\}_{t\geq 0}$ ergodic, the associated invariant probability measure $\overline{\pi}(\mathrm{d}x)$ satisfies*

$$\int_{\mathbb{R}^d} \big(\langle e, x\rangle^+\big)^{\alpha-1} \overline{\pi}(\mathrm{d}x) = \infty,$$

   and thus the queue is not stable. In addition, $\hat{\rho} = \int_{\mathbb{R}^d} \langle e, x\rangle^- \overline{\pi}(\mathrm{d}x)$.

(3) *For any constant control such that $\Gamma v \neq 0$, the conclusions of Theorem 3.5 apply for any $\theta < \alpha$. In addition, $\overline{\pi} \in \mathcal{P}_p(\mathbb{R}^d)$ for all $p < \alpha$ and, therefore, the queue is stable.*

PROOF. The assertion in (1) follows by Theorem 3.3 and Lemma 5.7(b). Item (2a) is a direct consequence of Theorem 3.2(i) and Theorem 3.4, while (2b) follows from Lemma 5.7(b) and Corollary 5.1. The assertion in (3) follows by Theorem 3.5. □

We also remark that when the arrival limit is a Brownian motion ($\alpha = 2$), the limit is a diffusion with piecewise linear drift. In this case, the conclusions in Corollary 4.1(2) hold for any $\theta \in [1, \infty)$, and those in Corollary 4.1(3) hold for any $\theta > 0$, and in both cases, we have exponential ergodicity. The basic reason behind this discontinuity at $\alpha = 2$ is the fact that the scaling constant $C(d, \alpha)$ of the fractional Laplacian given in (4.4) tends to 0 as $\alpha \nearrow 2$, and thus the singular integral in the generator $\mathcal{A}$ vanishes. Comparing the tails of the stationary distributions $\overline{\pi}(\mathrm{d}x)$, when $\alpha \in (1, 2)$, as shown in Theorem 3.4, $\overline{\pi}(\mathrm{d}x)$ does not have any absolute moments of order $\alpha - 1$ or larger in case (2), and that this is true under any Markov control $v = v(x)$. In case (3), $\overline{\pi}(\mathrm{d}x)$ does not have any absolute moments of order α or larger.

It is worth noting that the piecewise diffusion model $\hat{X}$ in (4.7) is more general than that considered in [17], as noted in Remark 3.4, and the rate of convergence is not identified there when $\Gamma = 0$. For the multiclass $M/M/N + M$ queues with abandonment, exponential ergodicity of the limiting diffusion under the constant control $v = (0, \ldots, 0, 1)'$ is established in [17], Theorem 3, and this is used in [4] to prove asymptotic optimality. Theorem 3.5 extends this result, by asserting exponential ergodicity under any constant control v such that $\Gamma v \neq 0$. We summarize these findings in the following corollary.

COROLLARY 4.2. *Assume $\alpha = 2$.*

(a) *If $\Gamma v = 0$, then $\hat{\rho} > 0$ is both necessary and sufficient for the process to be ergodic, and in such a case, (3.6) and (3.7) hold for for any $p > 0$.*

(b) *If $\Gamma v \neq 0$, then (3.11) and (3.12) hold for any $\theta > 0$.*

In particular, in either case, $\{\hat{X}(t)\}_{t \geq 0}$ is exponentially ergodic.

4.2. *Multiclass $G/M/n + M$ queues with service interruptions.* In [41], $G/M/n + M$ queues with service interruptions are studied in the H–W regime. It is shown that the limit queueing process is a one-dimensional Lévy-driven SDE if the interruption times are asymptotically negligible.

We consider a sequence of multiclass $G/M/n + M$ queues in the same renewal alternating (up-down, or on-off) random environment, where all the classes of customers are affected simultaneously. We make the same assumptions on the arrival, service and abandonment processes as well as the control processes as in Section 4.1. For the random environment, we assume that the system functions normally during up time periods, and a portion of servers stop functioning during down periods, while customers continue entering the system and may abandon while waiting in queue and those that have started service will wait for the system to resume. Here, we focus on the special case of all servers stopping functioning during down periods. Let $\{(u_k^n, d_k^n) : k \in \mathbb{N}\}$ be a sequence of i.i.d. positive random vectors representing the up-down cycles. Assume that

$$\{(u_k^n, n^{\frac{1}{\alpha}} d_k^n) : k \in \mathbb{N}\} \Rightarrow \{(u_k, d_k) : k \in \mathbb{N}\} \qquad \text{in } (\mathbb{R}^2)^{\infty} \text{ as } n \to \infty,$$

where (u_k, d_k), $k \in \mathbb{N}$, are i.i.d. positive random vectors and $\alpha \in (1, 2]$. This assumption is referred to as asymptotically negligible service interruptions. Define the counting process of down times, $N^n(t) := \max\{k \geq 0 : T_k^n \leq t\}$, where $T_k^n := \sum_{i=1}^{k}(u_i^n + d_i^n)$ for each $k \in \mathbb{N}$ and $T_0^n \equiv 0$. This assumption implies that $N^n \Rightarrow N$ in (D, J_1) as $n \to \infty$, where the limit process is defined as $N(t) := \max\{k \geq 0 : T_k \leq t\}$, $t \geq 0$, with $T_k := \sum_{i=1}^{k} u_i$ for $k \in \mathbb{N}$, and $T_0 \equiv 0$. Here, we assume that the process $\{N(t)\}_{t \geq 0}$ is Poisson.

Let $X^n = (X_1^n, \ldots, X_d^n)'$ be the processes counting the number of customers of each class in the system, and define the FCLT-scaled processes $\hat{X}^n$ as in (4.5). Following a similar argument as in [41] and [42], we can then show the following FCLT, whose proof is omitted for brevity.

THEOREM 4.2. *Under a fixed constant scheduling control $V \in \Delta$, if there exists $\hat{X}(0)$ such that $\hat{X}^n(0) \Rightarrow \hat{X}(0)$ as $n \to \infty$, then (4.6) holds, where the limit process $\hat{X}$ is a unique strong solution to the Lévy-driven SDE*

$$d\hat{X}(t) = \hat{b}(\hat{X}(t), V) \, dt + d\hat{A}(t) - \sigma_{\alpha} \, dW(t) + c \, d\hat{J}(t),$$

with initial condition $\hat{X}(0)$. *The drift takes the same form as in* (4.8) *with $\hat{\ell}_i$ in* (4.1), *the matrices σ_α and R are as given in Theorem* 4.1, $c = (\lambda_1, \dots, \lambda_d)'$, *and the process $\hat{J}$ is a compound Poisson process, defined by*

$$\hat{J}(t) := \sum_{k=1}^{N(t)} d_k, \qquad t \ge 0.$$

Observe that the jump component $\{\hat{J}(t)\}_{t\ge 0}$ is a one-dimensional spectrally positive pure-jump Lévy process. Hence, $\{c\hat{J}(t)\}_{t\ge 0}$ should be regarded as the component $\{L_2(t)\}_{t\ge 0}$ described in Theorem 3.1(i) and (iv). Let ϑ_d be the drift and $v_{\hat{j}}(\mathrm{d}u)$ be the Lévy measure of $\{\hat{J}(t)\}_{t\ge 0}$. Clearly, $\vartheta_d = \eta \int_{\mathbb{B}} u\delta(\mathrm{d}u)$, and $v_{\hat{j}}(\mathrm{d}u) = \eta\delta(\mathrm{d}u)$, where $\eta > 0$ is the rate of $\{N(t)\}_{t\ge 0}$ and $\delta(\mathrm{d}u)$ is the distribution of d_1. In this case, $\{L_2(t)\}_{t\ge 0}$ is determined by a Lévy measure $v_2(\mathrm{d}y)$ which is supported on $C := \{uc : u \ge 0\}$ and satisfies $v_2(\mathrm{d}(uc)) = v_{\hat{j}}(\mathrm{d}u)$, and drift

$$\hat{\vartheta} := \vartheta_d c + \int_{\mathbb{R}^d} y\big(\mathbb{1}_{\{y\in C:\ |y|\le 1\}}(y) - \mathbb{1}_{\{y\in C:\ |y|\le|c|\}}(y)\big)v_2(\mathrm{d}y).$$

Namely, we have

$$\mathbb{E}\big[e^{\iota\langle L_2(1),\xi\rangle}\big]$$

$$= \mathbb{E}\big[e^{\iota\hat{J}(1)\langle c,\xi\rangle}\big]$$

$$= \exp\left(\iota\vartheta_d\langle c,\xi\rangle + \int_{(0,\infty)} \big(e^{\iota\langle c,\xi\rangle u} - \iota u\langle c,\xi\rangle\mathbb{1}_{\mathbb{B}}(u) - 1\big)v_{\hat{j}}(\mathrm{d}u)\right)$$

$$= \exp\left(\iota\vartheta_d\langle c,\xi\rangle + \iota\int_{\mathbb{R}^d} y\big(\mathbb{1}_{\{y\in C:\ |y|\le 1\}}(y) - \mathbb{1}_{\{y\in C:\ |y|\le|c|\}}(y)\big)v_2(\mathrm{d}y)\right.$$

$$\left. + \int_C \big(e^{\iota\langle y,\xi\rangle} - \iota\langle y,\xi\rangle\mathbb{1}_{\{y\in C:\ |y|\le 1\}}(y) - 1\big)v_2(\mathrm{d}y)\right),$$

where $\iota = \sqrt{-1}$. When $\alpha = 2$, the arrival limit is a Brownian motion, and thus, we obtain a limit process as in case (C1). When $\alpha \in (1, 2)$, the arrival limit is an anisotropic Lévy process as in case (C3). In analogy to Theorems 3.2 and 3.5, we obtain the following corollary for cases (C1) and (C3). Here, the spare capacity takes the form

$$\tilde{\varrho} := \hat{\rho} - e'R^{-1}\left(\hat{\vartheta} + \int_{\mathbb{B}^c} yv_2(\mathrm{d}y)\right).$$

COROLLARY 4.3.   *Suppose that $\Gamma v = 0$, and $\tilde{\varrho} > 0$. In order for the process $\{\hat{X}(t)\}_{t\ge 0}$ to be ergodic, it is necessary and sufficient that*:

    (a) $\alpha \in (1, 2)$ *and $\mathbb{E}[d_1^\theta] < \infty$ for some $1 \le \theta < \alpha$, or*
    (b) $\alpha = 2$ *and $\mathbb{E}[d_1^\theta] < \infty$ for some $\theta \ge 1$.*

If any of these conditions are met, *the conclusions of Theorem* 3.2(i) *and Theorem* 3.4 *follow*, *that is*, *the process* $\{\hat{X}(t)\}_{t \geq 0}$ *is polynomially ergodic*, *and its rate of convergence is* $r(t) \approx t^{\theta_c - 1}$.

On the other hand, *if* $\Gamma v \neq 0$, *then under either* (a) *with* $0 < \theta < \alpha$, *or* (b) *with* $\theta > 0$, *the conclusions of Theorem* 3.5 *hold*.

4.3. *Other queueing models.* An FCLT is proved in [45, 46] for $GI/Ph/n$ queues with renewal arrival processes and phase-type service-time distributions in the H–W regime, where the limit processes tracking the numbers of customers in service at each phase form a multidimensional piecewise-linear diffusion. In [16], $G/Ph/n + GI$ queues with abandonment are studied and a multidimensional piecewise-linear diffusion limit is also proved in the H–W regime. When the arrival process is heavy-tailed, satisfying an FCLT as in (4.3), and/or when there are service interruptions, it can be shown that the limit processes are piecewise O–U processes with jumps as in (1.1), where in the drift function the constant coefficient ℓ is replaced by $-lv$ for a constant $l \in \mathbb{R}$ and $v \in \Delta$, and the vector Γv equals cv for some constant $c \in \mathbb{R}$. Our results include this limiting process as a special case.

5. Proofs of main theorems and other results. In this section we prove the main results with the exception of Theorem 3.1, the proof of which is in the Appendix.

5.1. *Technical lemmas.* This section concerns some estimates for nonlocal operators that we use in the proofs to establish Foster–Lyapunov equations.

For a σ-finite measure $\nu(\mathrm{d}y)$ on $\mathfrak{B}(\mathbb{R}^d_*)$, we let

$$\mathfrak{J}_{1,\nu}[\Phi](x) := \int_{\mathbb{R}^d_*} \mathfrak{d}_1 \Phi(x; y) \nu(\mathrm{d}y), \quad \text{and} \quad \mathfrak{J}_\nu[\Phi](x) := \int_{\mathbb{R}^d_*} \mathfrak{d}\Phi(x; y) \nu(\mathrm{d}y),$$

with $\mathfrak{d}_1 \Phi(x, y)$ as defined in (1.3), and

$$\mathfrak{d}f(x; y) := f(x + y) - f(x) - \langle y, \nabla f(x) \rangle, \qquad f \in C^1(\mathbb{R}^d).$$

Also define

$$\check{C}_0(r; \theta) := \int_{\mathcal{B}^c_r} |y|^\theta \nu(\mathrm{d}y), \qquad \widehat{C}_0 := \int_{\mathcal{B} \setminus \{0\}} |y|^2 \nu(\mathrm{d}y).$$

Note that $\check{C}_0(r; \theta) \to 0$ as $r \to \infty$.

LEMMA 5.1. *Suppose that* $\nu(\mathrm{d}y)$ *is a* σ-*finite measure on* $\mathfrak{B}(\mathbb{R}^d_*)$, *which satisfies* $\widehat{C}_0 + \check{C}_0(1; \theta) < \infty$ *for some* $\theta > 0$. *We have the following*:

(a) *If* $\Phi \in C^2(\mathbb{R}^d)$ *satisfies*

(5.1) $$\sup_{|x| \geq 1} |x|^{1-\theta} \max(|\nabla \Phi(x)|, |x| \|\nabla^2 \Phi(x)\|) < \infty,$$

then $\mathfrak{J}_\nu[\Phi]$ *vanishes at infinity when* $\theta \in [1, 2)$, *and the map* $x \mapsto (1 + |x|)^{2-\theta} \times \mathfrak{J}_\nu[\Phi](x)$ *is bounded when* $\theta \geq 2$.

(b) *If $\theta \in (0, 1)$, and $\Phi \in C^2(\mathbb{R}^d)$ satisfies*

$$\sup_{|x| \geq 1} |x|^{-\theta} \max(|\Phi(x)|, |x||\nabla\Phi(x)|, |x|^2\|\nabla^2\Phi(x)\|) < \infty,$$

then the function $x \mapsto \mathfrak{J}_{1,\nu}[\Phi](x)$ vanishes at infinity.

PROOF.   We first consider the case $\theta \in (1, 2)$. By (5.1), there exist positive constants c_0 ad c_1 such that

(5.2)
$$|\nabla\Phi(x)| \leq c_0 \mathbb{1}_{\mathcal{B}}(x) + c_1|x|^{\theta-1}\mathbb{1}_{\mathcal{B}^c}(x),$$

$$\|\nabla^2\Phi(x)\| \leq c_0 \mathbb{1}_{\mathcal{B}}(x) + c_1|x|^{\theta-2}\mathbb{1}_{\mathcal{B}^c}(x),$$

for all $x \in \mathbb{R}^d$. Let $z \colon [1, \infty) \to \mathbb{R}_+$ be defined by $z(r) := r(\check{C}_0(r; \theta))^{\frac{1}{2(1-\theta)}}$. Then $z(r)$ is a strictly increasing function, whose range is an interval of the form $[z_0, \infty)$, $z_0 > 0$. Let $r(z)$ denote the inverse of this map defined on the range of $z(r)$. Then of course $r(z) \to \infty$ as $z \to \infty$ and we have

(5.3) $\left(\dfrac{z}{r(z)}\right)^{\theta-1} \check{C}_0(r(z); \theta) = \sqrt{\check{C}_0(r(z); \theta)} \xrightarrow[z\to\infty]{} 0,$ and $\dfrac{r(z)}{z} \xrightarrow[z\to\infty]{} 0.$

We split the integral as follows:

(5.4)
$$\int_{\mathbb{R}_*^d} \eth\Phi(x; y)\nu(dy) = \int_{\mathcal{B}_{r(|x|)}\setminus\{0\}} \int_0^1 (1-t)\langle y, \nabla^2\Phi(x+ty)y\rangle dt\,\nu(dy)$$

$$+ \int_{\mathcal{B}_{r(|x|)}^c} \int_0^1 \langle y, \nabla\Phi(x+ty) - \nabla\Phi(x)\rangle dt\,\nu(dy).$$

Let $\bar{r} > 0$ be such that $z \geq r(z) + 1$ for all $z \geq \bar{r}$. We estimate the integrals in (5.4) for $x \in \mathcal{B}_{\bar{r}}^c$. For the first integral on the right-hand side of (5.4), we use the estimate in (5.2) which implies that

$$\langle y, \nabla^2\Phi(z)y\rangle \leq c_1|z|^{\theta-2}|y|^2, \qquad y \in \mathbb{R}^d, z \in \mathcal{B}^c,$$

to write

$$\langle y, \nabla^2\Phi(x+ty)y\rangle \leq c_1|x+ty|^{\theta-2}|y|^2$$

$$\leq c_1(|x| - r(|x|))^{\theta-2}|y|^2$$

$$= c_1\left(r(|x|)\left(\frac{|x|}{r(|x|)} - 1\right)\right)^{\theta-2}|y|^2$$

$$\leq c_1\left(\frac{|x|}{r(|x|)} - 1\right)^{\theta-2}|y|^\theta,$$

where the last inequality follows since $|y| \le r(|x|)$. So integrating with respect to $\nu(dy)$, we deduce that the first integral is bounded by

(5.5)
$$\frac{1}{2}c_1\left(\frac{|x|}{r(|x|)} - 1\right)^{\theta-2}\left(r(|x|)^{\theta-2}\int_{\mathcal{B}\setminus\{0\}}|y|^2\nu(dy) + \int_{\mathcal{B}_{r(|x|)}\setminus\mathcal{B}}|y|^\theta\nu(dy)\right)$$
$$\le \frac{1}{2}c_1\widehat{C}_0(|x| - r(|x|))^{\theta-2} + \frac{1}{2}c_1\check{C}_0(r(|x|);\theta)\left(\frac{|x|}{r(|x|)} - 1\right)^{\theta-2}.$$

We use the inequality $|y| \le (r(|x|))^{1-\theta}|y|^\theta$ on $\mathcal{B}^c_{r(|x|)}$, to derive the estimate

$$|\langle y, \nabla\Phi(x + ty) - \nabla\Phi(x)\rangle| \le |y|(c_0 + c_1|x + ty|^{\theta-1} + c_1|x|^{\theta-1})$$
$$\le |y|(c_0 + c_1(2|x|^{\theta-1} + |y|^{\theta-1}))$$
$$\le (r(|x|))^{1-\theta}|y|^\theta(c_0 + 2c_1|x|^{\theta-1}) + |y|^\theta.$$

Integrating this with respect to $\nu(dy)$, we obtain a bound for the absolute value of second integral on the right-hand side of (5.4), which takes the form

(5.6)
$$\left[c_0(r(|x|))^{1-\theta} + c_1\left(1 + 2\left(\frac{|x|}{r(|x|)}\right)^{\theta-1}\right)\right]\check{C}_0(r(|x|);\theta).$$

Combining (5.3), (5.5) and (5.6), we obtain a bound for $|\mathfrak{J}_\nu[\Phi](x)|$ that clearly vanishes as $|x| \to \infty$ when $\theta \in (1, 2)$.

For $\theta = 1$, we select $r(|x|) = \frac{1}{2}|x|$ and follow the same method. For $\theta \ge 2$, we select $r(z) = z$ and we use the bounds (for $t \in [0, 1]$, and $|x| \ge 2$)

$$\langle y, \nabla^2\Phi(x + ty)y\rangle \le c_1 2^{\theta-2}|x|^{\theta-2}|y|^2 \qquad \text{when } |y| \le |x| - 1,$$

and

$$\langle y, \nabla\Phi(x + ty)\rangle \le |y|(c_0 + 2^{2\theta-4} + 2^{\theta-2}(2^{\theta-2} + 1)c_1|y|^{\theta-1}),$$

when $|y| \ge |x| - 1$, to obtain the result as stated. This completes the proof of part (a).

We continue with part (b). Here, in addition to (5.2), we have the bound

$$|\Phi(x)| \le c_0\mathbb{1}_\mathcal{B}(x) + c_1|x|^\theta\mathbb{1}_{\mathcal{B}^c}(x) \qquad \forall x \in \mathbb{R}^d.$$

Further, since $\int_{\mathcal{B}^c}|x|^\theta\nu(dy) < \infty$, by the de la Vallée–Poussin theorem, there exists a nonnegative increasing convex function $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\frac{\phi(t)}{t} \to \infty$ as $t \to \infty$, such that $\int_{\mathcal{B}^c}\phi(|y|^\theta)\nu(dy) < \infty$. Without loss of generality, we may assume $\phi(t) > t$ for all $t \in \mathbb{R}_+$. Let $r(t) := (\phi^{-1}(t^\theta))^{\frac{1}{\theta}}$. Clearly, $r\colon \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, $r(t) \to \infty$, and $\frac{r(t)}{t} \to 0$ as $t \to \infty$. Now we have

$$\int_{\mathbb{R}^d_*}\mathfrak{d}_1\Phi(x; y)\nu(dy)$$

$$(5.7) \qquad = \int_{\mathcal{B}_{r(|x|)}\setminus\{0\}} \int_0^1 (1-t)\langle y, \nabla^2\Phi(x+ty)y\rangle \,\mathrm{d}t\,\nu(\mathrm{d}y)$$

$$+ \int_{\mathcal{B}_{r(|x|)}^c} (\Phi(x+y) - \Phi(x))\nu(\mathrm{d}y) + \int_{\mathcal{B}_{r(|x|)}\setminus\mathcal{B}} \langle y, \nabla\Phi(x)\rangle\nu(\mathrm{d}y).$$

For the first integral on the right-hand side of (5.7), we use the bound derived in part (a). For the last integral on the right-hand side of (5.7) (for $|x| \geq 1$), we use the bound $\langle y, \nabla\Phi(x)\rangle \leq |y|c_1|x|^{\theta-1}$. Thus,

$$\int_{\mathcal{B}_{r(|x|)}\setminus\mathcal{B}} \langle y, \nabla\Phi(x)\rangle\nu(\mathrm{d}y) \leq c_1|x|^{\theta-1}\int_{\mathcal{B}_{r(|x|)}\setminus\mathcal{B}} |y|\nu(\mathrm{d}y)$$

$$\leq c_1|x|^{\theta-1}r(|x|)^{1-\theta}\int_{\mathcal{B}_{r(|x|)}\setminus\mathcal{B}} |y|^\theta\nu(\mathrm{d}y)$$

$$\leq c_1|x|^{\theta-1}r(|x|)^{1-\theta}\check{C}_0(1;\theta),$$

which tends to 0 as $|x|$ tends to ∞. Lastly, for the second integral on the right-hand side of (5.7), we proceed as follows. First, in view of the bound of $\Phi(x)$ (for $|x| \geq 1$), we have

$$\int_{\mathcal{B}_{r(|x|)}^c} \Phi(x)\nu(\mathrm{d}y) \leq c_1|x|^\theta \int_{\mathcal{B}_{r(|x|)}^c} \nu(\mathrm{d}y) \leq c_1 \int_{\mathcal{B}_{r(|x|)}^c} \phi(|y|^\theta)\nu(\mathrm{d}y),$$

which tends to 0 as $|x|$ tends to ∞. Second, since

$$\Phi(x+y) \leq c_0\mathbb{1}_{\mathcal{B}}(x+y) + c_1|x+y|^\theta\mathbb{1}_{\mathcal{B}^c}(x+y) \leq c_0 + 2c_1\phi(|y|^\theta)$$

for $y \in \mathcal{B}_{r(|x|)}^c$, we obtain

$$\int_{\mathcal{B}_{r(|x|)}^c} \Phi(x+y)\nu(\mathrm{d}y) \leq c_0 \int_{\mathcal{B}_{r(|x|)}^c} \nu(\mathrm{d}y) + 2c_1 \int_{\mathcal{B}_{r(|x|)}^c} \phi(|y|^\theta)\nu(\mathrm{d}y),$$

which also tends to 0 as $|x|$ tends to ∞. This completes the proof. □

Recall the notation $\widetilde{V}_{Q,\theta}(x)$ from Notation 3.1.

LEMMA 5.2.   *Suppose that $\nu(\mathrm{d}y)$ satisfies*

$$\int_{\mathbb{R}_*^d} (|y|^2\mathbb{1}_{\mathcal{B}\setminus\{0\}}(y) + e^{\theta|y|}\mathbb{1}_{\mathcal{B}^c}(y))\nu(\mathrm{d}y) < \infty$$

for some $\theta > 0$. Then $x \mapsto (1 + \widetilde{V}_{Q,\theta}(x))^{-1}\mathfrak{J}_\nu[\widetilde{V}_{Q,\theta}](x)$ is bounded on $\mathbb{R}^d$.

PROOF.   We estimate $\mathfrak{J}_\nu[\widetilde{V}_{Q,\theta}](x)$ by using the first integral on the right-hand side of (5.4) for $y \in \mathcal{B}\setminus\{0\}$, while for $y \in \mathcal{B}^c$, we estimate the integral using the identity

$$\eth\widetilde{V}_{Q,\theta}(x;y) = \widetilde{V}_{Q,\theta}(x+y) + \left[\|x\|_Q^{-1}\left(\left\|\frac{\theta}{2}y\right\|_Q^2 - \left\|x+\frac{\theta}{2}y\right\|_Q^2\right) + \|x\|_Q - 1\right]\widetilde{V}_{Q,\theta}(x)$$

for $x \in \mathcal{B}^c$. □

In the proof of Theorems 3.2 and 3.5, we apply the results in Lemmas 5.1 and 5.2. It is worth noting that in the special case of SDEs driven by an isotropic α-stable processes alone, sharper estimates than Lemma 5.1 can be obtained (see the proof of Proposition 5.1 in [3]). We state such an estimate in Lemma 5.3 which follows. For $\Phi \in C^2(\mathbb{R}^d)$, and a positive vector $\eta = (\eta_1, \ldots, \eta_d)$, we define

$$\mathfrak{I}_\alpha[\Phi](x) := \int_{\mathbb{R}^d_*} \mathfrak{d}_1 \Phi(x; y) \frac{\mathrm{d}y}{|y|^{\alpha+d}},$$

$$\widehat{\mathfrak{I}}_\alpha[\Phi](x) := \sum_{i=1}^d \eta_i \int_{\mathbb{R}_*} \mathfrak{d}_1 \Phi(x; y_i e_i) \frac{\mathrm{d}y_i}{|y_i|^{\alpha+1}},$$

where $\mathfrak{d}_1 \Phi(x; y)$ is defined in (1.3), and e_i denotes a vector in $\mathbb{R}^d$ whose elements are all 0, except the ith element which equals 1. Recall the notation $V_{Q,\delta}(x)$ from Notation 3.1.

LEMMA 5.3. *The map* $x \mapsto |x|^{\alpha-\theta} \mathfrak{I}_\alpha[V_{Q,\theta}](x)$ *is bounded on* $\mathbb{R}^d$ *for any* $Q \in \mathcal{M}_+$ *and* $\theta \in (0, \alpha)$. *The same holds for the anisotropic operator* $\widehat{\mathfrak{I}}_\alpha$.

The following lemma, whose proof follows from a similar argument to the one used in Lemma 5.1, is not utilized in the proofs, but may be of independent interest.

LEMMA 5.4. *Assume the hypotheses of Lemma 5.1*(a), *but replace the bound of* $\nabla^2 \Phi(x)$ *in* (5.1) *by* $\sup_{x \in \mathbb{R}^d} \frac{\|\nabla^2 \Phi(x)\|}{1+|x|^\gamma} < \infty$ *for* $\gamma \in [0, \theta-1]$. *Then*

$$\limsup_{|x| \to \infty} |x|^{(1-\theta)(2-\theta+\gamma)} \int_{\mathbb{R}^d_*} \mathfrak{d}\Phi(x; y)\nu(\mathrm{d}y) < \infty.$$

PROOF. In the proof of Lemma 5.1(a), we set $r(z) = z^{\theta-1-\gamma}$. The rest of the proof is the same. $\square$

5.2. *Proofs of Theorems* 3.2 *to* 3.4. We first state two lemmas needed for the proof. The first part of the lemma that follows is in [17], Theorem 2.

LEMMA 5.5. *Let M be a nonsingular M-matrix such that* $M'e \geq 0$, *and* $v \in \Delta$. *There exists a positive definite matrix Q such that*

$$(5.8) \qquad QM + M'Q \succ 0, \quad \text{and} \quad QM(\mathbb{I} - ve') + (\mathbb{I} - ev')M'Q \succeq 0.$$

In addition,

$$(5.9) \qquad (\mathbb{I} - tev')M'Q + QM(\mathbb{I} - tve') \succ 0, \qquad t \in [0, 1).$$

PROOF.   We only need to prove (5.9). We argue by contradiction. Let $S :=$ $M'Q + QM$ and $T := ev'M'Q + QMve'$. Suppose that $x'(S - tT)x \le 0$ for some $t \in (0, 1)$ and $x \in \mathbb{R}^d$, $x \ne 0$. Then, since $S \succ 0$, we must have $x'(S - T)x < 0$ which contradicts the hypothesis.   □

Recall the constant $\tilde{\ell}$ in (1.5), and the cone $\mathcal{K}_\delta$ in (3.1). Define
$$\tilde{b}(x) := b(x) + \tilde{\ell} - \ell, \qquad x \in \mathbb{R}^d.$$

LEMMA 5.6.   Let $\bar{\kappa}_1 > 0$ be such that $\langle x, (QM + M'Q)x \rangle \ge 2\bar{\kappa}_1|x|^2$ for all $x \in \mathbb{R}^d$. Set $\delta = \frac{1}{4}\bar{\kappa}_1|QMv|^{-1}$ and $\zeta = -\langle \tilde{\ell}, Qv \rangle$. Then $\zeta > 0$, and for Q given in (5.8), we have
$$\langle \tilde{b}(x), \nabla V_{Q,2}(x) \rangle \le \begin{cases} \bar{\kappa}_0 - \bar{\kappa}_1|x|^2 & \text{if } x \in \mathcal{K}_\delta^c, \\ \bar{\kappa}_0 - \delta\zeta|x| & \text{if } x \in \mathcal{K}_\delta, \end{cases}$$
for some constant $\bar{\kappa}_0 > 0$. Consequently, there are positive constants κ_0 and κ_1, such that
$$(5.10) \qquad \langle \tilde{b}(x), \nabla V_{Q,2}(x) \rangle \le \kappa_0 - \kappa_1 V_{Q,2}(x)\mathbb{1}_{\mathcal{K}_\delta^c}(x) - \kappa_1 V_{Q,1}(x)\mathbb{1}_{\mathcal{K}_\delta}(x)$$
for all $x \in \mathbb{R}^d$.

PROOF.   Assume first that $x \in \mathcal{K}_\delta^c$. We have
$$\langle \tilde{b}(x), \nabla V_{Q,2}(x) \rangle = 2\langle \tilde{\ell}, Qx \rangle - \langle x, (QM + M'Q)x \rangle + 2\langle x, QMv \rangle\langle e, x \rangle^+$$
$$\le -2\bar{\kappa}_1|x|^2 + 2|Q\tilde{\ell}||x| + 2\delta|QMv||x|^2.$$
Thus, by the definition of δ, we obtain
$$\langle \tilde{b}(x), \nabla V_{Q,2}(x) \rangle \le \bar{\kappa}_0 - \bar{\kappa}_1|x|^2 \qquad \forall x \in \mathcal{K}_\delta^c,$$
for some constant $\bar{\kappa}_0 > 0$.

Now assume that $x \in \mathcal{K}_\delta$. We have
$$\langle \tilde{b}(x), \nabla V_{Q,2}(x) \rangle = 2\langle \tilde{\ell}, Qx \rangle - \langle x, (QM(\mathbb{I} - ve') + (\mathbb{I} - ev')M'Q)x \rangle.$$
We follow the technique in [17] by using the unique orthogonal decomposition $x = \eta v + z$, with $z \in \mathbb{R}^d$ such that $\langle z, v \rangle = 0$. In other words, $z = x - \langle x, v \rangle\frac{v}{|v|^2}$. As shown in (5.19) of [17], we have
$$(5.11) \qquad \langle x, (QM(\mathbb{I} - ve') + (\mathbb{I} - ev')M'Q)x \rangle \ge 2\hat{\kappa}_1|z|^2$$
for some $\hat{\kappa}_1 > 0$. Solving $v'QM(\mathbb{I} - ve') = 0$, which follows from (5.11), we obtain $v'Q = \langle v, QMv \rangle e'M^{-1}$. Thus $\langle \tilde{\ell}, Qv \rangle = \langle v, QMv \rangle\langle e, M^{-1}\tilde{\ell} \rangle$. Since $\langle v, (QM + M'Q)v \rangle > 0$, we have $\langle v, QMv \rangle > 0$. This implies that $\langle \tilde{\ell}, Qv \rangle < 0$. Note that $\langle e, x \rangle = \eta\langle e, v \rangle + \langle e, z \rangle$ and, therefore,
$$(5.12) \qquad \eta = \langle e, x \rangle - \langle e, z \rangle.$$

Using (5.11) and (5.12), and the orthogonal decomposition of x, we obtain

$$
\begin{aligned}
\langle \tilde{b}(x), \nabla V_{Q,2}(x)\rangle &\le -2\hat{\kappa}_1|z|^2 + \langle \tilde{\ell}, Qz\rangle + \eta\langle \tilde{\ell}, Qv\rangle \\
&= -2\hat{\kappa}_1|z|^2 + \langle \tilde{\ell}, Qz\rangle + \zeta\langle e, z\rangle - \zeta\langle e, x\rangle \\
&\le \hat{\kappa}_0 - \hat{\kappa}_1|z|^2 - \delta\zeta|x| \\
&\le \hat{\kappa}_0 - \delta\zeta|x|, \qquad x \in \mathcal{K}_\delta,
\end{aligned}
$$

for some constant $\hat{\kappa}_0 > 0$. It is clear that (5.10) follows from these estimates. $\qquad\square$

PROOF OF THEOREM 3.2. We start with part (i). Consider $V_{Q,\theta}(x)$, with Q is given in (5.8). Clearly, $V_{Q,\theta}(x)$ is an inf-compact function contained in $\mathcal{D}$. By (5.10), for any given $\theta > 0$, there exist positive constants κ_0' and κ_1', such that

$$
\begin{aligned}
\langle \tilde{b}(x), \nabla V_{Q,\theta}(x)\rangle &= \langle \tilde{b}(x), \nabla V_{Q,\theta}(x)\rangle \mathbb{1}_{\mathcal{B}}(x) \\
(5.13) \qquad &+ \frac{\theta}{2}\frac{V_{Q,\theta-1}(x)}{V_{Q,1}(x)}\langle \tilde{b}(x), \nabla V_{Q,2}(x)\rangle \mathbb{1}_{\mathcal{B}^c}(x) \\
&\le \kappa_0'\mathbb{1}_{\mathcal{B}}(x) - \kappa_1' V_{Q,\theta}(x)\mathbb{1}_{\mathcal{K}_\delta^c}(x) - \kappa_1' V_{Q,\theta-1}(x)\mathbb{1}_{\mathcal{K}_\delta}(x)
\end{aligned}
$$

for all $x \in \mathbb{R}^d$. By (3.2), there exists some compact set $K \supset \mathcal{B}$, independent of θ, such that

$$
(5.14) \qquad \mathrm{Tr}\big(a(x)\nabla^2 V_{Q,\theta}(x)\big) \le \langle \tilde{b}(x), \nabla V_{Q,\theta}(x)\rangle, \qquad x \in K^c.
$$

First, suppose $\theta \in [1, 2]$. Then $\mathfrak{J}[V_{Q,\theta}](x)$ is bounded by Lemma 5.1. Thus, (3.3) holds with $c_1 = \frac{\kappa_1'}{2}$, and for $c_0(\theta)$ we can use the sum of κ_0', the supremum of the left-hand side of (5.14) on K, and a bound of $\mathfrak{J}[V_{Q,\theta}](x)$. When $\theta > 2$, $(1 + |x|)^{2-\theta}\mathfrak{J}_\nu[V_{Q,\theta}](x)$ is bounded by Lemma 5.1, and the result follows by comparing $\mathfrak{J}_\nu[V_{Q,\theta}](x)$ to $\langle \tilde{b}(x), \nabla V_{Q,\theta}(x)\rangle$ in (5.13).

Equation (3.4) follows from [18], Theorems 3.2 and 3.4, and [40], Theorem 5.1 (for the case when $\theta = 1$).

We now turn to part (ii). Consider $\widetilde{V}_{Q,p}(x)$, where Q is given in (5.8), and $p > 0$ such that $p\|Q\|^{\frac{1}{2}} < \theta$. We have

$$
\langle \tilde{b}(x), \nabla \widetilde{V}_{Q,p}(x)\rangle = \langle \tilde{b}(x), \nabla \widetilde{V}_{Q,p}(x)\rangle \mathbb{1}_{\mathcal{B}}(x) + p e^{p\langle x, Qx\rangle^{\frac{1}{2}}}\frac{\langle \tilde{b}(x), Qx\rangle}{\langle x, Qx\rangle^{\frac{1}{2}}}\mathbb{1}_{\mathcal{B}^c}(x)
$$

for all $x \in \mathbb{R}^d$. By Lemma 5.2, it is clear that there exist constants $\tilde{\kappa}_0 > 0$ and $\tilde{\kappa}_1 > 0$, such that

$$
\langle \tilde{b}(x), \nabla \widetilde{V}_{Q,p}(x)\rangle \le \tilde{\kappa}_0\mathbb{1}_{\mathcal{B}}(x) - \tilde{\kappa}_1\widetilde{V}_{Q,p}(x), \qquad x \in \mathbb{R}^d.
$$

Thus we obtain (3.6). Finally, according to [40], Theorem 6.1 (see also [20], Theorem 5.2) we conclude that $\{X(t)\}_{t\ge 0}$ admits a unique invariant probability measure

$\bar{\pi}(\mathrm{d}y)$ such that for any $\delta > 0$ and $0 < \gamma < \delta\tilde{c}_1$,

$$\left\| \delta_x P_t^X(\mathrm{d}y) - \overline{\pi}(\mathrm{d}y) \right\|_{\widetilde{V}_{Q,p}} \leq C \widetilde{V}_{Q,p}(x) \mathrm{e}^{-\gamma \frac{t-\delta}{\delta}}, \qquad x \in \mathbb{R}^d, t \geq 0,$$

for some $C > 0$. $\square$

PROOF OF THEOREM 3.3.   We first consider the case $\tilde{\varrho} < 0$ (note that $\tilde{\ell}$ depends on the noise present. If the noise is only a Brownian motion, then $\tilde{\ell} = \ell$). We use a common test function for all three cases. In this manner, the result is established for any combination of the driving processes (a)–(c). We let

$$G(t) := \int_{-\infty}^{t} \frac{1}{|s|^\gamma + 1} \, \mathrm{d}s, \qquad t \in \mathbb{R},$$

for an appropriately chosen constant $\gamma > 1$, and define $\tilde{w} := (M^{-1})'e$, $\hat{h}(x) := \langle \tilde{w}, x \rangle$, and $V(x) := G(\hat{h}(x))$. Then $\tilde{\varrho} < 0$ is equivalent to $\langle \tilde{w}, \tilde{\ell} \rangle > 0$. Note that the second derivative of $G(t)$ takes the form

$$G''(t) = \pm \gamma \frac{|t|^{\gamma-1}}{(|t|^\gamma + 1)^2},$$

where we use the positive sign for $t \leq 0$, and the negative sign for $t \geq 0$.

Suppose (1.1) is driven by a Brownian motion. We select a constant $\beta > 0$ such that $\beta^{-1} > \gamma \langle \tilde{w}, \tilde{\ell} \rangle^{-1} \sup_{x \in \mathbb{R}^d} |\sigma'(x)\tilde{w}|^2$. Let $V_\beta(x) = V(\beta x)$ for $\beta > 0$. An easy calculation shows that

$$\mathcal{A}^X V_\beta(x) = \frac{1}{2} \mathrm{Tr}\big(a(x)\nabla^2 V_\beta(x)\big) + \langle \tilde{b}(x), \nabla V_\beta(x) \rangle$$

(5.15)
$$\geq -\beta^2 \gamma \frac{|\hat{h}(\beta x)|^{\gamma-1}}{(|\hat{h}(\beta x)|^\gamma + 1)^2} |\sigma'(x)\tilde{w}|^2 + \frac{\beta(\langle \tilde{w}, \tilde{\ell} \rangle + \langle e, x \rangle^-)}{|\hat{h}(\beta x)|^\gamma + 1}$$

$$> 0 \qquad \forall x \in \mathbb{R}^d.$$

Thus, $\{V(\beta X(t))\}_{t \geq 0}$ is a bounded submartingale, so it converges almost surely. Since $\{X(t)\}_{t \geq 0}$ is irreducible, it can be either recurrent or transient. If it is recurrent, then $V(x)$ should be constant a.e. in $\mathbb{R}^d$, which is not the case. Thus $\{X(t)\}_{t \geq 0}$ is transient (for a different argument, also based on the above calculation; see [53], Theorem 3.3).

We next turn to the case that $L(t)$ is an α-stable process (isotropic or not). Here, we select constants $0 < \delta < 1$, and $1 < \gamma < \delta\alpha$ (e.g., we can let $\delta = \frac{1+\alpha}{2\alpha}$, and $\gamma = \frac{3+\alpha}{4}$). We claim that, there exists a constant C such that

(5.16)
$$\left| \mathfrak{I}_\alpha[V](x) \right| \leq \frac{C}{|\hat{h}(x)|^\gamma + 1}, \qquad x \in \mathbb{R}^d,$$

and that the same is true for the anisotropic kernel $\widehat{\mathfrak{I}}_\alpha$. Since $\alpha > 1$ and $\mathfrak{I}_\alpha[V_\beta](x) = \beta^\alpha \mathfrak{I}_\alpha[V](\beta x)$, then given a bound as in (5.16) we may select $\beta > 0$

and sufficiently small, so that $\mathcal{A}^X V_\beta(x) > 0$, and the rest follows by the same argument used for the Brownian motion.

To obtain (5.16), we proceed as follows. First, note that the anisotropic case follows from the one-dimensional isotropic. This is because the generator of the anisotropic process is a sum of generators of one-dimensional isotropic processes. Second, observe that it suffices to prove (5.16) in the one-dimensional situation only. Namely, since $V \in C^{1,1}(\mathbb{R}^d)$ (recall that $\gamma > 1$), we have

$$\mathfrak{I}_\alpha[V](x) = \frac{1}{2} \int_{\mathbb{R}^d_*} (V(x+y) + V(x-y) - 2V(x)) \frac{dy}{|y|^{d+\alpha}}, \qquad x \in \mathbb{R}^d.$$

Here, $C^{1,1}(\mathbb{R}^d)$ denotes the class of C^1-functions whose partial derivatives are Lipschitz continuous. Since $V(x) = G(\langle \tilde{w}, x \rangle)$, it is constant on each set $\{x \in \mathbb{R}^d : \langle \tilde{w}, x \rangle = \text{constant}\}$. Thus, without loss of generality we may choose $x = \zeta \tilde{w}$, for $\zeta \in \mathbb{R}$, and $\tilde{w}$ to have unit length. Consider an orthonormal transformation of the coordinates via a unitary matrix S so that the first coordinate of $\hat{y} = Sy$ is along $\tilde{w}$. Due to the invariance of the kernel under orthonormal transformations, without loss of generality, we may choose $\tilde{w} = e_1'$. Then $\langle \tilde{w}, x \rangle = x_1$, and

$$\mathfrak{I}_\alpha[V](x) = \frac{1}{2} \int_{\mathbb{R}^d_*} (G(x_1+y_1) + G(x_1-y_1) - 2G(x_1)) \frac{dy}{|y|^{d+\alpha}}, \qquad x \in \mathbb{R}^d.$$

Finally, since

$$\int_{\mathbb{R}^{d-1}_*} \frac{dy_2 \cdots dy_d}{(y_1^2 + \cdots + y_d^2)^{\frac{(d+\alpha)}{2}}} = |y_1|^{-d-\alpha} \int_{\mathbb{R}^{d-1}_*} \frac{dy_2 \cdots dy_d}{(1 + \frac{y_2^2}{y_1^2} + \cdots + \frac{y_d^2}{y_1^2})^{\frac{(d+\alpha)}{2}}}$$
$$= \frac{C}{|y_1|^{1+\alpha}},$$

we conclude that

$$\mathfrak{I}_\alpha[V](x) = \frac{C}{2} \int_{\mathbb{R}^d_*} (G(x_1+y_1) + G(x_1-y_1) + 2G(x_1)) \frac{dy_1}{|y_1|^{1+\alpha}}, \qquad x \in \mathbb{R}^d.$$

Now let us prove (5.16) in the one-dimensional case. We decompose the integral as in Lemma 5.1, choosing a cutoff radius $r(t) = t^\delta \vee 1$ for this purpose. First, we write

$$\mathfrak{I}_\alpha[V](x) = \int_{\mathcal{B}\setminus\{0\}} \left(\int_0^1 (1-t)y^2 V''(x+ty) \, dt \right) \frac{dy}{|y|^{1+\alpha}}$$

(5.17)
$$+ \int_{\mathcal{B}_{r(|x|)}\setminus\mathcal{B}} \left(\int_0^1 y V'(x+ty) \, dt \right) \frac{dy}{|y|^{1+\alpha}}$$

$$+ \int_{\mathcal{B}^c_{r(|x|)}} \left(\int_0^1 y V'(x+ty) \, dt \right) \frac{dy}{|y|^{1+\alpha}}.$$

We first bound the third integral in (5.17). Provided $y \neq 0$ (and recall that without loss of generality we may assume that $\tilde{w} = 1$), we have

$$
\begin{aligned}
\left| \int_0^1 y V'(x + ty) \, dt \right| &\leq \int_0^1 \frac{|y|}{|x + ty|^\gamma + 1} \, dt \\
(5.18) \qquad\qquad &\leq \int_{-\infty}^\infty \frac{1}{|x + s|^\gamma + 1} \, ds = \|G\|_\infty.
\end{aligned}
$$

Using (5.18), the absolute value of the third integral in (5.17) has the bound

$$
(5.19) \qquad \|G\|_\infty \int_{\mathcal{B}^c_{r(|x|)}} \frac{dy}{|y|^{1+\alpha}} = \frac{\kappa_0}{(|x|^\delta \vee 1)^\alpha} \leq \frac{\kappa_1}{|x|^\gamma + 1} = \frac{\kappa_1}{|\hat{h}(x)|^\gamma + 1}
$$

for some positive constants κ_0 and κ_1. Next, we bound the second integral in (5.17), which we denote by $\mathfrak{I}_{\alpha,2}[V](x)$. For $|y| \leq |x|^\delta$ and $|x| \geq 2^{\frac{1}{1-\delta}}$, it holds that $2|x|^\delta \leq |x|$, and $2|y| \leq |x|$. Thus, $|x| \leq 2|x + ty|$ for all $t \in [0, 1]$. So we have

$$
\begin{aligned}
\left| \mathfrak{I}_{\alpha,2}[V](x) \right| &\leq \int_0^1 \left(\int_{\mathcal{B}_{r(|x|)} \setminus \mathcal{B}} \frac{|y|}{|x + ty|^\gamma + 1} \frac{dy}{|y|^{1+\alpha}} \right) dt \\
&\leq \int_0^1 \left(\int_{\mathcal{B}_{r(|x|)} \setminus \mathcal{B}} \frac{2|y|}{|x|^\gamma + 1} \frac{dy}{|y|^{1+\alpha}} \right) dt \\
(5.20) \qquad\qquad &\leq \frac{2}{|\hat{h}(x)|^\gamma + 1} \int_{\mathcal{B}^c} \frac{dy}{|y|^\alpha} \\
&= \frac{4}{(\alpha - 1)(|\hat{h}(x)|^\gamma + 1)}.
\end{aligned}
$$

For $|x| \leq 2^{\frac{1}{1-\delta}}$, we use the following bound:

$$
(5.21) \qquad \left| \mathfrak{I}_{\alpha,2}[V](x) \right| \leq \left\| \mathfrak{I}_{\alpha,2}[V] \right\|_\infty \leq \frac{(2^{\frac{\gamma}{1-\delta}} + 1) \|\mathfrak{I}_{\alpha,2}[V]\|_\infty}{|\hat{h}(x)|^\gamma + 1}.
$$

Finally, we bound the first integral in (5.17). We use the second derivative of $G(t)$, and the inequality

$$
\frac{|z|^{\gamma-1}}{(|z|^\gamma + 1)^2} \leq \frac{1}{|z|^\gamma + 1}, \qquad z \in \mathbb{R},
$$

to obtain

$$
\begin{aligned}
\left| \int_0^1 (1 - t) y^2 V''(x + ty) \, dt \right| &\leq \gamma \int_0^1 (1 - t) \frac{|x + ty|^{\gamma-1}}{(|x + ty|^\gamma + 1)^2} |y|^2 \, dt \\
&\leq \gamma \int_0^1 (1 - t) \frac{|y|^2}{|x + ty|^\gamma + 1} \, dt.
\end{aligned}
$$

For $x \in \mathbb{R}$, $y \in \mathcal{B} \setminus \{0\}$ and $t \in [0, 1]$, we have

$$
\begin{aligned}
|x|^\gamma + 1 &= |x + ty - ty|^\gamma + 1 \\
&\leq \kappa_0 |x + ty|^\gamma + \kappa_0 |ty|^\gamma + 1 \\
&\leq \kappa_0 |x + ty|^\gamma + \kappa_1 + 1 \\
&\leq \kappa_2 (|x + ty|^\gamma + 1)
\end{aligned}
$$

for some positive constants κ_0, κ_1 and κ_2. Thus,

(5.22)
$$
\begin{aligned}
\left| \int_{\mathcal{B} \setminus \{0\}} \left(\int_0^1 (1-t) y^2 V''(x+ty) \, dt \right) \frac{dy}{|y|^{1+\alpha}} \right| \\
\leq \frac{\gamma \kappa_2}{|\hat{h}(x)|^\gamma + 1} \int_{\mathcal{B} \setminus \{0\}} |y|^2 \frac{dy}{|y|^{1+\alpha}} \\
= \frac{\kappa_3}{|\hat{h}(x)|^\gamma + 1}
\end{aligned}
$$

for some positive constant κ_3. The inequality in (5.16) now follows by combining (5.19)–(5.22).

We now consider case (c). We follow the same approach, but here scaling with β has to be argued differently. Here, we can choose any constant $\gamma > 1$. We need to establish that

(5.23)
$$
\mathcal{A}^X V_\beta(x) = \mathfrak{J}_\nu[V_\beta](x) + \langle \tilde{b}(x), \nabla V_\beta(x) \rangle > 0, \qquad x \in \mathbb{R}^d.
$$

Recall that $\nu(dy)$ is supported on $\{tw : t \in [0, \infty)\}$. Let $\tilde{\nu}(dt) = \nu(d(tw))$, and define

$$
H_\beta(t, x) := \int_0^1 \left(\frac{|\beta \langle \tilde{w}, x \rangle|^\gamma + 1}{|\langle \tilde{w}, \beta(x + stw) \rangle|^\gamma + 1} - 1 \right) ds.
$$

We have

$$
\begin{aligned}
\mathfrak{J}_\nu[V_\beta](x) &= \int_{[0, \infty)} \left(V_\beta(x + tw) - V_\beta(x) \right) \tilde{\nu}(dt) - \int_{[0, \infty)} \langle tw, \nabla V_\beta(x) \rangle \tilde{\nu}(dt) \\
&= \int_{[0, \infty)} \int_0^1 \langle tw, \nabla V_\beta(x + stw) - \nabla V_\beta(x) \rangle \, ds \, \tilde{\nu}(dt) \\
&= \int_{[0, \infty)} \frac{\beta t \langle \tilde{w}, w \rangle}{|\beta \langle \tilde{w}, x \rangle|^\gamma + 1} H_\beta(t, x) \tilde{\nu}(dt).
\end{aligned}
$$

Clearly, if $\langle \tilde{w}, w \rangle = 0$, then (5.23) trivially holds. Assume now that $\langle \tilde{w}, w \rangle > 0$. According to (5.15), a sufficient condition for (5.23) is

(5.24)
$$
\limsup_{\beta \to 0} \sup_{x \in \mathbb{R}^d} \int_{[0, \infty)} t H_\beta(t, x) \tilde{\nu}(dt) = 0.
$$

Clearly, $-1 \leq H_\beta(t, x) \leq 0$ for $t \geq 0$ and $x \in \mathbb{R}^d$, $\langle \tilde{w}, x \rangle \geq 0$. Also,

$$
\int_0^1 \frac{-|\beta s t \langle \tilde{w}, w \rangle|^\gamma}{|\beta s t \langle \tilde{w}, w \rangle|^\gamma + 1} \, \mathrm{d}s
$$

$$
\leq \int_0^1 \left(\inf_{x \in \mathbb{R}^d, \langle \tilde{w}, x \rangle \geq 0} \frac{|\beta \langle \tilde{w}, x \rangle|^\gamma + 1}{|\langle \tilde{w}, \beta(x + s t w) \rangle|^\gamma + 1} - 1 \right) \mathrm{d}s
$$

$$
\leq \inf_{x \in \mathbb{R}^d, \langle \tilde{w}, x \rangle \geq 0} H_\beta(t, x)
$$

$$
\leq \sup_{x \in \mathbb{R}^d, \langle \tilde{w}, x \rangle \geq 0} H_\beta(t, x)
$$

$$
\leq \int_0^1 \left(\sup_{x \in \mathbb{R}^d, \langle \tilde{w}, x \rangle \geq 0} \frac{|\beta \langle \tilde{w}, x \rangle|^\gamma + 1}{|\langle \tilde{w}, \beta(x + s t w) \rangle|^\gamma + 1} - 1 \right) \mathrm{d}s = 0.
$$

This, together with reverse Fatou lemma, gives (5.24) on the set $\{x \in \mathbb{R}^d : \langle \tilde{w}, x \rangle \geq 0\}$. In particular, this means that there exists some $\beta_0 > 0$ such that

$$
(5.25) \qquad \int_{[0,\infty)} t H_{\beta_0}(t, x) \nu(\mathrm{d}t) \geq -\frac{\langle \tilde{w}, \tilde{\ell} \rangle}{2 \langle \tilde{w}, w \rangle}, \qquad x \in \mathbb{R}^d, \langle \tilde{w}, x \rangle \geq 0.
$$

On the other hand, if $\langle \tilde{w}, x \rangle \leq 0$, then $H_{\beta_0}(s, x) \geq H_{\beta_0}(s, -x)$, so that (5.25) holds for all $x \in \mathbb{R}^d$. In turn, (5.25) implies that

$$
\mathfrak{I}_\nu[V_{\beta_0}](x) + \langle \tilde{b}(x), \nabla V_{\beta_0}(x) \rangle \geq \frac{1}{2} \frac{\beta_0 (\langle \tilde{w}, \tilde{\ell} \rangle + 2 \langle e, x \rangle^-)}{|\hat{h}(\beta_0 x)|^\gamma + 1} > 0, \qquad x \in \mathbb{R}^d.
$$

Finally, if $\langle \tilde{w}, w \rangle < 0$, we proceed analogously. This completes the proof of case (c).

We now turn to the case $\tilde{\varrho} = 0$. Suppose that the process $\{X(t)\}_{t \geq 0}$ has an invariant probability measure $\pi(\mathrm{d}x)$. Let $h_{1,\beta}(x)$ and $h_{2,\beta}(x)$ denote the two terms on the right-hand side of (5.15), in the order they appear. Applying Itô's formula to (5.15), we have

$$
(5.26) \qquad \mathbb{E}^\pi \big[V \big(\beta X(t \wedge \hat{\tau}_r) \big) \big] - V(\beta x) \geq \sum_{i=1,2} \mathbb{E}^\pi \left[\int_0^{t \wedge \hat{\tau}_r} h_{i,\beta}\big(X(s) \big) \, \mathrm{d}s \right],
$$

where $\hat{\tau}_r$ denotes the first exit time from $\mathcal{B}_r$, $r > 0$. Note that $h_{1,\beta}(x)$ is bounded and $h_{2,\beta}(x)$ is nonnegative. Thus we can take limits in (5.26) as $r \to \infty$, using dominated and monotone convergence for the terms on the right-hand side, and obtain

$$
\mathbb{E}^\pi \big[V \big(\beta X(t) \big) \big] - V(\beta x) \geq t \sum_{i=1,2} h_{i,\beta}(x) \pi(\mathrm{d}x), \qquad t \geq 0.
$$

Now divide both the terms by t and β, and take limits as $t \to \infty$. Since $V(x)$ is bounded, we have

$$\int_{\mathbb{R}^d} \beta^{-1} h_{1,\beta}(x) \pi(\mathrm{d}x) + \int_{\mathbb{R}^d} \beta^{-1} h_{2,\beta}(x) \pi(\mathrm{d}x) \le 0.$$

Since $\beta^{-1} h_{1,\beta}(x)$ tends to 0 uniformly in x as $\beta \searrow 0$, and is bounded, we must have $\int_{\mathbb{R}^d} \beta^{-1} h_{2,\beta}(x) \pi(\mathrm{d}x) \to 0$ as $\beta \searrow 0$. However, since $\beta^{-1} h_{2,\beta}(x)$ is bounded away from 0 is the open set $\{x \in \mathbb{R}^d : \langle e, x \rangle^- > 1\}$, this is a contradiction in view of the fact that $\pi(\mathrm{d}x)$ has full support (due to open-set irreducibility of $\{X(t)\}_{t \ge 0}$). It is clear that the proof for the α-stable (isotropic or not) and the Lévy are exactly the same, since $\mathcal{A}^X V(\beta x)$ shares the same structural property in all these cases. This completes the proof of the theorem. $\quad\square$

REMARK 5.1. Since M is a nonsingular M-matrix, its eigenvalues have positive real part. According to this, it is well known that the so-called Lyapunov equation $SM + M'S = \mathbb{I}$ admits a unique positive definite symmetric solution S (which is given by $S = \int_0^\infty e^{-M't} e^{-Mt} \, \mathrm{d}t$). Further, recall the definition in (3.1), and assume that $a(x)$ satisfies (3.10). It is straightforward to show that for any $\theta \in \Theta_c$ there exist positive constants $\tilde{c}_i$, $i = 0, 1, 2$ and $\tilde{\delta}$, such that

$$\mathcal{A}^X V_{S,\theta}(x) \le \tilde{c}_0 - \tilde{c}_1 |x|^\theta \mathbb{1}_{\mathcal{K}_{\tilde{\delta}}^c}(x) + \tilde{c}_2 |x|^\theta \mathbb{1}_{\mathcal{K}_{\tilde{\delta}}}(x), \qquad x \in \mathbb{R}^d,$$

and over all Markov controls $v \in \mathfrak{U}_{\mathrm{SM}}$. This implies that any invariant probability measure $\overline{\pi}(\mathrm{d}x)$ of $\{X(t)\}_{t \ge 0}$ (if it exists) satisfies

$$(5.27) \qquad \int_{\mathbb{R}^d} |x|^\theta \overline{\pi}(\mathrm{d}x) \le \frac{\tilde{c}_0}{\tilde{c}_1} + \frac{\tilde{c}_2}{\tilde{\delta}\tilde{c}_1} \int_{\mathbb{R}^d} \left(\langle e, x \rangle^+ \right)^\theta \overline{\pi}(\mathrm{d}x).$$

Thus, if the integral on the right-hand side of (5.27) is finite, then $\overline{\pi} \in \mathcal{P}_\theta(\mathbb{R}^d)$.

We need to introduce some notation, which we fix throughout the rest of the paper.

NOTATION 5.1. We let $\chi(t)$ be a smooth concave function such that $\chi(t) = t$ for $t \le -1$, and $\chi(t) = -\frac{1}{2}$ for $t \ge 0$. Also $\check{\chi}(t) := -\chi(-t)$. Thus this is a convex function with $\check{\chi}(t) = t$ for $t \ge 1$ and $\check{\chi}(t) = \frac{1}{2}$ for $t \le 0$. We scale $\chi(t)$ to $\chi_R(t) := R + \chi(t - R)$, $R \in \mathbb{R}$. So $\chi_R(t) = t$ for $t \le R - 1$ and $\chi_R(t) = R - \frac{1}{2}$ for $t \ge R$.

We recall the definitions of $\tilde{w} = (M^{-1})'e$, and $\hat{h}(x) = \langle \tilde{w}, x \rangle$, from the proof of Theorem 3.3, and additionally define

$$F(x) := \check{\chi}(\hat{h}(x)), \quad \text{and} \quad F_{\kappa, R}(x) := \chi_R \circ F^\kappa(x), \qquad x \in \mathbb{R}^d, \kappa > 0, R > 0,$$

where $F^\kappa(x)$ denotes the κth power of $F(x)$.

Recall that Theorem 3.3, under the assumption that $1 \in \Theta_c$, shows that if $\tilde{\varrho} \le 0$, then the process $\{X(t)\}_{t \ge 0}$ does not admit an invariant probability measure under any Markov control satisfying $\Gamma v(x) = 0$ a.e. In Lemma 5.7 which follows, we show that the same is the case if $1 \notin \Theta_c$. Therefore, $\tilde{\varrho} > 0$ and $1 \in \Theta_c$ are both necessary conditions for the existence of an invariant probability measure.

LEMMA 5.7.   *Suppose that* (1.1) *is driven by either or both of*:

 (i) *An α-stable process (isotropic or not)*.

 (ii) *A Lévy process with finite Lévy measure $\nu(\mathrm{d}y)$ which is supported on a half-line in $\mathbb{R}^d$ of the form $\{tw \colon t \in [0, \infty)\}$, with $\langle e, M^{-1}w \rangle > 0$*.

A diffusion component may be present in the noise, in which case we assume the growth condition (3.2). *Under these assumptions, the following hold*:

 (a) *If $1 \notin \Theta_c$, then the process $\{X(t)\}_{t \ge 0}$ is not ergodic under any Markov control $v \in \mathfrak{U}_{\mathrm{SM}}$ satisfying $\Gamma v(x) = 0$ a.e.*

 (b) *Suppose $1 \in \Theta_c$, and that under a control $v \in \mathfrak{U}_{\mathrm{SM}}$ such that $\Gamma v(x) = 0$ a.e., the process $\{X(t)\}_{t \ge 0}$ has an invariant probability measure $\overline{\pi}(\mathrm{d}x)$ satisfying $\int_{\mathbb{R}^d} (\langle \tilde{w}, x \rangle^+)^{p-1} \overline{\pi}(\mathrm{d}x) < \infty$ for some $p > 1$. Then, necessarily $p \in \Theta_c$ and $\tilde{\varrho} > 0$.*

 (c) *In general, if an invariant probability measure $\overline{\pi}(\mathrm{d}x)$ (under some Markov control) satisfies $\int_{\mathbb{R}^d} (\langle \tilde{w}, x \rangle^+)^p \overline{\pi}(\mathrm{d}x) < \infty$ for some $p \ge 1$, then necessarily $p \in \Theta_c$.*

PROOF.   Recall Notation 5.1. Note that $F_{\kappa, R}(x)$ is smooth, bounded, and has bounded derivatives. Thus, if $\{X(t)\}_{t \ge 0}$ is positive recurrent with invariant probability measure $\overline{\pi}(\mathrm{d}x)$, we must have $\overline{\pi}(\mathcal{A}^X F_{\kappa, R}) = 0$. Note also that $F(x)$ is positive and bounded away from 0. For $f \in C^2(\mathbb{R}^d)$, let

$$(5.28) \qquad \mathcal{A}_\circ^X f(x) := \frac{1}{2} \mathrm{Tr}\big(a(x) \nabla^2 f(x)\big) + \begin{cases} \mathfrak{I}_\nu[f](x) & \text{if } 1 \in \Theta_c, \\ \mathfrak{I}_{1,\nu}[f](x) & \text{otherwise.} \end{cases}$$

We have

$$(5.29) \qquad \mathcal{A}^X F_{\kappa, R}(x) = \mathcal{A}_\circ^X F_{\kappa, R}(x) + \chi_R'\big(F^\kappa(x)\big)\langle \tilde{b}(x), \nabla F^\kappa(x) \rangle.$$

Let

$$h_\kappa(x) := \kappa \, \check{\chi}'\big(\hat{h}(x)\big) F^{\kappa-1}(x),$$

$$\tilde{h}_\kappa(x) := h_\kappa(x) \langle e, x \rangle^-,$$

$$\widetilde{F}_{\kappa, R}(x) := \frac{1}{2} \chi_R''\big(F^\kappa(x)\big)\big(h_\kappa(x)\big)^2 |\sigma'(x)\tilde{w}|^2.$$

A simple calculation shows that for any control $v(x)$ satisfying $\Gamma v(x) = 0$ a.e., it holds that

$$(5.30) \qquad \langle \tilde{b}(x), \nabla F^\kappa(x) \rangle = h_\kappa(x)\big(-\tilde{\varrho} + \langle e, x \rangle^-\big).$$

We also have

$$\mathcal{A}_\circ^X F_{\kappa,R}(x) = \frac{1}{2}\chi_R'(F^\kappa(x))\,\mathrm{Tr}(a(x)\nabla^2 F^\kappa(x)) + \widetilde{F}_{\kappa,R}(x)$$

$$+ \begin{cases} \mathfrak{I}_\nu[F_{\kappa,R}](x) & \text{if } 1 \in \Theta_c, \\ \mathfrak{I}_{1,\nu}[F_{\kappa,R}](x) & \text{otherwise.} \end{cases}$$

Integrating (5.29) with respect to $\overline{\pi}(\mathrm{d}x)$, and using (5.30) to rearrange terms, we obtain

(5.31)
$$\tilde{\varrho}\overline{\pi}(\chi_R'(F^\kappa)h_\kappa) = \overline{\pi}(\chi_R'(F^\kappa)\mathcal{A}_\circ^X F^\kappa) + \overline{\pi}(\chi_R'(F^\kappa)\tilde{h}_\kappa)$$
$$+ \overline{\pi}(\mathfrak{I}_R[F^\kappa]) + \overline{\pi}(\widetilde{F}_{\kappa,R}),$$

with

$$\mathfrak{I}_R[F^\kappa](x) := \begin{cases} \mathfrak{I}_\nu[F_{\kappa,R}](x) - \chi_R'(F^\kappa(x))\mathfrak{I}_\nu[F^\kappa](x) & \text{if } 1 \in \Theta_c, \\ \mathfrak{I}_{1,\nu}[F_{\kappa,R}](x) - \chi_R'(F^\kappa(x))\mathfrak{I}_{1,\nu}[F^\kappa](x) & \text{otherwise.} \end{cases}$$

Note that we can always select $\chi_R(t)$ so that $\chi_R''(t) \geq -\frac{1}{t+1}$, for $t \geq 0$. Thus, in view of (3.2), there exists some positive constant $\tilde{C}$ such that

(5.32)
$$|\widetilde{F}_{\kappa,R}(x)| \leq \tilde{C}(1 + F^{\kappa-1}(x)), \qquad x \in \mathbb{R}^d, R > 0.$$

Let $\kappa \in \Theta_c$, $\kappa \leq 1$. Then the functions $\mathfrak{I}_R[F^\kappa](x)$ and $\widetilde{F}_{\kappa,R}(x)$ are bounded, uniformly in R, and converge to 0, on compact sets as $R \to \infty$. Thus, we can take limits in (5.31) as $R \to \infty$, to obtain

(5.33)
$$\overline{\pi}([\mathcal{A}_\circ^X F^\kappa]^-) + \tilde{\varrho}\overline{\pi}(h_\kappa) = \overline{\pi}([\mathcal{A}_\circ^X F^\kappa]^+) + \overline{\pi}(\tilde{h}_\kappa).$$

We next prove part (a) of the lemma. First, consider the process in (ii). The map $x \mapsto \mathrm{Tr}(a(x)\nabla^2 F^\kappa(x))$ is clearly bounded on $\mathbb{R}^d$, uniformly in $\kappa \in (0,1)$, and

(5.34)
$$\mathfrak{I}_{1,\nu}[F^\kappa](x) \geq \int_{\mathcal{B}}\int_0^1 (1-t)\langle y, \nabla^2 F^\kappa(x+ty)y\rangle \,\mathrm{d}t\,\nu(\mathrm{d}y),$$

since $\langle \tilde{w}, w\rangle > 0$. Thus $[\mathcal{A}_\circ^X F^\kappa]^-(x)$ is bounded uniformly in $\kappa \in (0,1)$. If $\Theta_c = (0, \theta_c)$, then $\inf_{x\in\mathcal{B}} \mathcal{A}_\circ^X F^\kappa(x) \to \infty$ as $\kappa \nearrow \theta_c$, and contradicts (5.33), which is valid for all $\kappa \in \Theta_c$. In the event that $\Theta_c = (0, \theta_c]$, we express the integral of (5.29) as

(5.35)
$$\overline{\pi}([\mathcal{A}_\circ^X F_{\kappa,R}]^-) + \tilde{\varrho}\overline{\pi}(\chi_R'(F^\kappa)h_\kappa) = \overline{\pi}([\mathcal{A}_\circ^X F_{\kappa,R}]^+) + \overline{\pi}(\chi_R'(F^\kappa)\tilde{h}_\kappa),$$

and evaluate (5.35) at any $\kappa \in (\theta_c, 1]$. Again, $\mathfrak{I}_{1,\nu}[F_{\kappa,R}](x)$ has the bound in (5.34), implying that $[\mathcal{A}_\circ^X F_{\kappa,R}(x)]^-$ is uniformly bounded over $R \in (0, \infty)$. Thus, we can take limits in (5.35) as $R \to \infty$, to reach the same contradiction.

In the case of the process in (i), a straightforward calculation, using the estimates in the proof of Lemma 5.1, and the one-dimensional character of the singular integral as exhibited in the proof of Theorem 3.2, shows that there exists a positive

constant C_1 such that $[\mathfrak{I}_\alpha[F^\kappa]]^-(x) \leq \frac{C_1}{\alpha - \kappa}$ for all $x \in \mathbb{R}^d$. This bound can be easily obtained by using the first integral in (5.17) over $\mathbb{R}^d$ instead of $\mathcal{B}$, to compute $\mathfrak{I}_\alpha[F^\kappa](x)$. Let

$$D_{r_0} := \{x \in \mathbb{R}^d : 0 \leq \langle \tilde{w}, x \rangle \leq r_0\}, \qquad \tilde{D}_{r_0} := \{x \in \mathbb{R}^d : r_0 < \langle \tilde{w}, x \rangle\}.$$

We fix any $r_0 > 1$, and estimate $\mathfrak{I}_\alpha[F^\kappa](x)$ for $x \in D_{r_0}$. The part from the first integral in (5.17) over $\mathcal{B}$ is bounded below uniformly in $\kappa \in (0, \alpha)$ by some constant $-C_2$. We evaluate the remaining part by using the third integral in (5.17) with $r(|x|) = 1$. Taking advantage of the one-dimensional character of this integration, and assuming without loss of generality that $\langle \tilde{w}, e_1 \rangle = 1$, in order to simplify the notation, we obtain

$$\int_{-\infty}^{-1} \left(F^\kappa(x + te_1) - F^\kappa(x)\right) \frac{\mathrm{d}t}{|t|^{1+\alpha}} \geq -F^\kappa(x) \int_{-\infty}^{-1} \frac{\mathrm{d}t}{|t|^{1+\alpha}} \geq -C_3 r_0,$$

for a positive constant C_3 independent of r_0 or κ. Using the inequality, $|s + t|^\kappa - |s|^\kappa \geq 2^{-\frac{1}{\kappa}}|t|^\kappa - (1 - 2^{-\frac{1}{\kappa}})|s|^\kappa$, which is valid for any $0 \leq s \leq t$ in $\mathbb{R}$, we have

$$\int_1^\infty \left(F^\kappa(x + te_1) - F^\kappa(x)\right) \frac{\mathrm{d}t}{|t|^{1+\alpha}} \geq 2^{-\frac{1}{\kappa}} \int_1^\infty |t|^\kappa \frac{\mathrm{d}t}{|t|^{1+\alpha}}$$

$$- \left(1 - 2^{-\frac{1}{\kappa}}\right)r_0^\kappa \int_1^\infty \frac{\mathrm{d}t}{|t|^{1+\alpha}}$$

$$\geq \frac{2^{-\frac{1}{\alpha}}}{\alpha - \kappa} - C_3 r_0,$$

where C_3 is the same constant used earlier. Let r_0 be large enough so that $\overline{\pi}(\tilde{D}_{r_0}) \leq \frac{1}{2^{\frac{\alpha+1}{\alpha}}C_1}\overline{\pi}(D_{r_0})$. It is clear that $\mathfrak{I}_\alpha[F^\kappa](x)$ is nonnegative if $\langle \tilde{w}, x \rangle \leq 0$, for all κ sufficiently close to α. Therefore, combining the above, we have

$$\int_{\mathbb{R}^d} \mathfrak{I}_\alpha[F^\kappa](x)\overline{\pi}(\mathrm{d}x) \geq \int_{D_{r_0}} \mathfrak{I}_\alpha[F^\kappa](x)\overline{\pi}(\mathrm{d}x) + \int_{\tilde{D}_{r_0}} \mathfrak{I}_\alpha[F^\kappa](x)\overline{\pi}(\mathrm{d}x)$$

$$\geq \overline{\pi}(D_{r_0})\left(\frac{2^{-\frac{1}{\alpha}}}{\alpha - \kappa} - C_2 - 2C_3 r_0\right) - \overline{\pi}(\tilde{D}_{r_0})\frac{C_1}{\alpha - \kappa}$$

$$\geq \overline{\pi}(D_{r_0})\left(\frac{2^{-1-\frac{1}{\alpha}}}{\alpha - \kappa} - C_2 - 2C_3 r_0\right),$$

and we obtain a contradiction in (5.33) by letting $\kappa \nearrow \alpha$. Note that since $\tilde{w}$ has nonnegative components, the preceding argument applies to both the isotropic and anisotropic Lévy kernels. This completes the proof of part (a).

We now turn to part (b). Suppose first that $\int_{\mathbb{R}^d} |x|^{\theta_c - 1}\overline{\pi}(\mathrm{d}x) < \infty$, and $\Theta_c = (0, \theta_c)$. We apply $\mathcal{A}^X$ to $F_{\kappa, R}(x)$, for $\kappa \in (1, \theta_c)$, and note again that the function $\mathfrak{I}_R[F^\kappa](x)$ is bounded, uniformly in R, and converges to 0, on compact sets as

$R \to \infty$. Moreover, since $\kappa \geq 1$, and $\tilde{w}$ has nonnegative components we have $\mathcal{A}_\circ^X F^\kappa \geq 0$ by convexity. Thus, taking limits as $R \to \infty$, (5.33) takes the form

$$(5.36) \qquad \tilde{\varrho}\overline{\pi}(h_\kappa) = \overline{\pi}(\mathcal{A}_\circ^X F^\kappa) + \overline{\pi}(\tilde{h}_\kappa).$$

It is thus immediately clear that $\tilde{\varrho} > 0$. It also follows from (5.36) that $\overline{\pi}(h_\kappa) \to \infty$ as $\kappa \nearrow \theta_c$, which contradicts the original hypothesis that $\int_{\mathbb{R}^d} |x|^{\theta_c-1}\overline{\pi}(\mathrm{d}x) < \infty$.

Next, we consider the case $\Theta_c = (0, \theta_c]$. If the α-stable component is present, then necessarily $\theta_c < \alpha$, and we can follow the technique used for part (a). If a diffusion part is present we argue as follows. We suppose that $\int_{\mathbb{R}^d} |x|^{\theta_c-1+\epsilon}\overline{\pi}(\mathrm{d}x) < \infty$, for some $\epsilon > 0$, and select $\kappa = \theta_c + \epsilon$. Then $\overline{\pi}(\widetilde{F}_{\kappa,R})$ vanishes as $R \to \infty$ by (5.32) and the hypothesis. Thus, again we obtain from (5.36) that $\overline{\pi}(h_\kappa) \to \infty$ as $R \to \infty$, which contradicts the original hypothesis that $\int_{\mathbb{R}^d} |x|^{\theta_c-1+\epsilon}\overline{\pi}(\mathrm{d}x) < \infty$. This, together with Remark 5.1 proves (b).

Concerning part (c), in general, under a Markov control $v \in \mathfrak{U}_{\mathrm{SM}}$, (5.30) takes the form

$$\langle \tilde{b}(x), \nabla F^p(x) \rangle = p\check{\chi}'(\hat{h}(x))F^{p-1}(x)\big(-\tilde{\varrho} + \langle e, x \rangle^- - \langle \tilde{w}, \Gamma v(x) \rangle \langle e, x \rangle^+\big),$$

and following through the earlier calculations for $p \geq 1$, we see that (5.35) now takes the form

$$
\begin{aligned}
(5.37) \quad & \overline{\pi}(\mathcal{A}_\circ^X F_{p,R}) + \overline{\pi}(\chi_R'(F^p)\tilde{h}_p) \\
& \qquad \leq \tilde{\varrho}\overline{\pi}(\chi_R'(F^p)h_p) \\
& \qquad\quad + \int_{\mathbb{R}^d} \chi_R'(F^p(x)) p\check{\chi}'(\hat{h}(x))F^{p-1}(x)\langle \tilde{w}, \Gamma v(x) \rangle \langle e, x \rangle^+\overline{\pi}(\mathrm{d}x).
\end{aligned}
$$

Taking limits in (5.37) as $R \to \infty$, the right-hand side reaches a finite value by the hypothesis and Remark 5.1. In view of the decomposition of $\mathcal{A}_\circ^X F_{p,R}(x)$ in (5.31), and the growth estimate in (5.32), this implies that $\overline{\pi}(\mathcal{A}_\circ^X F^p) < \infty$, from which we deduce that $p \in \Theta_c$. $\quad\square$

REMARK 5.2. It can be seen from the proof of Lemma 5.7, that the conclusions of parts (b) and (c) are still valid if we replace the α-stable process in (i) with a subordinate Brownian motion, such that $\theta_c < \infty$.

REMARK 5.3. The hypothesis $\langle \tilde{w}, w \rangle = \langle e, M^{-1}w \rangle > 0$ in Lemma 5.7(ii) cannot, in general, be relaxed. The following example demonstrates this. Let $M = \mathbb{I}$, any constant control $v \in \Delta$, and $\langle \tilde{w}, w \rangle \leq 0$. Consider the function $V(x) = V_{Q,\theta}(x) + \tilde{V}(x)$, with $\tilde{V}(x) = \phi(\beta\langle e, x \rangle^+)$, $\beta > 0$, and $\phi(t) = \mathrm{e}^t$ for $t \geq 1$, and $\phi(t) = 1$ for $t \leq 0$, and smooth. It is straightforward to verify that $\mathcal{A}^X \tilde{V}(x) = 0$ if $\langle e, x \rangle \leq 0$, and that $\mathcal{A}^X \tilde{V}(x) \leq \kappa_0' - \kappa_1' \tilde{V}(x)$ if $\langle e, x \rangle \geq 0$, where κ_0' and κ_1' are positive constants. Adding this inequality to (3.3), and since for some positive constants C and R we have $\tilde{V}(x) \geq CV(x)$ for $x \in \mathcal{K}_\delta \cap \mathcal{B}_R^c$, we obtain $\mathcal{A}^X V(x) \leq \kappa_0'' - \kappa_1'' V(x)$ for some positive constants κ_0'' and κ_1'', and all $x \in \mathbb{R}^d$.

This shows that the process is exponentially ergodic. This is an example where the direction of jumps is beneficial to ergodicity.

COROLLARY 5.1.   *Assume the hypotheses of Theorem* 3.1, *and* (3.2). *Then, every invariant probability measure* $\overline{\pi}(\mathrm{d}x)$, *corresponding to a Markov control* $v \in \mathfrak{U}_{\mathrm{SM}}$ *such that* $\Gamma v(x) = 0$ *a.e., under which the process* $\{X(t)\}_{t \geq 0}$ *is ergodic, satisfies*

$$(5.38) \qquad \tilde{\varrho} = \int_{\mathbb{R}^d} \langle e, x \rangle^{-} \overline{\pi}(\mathrm{d}x).$$

PROOF.   We scale $\check{\chi}(t)$ by defining $\check{\chi}_r(t) := \check{\chi}(r + t) - r$ for $r \in \mathbb{R}$. Then we observe that (5.31) holds with $\kappa = 1$ and $F^\kappa(x)$ replaced by $\check{\chi}_r \circ \hat{h}(x)$ for any $r \in \mathbb{R}$. Note that when $\kappa = 1$, all the integrands in (5.31) are uniformly bounded in $r \in (0, \infty)$, and we have $\mathcal{A}_\circ^X \check{\chi}_r \circ \hat{h}(x) \geq 0$ due to convexity. Moreover, $\lim_{r \to \infty} \overline{\pi}(\mathcal{A}_\circ^X \check{\chi}_r \circ \hat{h}) = 0$, $\lim_{r \to \infty} \overline{\pi}(\check{\chi}_r \circ \hat{h}) = 1$, and $\lim_{r \to \infty} \overline{\pi}(\check{\chi}_r(\cdot)\langle e, \cdot \rangle^{-}) = \overline{\pi}(\langle e, \cdot \rangle^{-})$. Thus, taking limits as $r \to \infty$, we obtain (5.38).   □

REMARK 5.4.   Corollary 5.1 has important implications for queueing systems in the Halfin–Whitt regime. Suppose that $\Gamma = 0$, so that jobs do not abandon the queues. Let $v \in \mathfrak{U}_{\mathrm{SM}}$ be a scheduling control under which the process is ergodic. Then (5.38) asserts that the mean idleness in the servers equals the spare capacity. Note that there is a certain stiffness implied by this. The mean idleness does not depend on the particular Markov control.

PROOF OF THEOREM 3.4.   Part (b) follows from Theorem 3.2(i), Lemma 5.7, Remark 5.1, and Corollary 5.1.

We continue with part (a). The upper bounds follow from [18], Theorem 3.2. We next exhibit a lower bound for the rate of convergence. Consider first the case when the SDE is driven by an α-stable process. We apply [26], Theorem 5.1, and use the same notation to help the reader. We choose $G(x) = F^{\alpha-\epsilon}(x)$, for arbitrary $\epsilon \in (0, \alpha - 1)$. We have shown in Lemma 5.7 that $\overline{\pi}(F^{\alpha-\epsilon}) = \infty$. Further, from the Lyapunov equation in (3.3) by Itô's formula, and setting $\theta = \alpha - \epsilon$, we have

$$\mathbb{E}^x\big[V_{Q,\alpha-\epsilon}\big(X(t)\big)\big] - V_{Q,\alpha-\epsilon}(x) \leq c_0(\alpha - \epsilon)t, \qquad x \in \mathbb{R}^d.$$

Dominating $F^{\alpha-\epsilon}(x)$ with $V_{Q,\alpha-\epsilon}(x)$ and write

$$\mathbb{E}^x\big[F^{\alpha-\epsilon}\big(X(t)\big)\big] \leq C_1\big(c_0(\alpha - \epsilon)t + V_{Q,\alpha-\epsilon}(x)\big) =: g(x, t)$$

for some positive constant C_1. Next, we compute a lower bound for $\overline{\pi}(\{x : G(x) \geq t\})$. From (5.35), with $\kappa = 1$, we have

$$\tilde{\varrho}\overline{\pi}(\chi_t'(F)h_1) = \overline{\pi}(\mathcal{A}_\circ^X F_{1,t}) + \overline{\pi}(\chi_t'(F)\tilde{h}_1).$$

Subtracting this equation from (5.36), we obtain

$$\tilde{\varrho}\overline{\pi}(h_1 - \chi_t'(F)h_1) = \overline{\pi}(\mathcal{A}_\circ^X (F - F_{1,t})) + \overline{\pi}(\tilde{h}_1 - \chi_t'(F)\tilde{h}_1).$$

Note that all the terms are nonnegative. Moreover, $\mathcal{A}_\circ^X(F - F_{1,t})(x)$ is nonnegative by convexity, and thus

$$\overline{\pi}\big(\mathcal{A}_\circ^X(F - F_{1,t})\big) \geq \inf_{x \in \mathcal{B}} \big(\mathcal{A}_\circ^X(F - F_{1,t})(x)\big)\overline{\pi}(\mathcal{B})$$

$$\geq \mathcal{A}_\circ^X(F - F_{1,t})(0)\overline{\pi}(\mathcal{B}).$$

Recalling the definitions of the functions in Notation 5.1, it is then evident that

$$\overline{\pi}\big(\{x \colon \langle \tilde{w}, x \rangle > t\}\big) \geq \overline{\pi}(h_1 - \chi_t'(F)h_1)$$

(5.39)
$$\geq \tilde{\varrho}^{-1}\overline{\pi}(\mathcal{B})\mathcal{A}_\circ^X(F - F_{1,t})(0)$$

$$\geq C_2 t^{1-\alpha}.$$

Therefore, by (5.39), we have

$$\overline{\pi}\big(\{x \colon G(x) \geq t\}\big) = \overline{\pi}\big(\{x \colon (\langle \tilde{w}, x \rangle)^{\alpha-\epsilon} > t\}\big)$$

$$= \overline{\pi}\big(\{x \colon \langle \tilde{w}, x \rangle > t^{\frac{1}{\alpha-\epsilon}}\}\big)$$

$$\geq C_2 t^{\frac{1-\alpha}{\alpha-\epsilon}} =: f(t).$$

We solve $yf(y) = 2g(x, t)$ for $y = y(t)$, to obtain $y = (C_2^{-1}2g(x,t))^{\frac{\alpha-\epsilon}{1-\epsilon}}$, and

$$f(y) = C_2\big(C_2^{-1}2g(x,t)\big)^{\frac{1-\alpha}{1-\epsilon}} = C_3\big(c_0(\alpha - \epsilon)t + V_{Q,\alpha-\epsilon}(x)\big)^{\frac{1-\alpha}{1-\epsilon}},$$

with

$$C_3 := (2C_1)^{\frac{1-\alpha}{1-\epsilon}} C_2^{\frac{\alpha-\epsilon}{1-\epsilon}}.$$

Therefore, by [26], Theorem 5.1, and since ϵ is arbitrary, we have

(5.40)
$$\big\|\delta_x P_t^X(\mathrm{d}x) - \overline{\pi}(\mathrm{d}x)\big\|_{\mathrm{TV}} \geq f(y) - \frac{g(x, t)}{y}$$

$$= \frac{C_3}{2}\big(c_0(\alpha - \epsilon)t + V_{Q,\alpha-\epsilon}(x)\big)^{\frac{1-\alpha}{1-\epsilon}}$$

for all $t \geq 0$ and $\epsilon \in (0, \alpha - 1)$.

We next derive a suitable estimate for the constant $c_0(\alpha - \epsilon)$ as a function of ϵ, for a fixed choice of Q. First, evaluating at $x = 0$, it follows from (3.3) that $c_0(\alpha - \epsilon) \geq \mathfrak{I}_\alpha[V_{Q,\alpha-\epsilon}](0)$. Thus, for some positive constants κ_0 and κ_1 independent of ϵ, we have

$$c_0(\alpha - \epsilon) \geq \mathfrak{I}_\alpha[V_{Q,\alpha-\epsilon}](0) \geq \int_{\mathcal{B}^c} \big(V_{Q,\alpha-\epsilon}(y) - V_{Q,\alpha-\epsilon}(0)\big)\frac{\mathrm{d}y}{|y|^{d+\alpha}} \geq \frac{\kappa_0}{\epsilon} + \kappa_1.$$

On the other hand, as shown in the proof of Theorem 3.2, $c_0(\theta)$ can be selected as the sum of the supremum of $\mathfrak{I}_\nu[V_{Q,\theta}](x)$ on $\mathbb{R}^d$, and a constant that does not

depend on θ. An upper bound for $\mathfrak{I}_\alpha[V_{Q,\alpha-\epsilon}](x)$ can be obtained from the proof of Theorem 3.2 by using (5.3), (5.5) and (5.6), together with the fact that the radius $\bar{r}$ defined in the proof is bounded over the range of ϵ. However, we follow a more direct approach. Note that $\nabla V_{Q,\theta}(x)$ is Hölder continuous with exponent $\theta - 1$ on $\mathbb{R}^d$, that is, it satisfies $|\nabla V_{Q,\theta}(x) - \nabla V_{Q,\theta}(y)| \le \kappa |x - y|^{\theta-1}$ for a positive constant κ which is independent of $\theta \in (1, 2]$. Using this property, and the fact that the second derivatives of $V_{Q,\theta}(x)$ are uniformly bounded for $\theta \in (1, 2]$, decomposing the integral as in (5.4), we obtain

$$
\begin{aligned}
&\big|\mathfrak{I}_\alpha[V_{Q,\alpha-\epsilon}](x)\big| \\
&\quad \le \tilde{\kappa}_0 + \left| \int_{\mathcal{B}^c} \int_0^1 \langle y, \nabla V_{Q,\alpha-\epsilon}(x + ty) - \nabla V_{Q,\alpha-\epsilon}(x) \rangle \mathrm{d}t \, \frac{\mathrm{d}y}{|y|^{d+\alpha}} \right| \\
&\quad \le \tilde{\kappa}_0 + \frac{\tilde{\kappa}_1}{\epsilon}
\end{aligned}
$$

for some positive constants $\tilde{\kappa}_0$ and $\tilde{\kappa}_1$ which do not depend on ϵ. Using this estimate in (5.40), we obtain the lower bound in (3.8).

For a Lévy process in (ii), following (5.39), we obtain

$$
(5.41) \qquad \overline{\pi}\big(\{x : \langle \tilde{w}, x \rangle > t\}\big) \ge C_2 \int_{\{\langle \tilde{w}, x \rangle \ge t\}} |x| \nu(\mathrm{d}x).
$$

Since (5.41) does not give rise to an explicit estimate as in (5.39), we apply [26], Corollary 5.2. For $\epsilon \in (0, \frac{1}{3})$, we define

$$
W(x) := F^{\theta_c - 1 + \epsilon}(x), \qquad \widehat{F}(t) := t^{\frac{\theta_c - \epsilon}{\theta_c - 1 + \epsilon}}, \qquad h(t) := t^{-1 - \frac{\epsilon}{\theta_c - 1 + \epsilon}},
$$

and

$$
g(x, t) := C_1\big(c_0(\theta_c - \epsilon)t + V_{Q,\theta_c-\epsilon}(x)\big).
$$

Then the hypotheses in [26], Corollary 5.2, are satisfied. By the preceding definitions, we have $\widehat{F}(t)h(t) = t^{\frac{1-3\epsilon}{\theta_c - 1 + \epsilon}}$. Thus

$$
(\widehat{F} \cdot h)^{-1}(y) = y^{\frac{\theta_c - 1 + \epsilon}{1 - 3\epsilon}}, \quad \text{and} \quad h\big((\widehat{F} \cdot h)^{-1}(y)\big) = y^{-\frac{\theta_c - 1 + 2\epsilon}{1 - 3\epsilon}}.
$$

Therefore, by [26], Corollary 5.2, for every $x \in \mathbb{R}^d$, there exists a sequence $\{t_n\}_{n \in \mathbb{N}} \subset [0, \infty)$, $t_n \to \infty$, such that

$$
\big\| \delta_x P_{t_n}^X(\mathrm{d}x) - \overline{\pi}(\mathrm{d}x) \big\|_{\mathsf{TV}} \ge g(x, t_n)^{-\frac{\theta_c - 1 + 2\epsilon}{1 - 3\epsilon}},
$$

which establishes (3.9). This completes the proof. $\square$

5.3. *Proof of Theorem* 3.5. We start with the following lemma.

LEMMA 5.8. *Under the assumptions* (i) *or* (ii) *of Theorem* 3.5, *there exists* $Q \in \mathcal{M}_+$ *such that*

(5.42)
$$MQ + QM \succ 0, \quad and$$
$$\big(M - ev'(M - \Gamma)\big)Q + Q\big(M - (M - \Gamma)ve'\big) \succ 0.$$

PROOF. First, consider (i) of Theorem 3.5. Since M is a nonsingular M-matrix, $\bar{v} := M^{-1}(Mv - \Gamma v)$ is a nonnegative vector which satisfies $e'\bar{v} < 1$. The result then follows by (5.9).

Next, suppose that M is a diagonal matrix and $\Gamma v \neq 0$. If $d = 1$, the assertion is trivially satisfied. Assume $d \geq 2$ and define $\tilde{v} := M^{-1}\Gamma v$, $\hat{v} := \tilde{v} - v$ and $A_k := M^k(\mathbb{I} + \hat{v}e')$ for $k = 1, 2$. By assumption $\tilde{v} \neq 0$. Further, observe that $M - A_1 = (\Gamma - M)ve'$ has rank one. Thus, according to [30], Theorem 1, in order to assert the existence of a positive definite matrix Q satisfying (5.42), it suffices to show that the spectrum of A_1 lies in the open right half of the complex plane and that A_2 does not have real negative eigenvalues.

We first show that A_2 does not have real negative eigenvalues. Suppose that $-\lambda$, with $\lambda \geq 0$, is such an eigenvalue. Then

$$0 = \det\big((\lambda\mathbb{I} + M^2)^{-1}\big)\det(\lambda\mathbb{I} + A_2) = \det\big(\mathbb{I} + (\lambda\mathbb{I} + M^2)^{-1}M^2\hat{v}e'\big)$$
$$= 1 + e'(\lambda\mathbb{I} + M^2)^{-1}M^2\hat{v},$$

which implies that $e'(\lambda\mathbb{I} + M^2)^{-1}M^2\hat{v} = -1$. But

$$e'(\lambda\mathbb{I} + M^2)^{-1}M^2\hat{v} \geq e'(\lambda\mathbb{I} + M^2)^{-1}M^2\tilde{v} - e'(\lambda\mathbb{I} + M^2)^{-1}M^2v$$
$$> \left(\min_i \frac{m_i^2}{\lambda + m_i^2}\right)e'\tilde{v} - \left(\max_i \frac{m_i^2}{\lambda + m_i^2}\right) > -1,$$

which is a contradiction.

Next, we show that the spectrum of A_1 lies in the open right half of the complex plane. Suppose that $\iota\lambda$, $\lambda \in \mathbb{R}$, is an eigenvalue of A_1. Then, since

$$0 = \det(A_1 - \iota\lambda\mathbb{I}) = \det(M - \iota\lambda\mathbb{I})\det\big(\mathbb{I} + (M - \iota\lambda\mathbb{I})^{-1}M\hat{v}e'\big),$$

we have that

$$0 = \det\big(\mathbb{I} + (M - \iota\lambda\mathbb{I})^{-1}M\hat{v}e'\big) = 1 + e'(M - \iota\lambda\mathbb{I})^{-1}M\hat{v}$$
$$= 1 + \sum_{k=1}^d \frac{m_k^2\hat{v}_k}{m_k^2 + \lambda^2} + \iota\sum_{k=1}^d \frac{\lambda m_k\hat{v}_k}{m_k^2 + \lambda^2}.$$

However, it holds that

$$1 + \sum_{k=1}^{d} \frac{m_k^2 \hat{v}_k}{m_k^2 + \lambda^2} > 1 - \sum_{k=1}^{d} \frac{m_k^2 v_k}{m_k^2 + \lambda^2} \geq 1 - e'v = 0.$$

Thus we reach a contradiction. This shows that the matrix $A_1(t) := M(\mathbb{I} + (t\tilde{v} - v)e')$ cannot have any imaginary eigenvalues for any $t > 0$, nor does it have a zero eigenvalue. Moreover, for all small enough $t > 0$ the spectrum of $A_1(t)$ is in the open right half of the complex plane by Lemma 5.5. Hence, by the continuity of the spectrum of $A_1(t)$ as a function of t, it follows that the eigenvalues of $A_1(t)$ are in the open right half complex plane for all $t > 0$, which concludes the proof. $\square$

PROOF OF THEOREM 3.5. Consider the function $V_{Q,\theta}(x)$, $\theta > 0$, with Q as in (5.42). Let $\check{b}(x) := b(x) + \vartheta$, $x \in \mathbb{R}^d$. Thus $V_{Q,\theta}(x)$ is an inf-compact function, and satisfies

$$(5.43) \qquad \langle \check{b}(x), \nabla V_{Q,\theta}(x) \rangle \leq \kappa_0' - \kappa_1' V_{Q,\theta}(x), \qquad x \in \mathbb{R}^d,$$

for some constants $\kappa_0' > 0$ and $\kappa_1' > 0$ by Lemma 5.8. Then by Lemmas 5.1 and 5.3, and mimicking the proof of Theorem 3.2, we obtain (3.11), while (3.12) follows from [40], Theorem 6.1.

We now turn to the last statement of the theorem. It is well known that (3.11) implies that $\int_{\mathbb{R}^d} V_{Q,\theta}(x)\overline{\pi}(dx) \leq \frac{\bar{c}_0}{\bar{c}_1}$, see [40], Theorem 4.3. Thus $\int_{\mathbb{R}^d} |x|^\theta \overline{\pi}(dx) < \infty$. It remains to show that if $q > 0$ and $q \notin \Theta_c$, then $\int_{\mathbb{R}^d} |x|^q \overline{\pi}(dx) = \infty$. Recall the definition in (5.28). We write

$$(5.44) \quad \mathcal{A}^X V_{Q,\theta}(x) = \left(\mathcal{A}_\circ^X V_{Q,\theta}(x) \right)^+ - \left(\mathcal{A}_\circ^X V_{Q,\theta}(x) \right)^- + \langle \check{b}(x), \nabla V_{Q,\theta}(x) \rangle.$$

It is standard to show by using (5.43) and (5.44), together with Lemma 5.1, which holds for all $\theta \in \Theta_c$, and the arguments in the proof of [5], Lemma 3.7.2, that

$$(5.45) \begin{aligned} - \int_{\mathbb{R}^d} \langle \check{b}(x), \nabla V_{Q,\theta}(x) \rangle \overline{\pi}(dx) + \int_{\mathbb{R}^d} \left(\mathcal{A}_\circ^X V_{Q,\theta}(x) \right)^- \overline{\pi}(dx) \\ = \int_{\mathbb{R}^d} \left(\mathcal{A}_\circ^X V_{Q,\theta}(x) \right)^+ \overline{\pi}(dx) \end{aligned}$$

for all $x \in \mathbb{R}^d$ and $\theta \in \Theta_c$. Note that there exist positive constants C_0 and C_1 such that

$$(5.46) \qquad -C_0 - C_1^{-1} \langle \check{b}(x), \nabla V_{Q,\theta}(x) \rangle \leq |x|^\theta \leq C_0 - C_1 \langle \check{b}(x), \nabla V_{Q,\theta}(x) \rangle$$

for all $x \in \mathbb{R}^d$ and $\theta \in \Theta_c$.

First, we consider the case $\Theta_c = (0, \theta_c)$. A standard calculation shows that

$$(5.47) \qquad \inf_{x \in \mathcal{B}} \mathcal{A}_\circ^X V_{Q,\theta}(x) \xrightarrow[\theta \nearrow \theta_c]{} \infty.$$

Thus, combining (5.45)–(5.47), and since $(\mathcal{A}_\circ^X V_{Q,\theta}(x))^- \leq \kappa(1 + |x|^\theta)$, we obtain $\int_{\mathbb{R}^d} |x|^\theta \overline{\pi}(\mathrm{d}x) \to \infty$ as $\theta \nearrow \theta_c$. This implies the result by Remark 5.1.

It remains to consider the case $\Theta_c = (0, \theta_c]$. Suppose that $\int_{\mathbb{R}^d} |x|^\theta \overline{\pi}(\mathrm{d}x) < \infty$ for some $\theta > \theta_c$. Recall the function $\chi_R(t)$ from Notation 5.1, and let $V_R(x) := \chi_R \circ V_{Q,\theta}(x)$. Since $V_R(x) - R - 2$ is compactly supported, we have $\overline{\pi}(\mathcal{A}^X V_R) = 0$. Thus,

$$\mathcal{A}^X V_R(x) = \mathcal{A}_\circ^X V_R(x) + \chi_R'(V_{Q,\theta}(x))\langle \check{b}(x), \nabla V_{Q,\theta}(x)\rangle,$$

and integrating this with respect to $\overline{\pi}(\mathrm{d}x)$, and using (5.46), we obtain

$$
\begin{aligned}
(5.48) \quad & C_1 \int_{\mathbb{R}^d} \chi_R'(V_{Q,\theta}(x))(C_0 + |x|^\theta)\overline{\pi}(\mathrm{d}x) + \int_{\mathbb{R}^d} [\mathcal{A}_\circ^X V_R]^-(x)\overline{\pi}(\mathrm{d}x) \\
& \geq \int_{\mathbb{R}^d} [\mathcal{A}_\circ^X V_R]^+(x)\overline{\pi}(\mathrm{d}x).
\end{aligned}
$$

It is important to note that since $a(x)$ satisfies (3.10), the estimate in (5.32) here takes the form $|\widetilde{F}_{\kappa, R}(x)| \leq \tilde{C}(1 + F^\theta(x))$, and thus $\overline{\pi}(\widetilde{F}_{\kappa, R}) \to 0$ as $R \to \infty$, by hypothesis. Since have a similar bound for $(\mathfrak{J}_\nu[V_R])^-$, and $(\mathfrak{J}_{1,\nu}[V_R])^-$, the left-hand side of (5.48) is bounded uniformly in R, whereas the right-hand side diverges as $R \to \infty$, since $\theta > \theta_c$. Thus we reach a contradiction. This completes the proof. $\square$

5.4. *Some results on general drifts.* In this section we discuss ergodic properties of the solution to (1.1) in the case when it is governed by a more general drift function. We assume that $b(x)$ is locally Lipschitz continuous, and there exists $\kappa_0 > 0$ such that $\langle x, b(x)\rangle \leq \kappa_0(1 + |x|^2)$ for all $x \in \mathbb{R}^d$. Then (1.1) again admits a unique nonexplosive strong solution $\{X(t)\}_{t\geq 0}$ which is a strong Markov process and it satisfies the C_b-Feller property (see [1], Theorem 3.1, and Propositions 4.2 and 4.3). Furthermore, its infinitesimal generator $(\mathcal{A}^X, \mathcal{D}_{\mathcal{A}^X})$ satisfies $C_c^2(\mathbb{R}^d) \subseteq \mathcal{D}_{\mathcal{A}^X}$, and $\mathcal{A}^X|_{C_c^2(\mathbb{R}^d)}$ takes the form in (1.2). Therefore, the corresponding extended domain contains the set $\mathcal{D}$ (defined in (2.1)), and on this set for $\bar{\mathcal{A}}^X f(x)$ we can take exactly $\mathcal{A}^X f(x)$. Irreducibility and aperiodicity of $\{X(t)\}_{t\geq 0}$ with this general drift can be established as in Theorem 3.1. The following corollary provides sufficient conditions on the drift function such that the process $\{X(t)\}_{t\geq 0}$ exhibits subexponential or exponential ergodicity properties analogous to Theorems 3.2 and 3.5. The proof is similar to the proofs of those two theorems.

COROLLARY 5.2. *Suppose that $\{X(t)\}_{t\geq 0}$ satisfies the assumptions of Theorem 3.1, and $\theta \in \Theta_c$.*

(i) *If $\theta \geq 1$, $a(x)$ satisfies (3.2), and there exists $Q \in \mathcal{M}_+$ such that*

$$\limsup_{|x|\to\infty} \frac{\langle b(x) + r + \int_{\mathcal{B}^c} y\nu(\mathrm{d}y), Qx\rangle}{|x|} < 0,$$

then the conclusion of Theorem 3.2(i) holds with rate $r(t) \approx t^{\theta_c - 1}$.

(ii) *If $\theta \in (0, 1)$,*

$$\limsup_{|x| \to \infty} \frac{\|a(x)\|}{|x|^{1+\theta}} = 0, \quad and \quad \limsup_{|x| \to \infty} \frac{\langle b(x), Qx \rangle}{|x|^{1+\theta}} < 0$$

for some $Q \in \mathcal{M}_+$, then (3.4) holds with rate $r(t) = t^{\frac{\theta_c + \theta - 1 - \epsilon}{1 - \theta}}$ for $\epsilon \in (0, \theta_c + \theta - 1)$.

(iii) *If $a(x)$ satisfies (3.10), and $\limsup_{|x| \to \infty} \frac{\langle b(x), Qx \rangle}{|x|^2} < 0$ for some $Q \in \mathcal{M}_+$, then there exist positive constants c_0, c_1, such that*

$$\mathcal{A}^X V_{Q,\theta}(x) \le c_0 - c_1 V_{Q,\theta}(x), \qquad x \in \mathbb{R}^d.$$

The process $\{X(t)\}_{t \ge 0}$ admits a unique invariant probability measure $\overline{\pi} \in \mathcal{P}(\mathbb{R}^d)$, and for any $\gamma \in (0, c_1)$,

$$\lim_{t \to \infty} e^{\gamma t} \|\pi P_t(dy) - \overline{\pi}(dy)\|_{\mathsf{TV}} = 0, \qquad \pi \in \mathcal{P}_\theta(\mathbb{R}^d).$$

(iv) *Suppose that $\sigma(x)$ is bounded, and there exist $\theta > 0$ and $Q \in \mathcal{M}_+$ such that (3.5) holds and*

$$\limsup_{|x| \to \infty} \frac{\langle b(x) + r + \int_{\mathcal{B}^c} y\nu(dy), Qx \rangle}{|x|} < 0.$$

Then the conclusion of Theorem 3.2(ii) follows.

PROOF.   In cases (i)–(iii), we use $V_{Q,\theta}(x)$, while in case (iv) we use $\tilde{V}_{Q,p}(x)$ with $0 < p < \theta \|Q\|^{-\frac{1}{2}}$. The assertions now follow from Lemmas 5.1 and 5.2.   □

We next present some general criteria for the convergence rate to be no better than polynomial. These extend Theorem 3.4.

COROLLARY 5.3.   *We assume that $\{X(t)\}_{t \ge 0}$ satisfies the assumptions of Theorem 3.1, $\theta_c \in [1, \infty)$, and the drift satisfies, for some constant $\gamma \in (0, 1)$, one of the following:*

(i) *There exists some $x_0 \in \mathbb{R}^d$ and a positive constant C, such that*

$$\langle x_0, b(x) \rangle \ge -C(1 + \langle x_0, x \rangle^\gamma), \qquad \langle x_0, x \rangle \ge 0.$$

(ii) *There exists a positive definite symmetric matrix Q and a positive constant C, such that*

$$\langle Qx, b(x) \rangle \ge -C(1 + |x|^{1+\gamma}), \qquad x \in \mathbb{R}^d.$$

In addition, suppose that there exists an inf-compact function $\bar{V} \in C^2(\mathbb{R}^d)$ having strict polynomial growth of order $|x|^\beta$ for some $\beta > \theta_c + \gamma - 1$, such that $\mathcal{A}^X \bar{V}$ is bounded from above in $\mathbb{R}^d$. Then, if the process is ergodic, we have the following

lower bounds. In the case of the α-stable process (*isotropic or not*), *there exists a positive constant $\tilde{C}_1$ such that for all $\epsilon \in (0, 1 - \gamma)$ we have*

$$(5.49) \qquad \tilde{C}_1 \left(\frac{t}{\epsilon} + \bar{V}(x) \right)^{\frac{1-\gamma-\alpha}{1-\gamma-\epsilon}} \leq \left\| \delta_x P_t^X(\mathrm{d}y) - \overline{\pi}(\mathrm{d}y) \right\|_{\mathsf{TV}}$$

for all $t > 0$, and all $x \in \mathbb{R}^d$.

In the case of a Lévy process in Theorem 3.4(ii) we obtain a lower bound of the same type as in (3.9). There exists a positive constant $\tilde{C}_3(\epsilon)$ such that for all $0 < \epsilon < \frac{1}{2}(\beta - \theta_c - \gamma + 1)$, and all $x \in \mathbb{R}^d$, we have

$$(5.50) \qquad \left\| \delta_x P_{t_n}^X(\mathrm{d}y) - \overline{\pi}(\mathrm{d}y) \right\|_{\mathsf{TV}} \geq \tilde{C}_3(\epsilon) \left(t_n + \bar{V}(x) \right)^{-\frac{\theta_c + \gamma - 1 + 2\epsilon}{\beta - (\theta_c + \gamma - 1 + 2\epsilon)}}$$

for some sequence $\{t_n\}_{n \in \mathbb{N}} \subset [0, \infty)$, $t_n \to \infty$, depending on x.

PROOF. We use a test function of the form $F^\delta(x)$, $0 < \delta < 1 - \gamma$, with $F(x) = \chi(\langle x_0, x \rangle)$ ($\chi(t)$ is as in Notation 5.1) for case (i), or $F(x) = V_{Q,\delta}(x)$, with $V_{Q,\delta}(x)$ as in Notation 3.1, for case (ii). We proceed with the technique in Lemma 5.7, and show that $F^{\kappa - 1 + \gamma}(x)$ cannot be integrable under $\overline{\pi}(\mathrm{d}x)$, unless $\kappa \in \Theta_c$. We continue by mimicking the proof of Theorem 3.4. Using $\bar{V}$, we have

$$\mathbb{E}^x \left[F^\beta \left(X(t) \right) \right] \leq \left(C_1 t + \bar{V}(x) \right) =: g(x, t).$$

Note that necessarily $\beta \in \Theta_c$.

We estimate the tail of $\overline{\pi}(\mathrm{d}x)$ using $F_{(1-\gamma), R}(x)$ (instead of $F_{1,R}(x)$) as

$$\overline{\pi}\left(\{ y : \langle x, y \rangle > t \} \right) \geq C_2 t^{1-\gamma-\alpha}.$$

With $G(x) = F^{\alpha - \epsilon}(x)$, $0 < \epsilon < 1 - \gamma$, we have

$$\begin{aligned}
\overline{\pi}\left(\{ y : G(y) \geq t \} \right) &= \overline{\pi}\left(\{ y : (\langle x, y \rangle)^{\alpha - \epsilon} > t \} \right) \\
&= \overline{\pi}\left(\{ y : \langle x, y \rangle > t^{\frac{1}{\alpha - \epsilon}} \} \right) \\
&\geq C_2 t^{\frac{1-\gamma-\alpha}{\alpha-\epsilon}} =: f(t),
\end{aligned}$$

and solving $y f(y) = 2g(x, t)$ we obtain $f(y) = C_3 (t + \bar{V}(x))^{\frac{1-\gamma-\alpha}{1-\gamma-\epsilon}}$, and thus we obtain (5.49).

In the case of the Lévy, we define

$$W(x) := F^{\theta_c + \gamma - 1 + \epsilon}(x), \qquad \widehat{F}(t) := t^{\frac{\beta}{\theta_c + \gamma - 1 + \epsilon}}, \qquad h(t) := t^{-1 - \epsilon_1},$$

with $\epsilon_1 = \frac{\epsilon}{\theta_c + \gamma - 1 + \epsilon}$, and proceed as in the proof of Theorem 3.4, to establish (5.50). This completes the proof. $\square$

REMARK 5.5. If we combine Corollary 5.2(ii) and Corollary 5.3(ii), in the case of an α-stable process (isotropic or not), we obtain the following. First, note that the hypothesis in Corollary 5.2(ii) allows us to use the Lyapunov function $V_{Q,\alpha-\epsilon}$ for any $\epsilon > 0$, so that the assumption $\beta > \alpha + \gamma - 1$ in Corollary 5.3 comes for free. Using this Lyapunov function, and applying [18], Theorem 3.2, for the upper bound, then in combination with (5.49), we obtain

$$\tilde{C}_1 \left(\frac{t}{\epsilon} + |x|^{\alpha-\epsilon} \right)^{\frac{1-\gamma-\alpha}{1-\gamma-\epsilon}} \leq \left\| \delta_x P_t^X(\cdot) - \overline{\pi}(\cdot) \right\|_{\mathsf{TV}} \leq \tilde{C}_2(\epsilon)(t \vee 1)^{\frac{1+\epsilon-\alpha-\theta}{1-\theta}} |x|^{\alpha-\epsilon}.$$

Note that necessarily $\gamma \geq \theta$ by hypothesis.

APPENDIX: PROOF OF THEOREM 3.1

In this section we prove Theorem 3.1. The assertion for case (i) is shown in [36], proof of Proposition 3.1 (see also [33], Theorem 3.1). We prove the assertions in cases (ii), (iii) and (iv).

PROOF OF CASE (ii). First, observe that $P_t^X(x, O) > 0$ for any $t > 0$, $x \in \mathbb{R}^d$ and open set $O \subseteq \mathbb{R}^d$. Indeed, fix $0 < \rho \leq \frac{1}{4}$ and $0 < \varepsilon < \rho$, and let $x_0, y_0 \in \mathbb{R}^d$ be such that $|x_0 - y_0| = 2\rho$. Let $f \in C_c^2(\mathbb{R}^d)$ be such that $0 \leq f(x) \leq 1$, supp $f \subset \mathcal{B}_{\rho-\frac{\varepsilon}{2}}(y_0)$, and $f|_{\bar{\mathcal{B}}_{\rho-\varepsilon}(y_0)} = 1$. Recall that $\{X(t)\}_{t\geq0}$ is a C_b-Feller process with generator $(\mathcal{A}^X, \mathcal{D}_{\mathcal{A}^X})$ given in (1.2). Now, since $\lim_{t\to0} \| \frac{P_t^X f - f}{t} - \mathcal{A}^X f \|_\infty = 0$, we conclude that

$$\liminf_{t\searrow0} \inf_{x \in \mathcal{B}_{\rho-\frac{\varepsilon}{2}}(x_0)} \frac{P_t^X(x, \mathcal{B}_{\rho-\frac{\varepsilon}{2}}(y_0))}{t}$$

$$\geq \liminf_{t\searrow0} \inf_{x \in \mathcal{B}_{\rho-\frac{\varepsilon}{2}}(x_0)} \frac{P_t^X f(x)}{t}$$

$$= \liminf_{t\searrow0} \inf_{x \in \mathcal{B}_{\rho-\frac{\varepsilon}{2}}(x_0)} \left| \frac{P_t^X f(x)}{t} - \mathcal{A}^X f(x) + \mathcal{A}^X f(x) \right|$$

$$= \inf_{x \in \mathcal{B}_{\rho-\frac{\varepsilon}{2}}(x_0)} \left| \mathcal{A}^X f(x) \right|$$

$$\geq \inf_{x \in \mathcal{B}_{\rho-\frac{\varepsilon}{2}}(x_0)} \int_{\mathbb{R}_*^d} f(y+x)\nu(dy)$$

$$\geq \inf_{x \in \mathcal{B}_{\rho-\frac{\varepsilon}{2}}(x_0)} \nu\big(\mathcal{B}_{\rho-\varepsilon}(y_0-x)\big).$$

Observe that

$$\bigcup_{x\in\mathcal{B}_{\rho-\frac{\varepsilon}{2}}(x_0)} \mathcal{B}_{\rho-\varepsilon}(y_0-x) \subseteq \mathcal{B}\setminus\mathcal{B}_\varepsilon(0).$$

We claim that

$$\inf_{x\in\mathcal{B}_{\rho-\frac{\varepsilon}{2}}(x_0)} \nu\big(\mathcal{B}_{\rho-\varepsilon}(y_0-x)\big) > 0.$$

Suppose not. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}_{\rho-\frac{\varepsilon}{2}}(x_0)$ converging to some $x_\infty \in \bar{\mathcal{B}}_{\rho-\frac{\varepsilon}{2}}(x_0)$, such that $\lim_{n\to\infty} \nu(\mathcal{B}_{\rho-\varepsilon}(y_0-x_n)) = 0$. On the other hand, by the dominated convergence theorem (observe that $\bigcup_{n\in\mathbb{N}} \mathcal{B}_{\rho-\varepsilon}(y_0-x_n) \subseteq \mathcal{B}\setminus\mathcal{B}_\varepsilon(0)$ and $\nu(\mathcal{B}\setminus\mathcal{B}_\varepsilon(0)) < \infty$), we have that

$$\lim_{n\to\infty} \nu\big(\mathcal{B}_{\rho-\varepsilon}(y_0-x_n)\big) = \nu\big(\mathcal{B}_{\rho-\varepsilon}(y_0-x_\infty)\big),$$

which is strictly positive by hypothesis. Thus, we conclude that there exists $t_0 > 0$ such that $P_t^X(x, \mathcal{B}_{\rho-\varepsilon}(y_0)) > 0$ for all $t \in (0, t_0]$, and all $x \in \mathcal{B}_{\rho-\varepsilon}(x_0)$. The claim now follows by employing the Chapman–Kolmogorov equality.

Next, define $\nu_1(dy) := \nu(dy \cap \mathcal{B})$, and $\nu_2(dy) := \nu(dy \cap \mathcal{B}^c)$. Let $\{W(t)\}_{t\geq 0}$, $\{L_1(t)\}_{t\geq 0}$, and $\{L_2(t)\}_{t\geq 0}$ be a mutually independent standard Brownian motion, a Lévy process with drift ϑ and Lévy measure $\nu_1(dy)$ and a Lévy process with zero drift and Lévy measure $\nu_2(dy)$, respectively. Observe that $\{L(t)\}_{t\geq 0}$ and $\{L_1(t) + L_2(t)\}_{t\geq 0}$ have the same (finite-dimensional) distribution. Now define

$$d\bar{X}(t) := b\big(\bar{X}(t)\big)\,dt + \sigma\big(\bar{X}(t)\big)\,dW(t) + dL_1(t) + dL_2(t), \qquad \bar{X}(0) = x \in \mathbb{R}^d,$$

and

$$d\hat{X}(t) := b\big(\hat{X}(t)\big)\,dt + \sigma\big(\hat{X}(t)\big)\,dW(t) + dL_1(t), \qquad \hat{X}(0) = x \in \mathbb{R}^d.$$

It is clear that the processes $\{X(t)\}_{t\geq 0}$ and $\{\bar{X}(t)\}_{t\geq 0}$ have the same (finite-dimensional) distribution, and by the same reasoning as above, $P_t^{\bar{X}}(x, O) > 0$ and $P_t^{\hat{X}}(x, O) > 0$ for any $t > 0$, $x \in \mathbb{R}^d$ and open set $O \subseteq \mathbb{R}^d$. Next, define

$$\tau := \inf\big\{t \geq 0 \colon |\bar{X}(t) - \bar{X}(t-)| \geq 1\big\} = \inf\big\{t \geq 0 \colon |L_2(t) - L_2(t-)| \neq 0\big\}.$$

Now, by construction, we conclude that $\mathbb{P}^x(\tau > t) = e^{-\nu_2(\mathbb{R}^d)t} = e^{-\nu(\mathcal{B}^c)t}$, and $\{\bar{X}(t)\}_{t\geq 0}$ and $\{\hat{X}(t)\}_{t\geq 0}$ coincide on $[0, \tau)$. Consequently, for any $t > 0$, $x \in \mathbb{R}^d$ and $B \in \mathfrak{B}(\mathbb{R}^d)$, we have

$$\begin{aligned}
P_t^{\bar{X}}(x, B) &\geq \mathbb{P}^x\big(\bar{X}(t) \in B, \tau > t\big) \\
&= \mathbb{P}^x\big(\hat{X}(t) \in B, \tau > t\big) \\
&= \mathbb{E}^x\big[\mathbb{E}^x\big[\mathbb{1}_{\{\hat{X}(t)\in B\}}\mathbb{1}_{\{\tau>t\}} \,\big|\, \sigma\{L_2(t), t \geq 0\}\big]\big] \\
&= P_t^{\hat{X}}(x, B)\mathbb{P}^x(\tau > t).
\end{aligned}$$

Thus, according to [55], Theorem 3.2, in order to conclude open-set irreducibility and aperiodicity of $\{X(t)\}_{t\geq 0}$, it suffices to prove that $\{\hat{X}(t)\}_{t\geq 0}$ is a strong Feller process. Further, by [43], Lemma 2.2, for $\{\hat{X}(t)\}_{t\geq 0}$ to have he strong Feller property it is sufficient that that for any $t > 0$ there exists $c(t) > 0$ such that

$$(A.1) \qquad \left| P_t^{\hat{X}} f(x) - P_t^{\hat{X}} f(y) \right| \leq c(t, \kappa, \delta) \|f\|_\infty |x - y|$$

for all $f \in C_b^2(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$. This is what we show in the rest of the proof.

Let $\chi \in C_c^\infty(\mathbb{R}^d)$ satisfying $0 \leq \chi(x) \leq 1$ for all $x \in \mathbb{R}^d$, supp $\chi \subseteq \bar{\mathcal{B}}$ and $\int_{\mathbb{R}^d} \chi(x)\,\mathrm{d}x = 1$. For $\varepsilon > 0$, define $\chi_\varepsilon(x) := \varepsilon^{-d}\chi(\frac{x}{\varepsilon})$, $x \in \mathbb{R}^d$. By definition, $\chi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$, supp $\chi_\varepsilon \subseteq \bar{\mathcal{B}}_\varepsilon(0)$ and $\int_{\mathbb{R}^d} \chi_\varepsilon(x)\,\mathrm{d}x = 1$. The Friedrich's mollifiers $b_n(x)$ ad $\sigma_n(x)$ of $b(x)$ and $\sigma(x)$, respectively, are defined as

$$b_n(x) := n^d \int_{\mathbb{R}^d} \chi_{\frac{1}{n}}(x - y)b(y)\,\mathrm{d}y = \int_{\bar{B}_1(0)} \chi(y)b\left(x - \frac{y}{n}\right)\mathrm{d}y,$$

and analogously for σ_n. Let $\kappa > 0$ be larger than the Lipschitz constants of $b(x)$ and $\sigma(x)$. Since $b_n \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and $\sigma_n \in C^\infty(\mathbb{R}^d, \mathbb{R}^{d\times d})$, we have

$$\left| b_n^i(x) - b^i(x) \right| \leq \frac{1}{n}, \qquad \left| \sigma_n^{ij}(x) - \sigma^{ij} \right| \leq \frac{1}{n},$$

$$\left| \partial_i b_n^j(x) \right| \leq \kappa, \qquad \left| \partial_i \sigma_n^{jk}(x) \right| \leq \kappa, \qquad \left| \partial_{ij} b_n^k(x) \right| \leq \kappa_n,$$

and $\left| \partial_{ij} \sigma_n^{kl}(x) \right| \leq \kappa_n$, with $i, j, k, l \in \{1, \ldots, d\}$, $b_n(x) = (b_n^i(x))_{i=1,\ldots,d}$, and $\sigma_n(x) = (\sigma_n^{ij}(x))_{i,j=1,\ldots,d}$. Now define

$$\mathrm{d}\hat{X}_n(t) = b_n(\hat{X}(t))\,\mathrm{d}t + \sigma_n(\hat{X}_n(t))\,\mathrm{d}W(t) + \mathrm{d}L_1(t), \qquad \hat{X}_n(0) = x \in \mathbb{R}^d.$$

In [33], Lemma 2.3, it has been shown that for each fixed $t > 0$ there is a constant $c(t, \kappa, \delta) > 0$ such that

$$(A.2) \qquad \left| P_t^{\hat{X}_n} f(x) - P_t^{\hat{X}_n} f(y) \right| \leq c(t, \kappa, \delta) \|f\|_\infty |x - y|$$

for all $f \in C_b^2(\mathbb{R}^d)$, $x, y \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Recall that $\delta = \sup_{x \in \mathbb{R}^d} \|\sigma^{-1}(x)\| > 0$. As we have already commented, this automatically implies strong Feller property of $\{\hat{X}_n(t)\}_{t\geq 0}$. For any $t > 0$ and $x \in \mathbb{R}^d$, by employing Itô's formula, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}^x |\hat{X}_n(t) - \hat{X}(t)|^2$$

$$= \mathbb{E}^x \|\sigma_n(\hat{X}_n(t)) - \sigma(\hat{X}(t))\|^2 + 2\mathbb{E}^x \langle b_n(\hat{X}_n(t)) - b(\hat{X}(t)), \hat{X}_n(t) - \hat{X}(t)\rangle$$

$$\leq 2\mathbb{E}^x \|\sigma_n(\hat{X}_n(t)) - \sigma_n(\hat{X}(t))\|^2 + 2\mathbb{E}^x \|\sigma_n(\hat{X}(t)) - \sigma(\hat{X}(t))\|^2$$

$$\quad + 2\mathbb{E}^x \langle b_n(\hat{X}_n(t)) - b_n(\hat{X}(t)), \hat{X}_n(t) - \hat{X}(t)\rangle$$

$$\quad + 2\mathbb{E}^x \langle b_n(\hat{X}(t)) - b(\hat{X}(t)), \hat{X}_n(t) - \hat{X}(t)\rangle$$

$$\leq 2\kappa^2 \mathbb{E}^x |\hat{X}_n(t) - \hat{X}(t)|^2 + 2\mathbb{E}^x \|\sigma_n(\hat{X}(t)) - \sigma(\hat{X}(t))\|^2$$

$$+ 2\kappa \mathbb{E}^x |\hat{X}_n(t) - \hat{X}(t)|^2 + 2\mathbb{E}^x |b_n(\hat{X}(t)) - b(\hat{X}(t))|^2$$

$$\leq 2(\kappa + \kappa^2)\mathbb{E}^x |\hat{X}_n(t) - \hat{X}(t)|^2 + \frac{4}{n^2}.$$

By Gronwall's lemma, we obtain

$$\mathbb{E}^x |\hat{X}_n(t) - \hat{X}(t)|^2 \leq \frac{4}{n^2} t \mathrm{e}^{2(\kappa + \kappa^2)t}.$$

Hence, for each fixed $t > 0$ and $x \in \mathbb{R}^d$, $\hat{X}_n(t)$ converges to $\hat{X}(t)$ in $L^2(\Omega, \mathbb{P}^x)$. Now, for fixed $t > 0$ and $x, y \in \mathbb{R}^d$, using (A.2), we have

$$\left| P_t^{\hat{X}} f(x) - P_t^{\hat{X}} f(y) \right| \leq \left| P_t^{\hat{X}} f(x) - P_t^{\hat{X}_n} f(x) \right| + \left| P_t^{\hat{X}_n} f(x) - P_t^{\hat{X}_n} f(y) \right|$$

$$+ \left| P_t^{\hat{X}_n} f(y) - P_t^{\hat{X}} f(y) \right|$$

$$\leq \left| P_t^{\hat{X}} f(x) - P_t^{\hat{X}_n} f(x) \right| + \left| P_t^{\hat{X}_n} f(y) - P_t^{\hat{X}} f(y) \right|$$

$$+ c(t, \kappa, \delta) \|f\|_\infty |x - y|.$$

By letting $n \to \infty$, (A.1) follows and the proof is complete. $\square$

We next prove the assertions in Case (iii).

PROOF OF CASE (iii). By [55], Theorem 3.2, in order to prove open-set irreducibility and aperiodicity, it suffices to show that $\{X(t)\}_{t \geq 0}$ satisfies the strong Feller property and $P_t^X(x, O) > 0$ for all $t > 0$, all $x \in \mathbb{R}^d$, and all open sets $O \subseteq \mathbb{R}^d$. The strong Feller property of $\{X(t)\}_{t \geq 0}$ follows from [58], Theorem 2.1, and [56], Proposition 2.3. Recall that $\{X(t)\}_{t \geq 0}$ is a C_b-Feller process with generator $(\mathcal{A}^X, \mathcal{D}_{\mathcal{A}^X})$ given in (1.2). Now let $O \subseteq \mathbb{R}^d$ be an arbitrary open set and let $f \in C_c^2(\mathbb{R}^d)$ be such that $\mathrm{supp}\, f \subset O$ and $0 \leq f(x) \leq 1$. Since

$$\lim_{t \to 0} \left\| \frac{P_t^X f - f}{t} - \mathcal{A}^X f \right\|_\infty = 0,$$

we conclude that for any bounded set $B \subseteq O^c$,

$$\liminf_{t \searrow 0} \inf_{x \in B} \frac{P_t^X(x, O)}{t} \geq \liminf_{t \searrow 0} \inf_{x \in B} \frac{P_t^X f(x)}{t}$$

$$= \liminf_{t \searrow 0} \inf_{x \in B} \left| \frac{P_t^X f(x)}{t} - \mathcal{A}^X f(x) + \mathcal{A}^X f(x) \right|$$

$$= \inf_{x \in B} |\mathcal{A}^X f(x)| \geq \inf_{x \in B} \int_{\mathbb{R}_*^d} f(y + x) \nu(\mathrm{d}y).$$

Now, since $\{L(t)\}_{t \geq 0}$ has a subordinate Brownian motion component, we conclude that $\nu(\mathrm{d}y)$ has full support (see [50], Theorem 30.1). This automatically implies that the right-hand side in the above relation is strictly positive. Namely, if this was not the case then there would exist a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq B$ converging to some $x_\infty \in \mathbb{R}^d$ (recall that B is bounded), such that

$$\lim_{n \to \infty} \int_{\mathbb{R}_*^d} f(y + x_n) \nu(\mathrm{d}y) = 0.$$

Now by employing Fatou's lemma we conclude that $\int_{\mathbb{R}_*^d} f(y + x_\infty) \nu(\mathrm{d}y) = 0$, which is impossible. Hence, there exists $t_0 > 0$ such that $P_t^X(x, O) > 0$ for all $t \in (0, t_0]$, and all $x \in B$. The assertion now follows by employing the Chapman–Kolmogorov equality. $\square$

Lastly, we prove the assertions in Case (iv).

PROOF OF CASE (iv). Let us first show that the solution to

$$\mathrm{d}\hat{X}(t) = b(\hat{X}(t)) \, \mathrm{d}t + \mathrm{d}L_1(t), \qquad \hat{X}(0) = x \in \mathbb{R}^d,$$

is a strong Feller process. Clearly, $\{L_1(t)\}_{t \geq 0}$ admits a transition density function

$$P_t^{L_1}(x, \mathrm{d}y) = p^t(x, y) \, \mathrm{d}y = p^t(y - x) \, \mathrm{d}y, \qquad x, y \in \mathbb{R}^d, t > 0,$$

satisfying

$$p^t(x) = p_1^t(x_1) \cdots p_d^t(x_d), \qquad x = (x_1, \dots, x_d) \in \mathbb{R}^d, t > 0,$$

since the corresponding components are independent, where $\{p_i^t(u)\}_{u \in \mathbb{R}, t > 0}$ is a transition density of $\{A_1^i(t)\}_{t \geq 0}$, $i = 1, \dots, d$ (a one-dimensional symmetric α-stable Lévy process with scale parameter $\eta_i > 0$). Observe that $\{L_1^i(t)\}_{t \geq 0}$ is a subordinate Brownian motion with $\frac{\alpha}{2}$-stable subordinator $\{S_i(t)\}_{t \geq 0}$ with scale parameter $\eta_i > 0$, $i = 1, \dots, d$. According to [32], Corollary 3.5 and Example 4.4, $P_t^{L_1^i} f \in C_b^1(\mathbb{R})$,

$$\left| \frac{\partial}{\partial u} P_t^{L_1^i} f(u) \right| \leq c_1 t^{-\frac{\alpha}{4}} \|f\|_\infty \mathbb{E}\big[S_i(1)^{-\frac{1}{2}}\big]$$

and

$$|u| \left| \frac{\partial}{\partial u} P_t^{L_1^i} f(u) \right| \leq c_2 t^{-\frac{\alpha}{4}} \|f\|_\infty \mathbb{E}\big[S_i(1)^{-\frac{1}{2}}\big]$$

for all $u \in \mathbb{R}$, $t > 0$ and $f \in \mathcal{B}(\mathbb{R})$ with compact support. Here, the constant $c_1 > 0$ does not depend on u, t and $f(u)$, and $c_2 > 0$ depends on supp f only. According to [32], Proposition 3.11, we have $\mathbb{E}[S_i(1)^{-\frac{1}{2}}] < \infty$. Further, using the scaling

property and asymptotic behavior (at infinity) of one-dimensional symmetric stable densities (see [50], page 87), we deduce that for any fixed $t_0 > 0$, there exist positive constants c_3 and c_4, which depend only on t_0 and supp f, such that

$$|u||P_t^{L_1^i} f(u)| \leq t^{-\frac{1}{\alpha}} |u| \|f\|_\infty \int_{\text{supp } f} p_i^1 \left(t^{-\frac{1}{\alpha}}(v-u)\right) dv$$

$$\leq c_3 \|f\|_\infty |u|^{-\alpha} t \mathbb{1}_{\{|u| t_0^{-\frac{1}{\alpha}} \geq c_3\}}(u) + c_4 \|f\|_\infty \mathbb{1}_{\{|u| t_0^{-\frac{1}{\alpha}} \leq c_3\}}(u)$$

for all $u \in \mathbb{R}$, and $t \in (0, t_0]$.

According to [32], Lemma 2.4, a C_b-Feller semigroup $\{P_t\}_{t \geq 0}$ enjoys the strong Feller property if, and only if, $P_t f \in C_b(\mathbb{R}^d)$ for any $t > 0$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$ with compact support. Finally, fix $t > 0$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$ with compact support. By Duhamel's formula, we have

$$P_t^{\hat{X}} f(x) = P_t^{L_1} f(x) + \int_0^t P_{t-s}^{\hat{X}} \langle b(x), \nabla P_s^{L_1} f(x) \rangle ds.$$

Hence, it remains to prove that the second term in the above relation is continuous. We have

$$\langle b(x), \nabla P_s^{L_1} f(x) \rangle = \langle b(x)(1 + |x|)^{-1}, (1 + |x|) \nabla P_s^{L_1} f(x) \rangle$$

$$\leq (1 + |x|)^{-1} |b(x)| + (1 + |x|) |\nabla P_s^{L_1} f(x)|.$$

Clearly, $(1 + |x|)^{-1} |b(x)| < c$ for some $c > 0$. Since

$$\partial_{x_i} P_s^{L_1} f(x) = \partial_{x_i} \int_{\mathbb{R}^d} f(y_1, \ldots, y_d) \prod_{i=1}^d p_i^s(y_i - x_i) dy_i$$

$$= \int_{\mathbb{R}} \partial p_i^s(y_i - x_i) dy_i \int_{\mathbb{R}^{d-1}} f(y_1, \ldots, y_d) \prod_{j \neq i} p_j^s(y_j - x_j) dy_j,$$

the map

$$y_i \mapsto \int_{\mathbb{R}^{d-1}} f(y_1, \ldots, y_d) \prod_{j \neq i} p_j^s(y_j - x_j) dy_j$$

is bounded and has compact support. We use the estimates

$$|x_i| |\partial_{x_j} P_s^{L_1} f(x)| \left(\mathbb{E}[S_j(1)^{-\frac{1}{2}}]\right)^{-1} \leq \bar{c}_2 \|f\|_\infty s^{-\frac{\alpha}{4}} \qquad \text{if } i = j,$$

and

$$|x_i| |\partial_{x_j} P_s^{L_1} f(x)| \left(\mathbb{E}[S_j(1)^{-\frac{1}{2}}]\right)^{-1} \leq \bar{c}_3 \|f\|_\infty^2 |x_i|^{-\alpha} s^{1-\frac{\alpha}{4}} \mathbb{1}_{\{|x_i| t^{-\frac{1}{\alpha}} \geq \bar{c}_3\}}(x_i)$$

$$+ \bar{c}_4 \|f\|_\infty^2 s^{-\frac{\alpha}{4}} \mathbb{1}_{\{|x_i| t^{-\frac{1}{\alpha}} \leq \bar{c}_3\}}(x_i)$$

$$= (\bar{c}_3^{1-\alpha} + \bar{c}_4) \|f\|_\infty^2 s^{-\frac{\alpha}{4}} \qquad \text{if } i \neq j,$$

to obtain

(A.3) $\langle b(x), \nabla P_s^{L_1} f(x) \rangle \leq 2c + M(f) s^{-\frac{\alpha}{4}} \max_{i=1,\ldots,d} \mathbb{E}\big[S_i(1)^{-\frac{1}{2}}\big],$

with

$$M(f) := \bar{c}_1 \|f\|_\infty + \|f\|_\infty \sqrt{d\bar{c}_2^2 + d(d-1)(\bar{c}_3^{1-\alpha} + \bar{c}_4)^2 \|f\|_\infty^2}.$$

In (A.3), the constants $\bar{c}_2$, $\bar{c}_3$ and $\bar{c}_4$ depend on t and supp f only. Therefore, the map $x \mapsto \langle b(x), \nabla P_s^{L_1} f(x) \rangle$ is continuous and bounded for any $s \in (0, t]$. In particular, due to the C_b-Feller property, the map $x \mapsto P_{t-s}^{\hat{X}} \langle b(x), \nabla P_s^{L_1} f(x) \rangle$ is also continuous and bounded for any $s \in (0, t]$. Finally, according to the dominated convergence theorem and (A.3) we conclude that

$$x \mapsto \int_0^t P_{t-s}^{\hat{X}} \langle b(x), \nabla P_s^{L_1} f(x) \rangle \, \mathrm{d}s$$

is continuous, which concludes the proof.

Now let us show that $\{\hat{X}(t)\}_{t \geq 0}$ and the solution to

$$\mathrm{d}\bar{X}(t) = b\big(\bar{X}(t)\big) \, \mathrm{d}t + \mathrm{d}L_1(t) + \mathrm{d}L_2(t), \qquad \bar{X}(0) = x \in \mathbb{R}^d,$$

are irreducible and aperiodic. The one-dimensional case is covered by (iii). Assume $d \geq 2$. Define $\tau := \inf\{t \geq 0 \colon |L_2(t) - L_2(t-)| \neq 0\}$. By construction, we conclude that $\mathbb{P}^x(\tau > t) = \mathrm{e}^{-\nu_{L_2}(\mathbb{R}^d)t}$, and $\{\hat{X}(t)\}_{t \geq 0}$ and $\{\bar{X}(t)\}_{t \geq 0}$ coincide on $[0, \tau)$. Consequently,

$$\begin{aligned}
P_t^{\bar{X}}(x, B) &\geq \mathbb{P}^x\big(\bar{X}(t) \in B, \tau > t\big) \\
&= \mathbb{P}^x\big(\hat{X}(t) \in B, \tau > t\big) \\
&= \mathbb{E}^x\big[\mathbb{E}^x\big[\mathbb{1}_{\{\hat{X}(t) \in B\}} \mathbb{1}_{\{\tau > t\}} \mid \sigma\{L_2(t), t \geq 0\}\big]\big] \\
&= P_t^{\hat{X}}(x, B) \mathbb{P}^x(\tau > t)
\end{aligned}$$

for any $t > 0$, $x \in \mathbb{R}^d$ and $B \in \mathfrak{B}(\mathbb{R}^d)$. Thus, due to the previous observation, the fact that $\{\hat{X}(t)\}_{t \geq 0}$ is a strong Feller process and [55], Theorem 3.2, it suffices to show that $P_t^{\hat{X}}(x, B) > 0$ for any $t > 0$, $x \in \mathbb{R}^d$ and open set $O \subseteq \mathbb{R}^d$ containing 0. First, assume that $x_0, y_0 \in \mathbb{R}^d$ lie on the same coordinate axis. Fix $\rho > 0$ and $\varepsilon > 0$, and let $f \in C_c^2(\mathbb{R}^d)$ be such that $0 \leq f(x) \leq 1$, supp $f \subset \mathcal{B}_{\rho+\varepsilon}(y_0)$ and $f|_{\bar{\mathcal{B}}_{\rho+\frac{\varepsilon}{2}}(y_0)} = 1$. Now, since $\lim_{t \to 0} \|\frac{P_t^{\hat{X}} f - f}{t} - \mathcal{A}^{\hat{X}} f\|_\infty = 0$ $((\mathcal{A}^{\hat{X}}, \mathcal{D}_{\mathcal{A}^{\hat{x}}})$ is the generator of $\{\hat{X}(t)\}_{t \geq 0}$ given in (1.2)), we conclude that

$$\liminf_{t \searrow 0} \inf_{x \in \mathcal{B}_\rho(x_0)} \frac{P_t^{\hat{X}}(x, \mathcal{B}_{\rho+\varepsilon}(y_0))}{t}$$

$$\geq \liminf_{t \searrow 0} \inf_{x \in \mathcal{B}_\rho(x_0)} \frac{P_t^{\hat{X}} f(x)}{t}$$

$$= \liminf_{t \searrow 0} \inf_{x \in \mathcal{B}_\rho(x_0)} \left| \frac{P_t^{\hat{X}} f(x)}{t} - \mathcal{A}^{\hat{X}} f(x) + \mathcal{A}^{\hat{X}} f(x) \right|$$

$$= \inf_{x \in \mathcal{B}_\rho(x_0)} \left| \mathcal{A}^{\hat{X}} f(x) \right|$$

$$\geq \inf_{x \in \mathcal{B}_\rho(x_0)} \int_{\mathbb{R}_*^d} f(y + x) \nu(dy)$$

$$\geq \inf_{x \in \mathcal{B}_\rho(x_0)} \nu\big(\mathcal{B}_{\rho + \frac{\varepsilon}{2}}(y_0 - x)\big),$$

which is strictly positive. Hence, there is $t_0 > 0$ such that $P_t^{\hat{X}}(x, \mathcal{B}_{\rho+\varepsilon}(y_0)) > 0$ for all $t \in (0, t_0]$ and $x \in \mathcal{B}_\rho(x_0)$. Further, let $x \in \mathbb{R}^d$ be such that it does not lie on any coordinate axis, and let $\rho > 0$. Define $r_1 := |\langle x, e_1 \rangle| + \varepsilon_1$, where $\varepsilon_1 > 0$ is such that $r_1 < |x|$. Then, as above, we conclude that there is $t_1 > 0$ such that $P_t^{\hat{X}}(x, \mathcal{B}_{r_1}(0)) > 0$ for all $t \in (0, t_1]$. Next, inductively, define $r_n := \frac{r_{n-1}}{\sqrt{d}} + \varepsilon_n$, $n \geq 2$, where $\varepsilon_n > 0$ is such that $\varepsilon_n < r_{n-1}\sqrt{d - \frac{1}{2}}$. Clearly, $r_n \to 0$ as $n \to \infty$, and there is $t_n > 0$ such that $P_t^{\hat{X}}(x, \mathcal{B}_{r_n}(0)) > 0$ for all $t \in (0, t_n]$ and $x \in \mathcal{B}_{r_{n-1}}(0)$. The claim now follows by employing the Chapman–Kolmogorov equality. $\square$

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