Uniform polynomial rates of convergence for a class of Lévy–driven controlled SDEs arising in multiclass many-server queues

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Abstract We study the ergodic properties of a class of controlled stochastic differential equations (SDEs) driven by α-stable processes which arise as the limiting equations of multiclass queueing models in the Halfin–Whitt regime that have heavy–tailed arrival processes. When the safety staffing parameter is positive, we show that the SDEs are uniformly ergodic and enjoy a polynomial rate of convergence to the invariant probability measure in total variation, which is uniform over all stationary Markov controls resulting in a locally Lipschitz continuous drift. We also derive a matching lower bound on the rate of convergence (under no abandonment). On the other hand, when all abandonment rates are positive, we show that the SDEs are exponentially ergodic uniformly over the above-mentioned class of controls. Analogous results are obtained for Lévy–driven SDEs arising from multiclass many-server queues under asymptotically negligible service interruptions. For these equations, we show that the aforementioned ergodic properties are uniform over all stationary Markov controls. We also extend a key functional central limit theorem concerning diffusion approximations so as to make it applicable to the models studied here.

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1 Introduction

Lévy–driven controlled stochastic differential equations (SDEs) arise as scaling limits for multiclass many-server queues with heavy-tailed arrival processes and/or with asymptotically negligible service interruptions; see [4,12,13]. In these equations, the control appears only in the drift and corresponds to a work-conserving scheduling policy in multiclass many-server queues, that is, the allocation of the available service capacity to each class under a non-idling condition (no server idles whenever there are jobs in queue). For the limiting process, we focus on stationary Markov controls, namely time-homogeneous functions of the process. When the arrival process of each class is heavy-tailed (for example, with regularly varying interarrival times), the Lévy process driving the SDE is a multidimensional anisotropic α-stable process, $\alpha \in (1,2)$. When the system is subject to service interruptions (in an alternating renewal environment affecting the service processes only), the Lévy process is a combination of either a Brownian motion, or an anisotropic α-stable process, $\alpha \in (1,2)$, and an independent compound Poisson process.

Ergodic properties of these controlled SDEs are of great interest since they help to understand the performance of the queueing systems. In [4], the ergodic properties of the SDEs under constant controls are thoroughly studied. It is shown that when the safety staffing is positive, the SDEs have a polynomial rate of convergence to stationarity in total variation, while when the abandonment rates are positive, the rate of convergence is exponential. However, the technique developed in [4] does not equip us to investigate the ergodic properties of these SDEs beyond the constant controls, since the Lyapunov functions employed are modifications of the common quadratic functions that have been developed for piecewise linear diffusions [5].

It was recently shown in [7] that the Markovian multiclass many–server queues with positive safety staffing in the Halfin–Whitt regime are stable under any work-conserving scheduling policies. Motivated by this significant result, Arapostathis et al. (2018) [3] have developed a unified approach via a Lyapunov function method which establishes Foster-Lyapunov equations which are uniform under stationary Markov controls for the limiting diffusion and the prelimit diffusion-scaled queueing processes simultaneously. It is shown that the limiting diffusion is uniformly exponentially ergodic under any stationary Markov control.

In this paper we adopt and extend the approach in [3] to establish uniform ergodic properties for Lévy-driven SDEs. As done in [4], we distinguish two cases: (i) positive safety staffing, and (ii) positive abandonment rates. We focus primarily on the first case, which exhibits ergodicity at a polynomial rate, a result which is somewhat surprising. The second case always results in uniform exponential ergodicity. By employing a polynomial Lyapunov function instead of the exponential function used in [3], we first establish an upper bound on the rate of convergence which is polynomial. The drift inequalities carry over with slight modifications from [3], while the needed properties of the non-local part of the generator are borrowed from [2]. As in [4], we use the technique in [9] to establish a lower bound on the rate of convergence, which actually matches the upper bound. As a result, we establish that with positive safety staffing, the rate of convergence to stationarity in total variation is polynomial with a rate that is uniform over the family of Markov controls which result in a locally Lipschitz continuous drift.

When the SDE is driven by an α–stable process (isotropic or anisotropic), in order for the process to be open–set irreducible and aperiodic, it suffices to require that the controls are stationary Markov and the drift is locally Lipschitz continuous. However, the existing proof of the convergence of the scaled queueing processes of the multiclass many–server queues with heavy–tailed arrivals to this limit process, assumes that the drift is Lipschitz continuous [13]. In this paper, we extend this result on the continuity of the integral mapping (Theorem 1.1 in [13]) to drifts that are locally Lipschitz continuous with at most linear growth (see Lemma 4). Applying this, we also present an extended functional central limit theorem (FCLT) for multiclass many-server queues with heavy-tailed arrival processes (see Theorem 6).

On the other hand, when the Lévy process consists of a Brownian motion and a compound Poisson process, which arises in the multiclass many–server queues with asymptotically negligible interruptions under the $\sqrt{n}$ scaling, the SDE has a unique strong solution that is open–set irreducible and aperiodic under any stationary Markov control. To study uniform ergodic properties, we also need to account for the second order derivatives in the infinitesimal generator. For this reason we modify the Lyapunov function with suitable titling on the positive and negative half state spaces. We also discuss the model with a Lévy process consisting of a α-stable process and a compound Poisson process.

1.1 Organization of the paper

In Section 2, we present a class of SDEs driven by an α–stable process, whose ergodic properties are studied in Section 3. In Section 4, we study the ergodic properties of Lévy–driven SDEs arising from the multiclass queueing models with service interruptions. In Section 5, we provide a description of the multiclass many–server queues with heavy-tailed arrival processes, and establish the continuity of the integral mapping with a locally Lipschitz continuous function that has at most linear growth, as well as the associated FCLT.

1.2 Notation

We summarize some notation used throughout the paper. We use $\mathbb{R}^m$ (and $\mathbb{R}^m_+$), $m \geq 1$, to denote real-valued m-dimensional (nonnegative) vectors, and write $\mathbb{R}$ for $m = 1$. For $x, y \in \mathbb{R}$, we write $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. For a set $A \subseteq \mathbb{R}^m$, we use A^c, ∂A, and $\mathbb{1}_A$ to denote the complement, the boundary, and the indicator function of A, respectively. A ball of radius $r > 0$ in $\mathbb{R}^m$ around a point x is denoted by $\mathcal{B}_r(x)$, or simply as $\mathcal{B}_r$ if $x = 0$. We also let $\mathcal{B} \equiv \mathcal{B}_1$. The Euclidean norm on $\mathbb{R}^m$ is denoted by $|\cdot|$, and $\langle \cdot\,, \cdot \rangle$ stands for the inner product. For $x \in \mathbb{R}^m$, we let $\|x\|_1 := \sum_i |x_i|$, and we use x' to denote the transpose of x. We use the symbol e to denote the vector whose elements are all equal to 1, and e_i for the vector whose i^{th} element is equal to 1 and the rest are equal to 0.

We let $\mathscr{B}(\mathbb{R}^m)$, $\mathscr{B}_b(\mathbb{R}^m)$, and $\mathcal{P}(\mathbb{R}^m)$ denote the classes of Borel measurable functions, bounded Borel measurable functions, and Borel probability measures on $\mathbb{R}^m$, respectively. By $\mathcal{P}_p(\mathbb{R}^m)$, $p > 0$, we denote the subset of $\mathcal{P}(\mathbb{R}^m)$ containing all probability measures $\pi(\mathrm{d}x)$ with the property that $\int_{\mathbb{R}^m} |x|^p \pi(\mathrm{d}x) < \infty$. For a finite signed measure ν on $\mathbb{R}^m$, and a Borel measurable $f\colon \mathbb{R}^m \to [1,\infty)$, $\|\nu\|_f := \sup_{|g|\le f} \int_{\mathbb{R}^m} |g(x)|\, \nu(\mathrm{d}x)$, where the supremum is over all Borel measurable functions g satisfying this inequality.

2 The model

We consider an m-dimensional stochastic differential equation (SDE) of the form

$$\mathrm{d}X_t = b(X_t, U_t)\,\mathrm{d}t + \mathrm{d}\hat{A}_t, \qquad X_0 = x \in \mathbb{R}^m. \tag{1}$$

All random processes in (1) live in a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We have the following structural hypotheses.

(A1) The control process $\{U_t\}_{t\ge 0}$ lives in the $(m-1)$-simplex

$$\Delta := \{u \in \mathbb{R}^m : u \ge 0,\ \langle e, u\rangle = 1\},$$

and the drift $b\colon \mathbb{R}^m \times \Delta \to \mathbb{R}^m$ is given by

$$\begin{aligned} b(x,u) &= \ell - M\bigl(x - \langle e,x\rangle^+ u\bigr) - \langle e,x\rangle^+ \Gamma u \\ &= \begin{cases} \ell - \bigl(M + (\Gamma - M)ue'\bigr)x, & \langle e,x\rangle > 0, \\ \ell - Mx, & \langle e,x\rangle \le 0, \end{cases} \end{aligned} \tag{2}$$

where $\ell \in \mathbb{R}^m$, $M = \mathrm{diag}(\mu_1, \dots, \mu_m)$ with $\mu_i > 0$, and $\Gamma = \mathrm{diag}(\gamma_1, \dots, \gamma_m)$ with $\gamma_i \in \mathbb{R}_+$, $i = 1, \dots, m$.

(A2) The process $\{\hat{A}_t\}_{t\ge 0}$ is an anisotropic Lévy process with independent symmetric one-dimensional α-stable components for $\alpha \in (1,2)$.

Define

$$\mathcal{K}_+ := \bigl\{x \in \mathbb{R}^m : \langle e,x\rangle > 0\bigr\}, \quad \text{and } \mathcal{K}_- := \bigl\{x \in \mathbb{R}^m : \langle e,x\rangle \le 0\bigr\}.$$

A control U_t is called stationary Markov, if it takes the form $U_t = v(X_t)$ for a Borel measurable function $v\colon \mathcal{K}_+ \to \Delta$. We let $\mathfrak{U}_{\mathrm{sm}}$ denote the class of stationary Markov controls, and $\widetilde{\mathfrak{U}}_{\mathrm{sm}}$ its subset consisting of those controls under which

$$b_v(x) := b\bigl(x, v(x)\bigr)$$

is locally Lipschitz continuous. These controls can be identified with the function v. Note that if $v\colon \mathcal{K}_+ \to \Delta$ is Lipschitz continuous when restricted to any set $\mathcal{K}_+ \cap \mathcal{B}_R$, $R > 0$, then $v \in \widetilde{\mathfrak{U}}_{\mathrm{sm}}$, but this property is not necessary for membership in $\widetilde{\mathfrak{U}}_{\mathrm{sm}}$.

Clearly, for any $v \in \mathfrak{U}_{\mathrm{sm}}$, the drift $b_v(x)$ has at most linear growth. Therefore, if $v \in \widetilde{\mathfrak{U}}_{\mathrm{sm}}$, then using [1, Theorem 3.1, and Propositions 4.2 and 4.3], one can conclude that the SDE (1) admits a unique nonexplosive strong solution $\{X_t\}_{t\geq 0}$ which is a strong Markov process and it satisfies the C_b-Feller property. In addition, in the same reference, it is shown that the infinitesimal generator $(\mathcal{A}^v, \mathscr{D}_{\mathcal{A}^v})$ of $\{X_t\}_{t\geq 0}$ (with respect to the Banach space $(\mathscr{B}_b(\mathbb{R}^m), \|\cdot\|_\infty)$) satisfies $C_c^2(\mathbb{R}^m) \subseteq \mathscr{D}_{\mathcal{A}^v}$ and

$$\mathcal{A}^v\big|_{C_c^2(\mathbb{R}^m)} f(x) := \bigl\langle b_v(x), \nabla f(x)\bigr\rangle + \mathfrak{I}_\alpha f(x)\,, \tag{3}$$

where

$$\mathfrak{I}_\alpha f(x) := \sum_{i=1}^{d} \int_{\mathbb{R}_*} \mathfrak{d} f(x; y_i e_i)\, \frac{\xi_i\, \mathrm{d}y_i}{|y_i|^{1+\alpha}}\,,$$

for some positive constants $\xi_1, \dots, \xi_m$, and

$$\mathfrak{d} f(x;y) := f(x+y) - f(x) - \langle y, \nabla f(x)\rangle\,, \qquad f \in C^1(\mathbb{R}^m)\,. \tag{4}$$

Here, $\mathscr{D}_{\mathcal{A}^v}$ and $C_c^2(\mathbb{R}^m)$ denote the domain of $\mathcal{A}^v$ and the space of twice continuously differentiable functions with compact support, respectively.

We let $\mathbb{P}_x^v$ and $\mathbb{E}_x^v$ denote the probability measure and expectation operator on the canonical space of the solution of (1) under $v \in \widetilde{\mathfrak{U}}_{\mathrm{sm}}$ and starting at x. Also, $P_t^v(x, \mathrm{d}y)$ denotes its transition probability. From the proof of Theorem 3.1 (iv) in [4] we have the following result.

Theorem 1. *Under any $v \in \widetilde{\mathfrak{U}}_{\mathrm{sm}}$, $P_t^v(x,B) > 0$ for all $t > 0$, $x \in \mathbb{R}^m$ and $B \in \mathscr{B}(\mathbb{R}^m)$ with positive Lebesgue measure. In particular, under any $v \in \widetilde{\mathfrak{U}}_{\mathrm{sm}}$, the process $\{X_t\}_{t\geq 0}$ is open–set irreducible and aperiodic in the sense of [11].*

Remark 1. As far as the results in this paper are concerned we can replace the anisotropic non-local operator $\mathfrak{I}_\alpha$ with the isotropic operator

$$\int_{\mathbb{R}_*} \mathfrak{d} f(x;y)\, \frac{\mathrm{d}y}{|y|^{m+\alpha}}\,,$$

as done in [4].

We also define

$$\mathcal{A}^u f(x) := \bigl\langle b(x,u), \nabla f(x)\bigr\rangle + \mathfrak{I}_\alpha f(x)\,, \quad u \in \Delta\,.$$

In the next section we study the ergodic properties of $\{X_t\}_{t\geq 0}$. To facilitate the analysis, we define the *spare capacity*, or *safety staffing*, β as

$$\beta \coloneqq -\langle e, M^{-1}\ell \rangle\,. \tag{5}$$

Note that if we let $\zeta = \frac{\beta}{m}e + M^{-1}\ell$, with β as in (5), then a mere translation of the origin of the form $\tilde{X}_t = X_t - \zeta$ results in an SDE of the same form, with the only difference that the constant term ℓ in the drift equals $-\frac{\beta}{m}Me$. Since translating the origin does not alter the ergodic properties of the process, without loss of generality, we assume throughout the paper that the drift in (2) has the form

$$b(x,u) = -\frac{\beta}{m}Me - M(x - \langle e,x\rangle^+ u) - \langle e,x\rangle^+ \Gamma u\,. \tag{6}$$

3 Uniform ergodic properties

We recall some important definitions used in [3, Section 2.3].

Definition 1. We fix some convex function $\psi \in C^2(\mathbb{R})$ with the property that $\psi(t)$ is constant for $t \le -1$, and $\psi(t) = t$ for $t \ge 0$. The particular form of this function is not important. But to aid some calculations we fix this function as

$$\psi(t) \coloneqq \begin{cases} -\frac{1}{2}, & t \le -1\,,\\ (t+1)^3 - \frac{1}{2}(t+1)^4 - \frac{1}{2} & t \in [-1,0]\,,\\ t & t \ge 0\,. \end{cases}$$

Let $\mathcal{I} = \{1,\dots,m\}$. With δ and p positive constants, we define

$$\Psi(x) \coloneqq \sum_{i\in\mathcal{I}} \frac{\psi(x_i)}{\mu_i}\,, \quad \text{and} \quad V_p(x) \coloneqq \left(\delta\Psi(-x) + \Psi(x) + \frac{m}{\min_{i\in\mathcal{I}}\mu_i}\right)^p.$$

Note that the term inside the parenthesis in the definition of V_p, or in other words V_1, is bounded away from 0 uniformly in $\delta \in (0,1]$. The function V_p also depends on the parameter δ which is suppressed in the notation.

For $x \in \mathbb{R}^m$ we let $x^\pm \coloneqq (x_1^\pm,\dots,x_m^\pm)$. The results which follows is a corollary of Lemma 2.1 in [3], but we sketch the proof for completeness.

Lemma 1. *Assume $\beta > 0$, and let $\delta \in (0,1]$ satisfy*

$$\Bigl(\max_{i\in\mathcal{I}} \tfrac{\gamma_i}{\mu_i} - 1\Bigr)^+ \delta \le 1\,. \tag{7}$$

Then, the function V_p in Definition 1 satisfies, for any $p > 1$ and for all $u \in \Delta$,

$$\bigl\langle b(x,u), \nabla V_p(x)\bigr\rangle \le p\Bigl(\delta\beta + \frac{m}{2}(1+\delta) - \delta\|x\|_1\Bigr)V_{p-1}(x) \quad \forall x \in \mathcal{K}_-\,, \tag{8}$$

$$\bigl\langle b(x,u), \nabla V_p(x)\bigr\rangle \le -p\Bigl(\frac{\beta}{m} - \delta\beta - \delta\frac{m}{2} + \delta\|x^-\|_1\Bigr)V_{p-1}(x) \quad \forall x \in \mathcal{K}_+\,. \tag{9}$$

Proof. We have

$$\big\langle b(x,u),\nabla\Psi(x)\big\rangle = -\frac{\beta}{m}\sum_{i\in\mathcal{J}}\psi'(x_i)-\sum_{i\in\mathcal{J}}\psi'(x_i)\big(x_i-\langle e,x\rangle^+u_i\big) \\ -\langle e,x\rangle^+\sum_{i\in\mathcal{J}}\psi'(x_i)\tfrac{\gamma_i}{\mu_i}u_i\,, \tag{10}$$

and

$$\big\langle b(x,u),\nabla\Psi(-x)\big\rangle = \frac{\beta}{m}\sum_{i\in\mathcal{J}}\psi'(-x_i)+\sum_{i\in\mathcal{J}}\psi'(-x_i)x_i \\ -\langle e,x\rangle^+\sum_{i\in\mathcal{J}}\psi'(-x_i)\big(1-\tfrac{\gamma_i}{\mu_i}\big)^+u_i \\ +\langle e,x\rangle^+\sum_{i\in\mathcal{J}}\psi'(-x_i)\big(\tfrac{\gamma_i}{\mu_i}-1\big)^+u_i\,. \tag{11}$$

It is easy to verify that $\psi'(-1/2)=1/2$, from which we obtain

$$\sum_{i\in\mathcal{J}}\psi'(x_i)x_i \ge \|x^+\|_1-\frac{m}{2}\,,\quad\text{and}\quad -\sum_{i\in\mathcal{J}}\psi'(-x_i)x_i \ge \|x^-\|_1-\frac{m}{2}\,. \tag{12}$$

Therefore, (8) follows by using (12) in (10)–(11).

We next turn to the proof of (9). If $\gamma_i\le\mu_i$ for all $i\in\mathcal{J}$, then the proof is simple. This is because the inequality $\sum_{i\in\mathcal{J}}\psi'(x_i)x_i\ge\langle e,x\rangle$ and the fact that $\|\psi'\|_\infty\le 1$ implies that

$$\sum_{i\in\mathcal{J}}\psi'(x_i)\big(x_i-\langle e,x\rangle^+u_i\big)\ge 0\quad\text{for } x\in\mathcal{K}_+\,,$$

which together with (10) shows that

$$\big\langle b(x,u),\nabla\Psi(x)\big\rangle \le -\frac{\beta}{m}\sum_{i\in\mathcal{J}}\psi'(x_i)\le-\frac{\beta}{m}\quad\text{on } \mathcal{K}_+\,. \tag{13}$$

On the other hand, by (11) and (12) we obtain

$$\delta\big\langle b(x,u),\nabla\Psi(-x)\big\rangle \le \delta\frac{\beta}{m}\sum_{i\in\mathcal{J}}\psi'(-x_i)+\delta\sum_{i\in\mathcal{J}}\psi'(-x_i)x_i \\ \le \delta\beta+\delta\frac{m}{2}-\delta\|x^-\|_1\quad\text{on } \mathbb{R}^m\,. \tag{14}$$

Therefore, when $\gamma_i\le\mu_i$ for all $i\in\mathcal{J}$, (9) follows by adding (13) and (14).

Without assuming that $\gamma_i\le\mu_i$, a careful comparison of the terms in (10)–(11), shows that (see [3, Lemma 2.1])

$$\delta\langle e,x\rangle^+\sum_{i\in\mathcal{J}}\psi'(-x_i)\big(\tfrac{\gamma_i}{\mu_i}-1\big)^+u_i-\sum_{i\in\mathcal{J}}\psi'(x_i)\big(x_i-\langle e,x\rangle^+u_i\big) \\ -\langle e,x\rangle^+\sum_{i\in\mathcal{J}}\psi'(x_i)\tfrac{\gamma_i}{\mu_i}u_i\le 0\quad\forall\,(x,u)\in\mathcal{K}_+\times\Delta\,. \tag{15}$$

Thus (9) follows by using (13)–(15) in (10)–(11). This completes the proof. □

On the other hand, when $\Gamma > 0$, the proof of [3, Theorem 2.2] implies the following.

Lemma 2. *Assume that $\Gamma > 0$. Then there exists a positive constant δ such that for any $p > 1$,*

$$\big\langle b(x,u), \nabla V_p(x) \big\rangle \le c_0 - c_1 V_p(x) \quad \forall\,(x,u) \in \mathbb{R}^m \times \Delta\,,$$

for some positive constants c_0 and c_1 depending only on δ.

Another result that we borrow is Proposition 5.1 in [2], whose proof implies the following.

Lemma 3. *The map $x \mapsto |x|^{\alpha-p}\,\mathfrak{I}_\alpha V_p(x)$ is bounded on $\mathbb{R}^m$ for any $p \in (0,\alpha)$.*

Theorems 2 and 3 that follow establish ergodic properties which are uniform over controls in $\widetilde{\mathfrak{U}}_{\mathrm{sm}}$ in the case of positive safety staffing and positive abandonment rates, respectively.

Theorem 2. *Assume $\beta > 0$. In addition to (7), let*

$$\delta < \frac{\beta}{2m(2\beta+m)}\,. \tag{16}$$

We have the following.

(a) *For any $p \in (1,\alpha)$, the function $V_p(x)$ in Definition 1 satisfies the Foster–Lyapunov equation*

$$\mathcal{A}^u V_p(x) \le C_0(p) - p\left(\frac{\beta}{2m} + \delta\|x^-\|_1\right) V_{p-1}(x) \quad \forall\,(x,u) \in \mathbb{R}^m \times \Delta\,, \tag{17}$$

for some positive constant $C_0(p)$ depending only on p.

(b) *Under any $v \in \widetilde{\mathfrak{U}}_{\mathrm{sm}}$, the process $\{X_t\}_{t\ge 0}$ in (1) admits a unique invariant probability measure $\bar{\pi}_v \in \mathcal{P}(\mathbb{R}^m)$.*

(c) *There exists a constant $C_1(\varepsilon)$ depending only on $\varepsilon \in (0,\alpha)$, such that, under any $v \in \widetilde{\mathfrak{U}}_{\mathrm{sm}}$, the process $\{X_t\}_{t\ge 0}$ in (1) satisfies*

$$\big\|P_t^v(x,\cdot\,) - \bar{\pi}_v(\,\cdot\,)\big\|_{\mathsf{TV}} \le C_1(\varepsilon)(t \vee 1)^{1+\varepsilon-\alpha}|x|^{\alpha-\varepsilon} \quad \forall\,x \in \mathbb{R}^m\,. \tag{18}$$

Proof. Note that, since $\alpha > 1$, Lemma 3 implies that $\frac{\mathfrak{I}_\alpha V_p(x)}{1+|V_{p-1}(x)|}$ vanishes at infinity. Using δ as in (16), it is clear that $\delta\beta + \delta\frac{m}{2} \le \frac{\beta}{2m}$. Thus, (17) is a direct consequence of Lemmas 1 and 3 together with the definition in (3).

Clearly, (17) implies that

$$\mathcal{A}^v V_p(x) \le C_0(p) - p\frac{\beta}{2m} V_{p-1}(x) \quad \forall\,x \in \mathbb{R}^m\,, \tag{19}$$

and for any $v \in \widetilde{\mathfrak{U}}_{\mathrm{sm}}$. It is well known that the existence of an invariant probability measure $\overline{\pi}_v$ follows from the C_b-Feller property and (19), while the open-set irreducibility asserted in Theorem 1 implies its uniqueness.

Equation (18) is a direct result of (19), Theorem 1 and [6, Theorem 3.2]. This completes the proof. □

Theorem 3. *Assume that $\Gamma > 0$ and $p \in [1, \alpha)$. Then, there exists a positive constant δ such that*

$$\mathcal{A}^u V_p(x) \le \tilde{\kappa}_0 - \tilde{\kappa}_1 V_p(x) \quad \forall\, (x,u) \in \mathbb{R}^m \times \Delta\,.$$

for some positive constants $\tilde{\kappa}_0$ and $\tilde{\kappa}_1$. Moreover, under any $v \in \widetilde{\mathfrak{U}}_{\mathrm{sm}}$, the process $\{X_t\}_{t\ge 0}$ admits a unique invariant probability measure $\overline{\pi}_v \in \mathcal{P}(\mathbb{R}^m)$, and for any $\gamma \in (0, \tilde{\kappa}_1)$ there exists a positive constant C_γ such that

$$\big\| P_t^v(x, \cdot\,) - \overline{\pi}_v(\,\cdot\,) \big\|_{V_p} \le C_\gamma V_p(x)\, \mathrm{e}^{-\gamma t}\,, \qquad x \in \mathbb{R}^m\,,\ t \ge 0\,.$$

Remark 2. We limited our attention to controls in $\widetilde{\mathfrak{U}}_{\mathrm{sm}}$ only to take advantage of Theorem 1. However, if under some $v \in \mathfrak{U}_{\mathrm{sm}}$ the SDE in (1) has a unique weak solution which is an open-set irreducible and aperiodic C_b-Feller process, then it has a unique invariant probability measure $\overline{\pi}_v$, and the conclusions of Theorems 2 and 3 follow.

Concerning the lower bound on the rate of convergence, we need not restrict the controls in $\widetilde{\mathfrak{U}}_{\mathrm{sm}}$. The lack of integrability of functions that have strict polynomial growth of order α (or higher) under the Lévy measure of $\mathfrak{I}_\alpha$, plays a crucial role in determining this lower bound. Consider a $v \in \mathfrak{U}_{\mathrm{sm}}$ as in Remark 2, and suppose that $\beta > 0$.

Then it is shown in Lemma 5.7 (b) of [4] that

$$\int_{\mathbb{R}^m} \big(\langle e, M^{-1}x\rangle^+\big)^p\, \overline{\pi}_v(\mathrm{d}x) < \infty \quad \text{for some } p > 0 \quad \Longrightarrow \quad p < \alpha - 1\,. \tag{20}$$

We use this property in the proof of Theorem 4 which follows. To simplify the notation, for a function f which is integrable under $\overline{\pi}_v$, we let $\overline{\pi}_v(f) := \int_{\mathbb{R}^m} f(x)\overline{\pi}_v(\mathrm{d}x)$.

Theorem 4. *We assume $\beta > 0$. Suppose that under some $v \in \mathfrak{U}_{\mathrm{sm}}$ such that $\Gamma v = 0$ a.e. the SDE in (1) has a unique weak solution which is an open-set irreducible and aperiodic C_b-Feller process. Then the process $\{X_t\}_{t\ge 0}$ is polynomially ergodic. In particular, there exists a positive constant C_2 not depending on v, such that for all $\varepsilon > 0$ we have*

$$\big\| P_t^v(x, \cdot\,) - \overline{\pi}_v(\,\cdot\,) \big\|_{\mathsf{TV}} \ge C_2 \Big(\frac{t \vee 1}{\varepsilon} + |x|^{\alpha - \varepsilon} \Big)^{\frac{1-\alpha}{1-\varepsilon}} \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^m\,.$$

Proof. The proof uses [9, Theorem 5.1] and some results from [4]. Recall the function ψ, and define

$$\check{\chi}(t) := 1 + \psi(t)\,, \quad \text{and} \quad \chi(t) := -\check{\chi}(-t)\,.$$

Also, we scale $\chi(t)$ using $\chi_R(t) := R + \chi(t-R)$, $R \in \mathbb{R}$. Thus, $\chi_R(t) = t$ for $t \le R-1$ and $\chi_R(t) = R - \frac{1}{2}$ for $t \ge R$.

Let

$$F(x) := \breve{\chi}\bigl(\langle e, M^{-1}x\rangle\bigr)\,, \quad \text{and} \;\; F_{\kappa,R}(x) := \chi_R \circ F^{\kappa}(x)\,, \quad x \in \mathbb{R}^m\,,\; R > 0\,,$$

where $F^{\kappa}(x)$ denotes the κ^{th} power of $F(x)$, with $\kappa > 0$.

Using the same notation as in [9, Theorem 5.1] whenever possible, we define $G(x) := F^{\alpha-\varepsilon}(x)$, for $\varepsilon \in (0, \alpha-1)$. Then $\overline{\pi}_v(F^{\alpha-\varepsilon}) = \infty$ by (20). Applying the Itô formula to (19) we obtain

$$\mathbb{E}^{v}_{x}\bigl[V_{\alpha-\varepsilon}\bigl(X_t\bigr)\bigr] - V_{\alpha-\varepsilon}(x) \le C_0(\alpha-\varepsilon)\,t\,, \qquad x \in \mathbb{R}^m\,.$$

Since $F^{\alpha-\varepsilon} \le \overline{C}_0 V_{\alpha-\varepsilon}$ for some constant $\overline{C}_0 \ge 1$, the preceding inequality implies that

$$\mathbb{E}^{v}_{x}\bigl[F^{\alpha-\varepsilon}\bigl(X_t\bigr)\bigr] \le \overline{C}_0\bigl(C_0(\alpha-\varepsilon)t + V_{\alpha-\varepsilon}(x)\bigr) =: g(x,t)\,.$$

Next, we compute a suitable lower bound $f(t)$ for $\overline{\pi}_v\bigl(\{x\colon G(x) \ge t\}\bigr)$. We have

$$\begin{aligned} \mathcal{A}^{v} F_{1,R}(x) &= \mathfrak{I}_\alpha F_{1,R}(x) + \chi_R'\bigl(F(x)\bigr)\bigl\langle b_v(x), \nabla F(x)\bigr\rangle \\ &= \mathfrak{I}_\alpha F_{1,R}(x) + \chi_R'\bigl(F(x)\bigr)\breve{\chi}'\bigl(\langle e, M^{-1}x\rangle\bigr)\bigl(-\beta + \langle e, x\rangle^{-}\bigr)\,. \end{aligned} \tag{21}$$

Integrating (21) with respect to $\overline{\pi}_v$, and replacing the variable R with t, we obtain

$$\beta\,\overline{\pi}_v\bigl(\chi_t'(F)h\bigr) = \overline{\pi}_v\bigl(\mathfrak{I}_\alpha F_{1,t}\bigr) + \overline{\pi}_v\bigl(\chi_t'(F)\tilde{h}\bigr)\,, \tag{22}$$

where

$$h(x) := \breve{\chi}'\bigl(\langle e, M^{-1}x\rangle\bigr)\,, \quad \text{and} \quad \tilde{h}(x) := h(x)\,\langle e, x\rangle^{-}\,.$$

Taking limits as $t \to \infty$ in (22), we obtain

$$\beta\overline{\pi}_v(h) = \overline{\pi}_v(\mathfrak{I}_\alpha F) + \overline{\pi}_v(\tilde{h})\,. \tag{23}$$

Subtracting (22) from (23), gives

$$\beta\,\overline{\pi}_v(h - \chi_t'(F)h) = \overline{\pi}_v\bigl(\mathfrak{I}_\alpha(F - F_{1,t})\bigr) + \overline{\pi}_v\bigl(\tilde{h} - \chi_t'(F)\tilde{h}\bigr)\,. \tag{24}$$

Note that all the terms in this equation are nonnegative. Moreover, $\mathfrak{I}_\alpha(F - F_{1,t})(x)$ is nonnegative by convexity, and thus

$$\begin{aligned} \overline{\pi}_v\bigl(\mathfrak{I}_\alpha(F - F_{1,t})\bigr) &\ge \inf_{x \in \mathcal{B}} \bigl(\mathfrak{I}_\alpha(F - F_{1,t})(x)\bigr)\,\overline{\pi}_v(\mathcal{B}) \\ &\ge \mathfrak{I}_\alpha(F - F_{1,t})(0)\,\overline{\pi}_v(\mathcal{B})\,. \end{aligned} \tag{25}$$

It is straightforward to show that $\mathfrak{I}_\alpha(F - F_{1,t})(0) \ge \hat{\kappa} t^{1-\alpha}$ for some positive constant $\hat{\kappa}$. Therefore, by (24)–(25) and the definition of the functions F, $F_{1,R}$ and h, we obtain

$$\begin{aligned}\overline{\pi}_v\bigl(\{x\colon \langle e, M^{-1}x\rangle > t\}\bigr) &\ge \overline{\pi}_v(h - \chi_t'(F)h)\\ &\ge \beta^{-1}\,\overline{\pi}_v(\mathcal{B})\mathfrak{I}_\alpha(F - F_{1,t})(0)\\ &\ge \hat{\kappa}\, t^{1-\alpha}\,.\end{aligned} \tag{26}$$

Therefore, by (26), we have

$$\begin{aligned}\overline{\pi}_v\bigl(\{x\colon G(x) \ge t\}\bigr) &= \overline{\pi}_v\bigl(\{x\colon \bigl(\langle e, M^{-1}x\rangle\bigr)^{\alpha-\varepsilon} > t\}\bigr)\\ &= \overline{\pi}_v\bigl(\{x\colon \langle e, M^{-1}x\rangle > t^{\frac{1}{\alpha-\varepsilon}}\}\bigr)\\ &\ge \hat{\kappa}\, t^{\frac{1-\alpha}{\alpha-\varepsilon}} =: f(t)\,.\end{aligned}$$

Next we solve $y f(y) = 2g(x,t)$ for $y = y(t)$, and this gives us $y = \bigl(\hat{\kappa}^{-1} 2g(x,t)\bigr)^{\frac{\alpha-\varepsilon}{1-\varepsilon}}$, and

$$f(y) = \hat{\kappa}\bigl(\hat{\kappa}^{-1}2g(x,t)\bigr)^{\frac{1-\alpha}{1-\varepsilon}} = \overline{C}_1\bigl(C_0(\alpha-\varepsilon)t + V_{\alpha-\varepsilon}(x)\bigr)^{\frac{1-\alpha}{1-\varepsilon}}\,,$$

with

$$\overline{C}_1 := \bigl(2\overline{C}_0\bigr)^{\frac{1-\alpha}{1-\varepsilon}}\,\hat{\kappa}^{\frac{\alpha-\varepsilon}{1-\varepsilon}}\,.$$

Therefore, by [9, Theorem 5.1], and since ε is arbitrary, we have

$$\begin{aligned}\bigl\|P_t^v(x,\cdot\,) - \overline{\pi}_v(\,\cdot\,)\bigr\|_{\mathsf{TV}} &\ge f(y) - \frac{g(x,t)}{y}\\ &= \frac{\overline{C}_1}{2}\bigl(C_0(\alpha-\varepsilon)t + V_{\alpha-\varepsilon}(x)\bigr)^{\frac{1-\alpha}{1-\varepsilon}}\end{aligned} \tag{27}$$

for all $t \ge 0$ and $\varepsilon \in (0, \alpha-1)$.

As shown in the proof of [4, Theorem 3.4], there exists a positive constant κ_0', not depending on ε, such that

$$C_0(\alpha-\varepsilon) \ge \kappa_0'(1+\varepsilon^{-1})\,. \tag{28}$$

Thus the result follows by (27)–(28). □

4 Ergodic properties of the limiting SDEs arising from queueing models with service interruptions

The limiting equations of multiclass $G/M/n+M$ queues with asymptotically negligible service interruptions under the $\sqrt{n}$-scaling in the Halfin–Whitt regime are Lévy–driven SDEs of the form

$$\mathrm{d}X_t = b(X_t, U_t)\,\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}L_t\,, \qquad X_0 = x \in \mathbb{R}^m\,. \tag{29}$$

Here, the drift b is as in Section 2, σ is a nonsingular diagonal matrix, and $\{L_t\}_{t\ge 0}$ is a compound Poisson process, with a drift ϑ, and a finite Lévy measure $\eta(\mathrm{d}y)$ which is supported on a half-line of the form $\{tw: t \in [0,\infty)\}$, with $\langle e, M^{-1}w\rangle > 0$. This can be established as in Theorem 6 in Section 5, assuming that the control is of the form $U_t = v(X_t)$ for a map $v\colon \mathcal{K}_+ \to \Delta$, such that $b_v(x)$ is locally Lipschitz, when the scaling is of order $\sqrt{n}$ (see also Section 4.2 of [4]).

As we explain later, under any stationary Markov control, the SDE in (29) has a unique strong solution which is an open-set irreducible and aperiodic strong Feller process. Therefore, as far as the study of the process $\{X_t\}_{t\ge 0}$ is concerned, we do not need to impose a local Lipschitz continuity condition on the drift, but can allow the control to be any element of $\mathfrak{U}_{\mathrm{sm}}$.

There are two important parameters to consider. The first is the parameter θ_c, which is defined by

$$\theta_c := \sup\bigl\{\theta \in \Theta_c\bigr\}, \quad \text{with} \quad \Theta_c := \biggl\{\theta > 0 : \int_{\mathcal{B}^c} |y|^\theta \eta(\mathrm{d}y) < \infty\biggr\}.$$

The second is the *effective spare capacity*, defined as

$$\widetilde{\beta} := -\bigl\langle e, M^{-1}\tilde{\ell}\bigr\rangle,$$

where

$$\tilde{\ell} := \begin{cases} \ell + \vartheta + \int_{\mathcal{B}^c} y\,\eta(\mathrm{d}y), & \text{if } \int_{\mathcal{B}^c} |y|\eta(\mathrm{d}y) < \infty \\ \ell + \vartheta, & \text{otherwise.} \end{cases}$$

Suppose that $v \in \mathfrak{U}_{\mathrm{sm}}$ is such that $\Gamma v(x) = 0$ a.e. x in $\mathbb{R}^m$. Then as shown in Lemma 5.7 of [4], the process $\{X_t\}_{t\ge 0}$ controlled by v cannot have an invariant probability measure $\overline{\pi}_v$ unless $1 \in \Theta_c$ and $\widetilde{\beta} > 0$, and moreover,

$$\int_{\mathbb{R}^m} \bigl(\langle e, M^{-1}x\rangle^+\bigr)^p \,\overline{\pi}_v(\mathrm{d}x) < \infty \quad \text{for some } p > 0 \quad \Longrightarrow \quad p+1 \in \Theta_c\,.$$

In addition, $\widetilde{\beta} = \int_{\mathbb{R}^m} \langle e, x\rangle^- \,\overline{\pi}_v(\mathrm{d}x)$ [4, Theorem 3.4 (b)]. Conversely, $1 \in \Theta_c$ and $\widetilde{\beta} > 0$ are sufficient for $\{X_t\}_{t\ge 0}$ to have an invariant probability measure $\overline{\pi}_v$ under any constant control v, and $\overline{\pi}_v \in \mathcal{P}_p(\mathbb{R}^m)$ if $p+1 \in \Theta_c$ (see Theorems 3.2 and 3.4 (b) in [4]).

On the other hand, if $\Gamma > 0$, that is, it has positive diagonal elements, then $\{X_t\}_{t\ge 0}$ is geometrically ergodic under any constant Markov control, and $\overline{\pi}_v \in \mathcal{P}_\theta(\mathbb{R}^m)$ for any $\theta \in \Theta_c$ [4, Theorem 3.5]. This bound is tight since, in general, if under some Markov control v the process $\{X_t\}_{t\ge 0}$ has an invariant probability measure $\overline{\pi}_v \in \mathcal{P}_p(\mathbb{R}^m)$, then necessarily $p \in \Theta_c$.

We extend the results derived for constant Markov controls in [4] to all controls in $\mathfrak{U}_{\mathrm{sm}}$. Recall the definition in (4). Let

$$\tilde{b}(x,u) := b(x,u) + \tilde{\ell} - \ell\,,$$

and $\tilde{b}_v(x) = \tilde{b}\bigl(x, v(x)\bigr)$ for $v \in \mathfrak{U}_{\mathrm{sm}}$. As explained in Section 2, we assume, without loss of generality, that the constant term in $\tilde{b}$ is as in (6) with β replaced by $\widetilde{\beta}$.

We define the operator $\mathscr{A}^u$ on C^2 functions by

$$\mathscr{A}^u f(x) \coloneqq \mathcal{L}^u f(x) + \mathfrak{I}_\eta f(x)\,, \quad (x,u) \in \mathbb{R}^m \times \Delta\,,$$

where

$$\mathcal{L}^u f(x) = \frac{1}{2}\operatorname{trace}\bigl(\sigma\sigma'\nabla^2 f(x)\bigr) + \bigl\langle \tilde{b}(x,u), \nabla f(x)\bigr\rangle\,, \quad (x,u) \in \mathbb{R}^m \times \Delta\,, \tag{30}$$

and

$$\mathfrak{I}_\eta f(x) \coloneqq \int_{\mathbb{R}^m} \mathfrak{d} f(x;y)\,\eta(\mathrm{d}y)\,, \qquad x \in \mathbb{R}^m\,.$$

Also, $\mathcal{L}^v$ is defined as in (30) by replacing u with $v(x)$ for a control $v \in \mathfrak{U}_{\mathrm{sm}}$, and analogously for $\mathscr{A}^v$.

It follows from the results in [8] that, for any $v \in \mathfrak{U}_{\mathrm{sm}}$, the diffusion

$$\mathrm{d}\tilde{X}_t = \tilde{b}(\tilde{X}_t, v(\tilde{X}_t))\,\mathrm{d}t + \sigma(\tilde{X}_t)\,\mathrm{d}W_t\,, \quad \tilde{X}_0 = x \in \mathbb{R}^d \tag{31}$$

has a unique strong solution. Also, as shown in [14], since the the Lévy measure is finite, the solution of (29) can be constructed in a piecewise fashion using the solution of (31) (see also [10]). It thus follows that, under any stationary Markov control, (29) has a unique strong solution which is a strong Markov process. In addition, its transition probability $P_t^v(x,dy)$ satisfies $P_t^v(x,B) > 0$ for all $t > 0, x \in \mathbb{R}^m$ and $B \in \mathscr{B}(\mathbb{R}^m)$ with positive Lebesgue measure. Thus, under any $v \in \mathfrak{U}_{\mathrm{sm}}$, the process $\{X_t\}_{t\geq 0}$ is open–set irreducible and aperiodic.

Recall Definition 1. In order to handle the second order derivatives in $\mathscr{A}^u$ we need to scale the Lyapunov function V_p. This is done as follows. With ψ as in Definition 1, we define

$$\psi_\delta(t) \coloneqq \psi(\delta t)\,, \quad \text{and} \quad \Psi_\delta(x) \coloneqq \sum_{i\in\mathfrak{I}} \frac{\psi_\delta(x_i)}{\mu_i}\,, \quad \delta \in (0,1]\,,$$

and let

$$\mathcal{V}_{p,\delta}(x) \coloneqq \left(\delta^2\Psi(-x) + \Psi_\delta(x) + \frac{m}{\min_{i\in\mathfrak{I}}\mu_i}\right)^p.$$

Note that $\mathcal{V}_{1,\delta}$ is bounded away from 0 uniformly in $\delta \in (0,1]$. Here we use the inequality $\sum_{i\in\mathfrak{I}} \psi_\delta'(x_i)x_i \geq \delta\|x^+\|_1 - \frac{m}{2}$. Then, under the assumption that $\widetilde{\beta} > 0$, the drift inequalities take the form

$$\begin{aligned}
&\bigl\langle \tilde{b}(x,u), \nabla\mathcal{V}_{p,\delta}(x)\bigr\rangle \\
&\qquad \leq \begin{cases} p\delta\Bigl(\delta\widetilde{\beta} + \frac{m}{2\delta}(1+\delta^2) - \delta\|x\|_1\Bigr)\mathcal{V}_{p-1,\delta}(x) & \forall x \in \mathcal{K}_-\,, \\ -p\delta\bigl(\frac{\widetilde{\beta}}{m} - \delta\widetilde{\beta} - \delta\frac{m}{2} + \delta\|x^-\|_1\bigr)\mathcal{V}_{p-1,\delta}(x) & \forall (x,u) \in \mathcal{K}_+ \times \Delta\,. \end{cases}
\end{aligned} \tag{32}$$

The following result is analogous to Theorem 2.

Theorem 5. *Assume $\widetilde{\beta} > 0$, and $1 \in \Theta_c$. Let $p \in \Theta_c$ with $p > 1$. Then the following hold.*

(a) *There exists $\delta > 0$, a positive constant $\widetilde{C}_0$, and a compact set K such that*

$$\mathscr{A}^u \mathcal{V}_{p,\delta}(x) \le \widetilde{C}_0 \mathbb{1}_K(x) - p\delta \frac{\widetilde{\beta}}{2m} \mathcal{V}_{p-1,\delta}(x) \quad \forall (x,u) \in \mathbb{R}^m \times \Delta\,. \tag{33}$$

(b) *Under any $v \in \mathfrak{U}_{\mathrm{sm}}$, the process $\{X_t\}_{t\ge 0}$ in (1) admits a unique invariant probability measure $\overline{\pi}_v \in \mathcal{P}(\mathbb{R}^m)$.*

(c) *For any $\theta \in \Theta_c$ there exists a constant $\widetilde{C}_1(\theta)$ depending only on θ, such that, under any $v \in \mathfrak{U}_{\mathrm{sm}}$, the process $\{X_t\}_{t\ge 0}$ in (1) satisfies*

$$\big\| P_t^v(x,\cdot\,) - \overline{\pi}_v(\,\cdot\,) \big\|_{\mathsf{TV}} \le \widetilde{C}_1(\theta)(t \vee 1)^{1-\theta} |x|^\theta \quad \forall x \in \mathbb{R}^m\,.$$

Proof. It is straightforward to show that $\psi''_\delta(t) \le 2\delta^2$ and $\psi'_\delta(t) \le \delta$ for all $t \in \mathbb{R}$. An easy calculation then shows that there exists a positive constant C such that

$$\operatorname{trace}\big(\sigma\sigma'\nabla^2 \mathcal{V}_{p,\delta}(x)\big) \le C p^2 \delta^2 \big(\mathcal{V}_{p-1,\delta}(x) + \mathcal{V}_{p-2,\delta}(x)\big) \tag{34}$$

for all $p \ge 1$ and $x \in \mathbb{R}^m$. Recall that $\mathcal{V}_{1,\delta}$ is bounded away from 0 uniformly in $\delta \in (0,1]$. This of course implies that $\mathcal{V}_{p-2,\delta}$ is bounded by some fixed multiple of $\mathcal{V}_{p-1,\delta}$ for all $p \ge 1$. Therefore, (32) and (34) imply that for some small enough positive δ we can chose a positive constant $\widetilde{C}'_0$, and a compact set K' such that

$$\mathcal{L}^u \mathcal{V}_{p,\delta}(x) \le \widetilde{C}'_0 \mathbb{1}_{K'}(x) - p\delta \frac{3\widetilde{\beta}}{4m} \mathcal{V}_{p-1,\delta}(x) \quad \forall (x,u) \in \mathbb{R}^m \times \Delta\,. \tag{35}$$

If $p \in \Theta_c$, then [4, Lemma 5.1] asserts that $\mathfrak{J}_\eta \mathcal{V}_{p,\delta}$ vanishes at infinity for $p < 2$, and $\mathfrak{J}_\eta \mathcal{V}_{p,\delta}$ is of order $|x|^{p-2}$ for $p \ge 2$. This together with (35) implies (33). The rest are as in the proof of Theorem 2. □

If $\Gamma > 0$, then the arguments in the proof of Theorem 5 together with Lemma 2 show that the process $\{X_t\}_{t\ge 0}$ is geometrically ergodic uniformly over $v \in \mathfrak{U}_{\mathrm{sm}}$. Thus we obtain the analogous results to Theorem 3. We omit the details which are routine.

Note that the assumption that the Lévy measure $\eta(\mathrm{d}y)$ is supported on a half-line of the form $\{tw : t \in [0,\infty)\}$, with $\langle e, M^{-1}w\rangle > 0$ has not been used, and is not needed in Theorem 5. Under this assumption we can obtain a lower bound of the rate of convergence analogous to equation (3.9) in [4], by mimicking the arguments in that paper. We leave the details to the reader.

Remark 3. With heavy-tailed arrivals and asymptotically negligible service interruptions under the common $n^{1/\alpha}$-scaling for $\alpha \in (1,2)$, in the modified Halfin–Whitt regime, the limit process is an SDE driven by an anisotropic α-stable process (with independent α-stable components) as in (1), and a compound Poisson process with a finite Lévy measure as in (29). This can be established as in Theorem 6, under the same scaling assumptions in Section 4.2 of [4]. Thus the generator is given by

$$\hat{\mathcal{A}}^u f(x) := \big\langle \tilde{b}(x,u), \nabla f(x) \big\rangle + \mathfrak{I}_\eta f(x) + \mathfrak{I}_\alpha f(x)\,,$$

and $\hat{\mathcal{A}}^v$ is defined analogously by replacing u with $v(x)$ for $v \in \widetilde{\mathfrak{U}}_{\mathrm{sm}}$.

To study this equation, we use the Lyapunov function V_p in Definition 1, with $p \in [1,\alpha) \cap \Theta_c$. Following the proof of Theorem 5, and also using Lemma 3, it follows that there exists $\delta > 0$ sufficiently small, a constant $\widehat{C}_0$ and a compact set $\widehat{K}$ such that

$$\hat{\mathcal{A}}^u V_p(x) \le \widehat{C}_0 \mathbb{1}_{\widehat{K}}(x) - p\frac{\widetilde{\beta}}{2m} V_{p-1}(x) \quad \forall\,(x,u) \in \mathbb{R}^m \times \Delta\,.$$

Thus, (18) holds for any ε such that $\alpha - \varepsilon \in \Theta_c$. The results of Theorem 3 also follow provided we select $p \in [1,\alpha) \cap \Theta_c$. However the lower bound is not necessarily the one in Theorem 4. Instead we can obtain a lower bound in the form of equation (3.9) in [4].

5 Multiclass $G/M/n+M$ queues with heavy-tailed arrivals

As in [4, Subsection 4.1], consider $G/M/n+M$ queues with m classes of customers and one server pool of n parallel servers. Customers of each class form their own queue and are served in the first-come first-served (FCFS) service discipline. Customers of different classes are scheduled to receive service under the work conserving constraint, that is, non-idling whenever customers are in queue. We assume that the arrival process of each class is renewal with heavy-tailed interarrival times. The service and patience times are exponentially distributed with class-dependent rates. The arrival, service and abandonment processes of each class are mutually independent.

We consider a sequence of such queueing models indexed by n and let $n \to \infty$. Let A_i^n, $i = 1, \dots, m$, be the arrival process of class-i customers with arrival rate λ_i^n. Assume that A_i^n's are mutually independent. Define the FCLT-scaled arrival processes $\hat{A}^n = (\hat{A}_1^n, \dots, \hat{A}_m^n)'$ by $\hat{A}_i^n := n^{-1/\alpha}(A_i^n - \lambda_i^n \varpi)$, $i = 1, \dots, m$, where $\varpi(t) \equiv t$ for each $t \ge 0$, and $\alpha \in (1,2)$. We assume that

$$\lambda_i^n/n \to \lambda_i > 0, \quad \text{and} \quad \ell_i^n := n^{-1/\alpha}(\lambda_i^n - n\lambda_i) \to \ell_i \in \mathbb{R}\,, \tag{36}$$

for each $i = 1, \dots, m$, as $n \to \infty$, and that the arrival processes satisfy an FCLT

$$\hat{A}^n \Rightarrow \hat{A} = (\hat{A}_1, \dots, \hat{A}_m)' \qquad \text{in } (D_m, M_1),\ \text{as } n \to \infty\,,$$

where the limit processes $\hat{A}_i$, $i = 1, \dots, m$, are mutually independent symmetric α-stable processes with $\hat{A}_i(0) \equiv 0$, and $\Rightarrow$ denotes weak convergence and (D_m, M_1) is the space of $\mathbb{R}^m$-valued càdlàg functions endowed with the product M_1 topology [15]. The processes $\hat{A}_i$ have the same stability parameter α, with possibly different "scale" parameters ξ_i. Note that if the arrival process of each class is renewal with

regularly varying interarrival times of parameter α, then we obtain the above limit process. Let μ_i and γ_i be the service and abandonment rates for class-i customers, respectively.
The modified Halfin-Whitt regime. The parameters satisfy

$$n^{1-1/\alpha}(1-\rho^n) \xrightarrow[n\to\infty]{} \rho = -\sum_{i=1}^{m} \frac{\ell_i}{\mu_i},$$

where $\rho^n := \sum_{i=1}^m \frac{\lambda_i^n}{n\mu_i}$ is the aggregate traffic intensity. This follows from (36). Let $\rho_i := \lambda_i/\mu_i$ for $i \in \mathcal{I}$.

Let $X^n = (X_1^n,\dots,X_d^n)'$, $Q^n = (Q_1^n,\dots,Q_d^n)'$, and $Z^n = (Z_1^n,\dots,Z_d^n)'$ be the processes counting the number of customers of each class in the system, in queue, and in service, respectively. We consider work-conserving scheduling policies that are non-anticipative and allow preemption (namely, service of a customer can be interrupted at any time to serve some other class of customers and will be resumed at a later time). Scheduling policies determine the allocation of service capacity, i.e., the Z^n process, which must satisfy the condition that $\langle e, Z^n\rangle = \langle e, X^n\rangle \wedge n$ at each time, as well as the balance equations $X_i^n = Q_i^n + Z_i^n$ for each i.

Define the FCLT-scaled processes $\hat{X}^n = (\hat{X}_1^n,\dots,\hat{X}_d^n)'$, $\hat{Q}^n = (\hat{Q}_1^n,\dots,\hat{Q}_d^n)'$, and $\hat{Z}^n = (\hat{Z}_1^n,\dots,\hat{Z}_d^n)'$ by

$$\hat{X}_i^n := n^{-1/\alpha}(X_i^n - \rho_i n), \qquad \hat{Q}_i^n := n^{-1/\alpha} Q_i^n, \qquad \hat{Z}_i^n := n^{-1/\alpha}(Z_i^n - \rho_i n).$$

We need the following extension of Theorem 1.1 in [13]. Let $\phi\colon D([0,T],\mathbb{R}^m) \to D([0,T],\mathbb{R}^m)$ denote the mapping $x \mapsto y$ defined by the integral representation

$$y(t) = x(t) + \int_0^t h(y(s))\,\mathrm{d}s, \quad t \ge 0.$$

It is shown in [13, Theorem 1.1] that the mapping ϕ is continuous in the Skorohod M_1 topology when $m = 1$ and the function h is Lipschitz continuous. The lemma which follows extends this result to functions $h\colon \mathbb{R}^m \to \mathbb{R}^m$ which are locally Lipschitz continuous and have at most linear growth.

Lemma 4. *Assume that h is locally Lipschitz and has at most linear growth. Then the mapping ϕ defined above is continuous in (D_m, M_1), the space $D([0,T],\mathbb{R}^m)$ endowed with the product M_1 topology.*

Proof. Assume that $x_n \to x$ in D_m with the product M_1 topology as $n \to \infty$. Let x^i be the i^{th} component of x, and similarly for x_n^i. Let

$$G_x := \bigl\{(z,t) \in \mathbb{R}^m \times [0,T] \colon z^i \in [x^i(t-), x^i(t)] \quad \text{for each } i = 1,\dots,m\bigr\},$$

be the (weak) graph of x, and similarly, G_{x_n} for x_n; see Chapter 12.3.1 in [15]. Then following the proof of Theorem 1.2 in [13], it can be shown that there exist parametric representations (u,r) and (u_n,r_n) of x and x_n, that map $[0,1]$ onto the graphs G_x and G_{x_n} of x and x_n, respectively, and satisfy the properties below. In the construction of the time component as in Lemma 4.3 of [13], the discontinuity points of all the x^i components need to be included, and then the spatial component can be done similarly as in the proof of that lemma.

- The time (domain) components $r, r_n \in C([0,1],[0,T])$ are nondecreasing functions satisfying $r(0) = r_n(0) = 0$ and $r_n(1) = r_n(1) = T$, and such that r and r_n are absolutely continuous with respect to Lebesgue measure on $[0,1]$.
- The derivatives r' and r_n' exist for all n and satisfy $\|r'\|_\infty < \infty$, $\sup_n \|r_n'\|_\infty < \infty$, and $\|r_n' - r'\|_{L^1} \to 0$, where $\|r\|_\infty := \sup_{s\in[0,1]} |r(s)|$, and $\|\cdot\|_{L^1}$ denotes the L^1 norm.
- The spatial components $u = (u^1, \dots, u^m)$ and $u_n = (u_n^1, \dots, u_n^m)$, $n \in \mathbb{N}$, lie in $C([0,1],\mathbb{R}^m)$, and satisfy $u(0) = x(0)$, $u(1) = x(T)$, $u_n(0) = x_n(0)$, $u_n(1) = x_n(T)$, and $\|u_n - u\|_\infty \to 0$ as $n \to \infty$.

As shown in the proof of Theorem 1.1 in [13], there exist parametric representations (u_y, r_y) and (u_{y_n}, r_{y_n}) of y and y_n, respectively, with $r_y = r$ and $r_{y_n} = r_n$, satisfying

$$u_{y_n}(s) = u_n(s) + \int_0^s h\bigl(u_{y_n}(w)\bigr) r_n'(w)\,\mathrm{d}w\,, \quad s \in [0,1]\,, \tag{37}$$

and similarly for $u_y(s)$. Here, (u, r) and (u_n, r_n) are the parametric representations of x and x_n, respectively, whose properties are summarized above.

Since $x_n \to x$ in (D_m, M_1) as $n \to \infty$, we have $\sup_n \|u_n\|_\infty < \infty$. Taking norms in (37), and using also the property $\sup_n \|r_n'\|_\infty < \infty$, and the linear growth of h, an application of Gronwall's lemma shows that $\sup_n \|u_{y_n}\|_\infty \le R$ for some constant R. Enlarging this constant if necessary, we may also assume that $\|u_y\|_\infty \le R$. By the representation in (37), we have

$$\begin{aligned}|u_{y_n}(s) - u_y(s)| \le |u_n(s) - u(s)| &+ \left|\int_0^s \bigl(h(u_{y_n}(w)) - h(u_y(w))\bigr) r_n'(w)\,\mathrm{d}w\right| \\ &+ \left|\int_0^s h(u_y(w)) r_n'(w)\,\mathrm{d}w - \int_0^s h(u_y(w)) r'(w)\,\mathrm{d}w\right|.\end{aligned}$$

Let κ_R be a Lipschitz constant of h on the ball $\mathcal{B}_R$. Then, applying Gronwall's lemma once more, we obtain

$$\|u_{y_n} - u_y\|_\infty \le \Bigl(\|u_n - u\|_\infty + \|r_n' - r'\|_{L^1} \sup_{\mathcal{B}_R} h\Bigr) \mathrm{e}^{\kappa_R \|r_n'\|_\infty} \xrightarrow[n\to\infty]{} 0\,.$$

This completes the proof. □

Remark 4. Suppose h, x, x_n, and y are as in Lemma 4, but y_n satisfies

$$y_n(t) = x_n(t) + \int_0^t h_n(y_n(s))\,\mathrm{d}s\,, \quad t \ge 0\,,$$

for some sequence h_n which converges to h uniformly on compacta. Then a slight variation of the proof of Lemma 4, shows that $y_n \to y$ in D_m.

Control approximation. Given a continuous map $v\colon \mathcal{K}_+ \to \Delta$, we construct a stationary Markov control for the n-system which approximates it in a suitable manner.

Recall that $\langle e, \rho \rangle = 1$. Let

$$\mathcal{X}_n := \left\{ n^{-1/\alpha}(y - \rho n) \colon y \in \mathbb{Z}^m_+ , \langle e, y \rangle > n \right\},$$

and $\mathfrak{Z}_n = \mathfrak{Z}_n(\hat{x})$ denote the set of work-conserving actions at $\hat{x} \in \mathcal{X}_n$. It is clear that a work-conserving action $\hat{z}^n \in \mathfrak{Z}_n(\hat{x})$ can be parameterized via a map $\hat{U}^n \colon \mathcal{K}_+ \to \Delta$, satisfying

$$\hat{z}^n_i(\hat{x}) = \hat{x}_i - \langle e, \hat{x} \rangle^+ \hat{U}^n_i(\hat{x}) . \tag{38}$$

Consider the mapping defined in (38) from $\hat{z}^n \in \mathfrak{Z}_n(\hat{x})$ to $\hat{U}^n$, and denote its image as $\widehat{\mathfrak{U}}_n(\hat{x})$. Let

$$\hat{U}^n[v](\hat{x}) \in \operatorname*{Arg\,min}_{u \in \widehat{\mathfrak{U}}_n(\hat{x})} \bigl| \langle e, \hat{x} \rangle\, u - \langle e, \hat{x} \rangle\, v(\hat{x}) \bigr| , \quad \hat{x} \in \mathcal{X}_n . \tag{39}$$

The function $\hat{U}^n[v]$ has the following property. There exists a constant $\check{c}$ such that with $\check{\mathcal{B}}_n$ denoting the ball of radius $\check{c} n^{\check{\alpha}}$ in $\mathbb{R}^m$, with $\check{\alpha} := 1 - 1/\alpha$, then

$$\sup_{\hat{x} \in \check{\mathcal{B}}_n \cap \mathcal{X}_n} \Bigl| \langle e, \hat{x} \rangle\, \hat{U}^n[v](\hat{x}) - \langle e, \hat{x} \rangle\, v(\hat{x}) \Bigr| \le n^{-1/\alpha} . \tag{40}$$

We have the following functional limit theorem.

Theorem 6. *Let $v \in \widetilde{\mathfrak{U}}_{\mathrm{sm}}$. Under any stationary Markov control $\hat{U}^n[v]$ defined in (39), and provided there exists $X(0)$ such that $\hat{X}^n(0) \Rightarrow X(0)$ as $n \to \infty$, we have*

$$\hat{X}^n \Rightarrow X \qquad in \quad (D_m, M_1) \quad as \quad n \to \infty ,$$

where the limit process X is the unique strong solution to the SDE in (1). The parameters in the drift are given by ℓ_i in (36), μ_i, and γ_i, for $i = 1, \dots, m$.

Proof. The FCLT-scaled processes $\hat{X}^n_i$, $i = 1, \dots, m$, can be represented as

$$\hat{X}^n_i(t) = \hat{X}^n_i(0) + \ell^n_i t - \mu_i \int_0^t \hat{Z}^n_i(s)\, \mathrm{d}s - \gamma_i \int_0^t \hat{Q}^n_i(s)\, \mathrm{d}s + \hat{A}^n_i(t) - \hat{M}^n_{S,i}(t) - \hat{M}^n_{R,i}(t)$$

where ℓ^n_i is defined in (36), with

$$\hat{M}^n_{S,i}(t) = n^{-1/\alpha} \left(S^n_i \left(\mu_i \int_0^t Z^n_i(s)\, \mathrm{d}s \right) - \mu_i \int_0^t Z^n_i(s) \mathrm{d}s \right),$$

$$\hat{M}^n_{R,i}(t) = n^{-1/\alpha} \left(R^n_i \left(\gamma_i \int_0^t Q^n(s)\, \mathrm{d}s \right) - \gamma_i \int_0^t Q^n_i(s) \mathrm{d}s \right),$$

and S^n_i, R^n_i, $i = 1, \dots, m$, are mutually independent rate-one Poisson processes, representing the service and reneging (abandonment), respectively.

The result can be established by mimicking the arguments in the proof of in [4, Theorem 4.1], and applying Lemma 4 and Remark 4, using the function

$$h_n(x) := \ell^n + M\big(x - \langle e, x\rangle^+ \hat{U}^n[v](x)\big) - \langle e, x\rangle^+ \Gamma \hat{U}^n[v](x)\,,$$

and the bound in (40). □

6 Concluding remarks

We have extended some of the results in [4] stated for constant controls, to stationary Markov controls resulting in a locally Lipschitz drift in the case of SDEs driven by α-stable processes, and to all stationary Markov controls in the case of SDEs driven by a Wiener process and a compound Poisson process. The results in this paper can also be viewed as an extension of some results in [3]. However, the work in [3] also studies the prelimit process and establishes tightness of the stationary distributions. To the best of our knowledge, this is an open problem for systems with arrival processes which are renewal with heavy-tailed interarrival times (no second moments). This problem is very important and worth pursuing.

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