J. Math. Pures Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control

Ari Arapostathis [a,*], Anup Biswas [b], Subhamay Saha [c]

[a] *Department of ECE, The University of Texas at Austin, 2501 Speedway, EER 7.824, Austin, TX 78712, USA*
[b] *Department of Mathematics, Indian Institute of Science Education and Research, Dr. Homi Bhabha Road, Pune 411008, India*
[c] *Department of Mathematics, Indian Institute of Technology Guwahati, Assam 781039, India*

ARTICLE INFO

Article history:
Received 24 September 2017
Available online xxxx

MSC:
primary 35P15, 93E20
secondary 35B40, 35Q93, 60J60

Keywords:
Generalized principal eigenvalue
Recurrence and transience
Viscous Hamilton–Jacobi equations
Risk-sensitive control
Ergodic control
Semi-linear eigenvalue problems

ABSTRACT

This paper studies the eigenvalue problem on $\mathbb{R}^d$ for a class of second order, elliptic operators of the form $\mathcal{L}^f = a^{ij}\partial_{x_i}\partial_{x_j} + b^i\partial_{x_i} + f$, associated with non-degenerate diffusions. We show that strict monotonicity of the principal eigenvalue of the operator with respect to the potential function f fully characterizes the ergodic properties of the associated ground state diffusion, and the unicity of the ground state, and we present a comprehensive study of the eigenvalue problem from this point of view. This allows us to extend or strengthen various results in the literature for a class of viscous Hamilton–Jacobi equations of ergodic type with smooth coefficients to equations with measurable drift and potential. In addition, we establish the strong duality for the equivalent infinite dimensional linear programming formulation of these ergodic control problems. We also apply these results to the study of the infinite horizon risk-sensitive control problem for diffusions, and establish existence of optimal Markov controls, verification of optimality results, and the continuity of the controlled principal eigenvalue with respect to stationary Markov controls.

© 2018 Elsevier Masson SAS. All rights reserved.

RÉSUMÉ

Cet article étudie le problème des valeurs propres sur $\mathbb{R}^d$ pour une classe d'opérateurs elliptiques du second ordre $\mathcal{L}f = a^{ij}\partial x_i\partial x_j + b^i\partial x_i + f$, associés à des diffusions non-dégénérées. Nous montrons que la monotonicité stricte de la valeur propre principale de l'opérateur par rapport á la fonction potentielle f caractérise les propriétés ergodiques de la diffusion de l'état fondamental, et l'unicité de l'état fondamental, et nous présentons une étude complète du problème des valeurs propres de ce point de vue. Cela nous permet d'étendre ou renforcer des divers résultats trouvés dans la littérature pour une classe d'équations visqueuses Hamilton–Jacobi du type ergodique avec des coefficients lisses aux équations avec des coefficients mesurables. En outre, nous établissons la dualité forte pour la formulation équivalente de programmation linéaire infini-dimensionnelle de ces problèmes de contrôle ergodiques. Nous appliquons également nos résultats à l'étude

* Corresponding author.
E-mail addresses: ari@ece.utexas.edu (A. Arapostathis), anup@iiserpune.ac.in (A. Biswas), saha.subhamay@iitg.ernet.in (S. Saha).

https://doi.org/10.1016/j.matpur.2018.05.008
0021-7824/© 2018 Elsevier Masson SAS. All rights reserved.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

du problème de contrôle du type sensible au risque à l'horizon infini ; nous établissons ainsi l'existence de contrôles Markoviens optimaux, la vérification des résultats d'optimalité, et la continuité de la valeur propre principale contrôlée par rapport aux contrôles Markoviens stationnaires.

© 2018 Elsevier Masson SAS. All rights reserved.

1. Introduction

In this paper we study the eigenvalue problem on $\mathbb{R}^d$ for non-degenerate, second order elliptic operators $\mathcal{L}^f$ of the form

$$\mathcal{L}^f\varphi \;=\; \sum_{i,j=1}^{d} a^{ij}\frac{\partial^2\varphi}{\partial x_i\partial x_j} + \sum_{i=1}^{d} b^i\frac{\partial\varphi}{\partial x_i} + f\,\varphi\,. \tag{1.1}$$

Here $b, f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$, $a \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^d)$ and a, b satisfy a linear growth assumption in the outward radial direction (see (A2) in Subsection 1.1). In other words, a and b satisfy the usual assumptions for existence and uniqueness of a strong solution of the Itô equation

$$X_t \;=\; x + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \boldsymbol{\sigma}(X_s)\,\mathrm{d}W_s\,, \quad \text{with} \quad a \coloneqq \tfrac{1}{2}\,\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathsf{T}}, \tag{1.2}$$

where W is a standard Brownian motion.

We focus on certain properties of the principal eigenvalue of the operator $\mathcal{L}^f$ which play a key role in infinite horizon risk-sensitive control problems. When D is a smooth bounded domain, and a, b, f are regular enough, existence of a principal eigenvalue and corresponding eigenfunction under a Dirichlet boundary condition can be obtained by an application of Krein–Rutman theory (see for instance [1,2]). This eigenvalue is the bottom of the spectrum of $-\mathcal{L}^f$ with Dirichlet boundary condition. For non-smooth domains, a generalized notion of a principal eigenvalue was introduced in the seminal work of Berestycki, Nirenberg, and Varadhan [3]. An analogous theory for non-linear elliptic operators has been developed by Quaas and Sirakov in [4]. The principal eigenvalue plays a key role in the study of non-homogeneous elliptic operators and the maximum principle (see [3–6]). For some other definitions of the principal (or *critical*) eigenvalue we refer the reader to the works of Pinchover [7] and Pinsky [2, Chapter 4].

For unbounded domains, principal eigenvalue problems have been recently considered by Berestycki and Rossi in [5,8]. Not surprisingly, certain properties of the principal eigenvalue which hold in bounded domains may not be true for unbounded ones. For instance, when D is smooth and bounded it is well known that for the Dirichlet boundary value problem, the principal eigenvalue is simple, and the associated principal eigenfunction is positive. Moreover, it is the unique eigenvalue with a positive eigenfunction. But if D is unbounded and smooth, then there exists a constant $\lambda^* = \lambda^*(f)$ such that any $\lambda \in [\lambda^*, \infty)$ is an eigenvalue of $\mathcal{L}^f$ with a positive eigenfunction [5, Theorem 1.4] (see also [6] and [9, Theorem 2.6]). The lowest such value λ^* serves as a definition of the principal eigenvalue when D is not bounded. The principal eigenvalue is known to be strictly monotone as a function of the bounded domain D (the latter ordered with respect to set inclusion), and also strictly monotone in the coefficient f when the domain is bounded (see [5] and Lemma 2.1 below). These properties fail to hold in unbounded domains as remarked by Berestycki and Rossi [5, Remark 2.4]. Strict monotonicity of $f \mapsto \lambda^*(f)$ and its implications are a central theme in our study. We adopt a probabilistic approach in our investigation. One can view $\lambda^*(f)$ as a risk-sensitive average of f over the diffusion in (1.2). More precisely, since (1.2) has a unique solution which exists for all $t \in [0, \infty)$, then we can define

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

$$\mathcal{E}_x(f) \coloneqq \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}_x\Big[\mathrm{e}^{\int_0^T f(X_s)\,\mathrm{d}s}\Big], \quad x \in \mathbb{R}^d, \tag{1.3}$$

with 'log' denoting the natural logarithm. As shown in the proof of Lemma 2.3 in [10] we have $\lambda^*(f) \le \mathcal{E}_x(f)$, and equality is indeed the case in many important situations, although strictly speaking it is only a heuristic. This heuristic is based on the fact that for a bounded f, the operator $\mathcal{L}^f$ is the infinitesimal generator of a strongly continuous, positive semigroup with potential f, see for instance [11, Chapter IV]. If f is a bounded continuous function, and if the occupation measures of $\{X_t\}$ obey a large deviation principle, then one can express $\mathcal{E}_x(f)$ in terms of the large deviation rate function. This is known as the variational representation for the eigenvalue. See for instance the article by Donsker and Varadhan [12] where this representation is obtained for compact domains. But large deviation principles for $\{X_t\}$ are generally available only under strong hypotheses on the process (see [13]). In this paper we rely on the stochastic representation of the principal eigenfunction which can be established under very mild assumptions. This approach has been recently used by Arapostathis and Biswas in [10] to study the multiplicative Poisson equation when f is *near-monotone* (which includes the case of inf-compact f). By an *eigenpair* of $\mathcal{L}^f$ we mean a pair (Ψ, λ), with Ψ a positive function in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, for all $p \in [1,\infty)$, and $\lambda \in \mathbb{R}$, that satisfies

$$\mathcal{L}^f\Psi \;=\; a^{ij}\partial_{ij}\Psi + b^i\partial_i\Psi + f\,\Psi \;=\; \lambda\Psi\,. \tag{1.4}$$

We refer to λ as the *eigenvalue*, and to Ψ as the *eigenfunction*. In (1.4), and elsewhere in this paper, we adopt the notation $\partial_i \coloneqq \frac{\partial}{\partial x_i}$ and $\partial_{ij} \coloneqq \frac{\partial^2}{\partial x_i \partial x_j}$ for $i, j \in \mathbb{N}$, and use the standard summation rule that repeated subscripts and superscripts are summed from 1 through d.

As mentioned earlier, such a pair (Ψ, λ) exists only if $\lambda \ge \lambda^*(f)$ (see Corollary 2.1). Given an eigenpair (Ψ, λ), the associated *twisted diffusion* Y (a terminology used in [14]) is an Itô process as in (1.2), but with the drift b replaced by $b + 2a\nabla(\log\Psi)$. It is not generally the case that the twisted process has a strong solution which exists for all time. If $\lambda > \lambda^*(f)$ the twisted diffusion is always transient (see Lemma 2.6). When $\lambda = \lambda^*(f)$, the eigenfunction is denoted as Ψ^* and is called the *ground state* [2,15]. The corresponding twisted diffusion, denoted by Y^*, is referred to as the *ground-state diffusion*.

Let $C^+_{\mathrm{o}}(\mathbb{R}^d)$ ($C^+_{\mathrm{c}}(\mathbb{R}^d)$) denote the class of non-zero, nonnegative real valued continuous functions on $\mathbb{R}^d$ which vanish at infinity (have compact support). We say that $\lambda^*(f)$ is *strictly monotone at* f if there exists $h \in C^+_{\mathrm{o}}(\mathbb{R}^d)$ satisfying $\lambda^*(f-h) < \lambda^*(f)$. We also say that $\lambda^*(f)$ is *strictly monotone at* f *on the right* if $\lambda^*(f+h) > \lambda^*(f)$ for all $h \in C^+_{\mathrm{c}}(\mathbb{R}^d)$. In Theorem 2.1 we show that strict monotonicity at f implies strict monotonicity at f on the right. Our main results provide sharp characterizations of the ground state Ψ^* and the ground state process Y^* in terms of these monotonicity properties. Assume that $f\colon \mathbb{R}^d \to \mathbb{R}$ is a locally bounded, Borel measurable function, satisfying $\operatorname{ess\,inf}_{\mathbb{R}^d} f > -\infty$, and that $\lambda^*(f)$ is finite. We show that strict monotonicity of $\lambda^*(f)$ at f on the right implies the simplicity of $\lambda^*(f)$, i.e., the uniqueness of the ground state Ψ^*, and that this is also a necessary and sufficient condition for the ground state process to be recurrent (see Lemma 2.7 and Theorem 2.3). Another important result is that the ground state diffusion is exponentially ergodic (see Definition 2.2) if and only if $\lambda^*(f)$ is strictly monotone at f. These results are summarized in Theorem 2.1 in Section 2. Other results in Section 2 provide a characterization of the eigenvalue in terms of the long time behavior of the twisted process and stochastic representations of the ground state (see Lemmas 2.2, 2.3 and 2.7, and Theorem 2.6).

In [2], Pinsky uses the existence of a Green's measure to define the *critical* eigenvalue of a non-degenerate elliptic operator. This critical eigenvalue coincides with the principal eigenvalue when the boundary of the domain and the coefficients of $\mathcal{L}^f$ are smooth enough. He shows that for any bounded domain, and provided that the coefficients are in $C^{1,\alpha}(\mathbb{R}^d)$, $\alpha > 0$, and bounded, there exists a critical value λ_c such that for any $\lambda > \lambda_c$ we can find a Green's measure corresponding to the operator $\mathcal{L}^{(f-\lambda)}$ [2, Theorem 4.7.1]. The result in Theorem 2.3 in Section 2 extends this to $\mathbb{R}^d$ without assuming much regularity on the coefficients.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

Continuous dependence of λ^* on the coefficients of $\mathcal{L}$ has also been a topic of interest. It is not hard to see that $f \mapsto \lambda^*(f)$ is lower-semicontinuous in the $L^1_{\rm loc}(\mathbb{R}^d)$ topology for f. Continuity of this map is also established in [5, Proposition 9.2] with respect to the $L^\infty(\mathbb{R}^d)$ norm of f. In Theorem 2.6 and Remark 4.1 we study the continuity of $\lambda^*(f)$ for a class of functions f under the $L^1_{\rm loc}(\mathbb{R}^d)$ topology. We also obtain a pinned multiplicative ergodic theorem which is of independent interest, and show that $\mathcal{E}_x(f) = \lambda^*(f)$ for a large class of problems.

We next discuss the connection of this problem with a stochastic ergodic control problem. Defining $\breve\psi := \log \Psi^*$ we obtain from (1.4) that

$$a^{ij}\partial_{ij}\breve\psi + b^i\partial_i\breve\psi - \langle\nabla\breve\psi, a\nabla\breve\psi\rangle \;=\; a^{ij}\partial_{ij}\breve\psi + b^i\partial_i\breve\psi + \min_{u\in\mathbb{R}^d}\bigl[2\langle a\,u, \nabla\breve\psi\rangle + \langle u, au\rangle\bigr] \;=\; f - \lambda^*(f)\,. \tag{1.5}$$

It is easy to see that (1.5) is related to an ergodic control problem with controlled drift $b+2au$ and running cost $\langle u, au\rangle - f(x)$. The parameter $\lambda^*(f)$ can be thought of as the optimal ergodic value; see Ichihara [16]. Note then that the twisted process defined above corresponds to the optimally controlled diffusion. We refer to Ichihara [16,17] and Kaise and Sheu [9] for some important results in this direction. For a potential f that vanishes at infinity, Ichihara [16,18] considers the ergodic control problem in (1.5), with a more general Hamiltonian and under scaling of the potential. When f is nonnegative, it is shown that the value of the ergodic problem with potential βf, $\beta \in \mathbb{R}$, equals the eigenvalue $\lambda^*(\beta f)$, and $\nabla\psi^*$ is the optimal control when the parameter β exceeds a critical value β_c, while below that critical value a bifurcation occurs. Analogous are the results in [19] for viscous Hamilton–Jacobi equations with a the identity matrix and a Hamiltonian which is a power of the gradient term. Most of the above results are obtained for bounded, and Lipschitz continuous a, b, and f. In Theorems 2.7 and 2.8 we extend these results to measurable b and f, and possibly unbounded a and b.

Optimality for the ergodic problem is shown in [16,18] via the study of the optimal finite horizon problem (Cauchy parabolic problem). Inevitably, in doing so, optimality is shown in a certain class of controls. To overcome this limitation, we take a different approach to the ergodic control problem in (1.5). As well known, ergodic control problems can be cast as infinite dimensional linear programs [20,21]. Consider a controlled diffusion, with the control taking values in a space $\mathbb{U}$ with extended generator $\mathcal{A}$, where the 'action' $u \in \mathbb{U}$ enters implicitly as a parameter in $\mathcal{A}$. Let $\mathcal{R}\colon \mathbb{U} \to \mathbb{R}$ denote the running cost. The primal problem then can be written

$$\alpha_* \;=\; \biggl\{\inf \int_{\mathbb{R}^d\times\mathbb{U}} \mathcal{R}(x,u)\,\pi(\mathrm{d}x,\mathrm{d}u) \;:\; \mathcal{A}^*\pi = 0\,,\quad \pi \in \mathcal{P}(\mathbb{R}^d\times\mathbb{U})\biggr\}\,.$$

Here $\mathcal{P}(\mathbb{R}^d\times\mathbb{U})$ denotes the class of probability measures on the Borel σ-field of $\mathbb{R}^d\times\mathbb{U}$. Its elements are called *ergodic occupation measures* (see [20]). The dual problem takes the form

$$\alpha \;=\; \sup\Bigl\{c\in\mathbb{R} :\; \inf_{u\in\mathbb{U}}\bigl[\mathcal{A}g(x,u) + \mathcal{R}(x,u)\bigr] \ge c\,,\quad g\in\mathcal{D}(\mathcal{A})\Bigr\}\,,$$

where $\mathcal{D}(\mathcal{A})$ denotes the domain of $\mathcal{A}$. In other words the dual problem is a maximization over subsolutions of the Hamilton–Jacobi–Bellman (HJB) equation. For non-degenerate diffusions with a compact action space $\mathbb{U}$, under the hypothesis that $\mathcal{R}$ is near-monotone, or under uniform ergodicity conditions, it is well known that we have strong duality, i.e., $\alpha_* = \alpha$. To the best of our knowledge, this has not been established for problems with non-compact action spaces. In Theorem 2.9 we establish strong duality for the ergodic problem in (1.5). In this result, the coefficients b and f are bounded and measurable, and a is bounded, Lipschitz, and uniformly elliptic. Moreover, we establish the unicity of the optimal ergodic occupation measure, and as a result of this, the uniqueness of the optimal stationary Markov control. The methodology

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

is general enough that can be applied to various classes of ergodic control problems that are characterized by viscous HJB equations.

The results in [9,17] are obtained for smooth coefficients ($C^{2,\alpha}$), and under an assumption of exponential ergodicity (see (3.12) below). We provide a sufficient condition in (H2) under which strict monotonicity of the principal eigenvalue holds. It is also shown that the exponential ergodicity condition of [9,17] actually implies (H2); thus (H2) is weaker. Moreover, (3.12) cannot hold for bounded coefficients a and b. See Remark 3.4 for details. In Theorem 3.3 we cite a sufficient condition under which strict monotonicity of $\lambda^*(f)$ holds even when a and b are bounded. Let us also remark that the method of proof [9,17] utilizes the smoothness of the coefficients a, b and f. This is because a gradient estimate (Bernstein method) is required, which is not available under weaker regularity. But this amount of regularity might not be available in many situations, for instance in models with a measurable drift which are often encountered in stochastic control problems. Let us also mention the unpublished work of Kaise and Sheu in [22] that contains some results similar to ours, in particular, similar to the results in Section 3 and the pinned multiplicative ergodic theorems. These results are also obtained under sufficient smoothness of the coefficients a, b, and f.

In Section 4 we apply the above mentioned results to study the infinite horizon risk-sensitive control problem. We refer the reader to [10] where the importance of these control problems is discussed. Unfortunately, the development of the infinite horizon risk-sensitive control problem for controlled diffusions has not been completely satisfactory, and the same applies to controlled Markov chain models. Most of the available results have been obtained under restrictive settings, and a full characterization such as uniqueness of the solution to the risk-sensitive HJB equation, and verification of optimality results is lacking. Let us give a quick overview of the existing literature on risk-sensitive control in the context of controlled diffusions which is relevant to our problem. Risk-sensitive control for models with a constant diffusion matrix and *asymptotically flat* drift is studied by Fleming and McEneaney in [23]. Another particular setting is considered by Nagai [24], where the action space is the whole Euclidean space, and the running cost has a specific structure. Menaldi and Robin have considered models with periodic data [25]. Under the assumption of a near-monotone cost, the infinite horizon risk-sensitive control problem is studied in [10,26,27], whereas Biswas in [28] has considered this problem under the assumption of exponential ergodicity. Differential games with risk-sensitive type costs have been studied by Basu and Ghosh [29], Biswas and Saha [30], and Ghosh et al. [31]. All the above studies, have obtained existence of a pair (V, λ^*) that satisfies the risk-sensitive HJB equation, with λ^* the optimal risk-sensitive value, and show that any minimizing selector of the HJB is an optimal control. The works in [24,25] address the existence and uniqueness of a solution to the HJB equation, in their particular set up, but do not contain any verification of optimality results. Two main results that are missing from the existing literature, with the exception of [10], are (a) uniqueness of the solution to the HJB equation, and (b) verification for optimal control.

Following the ergodic control paradigm, we can identify two classes of models: (i) models with a near-monotone running cost and finite optimal value, and no other hypotheses on the dynamics, and (ii) models that enjoy a uniform exponential ergodicity. Near-monotone running cost models are studied in [10,24,26,27]; however, only [10] obtains a full characterization without imposing a blanket ergodicity hypothesis. Studies for models in class (ii) can be found in [23,28,29,31].

In this paper we study models in class (ii). The results developed in Sections 2 and 3 enable us to obtain a full characterization of the risk-sensitive control problem in Section 4. The main hypotheses are Assumptions 4.1 and 4.2. Another interesting result that we establish in Section 4 is the continuity of the controlled principal eigenvalue with respect to (relaxed) stationary Markov controls (see Theorem 4.3). This facilitates establishing the existence of an optimal stationary Markov control for risk-sensitive control problems under risk-sensitive type constraints. Let us also remark that this existence result is far from being obvious, since the controlled risk-sensitive value is lower-semicontinuous with respect to Markov controls and the equality $\lambda^*(f) = \mathcal{E}(f)$ is not true in general. Moreover, the usual technique of Lagrange multipliers does not work in this situation, because of the non-convex nature of the optimization criterion.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

To summarize the main contributions of the paper, we have established several characterizations of the property of strict monotonicity of the principal eigenvalue, and extended several results in the literature on viscous HJB equations with potentials f vanishing at infinity, and smooth data, to measurable potential and drift (Theorems 2.1, 2.2, 2.6 to 2.10 and 3.2). We have also studied a general class of risk-sensitive control problems under a uniform ergodicity hypothesis, and established the uniqueness of a solution to the HJB equation and verification of optimality results (Theorems 4.1 and 4.2). Equally interesting are the continuity results of the controlled principal eigenvalue with respect to stationary Markov controls (Theorems 4.3 and 4.5).

The paper is organized as follows. Subsection 1.1 states the assumptions on the coefficients of the operator $\mathcal{L}$, and Subsection 1.2 summarizes the notation used in the paper. The first three subsections of Section 2 contain the main results on the principal eigenvalue under minimal assumptions, while Subsection 2.4 is devoted to operators with potential f which vanishes at infinity. Section 3 improves on the results of Section 2, under the assumption that (1.2) is exponentially ergodic. Section 4 is dedicated to the infinite horizon, risk-sensitive optimal control problem.

1.1. Assumptions on the model

The following assumptions on the coefficients of $\mathcal{L}$ are in effect throughout the paper unless otherwise mentioned.

(A1) *Local Lipschitz continuity:* The function $\sigma = [\sigma^{ij}] : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is locally Lipschitz in x with a Lipschitz constant $C_R > 0$ depending on $R > 0$. In other words, with $\|\sigma\| := \sqrt{\operatorname{trace}(\sigma\sigma^{\mathsf{T}})}$, we have

$$\|\sigma(x) - \sigma(y)\| \le C_R\,|x-y| \qquad \forall\, x, y \in B_R\,.$$

We also assume that $b = [b^1, \dots, b^d]^{\mathsf{T}} : \mathbb{R}^d \to \mathbb{R}^d$ is locally bounded and measurable.

(A2) *Affine growth condition:* b and σ satisfy a global growth condition of the form

$$\langle b(x), x\rangle^+ + \|\sigma(x)\|^2 \le C_0\big(1+|x|^2\big) \qquad \forall\, x \in \mathbb{R}^d,$$

for some constant $C_0 > 0$.

(A3) *Nondegeneracy:* For each $R > 0$, it holds that

$$\sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \ge C_R^{-1}|\xi|^2 \qquad \forall\, x \in B_R\,,$$

and for all $\xi = (\xi_1, \dots, \xi_d)^{\mathsf{T}} \in \mathbb{R}^d$, where, as defined earlier, $a = \frac{1}{2}\sigma\sigma^{\mathsf{T}}$.

Let us remark that the assumptions (A1)–(A3) are not optimal, and can be weakened in many situations. For instance, if σ is continuous and its weak derivative lies in $L^{2(d+1)}_{\mathrm{loc}}(\mathbb{R}^d)$, then (2.1) has a unique strong solution (see [32]). The results in this paper can be extended to this setup as well.

1.2. Notation

The standard Euclidean norm in $\mathbb{R}^d$ is denoted by $|\cdot|$, and $\langle\,\cdot\,,\cdot\,\rangle$ denotes the inner product. The set of nonnegative real numbers is denoted by $\mathbb{R}_+$, $\mathbb{N}$ stands for the set of natural numbers, and $\mathbb{1}$ denotes the indicator function. Given two real numbers a and b, the minimum (maximum) is denoted by $a \wedge b$ ($a \vee b$),

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

respectively. The closure, boundary, and the complement of a set $A \subset \mathbb{R}^d$ are denoted by $\bar{A}$, ∂A, and A^c, respectively. We denote by $\uptau(A)$ the *first exit time* of the process $\{X_t\}$ from the set $A \subset \mathbb{R}^d$, defined by

$$\uptau(A) \coloneqq \inf\{t > 0 : X_t \notin A\}.$$

The open ball of radius r in $\mathbb{R}^d$, centered at the origin, is denoted by B_r, and we let $\uptau_r \coloneqq \uptau(B_r)$, and $\check{\uptau}_r \coloneqq \uptau(B_r^c)$.

The term *domain* in $\mathbb{R}^d$ refers to a nonempty, connected open subset of the Euclidean space $\mathbb{R}^d$. For a domain $D \subset \mathbb{R}^d$, the space $C^k(D)$ ($C^\infty(D)$), $k \ge 0$, refers to the class of all real-valued functions on D whose partial derivatives up to order k (of any order) exist and are continuous. Also, $C_b^k(D)$ ($C_b^\infty(D)$) is the class of functions whose partial derivatives up to order k (of any order) are continuous and bounded in D, and $C_{\mathrm{c}}^k(D)$ denotes the subset of $C^k(D)$, $0 \le k \le \infty$, consisting of functions that have compact support. In addition, $C_{\mathrm{o}}(\mathbb{R}^d)$ denotes the class of continuous functions on $\mathbb{R}^d$ that vanish at infinity. By $C_{\mathrm{c}}^+(\mathbb{R}^d)$ and $C_{\mathrm{o}}^+(\mathbb{R}^d)$ we denote the subsets of $C_{\mathrm{c}}(\mathbb{R}^d)$ and $C_{\mathrm{o}}(\mathbb{R}^d)$, respectively, consisting of all non-trivial nonnegative functions. We use the term *non-trivial* to refer to a function that is not a.e. equal to 0. The space $L^p(D)$, $p \in [1,\infty)$, stands for the Banach space of (equivalence classes of) measurable functions f satisfying $\int_D |f(x)|^p \,\mathrm{d}x < \infty$, and $L^\infty(D)$ is the Banach space of functions that are essentially bounded in D. The standard Sobolev space of functions on D whose generalized derivatives up to order k are in $L^p(D)$, equipped with its natural norm, is denoted by $\mathcal{W}^{k,p}(D)$, $k \ge 0$, $p \ge 1$. For a probability measure μ in $\mathcal{P}(\mathbb{R}^d)$ and a real-valued function f which is integrable with respect to μ we use the notation

$$\langle f, \mu \rangle \;=\; \mu(f) \;\coloneqq\; \int_{\mathbb{R}^d} f(x)\, \mu(\mathrm{d}x)\,.$$

In general, if $\mathcal{X}$ is a space of real-valued functions on Q, $\mathcal{X}_{\mathrm{loc}}$ consists of all functions f such that $f\varphi \in \mathcal{X}$ for every $\varphi \in C_{\mathrm{c}}^\infty(Q)$. In this manner we obtain for example the space $\mathcal{W}^{2,p}_{\mathrm{loc}}(Q)$.

We often use Krylov's extension of the Itô formula for functions in $\mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ [33, p. 122], which we refer to as the Itô–Krylov formula.

2. General results

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}, \mathbb{P})$ be a given filtered probability space with a complete, right continuous filtration $\{\mathfrak{F}_t\}$. Let W be a standard Brownian motion adapted to $\{\mathfrak{F}_t\}$. Consider the stochastic differential equation

$$X_t \;=\; X_0 + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \upsigma(X_s)\,\mathrm{d}W_s\,. \tag{2.1}$$

The third term on the right hand side of (2.1) is an Itô stochastic integral. We say that a process $X = \{X_t(\omega)\}$ is a solution of (2.1), if it is $\mathfrak{F}_t$-adapted, continuous in t, defined for all $\omega \in \Omega$ and $t \in [0,\infty)$, and satisfies (2.1) for all $t \in [0,\infty)$ a.s. It is well known that under (A1)–(A3), there exists a unique solution of (2.1) [34, Theorem 2.2.4]. We let $\mathbb{E}_x$ denote the expectation operator on the canonical space of the process with $X_0 = x$, and $\mathbb{P}_x$ the corresponding probability measure. Recall that $\uptau(D)$ denotes the first exit time of the process X from a domain D. The process X is said be *recurrent* if for any bounded domain D we have $\mathbb{P}_x(\uptau(D^c) < \infty) = 1$ for all $x \in \bar{D}^c$. Otherwise the process is called transient. A recurrent process is said to be *positive recurrent* if $\mathbb{E}_x[\uptau(D^c)] < \infty$ for all $x \in \bar{D}^c$. It is known that for a non-degenerate diffusion the property of recurrence (or positive recurrence) is independent of D and x, i.e., if it holds for some domain D and $x \in \bar{D}^c$, then it also holds for every domain D, and all points $x \in \bar{D}^c$ (see [34, Lemma 2.6.12 and Theorem 2.6.10]). We define the extended operator $\mathcal{L}\colon C^2(\mathbb{R}^d) \mapsto L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ associated to (2.1) by

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

$$\mathcal{L}g(x) \;\coloneqq\; a^{ij}(x)\,\partial_{ij}g(x) + b^{i}(x)\,\partial_{i}g(x)\,. \tag{2.2}$$

Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a locally bounded, Borel measurable function, which is bounded from below in $\mathbb{R}^d$, i.e., $\inf_{\mathbb{R}^d} f > -\infty$. We refer to a function f with these properties as a *potential*, and let $\mathcal{L}^f \coloneqq \mathcal{L} + f$.

2.1. Risk-sensitive value and Dirichlet eigenvalues

The following lemma summarizes some results from [3–5] on the eigenvalues of the Dirichlet problem for the operator $\mathcal{L}^f$. For simplicity, we state it for balls B_r, instead of more general domains.

Lemma 2.1. *For each $r \in (0,\infty)$ there exists a unique pair $(\widehat{\Psi}_r, \hat{\lambda}_r) \in \big(\mathcal{W}^{2,p}_{\mathrm{loc}}(B_r) \cap C(\bar{B}_r)\big) \times \mathbb{R}$, for any $p \in [1,\infty)$, satisfying $\widehat{\Psi}_r > 0$ on B_r, $\widehat{\Psi}_r = 0$ on ∂B_r, and $\widehat{\Psi}_r(0) = 1$, which solves*

$$\mathcal{L}\widehat{\Psi}_r(x) + f(x)\,\widehat{\Psi}_r(x) \;=\; \hat{\lambda}_r\,\widehat{\Psi}_r(x) \qquad \text{a.e. } x \in B_r\,, \tag{2.3}$$

with $\mathcal{L}$ as defined in (2.2). Moreover, $\hat{\lambda}_r$ has the following properties:

(a) *The map $r \mapsto \hat{\lambda}_r$ is continuous and strictly increasing.*
(b) *In its dependence on the function f, $\hat{\lambda}_r$ is nondecreasing, convex, and Lipschitz continuous (with respect to the L^∞ norm), with Lipschitz constant 1. In addition, if $f \lneqq f'$, then $\hat{\lambda}_r(f) < \hat{\lambda}_r(f')$.*

Proof. Existence and uniqueness of the solution follow by [4, Theorem 1.1] (see also [3]). Part (a) follows by [5, Theorem 1.10], and (iii)–(iv) of [5, Proposition 2.3], while part (b) follows by [3, Proposition 2.1]. □

We refer to $(\widehat{\Psi}_r, \hat{\lambda}_r)$ as the *eigensolution* of the Dirichlet problem, or the *Dirichlet eigensolution* of $\mathcal{L}^f$ on B_r. Correspondingly, $\hat{\lambda}_r$ and $\widehat{\Psi}_r$ are referred to as the *Dirichlet eigenvalue* and *Dirichlet eigenfunction*, respectively.

Lemma 2.1 (a) motivates the following definition.

Definition 2.1. Let f be a potential. The principal eigenvalue $\lambda^*(f)$ on $\mathbb{R}^d$ of the operator $\mathcal{L}^f$ given in (1.1) is defined as $\lambda^*(f) \coloneqq \lim_{r\to\infty} \hat{\lambda}_r(f)$.

For a potential f we also define

$$\mathcal{E}_x(f) \;\coloneqq\; \limsup_{T\to\infty} \frac{1}{T}\,\log \mathbb{E}_x\Big[\mathrm{e}^{\int_0^T f(X_s)\,\mathrm{d}s}\Big], \quad \text{and} \quad \mathcal{E}(f) \;\coloneqq\; \inf_{x\in\mathbb{R}^d} \mathcal{E}_x(f). \tag{2.4}$$

We refer to $\mathcal{E}(f)$ as the *risk-sensitive* average of f. This quantity plays a key role in our analysis.

We also compare Definition 2.1 with the following definition of the principal eigenvalue, commonly used in the pde literature [5].

$$\hat{\Lambda}(f) \;=\; \inf\Big\{\lambda \in \mathbb{R} :\; \exists\, \varphi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d),\, \varphi > 0,\, \mathcal{L}\varphi + (f-\lambda)\varphi \le 0,\ \text{a.e. in } \mathbb{R}^d\Big\}. \tag{2.5}$$

The following hypothesis is enforced throughout Section 2 without further mention, and it is repeated only for emphasis.

(H1) f is a potential, and $\lambda^*(f)$ is finite.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

Lemma 2.2. *The following hold*

(i) *For any $r > 0$, the Dirichlet eigensolutions $(\widehat{\Psi}_n, \hat{\lambda}_n)$ in (2.3) have the following stochastic representation*

$$\widehat{\Psi}_n(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check{\tau}_r}[f(X_t)-\hat{\lambda}_n]\,\mathrm{d}t}\,\widehat{\Psi}_n(X_{\check{\tau}_r})\,\mathbb{1}_{\{\check{\tau}_r<\tau_n\}}\Big] \qquad \forall\, x \in B_n \setminus \overline{B}_r\,, \tag{2.6}$$

for all large enough $n \in \mathbb{N}$.

(ii) *It holds that $\lambda^*(f) = \hat{\Lambda}(f)$.*

(iii) *Let Ψ^* be any limit point of the Dirichlet eigensolutions $(\widehat{\Psi}_n, \hat{\lambda}_n)$ as $n \to \infty$, and $\mathcal{B}$ be an open ball centered at 0 such that $\lambda^*(f-h) + \sup_{\mathcal{B}^c}|h| < \lambda^*(f) < \infty$ for some bounded function h. Then, with $\check{\tau}$ denoting the first hitting time of $\mathcal{B}$, we have*

$$\Psi^*(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check{\tau}}[f(X_t)-\lambda^*(f)]\,\mathrm{d}t}\,\Psi^*(X_{\check{\tau}})\,\mathbb{1}_{\{\check{\tau}<\infty\}}\Big] \qquad \forall\, x \in \mathcal{B}^c\,. \tag{2.7}$$

Proof. Part (i) follows from [10, Lemma 2.10 (i)].

Turning to part (ii), suppose that $\lambda^*(f)$ is finite. Then it is standard to show that there exists a positive $\Psi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ which satisfies

$$\mathcal{L}\Psi + f\,\Psi \;=\; \lambda^*(f)\,\Psi \quad \text{a.e. on } \mathbb{R}^d\,. \tag{2.8}$$

See [10,26] for instance. It is then clear that $\lambda^*(f) \ge \hat{\Lambda}(f)$.

To show the converse inequality, suppose that a pair $(\varphi, \lambda) \in \mathcal{W}^{2,d}_{\mathrm{loc}} \times \mathbb{R}$, with $\varphi > 0$, satisfies

$$\mathcal{L}\varphi + (f-\lambda)\varphi \;\le\; 0\,, \qquad \text{and} \quad \lambda \ge \hat{\Lambda}(f)\,. \tag{2.9}$$

We claim that $\lambda^*(f) \le \lambda$. If not, then we can find a pair $(\widehat{\Psi}_r, \hat{\lambda}_r)$ as in by Lemma 2.1, satisfying (2.3) and $\hat{\lambda}_r > \lambda$. By the Itô–Krylov formula [33, p. 122] we have

$$\varphi(x) \;\ge\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check{\tau}_r}[f(X_t)-\lambda]\,\mathrm{d}t}\,\varphi(X_{\check{\tau}_r})\,\mathbb{1}_{\{\check{\tau}_r<\infty\}}\Big]\,. \tag{2.10}$$

Since φ is positive, (2.6) and (2.10) imply that we can scale it by multiplying with a constant $\kappa > 0$ so that $\kappa\varphi - \widehat{\Psi}_r$ attains it minimum in $\bar{B}_r$ and this minimum value is 0. Combining (2.3) and (2.9), we obtain

$$\mathcal{L}(\kappa\varphi - \widehat{\Psi}_r) - (f-\hat{\lambda}_r)^-(\kappa\varphi - \widehat{\Psi}_r) \;\le\; -(f-\hat{\lambda}_r)^+(\kappa\varphi - \widehat{\Psi}_r) + (-\hat{\lambda}_r + \lambda)\kappa\varphi \;\le\; 0 \quad \text{in } B_r\,.$$

It then follows by the strong maximum principle [35, Theorem 9.6] that $\kappa\varphi - \widehat{\Psi}_r = 0$ in $\bar{B}_r$, which is not possible since $\varphi > 0$ on $\mathbb{R}^d$. This proves the claim. Since λ was arbitrary, this implies that $\hat{\Lambda}(f) \ge \lambda^*(f)$, and thus we have equality.

It remains to prove (2.7). We follow the same argument as in [10, Lemma 2.10]. We fix $\mathcal{B} = B_r$. Letting $n \to \infty$ in (2.6) and applying Fatou's lemma we obtain

$$\Psi^*(x) \;\ge\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check{\tau}}[f(X_t)-\lambda^*(f)]\,\mathrm{d}t}\,\Psi^*(X_{\check{\tau}})\,\mathbb{1}_{\{\check{\tau}<\infty\}}\Big]\,. \tag{2.11}$$

Thus, with $\tilde{\Psi}^*$ denoting a solution of (2.8), with f replaced by $f-h$ and $\lambda = \lambda^*(f-h)$, we also have

$$\tilde{\Psi}^*(x) \;\ge\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check{\tau}}[f(X_t)-h(X_t)-\lambda^*(f-h)]\,\mathrm{d}t}\,\tilde{\Psi}^*(X_{\check{\tau}})\,\mathbb{1}_{\{\check{\tau}<\infty\}}\Big]\,,$$

which implies that

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[f(X_t)-h(X_t)-\lambda^*(f-h)]\,\mathrm{d}t}\,\mathbb{1}_{\{\check\tau<\infty\}}\Big] \;<\; \infty \qquad \forall\, x\in\mathcal{B}^c\,, \tag{2.12}$$

since $\tilde\Psi^*>0$ in $\mathbb{R}^d$. We write (2.6) as

$$\widehat\Psi_n(x) \;\le\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[f(X_t)-\hat\lambda_n]\,\mathrm{d}t}\,\Psi^*(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\tau_n\}}\Big] \;+\; \Big(\sup_{\mathcal{B}}\big|\Psi^*-\widehat\Psi_n\big|\Big)\;\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[f(X_t)-\hat\lambda_n]\,\mathrm{d}t}\,\mathbb{1}_{\{\check\tau<\tau_n\}}\Big]\,. \tag{2.13}$$

Note that since $\hat\lambda_n\nearrow\lambda^*(f)$, the first term on the right hand side of (2.13) is finite by (2.12) for all large enough n. Let

$$\kappa_n \;:=\; \Big(\inf_{\mathcal{B}}\widehat\Psi_n\Big)^{-1}\sup_{\mathcal{B}}\big|\Psi^*-\widehat\Psi_n\big|\,.$$

The second term on the right hand side of (2.13) has the bound

$$\Big(\sup_{\mathcal{B}}\big|\Psi^*-\widehat\Psi_n\big|\Big)\;\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[f(X_t)-\hat\lambda_n]\,\mathrm{d}t}\,\mathbb{1}_{\{\check\tau<\tau_n\}}\Big] \;\le\; \kappa_n\;\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[f(X_t)-\hat\lambda_n]\,\mathrm{d}t}\,\widehat\Psi_n(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\tau_n\}}\Big] \;=\; \kappa_n\;\widehat\Psi_n(x)\,.$$

By the convergence of $\widehat\Psi_n\to\Psi^*$ as $n\to\infty$, uniformly on compact sets, and since $\widehat\Psi_n$ is bounded away from 0 in $\mathcal{B}$, uniformly in $n\in\mathbb{N}$, by Harnack's inequality, we have $\kappa_n\to0$ as $n\to\infty$. Therefore, the second term on the right hand side of (2.13) vanishes as $n\to\infty$. Also, since $\hat\lambda_n$ is nondecreasing in n, and $\hat\lambda_n\nearrow\lambda^*(f)$, we obtain

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[f(X_t)-\hat\lambda_n]\,\mathrm{d}t}\,\Psi^*(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\tau_n\}}\Big] \;\xrightarrow[n\to\infty]{}\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[f(X_t)-\lambda^*(f)]\,\mathrm{d}t}\,\Psi^*(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\infty\}}\Big]\,, \tag{2.14}$$

by (2.12) and dominated convergence. Thus taking limits in (2.13) as $n\to\infty$, and using (2.11) and (2.14), we obtain (2.7). This completes the proof. □

Combining Lemma 2.2 (ii) and [5, Theorem 1.4] we have the following result.

Corollary 2.1. *There exists a positive* $\Psi\in\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p\ge1$, *satisfying*

$$\mathcal{L}\Psi+f\,\Psi \;=\; \lambda\Psi \quad \text{a.e. on } \mathbb{R}^d\,, \tag{2.15}$$

if and only if $\lambda\ge\lambda^*(f)$.

As also mentioned in the introduction, throughout the rest of the paper, by an eigenpair (Ψ,λ) of $\mathcal{L}^f$ we mean a positive function $\Psi\in\mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ and a scalar $\lambda\in\mathbb{R}$ that satisfy (2.15). In addition, the eigenfunction Ψ is assumed to be normalized as $\Psi(0)=1$, unless indicated otherwise. When λ is the principal eigenvalue, we refer to (Ψ,λ) as a *principal eigenpair*. Note, that in view of the assumptions on the coefficients, any $\Psi\in\mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ which satisfies (2.15) belongs to $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, for all $p\in[1,\infty)$. Therefore, in the interest of notational economy, we refrain from mentioning the function space of solutions Ψ of equations of the form (2.15), and any such solution is meant to be in $\mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$. Moreover, since these are always strong solutions, we often suppress the qualifier 'a.e.', and unless a different domain is specified, such equations or inequalities are meant to hold on $\mathbb{R}^d$.

2.2. Summary of the results

A major objective in this paper is to relate the properties of the eigenvalues λ in (2.15) to the recurrence properties of the *twisted process* which is defined as follows. For an eigenfunction Ψ satisfying (2.15) we let $\psi:=\log\Psi$. Then we can write (2.15) as

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

$$\mathcal{L}\psi + \langle \nabla\psi, a\nabla\psi\rangle + f \;=\; \lambda\,. \tag{2.16}$$

The twisted process corresponding to an eigenpair (Ψ, λ) of $\mathcal{L}^f$ is defined by the SDE

$$\mathrm{d}Y_s \;=\; b(Y_s)\mathrm{d}s + 2a(Y_s)\nabla\psi(Y_s)\,\mathrm{d}s + \upsigma(Y_s)\,\mathrm{d}W_s\,. \tag{2.17}$$

Since $\psi \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p > d$, it follows that $\nabla\psi$ is locally bounded (in fact it is locally Hölder continuous), and therefore (2.17) has a unique strong solution up to its explosion time. We let $\widetilde{\mathcal{L}}^{\psi}$ denote the extended generator of (2.17), and $\widetilde{\mathbb{E}}^{\psi}_x$ the associated expectation operator. The reader might have observed that the twisted process corresponds to Doob's h-transformation of the operator $\mathcal{L}^{(f-\lambda)}$ with $h = \Psi$.

With Ψ^* denoting a principal eigenfunction, i.e., an eigenfunction associated with $\lambda^*(f)$, we let $\psi^* := \log\Psi^*$, and denote by Y^* the corresponding twisted process. A twisted process corresponding to a principal eigenpair is called a *ground state process*, and the eigenfunction Ψ^* is called a *ground state*.

Recall that $C^+_{\mathrm{o}}(\mathbb{R}^d)$ denotes the collection of all non-trivial, nonnegative, continuous functions which vanish at infinity. We consider the following two properties of $\lambda^*(f)$.

(P1) Strict monotonicity at f. For some $h \in C^+_{\mathrm{o}}(\mathbb{R}^d)$ we have $\lambda^*(f-h) < \lambda^*(f)$.
(P2) Strict monotonicity at f on the right. For all $h \in C^+_{\mathrm{o}}(\mathbb{R}^d)$ we have $\lambda^*(f) < \lambda^*(f+h)$.

It follows by the convexity of $f \mapsto \lambda^*(f)$ that (P1) implies (P2).

Later, in Section 3, we provide sufficient conditions under which (P1) holds. Also, the finiteness of $\lambda^*(f)$ and $\lambda^*(f-h)$ is implicit in (P1). Indeed, since for every positive $\varphi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$ we have

$$\mathcal{L}\varphi + (f - \lambda - \|h\|_\infty)\varphi \;\le\; \mathcal{L}\varphi + (f-h-\lambda)\varphi \;\le\; \mathcal{L}\varphi + (f-\lambda)\varphi\,,$$

it follows that $\lambda^*(f-h)$ and $\lambda^*(f)$ are either both finite, or both equal to $\pm\infty$. It is also clear that $\lambda^*(f-h) \le \lambda^*(f)$ always hold. As shown in Theorem 2.2, (P1) implies that $\lambda^*(f-h) < \lambda^*(f)$ for all $h \in C^+_{\mathrm{o}}(\mathbb{R}^d)$.

We introduce the following definition of exponential ergodicity which we often use.

Definition 2.2 *(exponential ergodicity).* The process X governed by (1.2) is said to be exponentially ergodic if for some compact set $\mathcal{B}$ and $\delta > 0$ we have $\mathbb{E}_x\bigl[\mathrm{e}^{\delta\,\uptau(\mathcal{B}^c)}\bigr] < \infty$, for all $x \in \mathcal{B}^c$.

The main results of this section center around the following theorem.

Theorem 2.1. *Under* (H1)*, the following hold:*

(a) *A ground state process is recurrent if and only if $\lambda^*(f)$ is strictly monotone at f on the right, in which case the principal eigenvalue $\lambda^*(f)$ is also simple, and the ground state Ψ^* satisfies*

$$\Psi^*(x) \;=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_r}[f(X_s)-\lambda^*(f)]\,\mathrm{d}s}\,\Psi^*(X_{\uptau})\,\mathbb{1}_{\{\uptau_r<\infty\}}\Bigr] \qquad \forall\, x \in \bar{B}_r^c\,, \quad \forall\, r > 0\,. \tag{2.18}$$

(b) *The ground state process is exponentially ergodic if and only if $\lambda^*(f)$ is strictly monotone at f.*
(c) *If $\lambda > \lambda^*(f)$, the twisted process (2.17) corresponding to any solution ψ of (2.16) is transient.*

Proof. Part (a) follows by Lemma 2.7, Theorem 2.3, and Corollary 2.3. Part (b) is the statement of Theorem 2.2, while part (c) is shown in Lemma 2.6. □

Theorem 2.1 should be compared with the results in [17, Theorem 2.2] and [9, Theorem 3.2 and 3.7]. The results in [9,17] are obtained under a stronger hypothesis (same as (3.12) below) and for sufficiently

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

regular coefficients. For a similar result in a bounded domain we refer the reader to [2, Theorem 4.2.4], where results are obtained for a certain class of operators with regular coefficients.

We remark that (P1) does not imply that the underlying process in (2.1) is recurrent. Indeed consider a one-dimensional diffusion with $b(x) = \frac{3}{2}x$ and $\sigma = 1$, and let $f(x) = x^2$. Then (2.15) holds with $\Psi(x) = \mathrm{e}^{-x^2}$ and $\lambda = -1$. But $b(x) + 2a\nabla\psi = -\frac{1}{2}x$, so the twisted process is exponentially ergodic, while the original diffusion is transient.

The proof of Theorem 2.1 is divided in several lemmas which also contain results of independent interest. These occupy the next section.

2.3. Proof of Theorem 2.1 and other results

In the sequel, we often use the following finite time representation. This also appears in [10, Lemma 2.4] but in a slightly different form. Let $\tau_\infty := \lim_{n\to\infty} \tau_n$ where τ_n denotes the exit time from the ball B_n. Recall that if (Ψ, λ) is an eigenpair of $\mathcal{L}^f$, and $\psi = \log \Psi$, then $\widetilde{\mathbb{E}}^{\psi}_x$ denotes the expectation operator associated with the twisted process Y in (2.17).

Lemma 2.3. *If (Ψ, λ) is an eigenpair of $\mathcal{L}^f$, then*

$$\Psi(x)\,\widetilde{\mathbb{E}}^{\psi}_x\bigl[g(Y_T)\,\Psi^{-1}(Y_T)\,\mathbb{1}_{\{T<\tau_\infty\}}\bigr] \;=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^T [f(X_t)-\lambda]\,\mathrm{d}t}\,g(X_T)\Bigr] \qquad \forall\, T>0\,, \quad \forall\, x\in\mathbb{R}^d\,, \tag{2.19}$$

and for any function $g \in C_{\mathrm{c}}(\mathbb{R}^d)$, where Y is the corresponding twisted process defined by (2.17).

Proof. The equation in (2.19) can be obtained by applying the Cameron–Martin–Girsanov theorem [36, p. 225]. Since ψ and f are not bounded, we need to localize the martingale. We use the first exit times τ_n from B_n as localization times. It is well-known that assumption (A2) implies that $\tau_n \to \infty$ as $n \to \infty$ $\mathbb{P}_x$-a.s. Applying the Itô–Krylov formula and using (2.16), we obtain

$$\begin{aligned}
\psi(X_{T\wedge\tau_n}) - \psi(x) \;&=\; \int_0^{T\wedge\tau_n} \mathcal{L}\psi(X_s)\,\mathrm{d}s + \int_0^{T\wedge\tau_n} \langle\nabla\psi(X_s), \sigma(X_s)\,\mathrm{d}W_s\rangle \\
&=\; \int_0^{T\wedge\tau_n} \Bigl(\lambda - f(X_s) - \langle\nabla\psi, a\nabla\psi\rangle(X_s)\Bigr)\,\mathrm{d}s + \int_0^{T\wedge\tau_n} \langle\nabla\psi(X_s), \sigma(X_s)\mathrm{d}W_s\rangle\,.
\end{aligned} \tag{2.20}$$

Let g be any nonnegative, continuous function with compact support. Then from (2.20) we obtain

$$\begin{aligned}
\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{T\wedge\tau_n}[f(X_s)-\lambda]\,\mathrm{d}s}\,g(X_{T\wedge\tau_n})\Bigr] \;&=\; \mathbb{E}_x\Biggl[g(X_{T\wedge\tau_n})\exp\Biggl(-\psi(X_{T\wedge\tau_n}) + \psi(x) \\
&\qquad + \int_0^{T\wedge\tau_n} \langle\nabla\psi(X_s), \sigma(X_s)\mathrm{d}W_s\rangle - \int_0^{T\wedge\tau_n} \langle\nabla\psi, a\nabla\psi\rangle(X_s)\,\mathrm{d}s\Biggr)\Biggr] \\
&= \Psi(x)\,\widetilde{\mathbb{E}}^{\psi}_x\bigl[g(Y_{T\wedge\tau_n})\,\Psi^{-1}(Y_{T\wedge\tau_n})\bigr]\,,
\end{aligned} \tag{2.21}$$

where in the last line we use Girsanov's theorem. Given any bounded ball $\mathcal{B}$, by Itô's formula and Fatou's lemma, we obtain from (2.15) that

$$\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^T [f(X_t)-\lambda]\,\mathrm{d}t}\,\mathbb{1}_{\mathcal{B}}(X_T)\Bigr] \;\le\; \Bigl(\inf_{\mathcal{B}}\,\Psi\Bigr)^{-1}\Psi(x) \qquad \forall\, T>0\,, \quad \forall\, x\in\mathbb{R}^d\,. \tag{2.22}$$

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

Therefore, if we write

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{T\wedge\tau_n}[f(X_s)-\lambda]\,\mathrm{d}s}\,g(X_{T\wedge\tau_n})\Big] \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\tau_n}[f(X_s)-\lambda]\,\mathrm{d}s}\,g(X_{\tau_n})\,\mathbb{1}_{\{T\ge\tau_n\}}\Big] + \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{T}[f(X_s)-\lambda]\,\mathrm{d}s}\,g(X_T)\,\mathbb{1}_{\{T<\tau_n\}}\Big],$$

we deduce that the first term on the right hand side is equal to 0 for all n sufficiently large since g is compactly supported, while the second term converges as $n\to\infty$ to the right hand side of (2.19) by (2.22) and dominated convergence. In addition, since g has compact support, the term inside the expectation in the right hand side of (2.21) is bounded uniformly in n. Since also $\widetilde{\mathbb{E}}^{\psi}_x\big[g(Y_{\tau_n})\,\Psi^{-1}(Y_{\tau_n})\big]=0$ for all sufficiently large n, letting $n\to\infty$ in (2.21), we obtain

$$\begin{aligned}\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{T}[f(X_s)-\lambda]\,\mathrm{d}s}\,g(X_T)\Big] &= \lim_{n\to\infty}\;\Psi(x)\,\widetilde{\mathbb{E}}^{\psi}_x\Big[g(Y_T)\,\Psi^{-1}(Y_T)\,\mathbb{1}_{\{T<\tau_n\}}\Big]\\ &= \Psi(x)\,\widetilde{\mathbb{E}}^{\psi}_x\Big[g(Y_T)\,\Psi^{-1}(Y_T)\,\mathbb{1}_{\{T<\tau_\infty\}}\Big]\qquad\forall\,T>0\,.\end{aligned}$$

This proves (2.19). □

Recall that $\boldsymbol{\tau}_\infty \coloneqq \lim_{n\to\infty}\boldsymbol{\tau}_n$. An immediate corollary to Lemma 2.3 is the following.

Corollary 2.2. *With (Ψ,λ) as in Lemma 2.3, we have*

$$\Psi(x)\,\widetilde{\mathbb{P}}^{\psi}_x(T<\boldsymbol{\tau}_\infty) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{T}[f(X_t)-\lambda]\,\mathrm{d}t}\,\Psi(X_T)\Big]\qquad\forall\,T>0\,,\quad\forall\,x\in\mathbb{R}^d\,.$$

Proof. Choose a sequence of cut-off functions g_n that approximates unity from below. Then (2.19) holds with g replaced by $g_n\Psi$. Thus the result follows by letting $n\to\infty$ and applying the monotone convergence theorem. □

We are now ready to prove uniqueness of the principal eigenfunction.

Lemma 2.4. *Under* (P1) *there exists a unique ground state Ψ^* for $\mathcal{L}^f$, i.e., a positive $\Psi^*\in\mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$, $\Psi^*(0)=1$, which solves*

$$\mathcal{L}\Psi^* + f\,\Psi^* \;=\; \lambda^*(f)\,\Psi^*\,. \tag{2.23}$$

Proof. Let Ψ^* be a solution of (2.23) obtained as a limit of $\widehat{\Psi}_r$ (see Lemma 2.2). Thus by Lemma 2.2 (iii) we can find an open ball $\mathcal{B}$ such that

$$\Psi^*(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[f(X_s)-\lambda^*(f)]\,\mathrm{d}s}\,\Psi^*(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\infty\}}\Big]\,,\quad x\in\mathcal{B}^c\,,$$

with $\check\tau=\tau(\mathcal{B}^c)$. Suppose that $\tilde\Psi$ is another principal eigenfunction of $\mathcal{L}^f$. By the Itô–Krylov formula and Fatou's lemma, and since Ψ^* is positive on $\bar{\mathcal{B}}$, we obtain

$$\begin{aligned}\tilde\Psi(x) &\ge \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[f(X_s)-\lambda^*(f)]\,\mathrm{d}s}\,\tilde\Psi(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\infty\}}\Big]\\ &\ge \Big(\min_{\bar{\mathcal{B}}}\tfrac{\tilde\Psi}{\Psi^*}\Big)\,\Psi^*(x)\qquad\forall\,x\in\mathcal{B}^c\,.\end{aligned}\tag{2.24}$$

It is clear by (2.24) that if $\tilde\Psi>\Psi^*$ on $\bar{\mathcal{B}}$, then $\tilde\Psi-\Psi^*>0$ on $\mathbb{R}^d$. Therefore, we can scale Ψ^* by multiplying it with $\min_{\bar{\mathcal{B}}}\frac{\tilde\Psi}{\Psi^*}$ so that $\tilde\Psi$ touches Ψ^* from above in $\bar{\mathcal{B}}$ at the points $\arg\min_{\bar{\mathcal{B}}}\frac{\tilde\Psi}{\Psi^*}$. Denoting this scaled Ψ^*

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

also as Ψ^*, it follows from (2.24) that $\tilde\Psi - \Psi^*$ is nonnegative in $\mathbb{R}^d$, and its minimum is 0 and attained in $\bar{\mathcal{B}}$. On the other hand, we have

$$\mathcal{L}(\tilde\Psi - \Psi^*) - \big(f - \lambda^*(f)\big)^-(\tilde\Psi - \Psi^*) \;=\; -\big(f - \lambda^*(f)\big)^+(\tilde\Psi - \Psi^*) \;\le\; 0\,.$$

Thus $\tilde\Psi - \Psi^* = 0$ by the strong maximum principle [35, Theorem 9.6], and this proves the result. □

We next show that (P1) implies the exponential ergodicity of Y^*.

Lemma 2.5. *Assume* (P1). *Let Ψ^* be the ground state of $\mathcal{L}^f$, and $\psi^* = \log \Psi^*$. Then the ground state process Y^* governed by*

$$\mathrm{d}Y^*_s \;=\; b(Y^*_s)\,\mathrm{d}s + 2a(Y^*_s)\nabla\psi^*(Y^*_s)\,\mathrm{d}s + \boldsymbol{\sigma}(Y^*_s)\,\mathrm{d}W_s\,, \tag{2.25}$$

is exponentially ergodic. In particular, Y^* is positive recurrent.

Proof. We first show that the finite time representation of Ψ^* holds. Let $\tilde\lambda^* := \lambda^*(f-h)$, and $\mathcal{B}$ be a ball as in Lemma 2.2 (iii). Recall that $\hat\lambda_n \to \lambda^*(f)$ as $n\to\infty$, and therefore, we have $\hat\lambda_n > \tilde\lambda^* + \sup_{\mathcal{B}^c}|h|$ for all sufficiently large n. Consider the following equations

$$\begin{aligned}
\mathcal{L}\widehat\Psi_n + f\widehat\Psi_n \;&=\; \hat\lambda_r\widehat\Psi_n\,, \quad \widehat\Psi_n > 0, \quad \widehat\Psi_n(0) = 1, \quad \widehat\Psi_n = 0 \;\text{ on } \partial B_n\,,\\
\mathcal{L}\tilde\Psi^* + (f-h)\tilde\Psi^* \;&=\; \tilde\lambda^*\tilde\Psi^*, \quad \tilde\Psi(0) = 1\,.
\end{aligned}$$

Choose n large enough so that $\mathcal{B}\subset B_n$. We can scale $\tilde\Psi^*$, by multiplying it with a positive constant, so that $\tilde\Psi^*$ touches $\widehat\Psi_n$ from above. Next we show that it can only touch $\widehat\Psi_n$ in $\mathcal{B}$. Note that in $B_n\setminus\mathcal{B}$ we have

$$\mathcal{L}(\tilde\Psi^* - \widehat\Psi_n) - (f-h-\tilde\lambda^*)^-(\tilde\Psi^* - \widehat\Psi_n) \;=\; -(f-h-\tilde\lambda^*)^+(\tilde\Psi^* - \widehat\Psi_n) - (\hat\lambda_n - \tilde\lambda^* - h)\,\widehat\Psi_n \;\le\; 0\,.$$

Therefore, by the strong maximum principle, if $(\tilde\Psi^* - \widehat\Psi_n)$ attains its minimum in $B_n\setminus\mathcal{B}$, then $(\tilde\Psi^* - \widehat\Psi_n) = 0$ in B_n, which is not possible. Thus $\tilde\Psi^*$ touches $\widehat\Psi_n$ in $\mathcal{B}$. Thus, applying Harnack's inequality we can find a constant κ_1 such that $\kappa_1\tilde\Psi^* \ge \widehat\Psi_n$ for all sufficiently large n. On the other hand, by the Itô–Krylov formula and Fatou's lemma we know that

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^T [f(X_s) - h(X_s) - \tilde\lambda^*]\,\mathrm{d}s}\,\tilde\Psi^*(X_T)\Big] \;\le\; \tilde\Psi^*(x) \qquad \forall\, T>0\,. \tag{2.26}$$

Applying the Itô–Krylov formula to (2.3) we have

$$\widehat\Psi_n(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^T [f(X_s) - \hat\lambda_n]\,\mathrm{d}s}\,\widehat\Psi_n(X_T)\,\mathbb{1}_{\{T<\tau_n\}}\Big]\,,$$

and letting $n\to\infty$, using (2.26) and the dominated convergence theorem, we obtain

$$\Psi^*(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^T [f(X_s) - \lambda^*]\,\mathrm{d}s}\,\Psi^*(X_T)\Big]\,,$$

where Ψ^* is the unique solution of (2.23). This proves the finite time representation. Thus it follows from Corollary 2.2 that (2.25) is regular, i.e., $\widetilde{\mathbb{P}}_x^{\psi^*}(\tau_\infty < \infty) = 0$.

If we define $\Phi := \frac{\tilde\Psi^*}{\Psi^*}$, a straightforward calculation shows that

$$\widetilde{\mathcal{L}}^{\psi^*}\Phi \;=\; \mathcal{L}\Phi + 2\langle a\nabla\psi^*, \nabla\Phi\rangle \;=\; (\lambda^*(f-h) - \lambda^*(f) + h)\Phi \;\le\; C\mathbb{1}_{\mathcal{B}} - \epsilon\Phi \tag{2.27}$$

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

for some positive constants C and ϵ. Recall $\mathcal{B}$ from Lemma 2.2 (iii). It is easy to see from (2.7) that

$$\Phi(x) \;\ge\; \frac{\min_{\mathcal{B}} \tilde\Psi^*}{\max_{\mathcal{B}} \Psi^*}\, \frac{\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\breve\tau}[f(X_t)-h(X_t)-\lambda^*(f-h)]\,\mathrm{d}t}\,\mathbb{1}_{\{\breve\tau<\infty\}}\Big]}{\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\breve\tau}[f(X_t)-\lambda^*(f)]\,\mathrm{d}t}\,\mathbb{1}_{\{\breve\tau<\infty\}}\Big]} \;\ge\; \frac{\min_{\mathcal{B}} \tilde\Psi^*}{\max_{\mathcal{B}} \Psi^*} \quad \forall\, x\in\mathcal{B}^c\,.$$

Thus Φ is uniformly bounded from below by a positive constant. Since Y^* in (2.25) is regular, the Foster–Lyapunov inequality in (2.27) implies that Y^* is exponentially ergodic. □

We denote the invariant measure of (2.25) by μ^*. The following lemma shows that the twisted process is transient for any $\lambda > \lambda^*(f)$.

Lemma 2.6. *Let Ψ be an eigenfunction of $\mathcal{L}^f$ for an eigenvalue $\lambda > \lambda^*(f)$. Then the corresponding twisted process Y is transient.*

Proof. Let $\psi = \log\Psi$. If $\widetilde{\mathbb{P}}^{\psi}_x(\tau_\infty < \infty) > 0$, then there is nothing to prove. So we assume the contrary. Hence from Lemma 2.3 we have

$$\Psi(x)\,\widetilde{\mathbb{E}}^{\psi}_x\big[g(Y_T)\,\Psi^{-1}(Y_T)\big] \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^T[f(X_t)-\lambda]\,\mathrm{d}t}\,g(X_T)\Big] \qquad \forall\, T>0\,, \tag{2.28}$$

for any continuous g with compact support. Let $g\in C_{\mathrm{c}}^+(\mathbb{R}^d)$. By the Itô–Krylov formula and Fatou's lemma, we have

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^T[f(X_t)-\lambda^*(f)]\,\mathrm{d}t}\,g(X_T)\Big] \;\le\; \Big(\sup_{\mathbb{R}^d}\frac{g}{\Psi^*}\Big)\,\mathbb{E}_x\Big[\mathrm{e}^{\int_0^T[f(X_t)-\lambda^*(f)]\,\mathrm{d}t}\,\Psi^*(X_T)\Big] \;\le\; \Big(\sup_{\mathbb{R}^d}\frac{g}{\Psi^*}\Big)\,\Psi^*(x)\,.$$

Thus, for $\delta = \lambda - \lambda^*(f) > 0$, we obtain

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^T[f(X_t)-\lambda]\,\mathrm{d}t}\,g(X_T)\Big] \;\le\; \Big(\sup_{\mathbb{R}^d}\frac{g}{\Psi^*}\Big)\,\mathrm{e}^{-\delta T}\,\Psi^*(x)\,, \quad T>0\,. \tag{2.29}$$

Combining (2.28) and (2.29), we have

$$\Big(\sup_{\text{support}(g)}\Psi\Big)^{-1}\int_0^\infty \widetilde{\mathbb{E}}^{\psi}_x\big[g(Y_t)\big]\,\mathrm{d}t \;\le\; \int_0^\infty \widetilde{\mathbb{E}}^{\psi}_x\Big[\tfrac{g(Y_t)}{\Psi(Y_t)}\Big]\,\mathrm{d}t \;\le\; \frac{1}{\delta}\Big(\sup_{\mathbb{R}^d}\frac{g}{\Psi^*}\Big)\,\Psi^*(x)\,\Psi^{-1}(x)\,.$$

Therefore, Y is transient. □

Theorem 2.2. *The following are equivalent.*

(i) *The process Y^*, defined in (2.25), corresponding to some principal eigenpair $\big(\Psi^*,\lambda^*(f)\big)$ is exponentially ergodic.*
(ii) *It holds that $\lambda^*(f-h)<\lambda^*(f)$ for all $h\in C_{\mathrm{o}}^+(\mathbb{R}^d)$.*
(iii) *It holds that $\lambda^*(f-h)<\lambda^*(f)$ for some $h\in C_{\mathrm{o}}^+(\mathbb{R}^d)$.*

Proof. (iii) ⇒ (i) follows from Lemma 2.5, and (ii) ⇒ (iii) is obvious.

We show that (i) ⇒ (ii). If Y^* is exponentially ergodic, then there exists a ball $\mathcal{B}$ and $\delta>0$ such that

$$\widetilde{\mathbb{E}}^{\psi^*}_x\big[e^{\delta\breve\tau}\big] \;<\; \infty\,, \qquad \breve\tau=\breve\tau(\mathcal{B}^c)\,.$$

Mimicking the calculations in the proof of Lemma 2.3, we obtain that

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{T\wedge\check{\tau}}[f(X_s)-\lambda^*(f)+\delta]\,\mathrm{d}s}\,g(X_{T\wedge\check{\tau}})\Psi^*(X_{T\wedge\check{\tau}})\,\mathbb{1}_{\{\check{\tau}<\infty\}}\Big] \;=\; \Psi^*(x)\,\widetilde{\mathbb{E}}_x^{\psi^*}\big[e^{\delta(T\wedge\check{\tau})}g(Y^*_{T\wedge\check{\tau}})\big] \qquad \forall\, T>0\,,$$

for $g\in C_{\mathrm{c}}(\mathbb{R}^d)$. We apply this equation to an increasing sequence $\{g_m\}\subset C_{\mathrm{c}}(\mathbb{R}^d)$ which converges to 1, and let first $m\to\infty$, and then $T\to\infty$, using Fatou's lemma and the exponential ergodicity of Y^*, to obtain

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check{\tau}}[f(X_s)-\lambda^*(f)+\delta]\,\mathrm{d}s}\,\mathbb{1}_{\{\check{\tau}<\infty\}}\Big] \;<\; \infty\,, \quad x\in\mathcal{B}^c. \tag{2.30}$$

Let $h\in C_{\mathrm{o}}^+(\mathbb{R}^d)$. Since h is bounded, it is easy to see that $\lambda^*(f-h)$ is finite. Let $\tilde f:=f-h$, and $\big(\tilde\Psi^*,\lambda^*(\tilde f)\big)$ be a solution of

$$\mathcal{L}\tilde\Psi^*+(f-h)\tilde\Psi^* \;=\; \lambda^*(\tilde f)\tilde\Psi^*, \quad \tilde\Psi^*>0\,, \tag{2.31}$$

which is obtained as a limit of Dirichlet eigensolutions as in Lemma 2.2. If $\lambda^*(\tilde f)=\lambda^*(f)$, then in view of (2.30) and the calculations in the proof of Lemma 2.2 (iii), we have

$$\tilde\Psi^*(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check{\tau}}[f(X_t)-h(X_t)-\lambda^*(\tilde f)]\,\mathrm{d}t}\,\tilde\Psi^*(X_{\check{\tau}})\,\mathbb{1}_{\{\check{\tau}<\infty\}}\Big] \qquad \forall\, x\in\mathcal{B}^c\,. \tag{2.32}$$

Applying the Itô–Krylov formula and Fatou's lemma to (2.23), we obtain

$$\Psi^*(x) \;\ge\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check{\tau}}[f(X_s)-\lambda^*(f)]\,\mathrm{d}s}\,\Psi^*(X_{\check{\tau}})\,\mathbb{1}_{\{\check{\tau}<\infty\}}\Big], \quad x\in\mathcal{B}^c. \tag{2.33}$$

It follows by (2.32) and (2.33) that we can multiply Ψ^* with a suitable positive constant so that $\Psi^*-\tilde\Psi^*$ attains a minimum of 0 in $\mathcal{B}$. On the other hand, from (2.23) and (2.31) we have

$$\mathcal{L}(\Psi^*-\tilde\Psi^*)-\big(f-\lambda^*(f)\big)^-(\Psi^*-\tilde\Psi^*) \;=\; -\big(f-\lambda^*(f)\big)^+(\Psi^*-\tilde\Psi^*)-h\,\tilde\Psi^* \;\le\; 0\,. \tag{2.34}$$

Thus by strong maximum principle we have $\Psi^*=\tilde\Psi^*$. This, in turn, implies that $h\,\tilde\Psi^*=0$ by (2.34). But this is not possible. Hence we have $\lambda^*(\tilde f)<\lambda^*(f)$, and the proof is complete. □

We define the Green's measure G_λ, $\lambda\in\mathbb{R}$, by

$$G_\lambda(g) \;:=\; \mathbb{E}_0\bigg[\int_0^\infty \mathrm{e}^{\int_0^t[f(X_s)-\lambda]\,\mathrm{d}s}\,g(X_t)\,\mathrm{d}t\bigg] \quad \text{for all } g\in C_c^+(\mathbb{R}^d)\,.$$

The density of the Green's measure with respect to the Lebesgue measure is called the Green's function. Existence of a Green's function (and Green's measure) is used by Pinsky [2, Chapter 4.3] in his definition of the generalized principal eigenvalue of $\mathcal{L}^f$. A number $\lambda\in\mathbb{R}$ is said to be *subcritical* if G_λ possesses a density, *critical* if it is not subcritical and $\mathcal{L}^{f-\lambda}V=0$ has a positive solution V, and *supercritical* if it is neither subcritical nor critical.

The lemma which follows is an extension of [2, Theorem 4.3.4] where, under a regularity assumption on the coefficients, it is shown that a critical eigenvalue λ is always simple. This result establishes several equivalences of the notion of criticality of λ.

Lemma 2.7. *The following are equivalent.*

(i) *The twisted process Y corresponding to the eigenpair (Ψ,λ) is recurrent.*

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

(ii) *$G_\lambda(g)$ is infinite for some $g \in C_{\rm c}^+(\mathbb{R}^d)$.*
(iii) *For some open ball $\mathcal{B}$, and with $\breve\tau = \breve\tau(\mathcal{B})$, we have*

$$\Psi(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\breve\tau}[f(X_s)-\lambda]\,\mathrm{d}s}\,\Psi(X_{\breve\tau})\,\mathbb{1}_{\{\breve\tau<\infty\}}\Big]\,, \quad x\in\bar{\mathcal{B}}^c\,,$$

where Ψ is an eigenfunction corresponding to the eigenvalue λ.

In addition, in (ii)–(iii) "some" may be replaced by "all", and if any one of (i)–(iii) holds, then λ is a simple eigenvalue.

Proof. The argument of this proof is inspired from [10, Theorem 2.8]. By Corollary 2.1 we have $\lambda \ge \lambda^*(f)$. Assume that (i) holds for some $\lambda \ge \lambda^*(f)$. Let (Ψ,λ) be an eigenpair of $\mathcal{L}^f$. Then for any $g \in C_{\rm c}^+(\mathbb{R}^d)$ we have from Lemma 2.3 that

$$\Psi(x)\,\widetilde{\mathbb{E}}_x^{\psi}\big[g(Y_T)\,\Psi^{-1}(Y_T)\big] \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^T[f(X_t)-\lambda]\,\mathrm{d}t}\,g(X_T)\Big] \qquad \forall\, T>0\,. \tag{2.35}$$

On the other hand, if Y is recurrent, then

$$\int_0^\infty \widetilde{\mathbb{E}}_x^{\psi}\big[g(Y_t)\,\Psi^{-1}(Y_t)\big]\mathrm{d}t \;=\; \infty\,.$$

Combining this with (2.35) we have $G_\lambda(g)=\infty$. Hence (ii) follows.

Next suppose that (ii) holds, i.e., $G_\lambda(g)=\infty$ for some $g\in C_{\rm c}^+(\mathbb{R}^d)$ and $\lambda\ge\lambda^*(f)$. Applying the Itô–Krylov formula to $\mathcal{L}\Psi+(f-\lambda)\Psi=0$, we have

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^t[f(X_s)-\lambda]\,\mathrm{d}s}\,\mathbb{1}_{\mathcal{B}}(X_t)\Big] \;\le\; \frac{1}{\min_{\mathcal{B}}\Psi}\;\mathbb{E}_x\Big[\mathrm{e}^{\int_0^t[f(X_s)-\lambda]\,\mathrm{d}s}\,\Psi(X_t)\Big] \;\le\; \frac{1}{\min_{\mathcal{B}}\Psi}\,\Psi(x) \tag{2.36}$$

for all $t\ge0$, and for any bounded ball $\mathcal{B}$. Define $F_\alpha(x) \coloneqq f(x)-\lambda-\alpha$, and

$$\Gamma_\alpha \;\coloneqq\; \mathbb{E}_0\bigg[\int_0^\infty \mathrm{e}^{\int_0^t F_\alpha(X_s)\,\mathrm{d}s}\,g(X_t)\,\mathrm{d}t\bigg]\,,$$

for $\alpha>0$, and some $g\in C_{\rm c}^+(\mathbb{R}^d)$. From (2.36) we have $\Gamma_\alpha<\infty$ for all $\alpha>0$. Moreover, $\Gamma_\alpha\to\infty$ as $\alpha\searrow0$ by hypothesis. Choose n_0 large enough so that $\mathrm{support}(g)\subset B_{n_0}$. Following [10, Theorem 2.8] we consider the positive solution $\varphi_{\alpha,n}\in\mathcal{W}^{2,p}_{\rm loc}(B_n)\cap C(\bar B_n)$ of

$$\mathcal{L}\varphi_{\alpha,n}+F_\alpha\,\varphi_{\alpha,n} \;=\; -\Gamma_\alpha^{-1}g \quad \text{in } B_n\,, \quad \varphi_{\alpha,n}=0 \quad \text{on } \partial B_n\,, \tag{2.37}$$

for $n\ge n_0$. Since for every fixed n we have

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^T F_\alpha(X_s)\,\mathrm{d}s}\,\varphi_{\alpha,n}(X_T)\,\mathbb{1}_{\{T\le\tau_n\}}\Big] \;\le\; \Big(\max_{B_n}\varphi_{\alpha,n}\Big)\,\mathrm{e}^{-\alpha T}\;\mathbb{E}_x\Big[\mathrm{e}^{\int_0^T F_0(X_s)\,\mathrm{d}s}\,\mathbb{1}_{B_n}(X_T)\mathbb{1}_{\{T\le\tau_n\}}\Big] \;\xrightarrow[T\to\infty]{}\; 0$$

by (2.36), applying the Itô–Krylov formula to (2.37), we obtain by [10, Theorem 2.8] that

$$\varphi_{\alpha,n}(0) \;=\; \Gamma_\alpha^{-1}\,\mathbb{E}_0\bigg[\int_0^{\tau_n}\mathrm{e}^{\int_0^t F_\alpha(X_s)\,\mathrm{d}s}\,g(X_t)\,\mathrm{d}t\bigg] \;\le\; \Gamma_\alpha^{-1}\Gamma_\alpha=1\,. \tag{2.38}$$

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

Since Γ_α^{-1} is bounded uniformly on $\alpha \in (0,1)$ by hypothesis, we can apply Harnack's inequality for a class of superharmonic functions [37, Corollary 2.2] to conclude that $\{\varphi_{\alpha,n}\,, n \in \mathbb{N}\}$ is locally bounded, and therefore also uniformly bounded in $\mathcal{W}^{2,p}_{\mathrm{loc}}(B_R)$, $p > d$, for any $R > 0$. Thus, we have that $\varphi_{\alpha,n} \to \varphi_\alpha$ weakly in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$ along some subsequence, and that φ_α satisfies

$$\mathcal{L}\varphi_\alpha + F_\alpha\,\varphi_\alpha \;=\; -\Gamma_\alpha^{-1} g \quad \text{in } \mathbb{R}^d \tag{2.39}$$

by (2.37). Let $\mathcal{B}$ be an open ball centered at 0 such that $\mathrm{support}(g) \subset \mathcal{B}$. Applying the Itô–Krylov formula to (2.37) we obtain

$$\varphi_{\alpha,n}(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau\wedge T} F_\alpha(X_s)\,\mathrm{d}s}\,\varphi_{\alpha,n}(X_{\check\tau\wedge T})\,\mathbb{1}_{\{\check\tau\wedge T<\tau_n\}}\Big]\,, \quad x \in B_n\setminus\bar{\mathcal{B}}, \quad \forall\, T>0\,,$$

with $\check\tau = \check\tau(\mathcal{B}^c)$. As in the derivation of (2.38), using (2.36) and a similar argument we obtain

$$\varphi_{\alpha,n}(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau} F_\alpha(X_s)\,\mathrm{d}s}\,\varphi_{\alpha,n}(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\tau_n\}}\Big]\,, \quad x \in B_n\setminus\bar{\mathcal{B}}\,. \tag{2.40}$$

Letting $n \to \infty$ along some subsequence, and arguing as above, we obtain a function φ_α which satisfies (2.39) and

$$\varphi_\alpha(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau} F_\alpha(X_s)\,\mathrm{d}s}\,\varphi_\alpha(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\infty\}}\Big]\,, \quad x \in \bar{\mathcal{B}}^c, \tag{2.41}$$

where (2.41) follows from (2.40). From (2.38) we have $\varphi_\alpha(0) = 1$ for all $\alpha \in (0,1)$. Now applying Harnack's inequality once again and letting $\alpha \searrow 0$, we deduce that φ_α converges weakly in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p > d$, to some positive function Ψ which satisfies $\mathcal{L}\Psi + F_0\,\Psi = 0$ in $\mathbb{R}^d$, and

$$\Psi(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau} F_0(X_s)\,\mathrm{d}s}\,\Psi(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\infty\}}\Big]\,, \quad x \in \bar{\mathcal{B}}^c.$$

This implies (iii).

Lastly, suppose that (iii) holds. In other words, there exists an eigenpair (Ψ,λ) and an open ball $\mathcal{B}$ such that

$$\Psi(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau} F_0(X_s)\,\mathrm{d}s}\,\Psi(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\infty\}}\Big]\,, \quad x \in \mathcal{B}^c. \tag{2.42}$$

We first show that λ is a simple eigenvalue, which implies that there is a unique twisted process Y corresponding to λ. To establish the simplicity of λ consider another eigenpair $(\tilde\Psi,\lambda)$ of $\mathcal{L}^f$. By the Itô–Krylov formula we obtain

$$\tilde\Psi(x) \;\ge\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau} F_0(X_s)\,\mathrm{d}s}\,\tilde\Psi(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\infty\}}\Big]\,, \quad x \in \mathcal{B}^c\,.$$

Thus using (2.42) and an argument similar to Lemma 2.4 we can show that $\Psi = \tilde\Psi$. Then (iii) $\Rightarrow$ (i) follows from [10, Lemma 2.6].

Uniqueness of the eigenfunction Ψ follows from the stochastic representation in (2.42) and the proof of (iii) $\Rightarrow$ (i). □

As an immediate corollary to Lemmas 2.6 and 2.7 we have the following.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

Corollary 2.3. *Let (Ψ, λ) be an eigenpair of $\mathcal{L}^f$ which satisfies*

$$\Psi(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[f(X_s)-\lambda]\,\mathrm{d}s}\,\Psi(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\infty\}}\Big] \quad \forall\, x\in\mathcal{B}^c\,,$$

for some bounded open ball $\mathcal{B}$ in $\mathbb{R}^d$. Then $\lambda=\lambda^*(f)$, and it is a simple eigenvalue.

Theorem 2.3 below is a generalization of [2, Theorem 4.7.1] in $\mathbb{R}^d$, which is stated in bounded domains, and for bounded and smooth coefficients. It is shown in [2] that for smooth bounded domains, the Green's measure is not defined at the critical value λ^* [2, Theorem 3.2]. But by Theorem 2.3 below we see that this is not the case on $\mathbb{R}^d$. In fact, [2, Theorem 4.3.2] shows that λ^* could be either subcritical or critical in the sense of Pinsky. We show that the criticality of λ^* is equivalent to the strict monotonicity of $\lambda^*(f)$ on the right, i.e., $\lambda^*(f)<\lambda^*(f+h)$ for all $h\in C_{\mathrm{o}}^+(\mathbb{R}^d)$.

Theorem 2.3. *A ground state process is recurrent if and only if $\lambda^*(f)<\lambda^*(f+h)$ for all $h\in C_{\mathrm{o}}^+(\mathbb{R}^d)$.*

Proof. Suppose first that a ground state process corresponding to $\lambda^*(f)$ is recurrent. Then $G_{\lambda^*}(g)=\infty$ for all $g\in C_{\mathrm{c}}^+(\mathbb{R}^d)$ by Lemma 2.7. Let $\tilde f=f+h$ and $\tilde\lambda^*:=\lambda^*(f+h)$. Suppose that $\lambda^*=\tilde\lambda^*$. Let $\tilde\Psi$ be a principal eigenfunction of $\mathcal{L}^{\tilde f}$, i.e.,

$$\mathcal{L}\tilde\Psi+\tilde f\,\tilde\Psi \;=\; \tilde\lambda^*\tilde\Psi. \tag{2.43}$$

Writing (2.43) as $\mathcal{L}\tilde\Psi+(f-\lambda^*)\tilde\Psi=-h\tilde\Psi$, and applying the Itô–Krylov formula, followed by Fatou's lemma, we obtain

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^T[f(X_s)-\lambda^*]\,\mathrm{d}s}\,\tilde\Psi(X_T)\Big]+\int_0^T\mathbb{E}_x\Big[\mathrm{e}^{\int_0^t[f(X_s)-\lambda^*]\,\mathrm{d}s}\,h(X_t)\tilde\Psi(X_t)\Big]\,\mathrm{d}t \;\le\; \tilde\Psi(x)\,,$$

which contradicts the property that $G_{\lambda^*}(g)=\infty$ for all $g\in C_{\mathrm{c}}^+(\mathbb{R}^d)$. Therefore, $\lambda^*(f)<\lambda^*(f+h)$ for all $h\in C_{\mathrm{o}}^+(\mathbb{R}^d)$.

To prove the converse, suppose that Y^* is transient. Then for $g\in C_{\mathrm{c}}^+(\mathbb{R}^d)$ with $B_1\subset\mathrm{support}(g)$ we have $G_{\lambda^*}(g)<\infty$. Following the arguments in the proof of (ii)$\Rightarrow$(iii) in Lemma 2.7, we obtain a positive Φ satisfying

$$\mathcal{L}\Phi+(f-\lambda^*)\,\Phi \;=\; -\Gamma_0^{-1}g\,. \tag{2.44}$$

Let $\varepsilon=\Gamma_0^{-1}\min_{B_1}\frac{g}{\Phi}$. Then from (2.44) we have

$$\mathcal{L}\Phi+(f+\varepsilon\mathbb{1}_{B_1}-\lambda^*)\,\Phi \;\le\; 0\,.$$

This implies that $\lambda^*(f+\varepsilon\mathbb{1}_{B_1})\le\lambda^*(f)$ by Lemma 2.2 (ii). Thus $\lambda^*(f+\varepsilon\mathbb{1}_{B_1})=\lambda^*(f)$. Therefore, if $\lambda^*(f)<\lambda^*(f+h)$ for all $h\in C_{\mathrm{o}}^+(\mathbb{R}^d)$, then Y^* has to be recurrent. This completes the proof. □

It is well known that a (null) recurrent diffusion $\{X_t\}$ with locally uniformly elliptic and Lipschitz continuous a, and locally bounded measurable drift, admits a σ-finite invariant probability measure ν which is a Radon measure on the Borel σ-field of $\mathbb{R}^d$ [38]. This measure is equivalent to the Lebesgue measure and is unique up to a multiplicative constant. Theorem 8.1 in [38] states that if g and h are real-valued functions which are integrable with respect to the measure ν then

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

$$\mathbb{P}_x\left(\lim_{T\to\infty}\frac{\int_0^T g(X_t)\,\mathrm{d}t}{\int_0^T h(X_t)\,\mathrm{d}t} = \frac{\int_{\mathbb{R}^d} g(x)\,\nu(\mathrm{d}x)}{\int_{\mathbb{R}^d} h(x)\,\nu(\mathrm{d}x)}\right) = 1\,. \tag{2.45}$$

Suppose $g\colon \mathbb{R}^d \to \mathbb{R}_+$ is a non-trivial function. Select h as the indicator function of some open ball. Then it is well known that the expectation of $Y^h_t := \int_0^t h(X_t)\,\mathrm{d}t$ tends to ∞ as $t\to\infty$. Adopt the analogous notation Y^g_t, and let $\alpha = \frac{\nu(g)}{2\nu(h)}$. Let $M>0$ be arbitrary, and select t_0 large enough such that $\mathbb{E}\big[Y^h_{t_0}\big] \ge 2M$. Then of course we may find a positive constant κ such $\mathbb{E}\big[Y^h_{t_0}\mathbb{1}_{\{Y^h_{t_0}\le\kappa\}}\big] \ge M$. Since Y^h_t and Y^g_t are nondecreasing in t, it follows by (2.45) that

$$\mathbb{P}_x\left(\lim_{t\to\infty}\big(Y^g_t - \alpha Y^h_{t_0}\big)^-\mathbb{1}_{\{Y^h_{t_0}\le\kappa\}} \ge 0\right) = 0\,.$$

This of course implies, using dominated convergence, that $\liminf_{t\to\infty}\mathbb{E}\big[Y^g_t\mathbb{1}_{\{Y^h_{t_0}\le\kappa\}}\big] \ge \alpha M$. Since M was arbitrary, this shows that $\mathbb{E}\big[Y^g_t\big]\to\infty$ as $t\to\infty$, or equivalently that $\int_0^\infty \mathbb{E}_x[g(X_t)]\,\mathrm{d}t = \infty$. Using this property in the proof of Theorem 2.3 we obtain the following corollary.

Corollary 2.4. *For $\lambda^*(f)$ to be strictly monotone at f on the right it is sufficient that there exists some non-trivial Borel measurable bounded function $g\colon \mathbb{R}^d\to\mathbb{R}_+$ with compact support satisfying $\lambda^*(f+\epsilon g) > \lambda^*(f)$ for all $\epsilon>0$.*

2.3.1. Minimal growth at infinity

We next discuss the property known as *minimal growth at infinity* [5, Definition 8.2]. As shown in [5, Proposition 8.4], minimal growth at infinity implies that the eigenspace corresponding to the eigenvalue $\lambda^*(f)$ is one dimensional, i.e., $\lambda^*(f)$ is simple. We start with the following definition, which is a variation of [5, Definition 8.2].

Definition 2.3. A positive function $\varphi\in\mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ is said to be a solution of minimal growth at infinity of $\mathcal{L}^f\varphi - \lambda\varphi = 0$, if for any $r>0$ and any positive function $v\in\mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d\setminus B_r)$ satisfying $\mathcal{L}^f v - \lambda v \le 0$ a.e., in B_r^c, there exists $R>r$ and $k>0$ such that $k\varphi\le v$ in B_R^c.

Define the generalized principal eigenvalue of $\mathcal{L}^f$ in the domain D by

$$\lambda_1(f,D) := \inf\Big\{\lambda\,:\,\exists\varphi\in\mathcal{W}^{2,d}_{\mathrm{loc}}(D),\ \varphi>0,\ \mathcal{L}\varphi + (f-\lambda)\varphi \le 0 \text{ a.e. in } D\Big\}\,.$$

Note that $\lambda_1(f,\mathbb{R}^d) = \hat\Lambda(f) = \lambda^*(f)$. It is also clear from this definition that for $D_1\subset D_2$ we have $\lambda_1(f,D_1)\le\lambda_1(f,D_2)$.

It is shown in [5, Theorem 8.5] that the hypothesis

(A1) $\lim_{r\to\infty}\lambda_1(f,B_r^c) < \lambda^*(f)$

implies that the ground state Ψ^* of $\mathcal{L}^f$ is a solution of minimal growth at infinity.

On the other hand, the following result has been established in [39, Theorem 2.1].

Theorem 2.4. *The ground state Ψ^* of $\mathcal{L}^f$ is a solution of minimal growth at infinity of $\mathcal{L}^f\Psi^* - \lambda^*(f)\Psi^* = 0$ if and only if $\lambda^*(f)$ is strictly monotone at f on the right.*

It thus follows by the above results that (A1) is a sufficient condition for strict monotonicity of $\lambda^*(f)$ on the right. It turns out that (A1) is equivalent to strict monotonicity and, moreover, the map $r\mapsto\lambda_1(f,B_r^c)-\lambda^*(f)$ is either negative on $(0,\infty)$ or identically equal to 0. This is the subject of the following theorem.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

Theorem 2.5. *The following are equivalent.*

(a) $\exists\, r > 0 :\ \lambda_1(f, B_r^c) < \lambda_1(f, \mathbb{R}^d)$.
(b) $\lambda^*(f)$ *is strictly monotone at* f.
(c) $\lambda_1(f, B_r^c) < \lambda_1(f, \mathbb{R}^d) \quad \forall\, r > 0$.

Proof. It easily follows by Lemma 2.5 and Theorem 2.2 and the definition of λ_1 that (b) $\Rightarrow$ (c). Thus it remains to prove that (a) $\Rightarrow$ (b). Suppose that $\lambda \equiv \lambda_1(f, B_{\bar r}^c) < \lambda_1(f, \mathbb{R}^d) = \lambda^*(f)$ for some $\bar r > 0$. Using the Dirichlet eigenvalues for the annulus $\mathcal{B}_r \setminus \bar B_{\bar r}$, for $r > \bar r$, and letting $r \to \infty$, we can construct a solution $\psi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\bar B_{\bar r}^c)$ of $\mathcal{L}\psi + f\psi = \lambda\psi$ on $B_{\bar r}^c$, with $\psi > 0$ on $\bar B_{\bar r}^c$, and $\psi = 0$ on $\partial B_{\bar r}$. Then ψ is bounded away from 0 on $\partial B_{r'}$ for all $r' > r$. Using any $r' > r$, we extend ψ smoothly inside $B_{r'}$ to obtain some function $\varphi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ which is strictly positive on $\mathbb{R}^d$ and agrees with ψ on $B_{r'}^c$. Let $h \coloneqq \lambda\varphi - \mathcal{L}\varphi - f\varphi$, and $\tilde f \coloneqq f + \frac{h}{\varphi}$. Then $\mathcal{L}\varphi + \tilde f\varphi = \lambda\varphi$, and therefore, we have

$$\lambda^*\big(f + \tfrac{h^-}{\varphi}\big) \ \le\ \lambda^*(\tilde f) \ \le\ \lambda \ <\ \lambda^*(f)\,,$$

which implies strict monotonicity at f, and completes the proof. □

2.4. Potentials f vanishing at infinity

Let $\mathcal{B}_{\mathrm{o}}(\mathbb{R}^d)$ denote the class of bounded Borel measurable functions which are vanishing at infinity, i.e., satisfying $\lim_{R\to\infty} \sup_{B_R^c} |f| = 0$, and $\mathcal{B}_{\mathrm{o}}^+(\mathbb{R}^d)$ the class of nonnegative functions in $\mathcal{B}_{\mathrm{o}}(\mathbb{R}^d)$ which are not a.e. equal to 0.

Theorem 2.6 which follows is a (pinned) multiplicative ergodic theorem (compare with [22, Theorem 7.1]). Note that the continuity result in this theorem is stronger than that of [5, Proposition 9.2]. See also Remark 4.1 on the continuity of $\lambda^*(f)$ for a larger class of f. We introduce the eigenvalue $\lambda''(f)$ defined by

$$\lambda''(f) \ \coloneqq\ \inf\Big\{\lambda :\ \exists\, \varphi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d),\ \inf_{\mathbb{R}^d} \varphi > 0,\ \mathcal{L}\varphi + (f-\lambda)\varphi \ \le\ 0 \text{ a.e. in } \mathbb{R}^d\Big\}\,. \tag{2.46}$$

Theorem 2.6. *Let* $f \in \mathcal{B}_{\mathrm{o}}(\mathbb{R}^d)$. *If the solution of* (2.1) *is recurrent, then* $\lambda^*(f) = \lambda''(f) = \mathcal{E}(f)$. *In addition, if the solution of* (2.1) *is positive recurrent with invariant measure* μ, *and* $\int_{\mathbb{R}^d} f\,\mathrm{d}\mu > 0$, *the following hold:*

(a) *for any measurable* g *with compact support we have*

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^T [f(X_s) - \lambda^*(f)]\,\mathrm{d}s}\, g(X_T)\Big] \ \xrightarrow[T\to\infty]{}\ C_g \Psi^*(x) \tag{2.47}$$

for some positive constant C_g. *Moreover, the corresponding twisted process* Y^* *is exponentially ergodic.*

(b) *If* f_n *is a sequence of functions in* $\mathcal{B}_{\mathrm{o}}(\mathbb{R}^d)$ *satisfying* $\sup_n \|f_n\|_\infty < \infty$, *and converging to* f *in* $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, *and also uniformly outside some compact set* $K \subset \mathbb{R}^d$, *then* $\lambda^*(f_n) \to \lambda^*(f)$.

Proof. Applying the Itô–Krylov formula to $\mathcal{L}\varphi + (f-\lambda)\varphi \le 0$, it is easy to see that $\mathcal{E}(f) \le \lambda''(f)$. Also, from [10, Lemma 2.3] we have $\lambda^*(f) \le \mathcal{E}(f)$. Thus we obtain $\lambda^*(f) \le \mathcal{E}(f) \le \lambda''(f)$. If $\lambda^*(f) \ge \lim_{|x|\to\infty} f(x)$, then by [5, Theorem 1.9 (iii)] we have $\lambda^*(f) = \lambda''(f)$ which in turn implies that $\lambda^*(f) = \mathcal{E}(f) = \lambda''(f)$. On the other hand, if $\lambda^*(f) < \lim_{|x|\to\infty} f(x)$, then f is near-monotone, relative to $\lambda^*(f)$, in the sense of [10]. Applying [10, Lemma 2.1] we again obtain $\lambda^*(f) = \mathcal{E}(f) = \lambda''(f)$.

We now turn to part (a). Applying Jensen's inequality it is easy to see that $\mathcal{E}(f) \ge \int f\,\mathrm{d}\mu > 0$. Therefore, $\lambda^*(f - f^+) \le 0 < \lambda^*(f)$. Taking $h = f^+$ and mimicking the arguments of Theorem 2.1 we see that Y^* is

exponentially ergodic. Let μ^* be the unique invariant measure of Y^*. Then (2.47) follows from (2.28) and [40, Theorem 1.3.10] with $C_g = \int \frac{g}{\Psi^*}\,\mathrm{d}\mu^*$.

Next we prove part (b). By the first part of the theorem we have $\lambda^*(f_n) = \mathcal{E}(f_n)$ for all n, and by the lower-semicontinuity property of λ^* it holds that $\liminf_{n\to\infty} \lambda^*(f_n) \ge \lambda^*(f)$. Let $h \in C_{\mathrm{c}}^+(\mathbb{R}^d)$ and $\tilde f = f - h$. Then by Theorem 2.2 we have $2\delta := \lambda^*(f) - \lambda^*(\tilde f) > 0$. Choose a open ball $\mathcal{B}$, containing K, such that $\sup_{x\in\mathcal{B}^c} |f_n - f| < \delta$ and $\lambda^*(f_n) > \lambda^*(f) - \delta$ for all sufficiently large n. Let $(\Psi_n^*, \lambda^*(f_n))$ denote the principal eigenpair. Then

$$\mathcal{L}\Psi_n^* + f_n\,\Psi_n^* \;=\; \lambda^*(f_n)\,\Psi_n^*\,. \tag{2.48}$$

We can choose $\mathcal{B}$ large enough such that

$$\Psi_n^*(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\breve\tau}[f_n(X_t)-\lambda^*(f_n)]\,\mathrm{d}t}\,\Psi_n^*(X_{\breve\tau})\Big]\,, \quad x\in\mathcal{B}^c, \quad \forall\, n\in\mathbb{N}\,, \tag{2.49}$$

where $\breve\tau = \breve\tau(\mathcal{B})$. Suppose $\limsup_{n\to\infty} \lambda^*(f_n) = \Lambda$. It is standard to show that for some positive Ψ, it holds that $\Psi_n^* \to \Psi$ weakly in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p > d$, as $n\to\infty$, and therefore, from (2.48) we have

$$\mathcal{L}\Psi + f\,\Psi \;=\; \Lambda\,\Psi\,.$$

Therefore, $\Lambda \ge \lambda^*(f)$. Note that on $\mathcal{B}^c$ we have

$$f_n - \lambda^*(f_n) \;\le\; f + \delta - \lambda^*(f_n) \;\le\; f - \lambda^*(f) + 2\delta \;=\; f - \lambda^*(\tilde f)$$

for all n sufficiently large. Since $\mathbb{E}_x\big[\mathrm{e}^{\int_0^{\breve\tau}[f(X_t)-\lambda^*(\tilde f)]\,\mathrm{d}t}\big] < \infty$, passing to the limit in (2.49), and using the dominated convergence theorem, we obtain that

$$\Psi(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\breve\tau}[f(X_t)-\Lambda]\,\mathrm{d}t}\,\Psi(X_{\breve\tau})\Big]\,, \quad x\in\mathcal{B}^c\,.$$

Therefore, $\Lambda = \lambda^*(f)$ by Corollary 2.3. This completes the proof. □

We pause for a moment to provide an example where (P2) holds but (P1) fails.

Example 2.1. Let $d = 2$ and $\mathcal{L} = \Delta$. If $f = 0$, then the ground state is a constant function, and in turn, the ground state diffusion is a two dimensional Brownian motion, hence recurrent. It follows that λ^* is strictly monotone on the right at 0. Now let f a non-trivial non-negative continuous function with compact support. It is clear that $\lambda^*(\beta f) \le 0$ for $\beta \le 0$. On the other hand, by Theorem 2.6, we have $\lambda^*(\beta f) = \mathcal{E}(\beta f)$ for all $\beta\in\mathbb{R}$. Therefore, for $\beta \le 0$, we have

$$\begin{aligned} 0 \;\ge\; \lambda^*(\beta f) \;=\; \mathcal{E}(\beta f) \;&=\; \limsup_{T\to\infty}\,\frac{1}{T}\,\log\,\mathbb{E}_x\Big[\mathrm{e}^{\int_0^T \beta f(X_s)\,\mathrm{d}s}\Big] \\ &\ge\; \limsup_{T\to\infty}\,\frac{1}{T}\,\mathbb{E}\Big[\int_0^T \beta f(X_s)\,\mathrm{d}s\Big] \;=\; 0\,. \end{aligned}$$

Thus $\lambda^*(\beta f) = 0$ for all $\beta \le 0$, which implies that λ^* it is not strictly monotone at 0.

In the rest of this section we show how the previous development can be used to obtain results analogous to those reported in [16], without imposing any smoothness assumptions on the coefficients. With $\breve\psi = -\log\Psi^* = -\psi^*$, we have

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

$$-a^{ij}\partial_{ij}\breve{\psi} - b^i\,\partial_i\breve{\psi} + \langle\breve{\psi}, a\breve{\psi}\rangle + f \;=\; \lambda^*(f)\,. \tag{2.50}$$

Note that (2.50) is a particular form of a more general class of quasilinear pdes of the form

$$-a^{ij}\partial_{ij}\breve{\psi} + H(x,\nabla\breve{\psi}) + f \;=\; \lambda^*(f)\,, \tag{2.51}$$

where the function $H(x,p)$, with $(x,p)\in\mathbb{R}^d\times\mathbb{R}^d$, serves as a Hamiltonian. Let f be a non-constant, nonnegative continuous function satisfying $\lim_{|x|\to\infty} f(x) = 0$, and define $\Lambda_\beta \coloneqq \lambda^*(\beta f)$, $\beta\in\mathbb{R}$. Then by [5, Proposition 2.3 (vii)] we know that $\beta\mapsto\Lambda_\beta$ is non-decreasing and convex. For the diffusion matrix a equal the identity, Ichihara studies some qualitative properties of Λ_β in [16] associated to the pde (2.51), and their relation to the recurrence and transience behavior of the process with generator

$$\mathcal{A}^{\breve{\psi}}g \;=\; \Delta g - \langle\nabla_p H(x,\nabla\breve{\psi}),\,\nabla g\rangle, \quad g\in C_c^2(\mathbb{R}^d)\,.$$

It is clear that if $H(x,p) = -\langle b(x),p\rangle + \langle p, a(x)p\rangle$, then $\mathcal{A}^{\breve{\psi}}$ is the generator of the twisted process Y^* corresponding to Ψ^*. One of the key assumptions in [16, Assumption (H1) (i)] is that $H(x,p)\ge H(x,0) = 0$ for all x and p. Note that this forces b to be 0.

Let

$$\beta_c \;\coloneqq\; \inf\Big\{\beta\in\mathbb{R} : \Lambda_\beta > \lim_{\beta\to-\infty}\Lambda_\beta\Big\}\,.$$

It is easy to see that $\beta_c\in[-\infty,\infty]$. The following result is an extension of [16, Theorems 2.2 and 2.3] to measurable drifts b and potentials f.

Theorem 2.7. *Let $f\in\mathcal{B}_{\rm o}^+(\mathbb{R}^d)$. Then the twisted process $Y^* = Y^*(\beta)$ corresponding to the eigenpair $(\Psi^*_\beta,\Lambda_\beta)$ is transient for $\beta<\beta_c$, exponentially ergodic for $\beta>\beta_c$, and, provided $f=0$ a.e. outside some compact set, it is recurrent for $\beta=\beta_c$. In addition, the following hold.*

(i) *If $\mathcal{L}^0$ is self-adjoint (i.e., $\mathcal{L}^0 = \partial_i(a^{ij}\partial_j)$), with the matrix a bounded, uniformly elliptic and radially symmetric in $\mathbb{R}^d$, and the solution of (2.1) is transient, then $\beta_c\ge 0$. Also $\Lambda_\beta\ge 0$ for all $\beta\in\mathbb{R}$.*

(ii) *Provided that the solution of (2.1) is recurrent, then $\beta_c<0$ if it is exponentially ergodic, and $\beta_c = 0$ otherwise.*

(iii) *Assume that $\beta>\beta_c$, and that (2.1) is recurrent in the case that $\Lambda_\beta\le 0$. Let Ψ^*_β and μ^*_β denote the ground state and the invariant probability measure of the ground state diffusion, respectively, corresponding to Λ_β. Then it holds that*

$$\Lambda_\beta \;=\; \mu^*_\beta\big(\beta f - \langle\nabla\psi^*_\beta, a\nabla\psi^*_\beta\rangle\big)\,, \tag{2.52}$$

where, as usual, $\psi^*_\beta = \log\Psi^*_\beta$.

Proof. The first part of the proof follows from Theorems 2.1 and 2.3, and Corollary 2.4. Next we proceed to prove (i). Suppose $\beta_c<0$. Then $Y^* = Y^*(0)$, i.e., the twisted process corresponding to Λ_0, is exponentially ergodic. By [5, Theorem 1.9 (i)–(ii)] we have $\Lambda_0 = \mathcal{E}(0) = 0$. Moreover, $\Psi^*_0 = 1$ is a ground state. Therefore, the twisted process must be given by (2.1), which is transient by hypothesis. This is a contradiction. Hence $\beta_c\ge 0$. Since $\beta\mapsto\Lambda_\beta$ is convex, it follows that Λ_β is constant in $(-\infty,\beta_c]\ni 0$. Hence $\Lambda_\beta = \Lambda_0 = 0$ for $\beta\le\beta_c$. This proves (i).

We now turn to part (ii). By Theorem 2.6 we have $\Lambda_\beta = \mathcal{E}(\beta f)$. We claim that if the solution of (2.1) is recurrent then $\lambda^*(\beta f)>0$, whenever $\beta>0$. Indeed, arguing by contradiction, if $\lambda^*(\beta f) = 0$ for some $\beta>0$, then $\mathcal{L}\Psi^*_\beta = -\beta f\Psi^*_\beta$ on $\mathbb{R}^d$, which implies that $\Psi^*_\beta(X_t)$ is a nonnegative supermartingale, and since it is integrable, it converges a.s. Since the process X is recurrent, this implies that Ψ^*_β must equal to a constant,

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

which, in turn, necessitates that $f = 0$, a contradiction. This proves the claim, which in turn implies that if the solution of (2.1) is recurrent then $\beta_c \le 0$. Now suppose that β_c is negative. Then the twisted process corresponding to $\beta = 0$ is exponentially ergodic by Theorem 2.1. Since $\Psi_0^* = 1$, the ground state diffusion for $\beta = 0$ agrees with (2.1), which implies that the latter is exponentially ergodic.

Next, suppose that X, and therefore also $Y^*(0)$ is exponentially ergodic. It then follows from Theorem 2.2 that $\beta \mapsto \Lambda_\beta$ is strictly monotone at 0. This of course implies that $\beta_c < 0$. The proof of part (ii) is now complete.

Next we prove part (iii). We distinguish two cases.

Case 1. Suppose $\Lambda_\beta > 0$. Let $\breve\Psi = \breve\Psi_\beta := (\Psi_\beta^*)^{-1}$ and $\breve\psi := \log \breve\Psi$. Then $\breve\Psi$ satisfies

$$\widetilde{\mathcal{L}}^{\psi_\beta^*} \breve\Psi \;=\; \bigl(\beta f - \Lambda_\beta\bigr)\breve\Psi\,. \tag{2.53}$$

Since $\beta f \in \mathcal{B}_{\mathrm{o}}(\mathbb{R}^d)$, there exists $\epsilon_\circ > 0$ and a ball $\mathcal{B}$ such that $\beta f - \Lambda_\beta < -\epsilon_\circ$ for all $x \in \mathcal{B}^c$. Applying the Feynman–Kac formula, it follows from [10, Lemma 2.1] that $\inf_{\mathbb{R}^d} \breve\Psi = \min_{\bar{\mathcal{B}}} \breve\Psi$. Thus $\breve\Psi$ is bounded away from 0 on $\mathbb{R}^d$. Let Y^* denote the ground state process corresponding to the eigenvalue Λ_β. Simplifying the notation we let $\widetilde{\mathbb{E}}_x^* := \widetilde{\mathbb{E}}_x^{\psi_\beta^*}$. By the exponential Foster–Lyapunov equation (2.53) we have that (see [34, Lemma 2.5.5])

$$\widetilde{\mathbb{E}}_x^*\bigl[\breve\Psi(Y_t^*)\bigr] \;\le\; C_0 + \breve\Psi(x)\,\mathrm{e}^{-\epsilon_\circ t} \qquad \forall\, t \ge 0\,. \tag{2.54}$$

Using this estimate together with the fact that $\inf_{\mathbb{R}^d} \breve\Psi > 0$, we obtain

$$\lim_{t\to\infty} \frac{1}{t}\, \widetilde{\mathbb{E}}_x^*\bigl[\breve\psi(Y_t^*)\bigr] \;=\; 0\,. \tag{2.55}$$

Next, we show that

$$\lim_{R\to\infty} \widetilde{\mathbb{E}}_x^*\bigl[\breve\psi(Y_{t\wedge\uptau_R}^*)\bigr] \;=\; \widetilde{\mathbb{E}}_x^*\bigl[\breve\psi(Y_t^*)\bigr]\,, \tag{2.56}$$

where $\uptau_R$ denotes the exit time from the ball B_R. First, there exists some constant k_0 such that $\bigl(\beta f - \Lambda_\beta\bigr)\breve\Psi \le k_0$ on $\mathbb{R}^d$. Thus $\widetilde{\mathbb{E}}_x^*\bigl[\breve\Psi(Y_t^*)\bigr] \le k_0 t + \breve\Psi(x)$ by (2.53), and of course also $\widetilde{\mathbb{E}}_x^*\bigl[\breve\Psi(Y_{t\wedge\uptau_R}^*)\bigr] \le k_0 t + \breve\Psi(x)$ for all $R > 0$. Let

$$\Gamma(R,m) \;:=\; \bigl\{x \in \partial B_R : |\breve\psi(x)| \ge m\bigr\} \qquad \text{for}\;\; m \ge 1\,.$$

Then

$$\begin{aligned}
\widetilde{\mathbb{E}}_x^*\bigl[\breve\psi(Y_{\uptau_R}^*)\,\mathbb{1}_{\{t\ge\uptau_R\}}\bigr] \;&\le\; m\,\widetilde{\mathbb{P}}_x^*(t \ge \uptau_R) + \widetilde{\mathbb{E}}_x^*\bigl[\breve\psi(Y_{\uptau_R}^*)\,\mathbb{1}_{\Gamma(R,m)}(Y_{\uptau_R}^*)\,\mathbb{1}_{\{t\ge\uptau_R\}}\bigr] \\
&\le\; m\,\widetilde{\mathbb{P}}_x^*(t \ge \uptau_R) + \bigl(k_0 t + \breve\Psi(x)\bigr) \sup_{\Gamma(R,m)} \frac{\breve\psi}{\breve\Psi} \\
&\le\; m\,\widetilde{\mathbb{P}}_x^*(t \ge \uptau_R) + \frac{m}{\mathrm{e}^m}\,\bigl(k_0 t + \breve\Psi(x)\bigr)\,.
\end{aligned}$$

Taking limits as $R \to \infty$, and since $m \in \mathbb{R}_+$ is arbitrary, it follows that

$$\lim_{R\to\infty} \widetilde{\mathbb{E}}_x^*\bigl[\breve\psi(Y_{\uptau_R}^*)\,\mathbb{1}_{\{t\ge\uptau_R\}}\bigr] \;=\; 0\,. \tag{2.57}$$

Write

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

$$\widetilde{\mathbb{E}}^*_x\big[\breve{\psi}(Y^*_{t\wedge\tau_R})\big] \;=\; \widetilde{\mathbb{E}}^*_x\big[\breve{\psi}(Y^*_t)\,\mathbb{1}_{\{t<\tau_R\}}\big] + \widetilde{\mathbb{E}}^*_x\big[\breve{\psi}(Y^*_{\tau_R})\,\mathbb{1}_{\{t\ge\tau_R\}}\big]\,. \tag{2.58}$$

Without loss of generality we assume $\breve{\Psi}\ge 1$. Since $|\breve{\psi}|\le\breve{\Psi}$, an application of Fatou's lemma shows that

$$\begin{aligned}\widetilde{\mathbb{E}}^*_x\big[\breve{\psi}(Y^*_t)\big] \;&\le\; \liminf_{R\to\infty}\, \widetilde{\mathbb{E}}^*_x\big[\breve{\psi}(Y^*_t)\,\mathbb{1}_{\{t<\tau_R\}}\big]\\ &\le\; \limsup_{R\to\infty}\, \widetilde{\mathbb{E}}^*_x\big[\breve{\psi}(Y^*_t)\,\mathbb{1}_{\{t<\tau_R\}}\big] \;\le\; \widetilde{\mathbb{E}}^*_x\big[\breve{\psi}(Y^*_t)\big]\,.\end{aligned}$$

We use this together with (2.57) and (2.58) to obtain (2.56).

We write (2.50) as

$$\begin{aligned}0 \;&=\; a^{ij}\partial_{ij}\breve{\psi} + b^i\partial_i\breve{\psi} + \min_{u\in\mathbb{R}^d}\big[2\langle a\,u,\nabla\breve{\psi}\rangle + \langle u,au\rangle\big] - \beta f + \Lambda_\beta\\ &=\; a^{ij}\partial_{ij}\breve{\psi} + b^i\partial_i\breve{\psi} - 2\langle\nabla\breve{\psi},a\nabla\breve{\psi}\rangle + \langle\nabla\breve{\psi},a\nabla\breve{\psi}\rangle - \beta f + \Lambda_\beta\\ &=\; \widetilde{\mathcal{L}}^{\psi^*_\beta}\breve{\psi} + \langle\nabla\breve{\psi},a\nabla\breve{\psi}\rangle - \beta f + \Lambda_\beta\,.\end{aligned} \tag{2.59}$$

Let $F := \langle\nabla\breve{\psi},a\nabla\breve{\psi}\rangle - \beta f = \langle\nabla\psi^*_\beta,a\nabla\psi^*_\beta\rangle - \beta f$. Applying the Itô–Krylov formula to (2.59), we obtain

$$\widetilde{\mathbb{E}}^*_x\big[\breve{\psi}(Y^*_{t\wedge\tau_R})\big] - \breve{\psi}(x) + \widetilde{\mathbb{E}}^*_x\bigg[\int_0^{t\wedge\tau_R} F(Y^*_s)\,\mathrm{d}s\bigg] + \Lambda_\beta\widetilde{\mathbb{E}}^*_x\big[t\wedge\tau_R\big] \;=\; 0\,. \tag{2.60}$$

Letting $R\to\infty$ in (2.60), using (2.56), then dividing by t and letting $t\to\infty$, using (2.55) and Birkhoff's ergodic theorem, we obtain

$$\mu^*_\beta\big(\langle\nabla\psi^*_\beta,a\nabla\psi^*_\beta\rangle - \beta f\big) + \Lambda_\beta \;=\; 0\,,$$

which is the assertion in part (iii).

Case 2. Suppose $\Lambda_\beta\le 0$ and (2.1) is recurrent. The case $\Lambda_\beta = 0$ is then trivial, since $\nabla\psi^*_0 = 0$, so we assume that $\Lambda_\beta < 0$. Then (2.1) is exponentially ergodic by part (ii), and thus Ψ^*_β is bounded below in $\mathbb{R}^d$ by [10, Lemma 2.1]. With $\psi^* = \psi^*_\beta = \log\Psi^*_\beta$, in analogy to (2.59) we have

$$\widetilde{\mathcal{L}}^{\psi^*}\psi^* - \langle\nabla\psi^*,a\nabla\psi^*\rangle + \beta f - \Lambda_\beta \;=\; 0\,. \tag{2.61}$$

We claim that

$$\lim_{t\to\infty}\frac{1}{t}\,\widetilde{\mathbb{E}}^*_x\big[\psi^*(Y^*_t)\big] \;=\; 0\,,\quad\text{and}\quad \lim_{R\to\infty}\widetilde{\mathbb{E}}^*_x\big[\psi^*(Y^*_{t\wedge\tau_R})\big] \;=\; \widetilde{\mathbb{E}}^*_x\big[\psi^*(Y^*_t)\big]\,, \tag{2.62}$$

where as defined earlier, $\widetilde{\mathbb{E}}^*_x = \widetilde{\mathbb{E}}^{\psi^*}_x$, and Y^* denotes the ground state process. Assuming (2.62) is true, we first apply the Itô–Krylov formula to (2.61) to obtain the analogous equation to (2.60), and then take limits and use Birkhoff's ergodic theorem to establish (2.52).

It remains to prove (2.62). Choose $\epsilon>0$ so that $\beta>\beta-\epsilon>\beta_c$, and let $\Psi^*_{\beta-\epsilon}$ denote the ground state corresponding to $\Lambda_{\beta-\epsilon}$. We choose a ball $\mathcal{B}$ such that

$$\epsilon\, f(x) \;<\; \frac{1}{2}(\Lambda_\beta - \Lambda_{\beta-\epsilon}) \qquad \forall\, x\in\mathcal{B}^c\,. \tag{2.63}$$

Since f vanishes at infinity, and $\Lambda_\beta>\Lambda_{\beta-\epsilon}$, there exists a constant $\alpha>1$ and a ball also denoted as $\mathcal{B}$, such that

$$\alpha\big(\beta f(x)-\Lambda_\beta\big) \;<\; (\beta-\epsilon)f(x)-\Lambda_{\beta-\epsilon} \qquad \forall\, x\in\mathcal{B}^c\,. \tag{2.64}$$

Since the ground state processes corresponding to the principal eigenvalues Λ_β and $\Lambda_{\beta-\epsilon}$ are ergodic we have from Lemma 2.7 that

$$\begin{aligned} \Psi^*_\beta(x) \;&=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[\beta f(X_s)-\Lambda_\beta]\,\mathrm{d}s}\,\Psi^*_\beta(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\infty\}}\Big]\,,\\ \Psi^*_{\beta-\epsilon}(x) \;&=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[(\beta-\epsilon)f(X_s)-\Lambda_{\beta-\epsilon}]\,\mathrm{d}s}\,\Psi^*_{\beta-\epsilon}(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\infty\}}\Big] \end{aligned} \tag{2.65}$$

for all $x\in\mathcal{B}^c$, where $\check\tau=\tau(\mathcal{B}^c)$. By (2.27), the function $\widetilde\Psi_\epsilon:=\frac{\Psi^*_{\beta-\epsilon}}{\Psi^*_\beta}$ satisfies

$$\widetilde{\mathcal{L}}^{\psi^*}\widetilde\Psi_\epsilon \;=\; \big(\Lambda_{\beta-\epsilon}-\Lambda_\beta+\epsilon f\big)\,\widetilde\Psi_\epsilon\,. \tag{2.66}$$

Applying the Feynman–Kac formula to (2.66), using (2.63), it follows as in [10, Lemma 2.1] that $\inf_{\mathbb{R}^d}\widetilde\Psi_\epsilon=\min_{\bar{\mathcal{B}}}\widetilde\Psi_\epsilon$. Thus $\widetilde\Psi_\epsilon$ is bounded away from 0 on $\mathbb{R}^d$.

Let $\kappa:=\min_{\mathcal{B}}\frac{\Psi^*_{\beta-\epsilon}}{(\Psi^*_\beta)^\alpha}$. Then by (2.64) and (2.65) we obtain

$$\begin{aligned} \widetilde\Psi_\epsilon(x) \;&\ge\; \frac{\kappa}{\Psi^*_\beta(x)}\,\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}\alpha[\beta f(X_s)-\Lambda_\beta]\,\mathrm{d}s}\,\big(\Psi^*_\beta(X_{\check\tau})\big)^\alpha\,\mathbb{1}_{\{\check\tau<\infty\}}\Big]\\ &\ge\; \frac{\kappa}{\Psi^*_\beta(x)}\Big(\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[\beta f(X_s)-\Lambda_\beta]\,\mathrm{d}s}\,\Psi^*_\beta(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\infty\}}\Big]\Big)^\alpha \;\ge\; \kappa\,\big(\Psi^*_\beta(x)\big)^{\alpha-1} \qquad \forall\, x\in\mathcal{B}^c\,. \end{aligned}$$

Therefore, for some constant κ_1 we have

$$\psi^* \;\le\; \kappa_1+\frac{1}{\alpha-1}\,\log\widetilde\Psi_\epsilon \quad \text{on } \mathbb{R}^d\,. \tag{2.67}$$

Let $\epsilon_\circ:=\frac12(\Lambda_\beta-\Lambda_{\beta-\epsilon})$. From (2.63) and exponential Foster–Lyapunov equation (2.66) we deduce that (2.54) holds for $\widetilde\Psi_\epsilon$. Thus the first equation in (2.62) follows directly from (2.54) and (2.67) and the fact that $\inf_{\mathbb{R}^d}\psi^*>-\infty$, while the second one follows by repeating the argument leading to (2.57). This completes the proof. □

Remark 2.1. The assumption that (2.1) is recurrent in the case that $\Lambda_\beta<0$ in Theorem 2.7 (iii) is equivalent to the statement that $\lambda^*(0)=0$. Note that as shown in [18, Theorem 2.1], unless $\lambda^*(0)=0$, then (2.52) does not hold if $\Lambda_\beta<0$.

If (2.1) is not recurrent, then it is possible that $\beta_c<0$ and also that $\Lambda_\beta<0$ for $\beta\ge0$. Consider a diffusion with $d=1$, $b(x)=2x$, and $\sigma(x)=\sqrt{2}$. Then, we have $\mathcal{L}\varphi=-\varphi$ for $\varphi(x)=\frac12 e^{-x^2}$. Thus $\hat\Lambda_0\le-1$, where $\hat\Lambda_0$ denotes the eigenvalue in (2.5) for $f=0$. Thus $\lambda^*(0)\le-1$ by Lemma 2.2 (b). Since the twisted process corresponding to φ is exponentially ergodic, we must have $\lambda^*(0)=-1$ by Theorem 2.1 (c), and thus φ is the ground state. Theorem 2.1 (b) then asserts that $\beta\mapsto\Lambda_\beta$ is strictly increasing at $\beta=0$. Thus $\beta_c<0$. Observe that the ground state diffusion is an Ornstein–Uhlenbeck process having a Gaussian stationary distribution of mean 0 and variance 1/2. An easy computation reveals that $\mu^*\big(-\langle\nabla\psi^*,a\nabla\psi^*\rangle\big)=-2$ which is smaller than $\lambda^*(0)$.

The conclusion of Theorem 2.7 (iii) can be sharpened. Consider the controlled diffusion

$$\mathrm{d}Z_t \;=\; \big(b(Z_t)+2a(Z_t)v(Z_t)\big)\,\mathrm{d}t+\sigma(Z_t)\,\mathrm{d}W_t\,. \tag{2.68}$$

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

Here $v\colon \mathbb{R}^d \to \mathbb{R}^d$ is a locally bounded Borel measurable map. Let $\widehat{\mathfrak{U}}_{\rm SM}$ denote the class of such maps. These are identified with the class of locally bounded stationary Markov controls. Let $\widehat{\mathfrak{U}}_{\rm SSM} \subset \widehat{\mathfrak{U}}_{\rm SM}$ be the collection of those v under which the diffusion in (2.68) is ergodic, and denote by $\widehat{\mu}_v$ the associated invariant probability measure. We let $\mathcal{A}_v \coloneqq \mathcal{L} + 2\langle av, \nabla\rangle$, and use the symbol $\widehat{\mathbb{E}}^v_x$ to denote the expectation operator associated with (2.68).

In order to simplify the notation, we use the norm $\|v\|_a \coloneqq \sqrt{\langle v, av\rangle}$. For $v \in \widehat{\mathfrak{U}}_{\rm SM}$ we define

$$F_v(z) \;\coloneqq\; \|v(z)\|^2_{a(z)} - \beta f(z)\,,$$

$$\mathcal{J}_x(v) \;\coloneqq\; \limsup_{T\to\infty}\; \frac{1}{T}\,\widehat{\mathbb{E}}^v_x\biggl[\int_0^T F_v(Z_s)\,\mathrm{d}s\biggr]\,,$$

and $\overline{\mathcal{J}}_x \coloneqq \inf_{v\in\widehat{\mathfrak{U}}_{\rm SM}} \mathcal{J}_x(v)$.

Theorem 2.8. *Assume that $f \in \mathcal{B}^+_{\rm o}(\mathbb{R}^d)$ and $\beta > \beta_c$. Then the following hold*

(a) *If $\Lambda_\beta > 0$, then we have*

$$\overline{\mathcal{J}}_x \;=\; \mathcal{J}_x(\nabla\psi^*_\beta) \;=\; -\Lambda_\beta \qquad \forall\, x \in \mathbb{R}^d\,. \tag{2.69}$$

In addition, if $v \in \widehat{\mathfrak{U}}_{\rm SM}$ satisfies $\mathcal{J}_x(v) = \overline{\mathcal{J}}_x$, then $v = \nabla\psi^*_\beta$ a.e.

(b) *If $\Lambda_\beta \le 0$ and (2.1) is recurrent then (2.69) holds, and $v = \nabla\psi^*_\beta$ is the a.e. unique control in $\widehat{\mathfrak{U}}_{\rm SSM}$ which satisfies $\mathcal{J}_x(v) = \overline{\mathcal{J}}_x$.*

(c) *If $\Lambda_\beta < 0$ and (2.1) is not recurrent, then $\overline{\mathcal{J}}_x = 0$ for all $x \in \mathbb{R}^d$.*

Proof. We start with part (a). By Theorem 2.7 (iii), we have $\mathcal{J}_x(\nabla\psi^*_\beta) = -\Lambda_\beta$ in both of cases (a) and (b). It suffices then to show that if $\mathcal{J}_x(v) \le -\Lambda_\beta$ for some $v \in \widehat{\mathfrak{U}}_{\rm SM}$, then $v = \nabla\psi^*_\beta$ a.e. in $\mathbb{R}^d$. Let such a control v be given. Then (2.68) must be positive recurrent under v, for otherwise we must have $\mathcal{J}_x(v) \ge \limsup_{T\to\infty} \frac{1}{T}\,\widehat{\mathbb{E}}^v_x\bigl[\int_0^T -\beta f(Z_s)\,\mathrm{d}s\bigr] \ge 0$. Therefore,

$$\mathcal{J}_x(v) \;=\; \int_{\mathbb{R}^d} F_v(z)\,\widehat{\mu}_v(\mathrm{d}z) \;\le\; -\Lambda_\beta \;<\; 0\,, \tag{2.70}$$

where $\widehat{\mu}_v$, as defined earlier, denotes the invariant probability measure associated with $\mathcal{A}_v \coloneqq \mathcal{L} + 2\langle v, a\nabla\rangle$. Thus, $\mathcal{J}_x(v)$ does not depend on x, and dropping this dependence in the notation we let $\mathcal{J}(v) = \mathcal{J}_x(v)$. Since $f \in \mathcal{B}_{\rm o}(\mathbb{R}^d)$, it follows by (2.70) and the definition of F_v that there exists a ball $\mathcal{B}$ such that

$$\mathcal{J}(v) - F_v(x) \;\le\; -\frac{1}{2}\,\Lambda_\beta \;<\; 0 \qquad \forall x \in \mathcal{B}^c\,. \tag{2.71}$$

By (2.71), and since v is locally bounded, and F_v is integrable with respect to $\widehat{\mu}_v$, we can assert the existence of a solution $\breve{\varphi} \in \mathcal{W}^{2,d}_{\rm loc}(\mathbb{R}^d)$ to the Poisson equation

$$\mathcal{L}\breve{\varphi}(x) + 2\bigl\langle a(x)v(x), \nabla\breve{\varphi}(x)\bigr\rangle + F_v(x) \;=\; \mathcal{J}(v)\,, \tag{2.72}$$

which is bounded below in $\mathbb{R}^d$ (see Lemma 3.7.8 (d) in [34]). It follows by (2.72) that $\Phi \coloneqq \mathrm{e}^{\varphi}$, $\varphi = -\breve{\varphi}$, satisfies

$$\mathcal{L}\Phi + \bigl(\beta f - \|v - \nabla\varphi\|^2_a\bigr)\,\Phi \;=\; -\mathcal{J}(v)\,\Phi\,. \tag{2.73}$$

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

This shows that $\big(\Phi, -\jmath(v)\big)$ is an eigenpair for $\mathcal{L}^{\breve F}$, with $\breve F := \beta f - \|v - \nabla\varphi\|_a^2$. The corresponding twisted process with generator $\tilde{\mathcal{L}} = \mathcal{L} + 2\langle a\nabla\varphi, \nabla\rangle$ then satisfies

$$\tilde{\mathcal{L}}\breve\varphi + \|v - \nabla\varphi\|_a^2 - \beta f \;=\; \jmath(v)\,. \tag{2.74}$$

Since $\breve\varphi$ is bounded below in $\mathbb{R}^d$ and $\jmath(v) < 0$, (2.74) shows that the twisted process is positive recurrent. We claim that $-\jmath(v)$ is the principal eigenvalue of $\mathcal{L}^{\breve F}$. Indeed, if $\lambda^*(\breve F) < -\jmath(v)$ then by the proof of Lemma 2.3 and for any $g \in C_c^+(\mathbb{R}^d)$ we obtain

$$\begin{aligned}\Phi(x)\,\widetilde{\mathbb{E}}_x^{\varphi}\big[g(Y_T)\Phi^{-1}(Y_T)\mathbb{1}_{\{T\le\tau_n\}}\big] \;&=\; \Psi^*(x)\,\widetilde{\mathbb{E}}_x^{\psi^*}\Big[e^{[\lambda^*(\breve F)+\jmath(v)]T} g(Y_T^*)(\Psi^*)^{-1}(Y_T^*)\mathbb{1}_{\{T\le\tau_n\}}\Big]\\ &\le\; \Psi^*(x)\,\Big(\sup_{\mathbb{R}^d}\frac{g}{\Psi^*}\Big)e^{[\lambda^*(\breve F)+\jmath(v)]T}\end{aligned}$$

for all sufficiently large n, where τ_n denotes the first exit time from B_n. By first letting $n\to\infty$, and then integrating with respect to T we obtain

$$\int_0^\infty \widetilde{\mathbb{E}}^{\varphi}\big[g(Y_t)\Phi^{-1}(Y_t)\big]\,\mathrm{d}t \;<\; \infty\,.$$

But this contradicts the positive recurrence of the twisted process corresponding to $\tilde{\mathcal{L}}$. Therefore, $-\jmath(v)$ must be the principal eigenvalue of $\mathcal{L}^{\breve F}$, which implies that

$$-\jmath(v) \;=\; \lambda^*\big(\beta f - \|v - \nabla\varphi\|_a^2\big) \;\le\; \Lambda_\beta\,. \tag{2.75}$$

Thus we have shown that $\jmath_x(v) = \jmath(v) = -\Lambda_\beta$. The strict monotonicity of λ^* at βf together with (2.75) imply that $v = \nabla\varphi$ a.e. in $\mathbb{R}^d$. In turn, (2.73) and the uniqueness of the ground state imply that $\Phi = \Psi_\beta^*$, up to a multiplication by a positive constant. Therefore, we have $v = \nabla\psi_\beta^*$ a.e. in $\mathbb{R}^d$, and this completes the proof of part (a).

We continue with part (b). The case $\Lambda_\beta = 0$ is trivial, so assume that $\Lambda_\beta < 0$. Then (2.1) is exponentially ergodic by Theorem 2.7(ii). Thus Ψ_β^* is bounded away from 0 in $\mathbb{R}^d$ by [10, Lemma 2.1]. Let $v \in \widehat{\mathfrak{U}}_{\mathrm{SM}}$, and $\breve\psi = -\psi_\beta^*$. We have

$$\mathcal{L}\breve\psi + 2\langle v, a\nabla\breve\psi\rangle - \|v + \nabla\breve\psi\|_a^2 + F_v \;=\; -\Lambda_\beta\,. \tag{2.76}$$

Since $\breve\psi$ is bounded above in $\mathbb{R}^d$, it follows from (2.76) by a standard argument that

$$\limsup_{T\to\infty}\;\frac{1}{T}\,\widehat{\mathbb{E}}_x^v\bigg[\int_0^T F_v(Z_s)\,\mathrm{d}s\bigg] \;\ge\; -\Lambda_\beta\,.$$

We next show uniqueness in $\widehat{\mathfrak{U}}_{\mathrm{SSM}}$ of the optimal control $\nabla\psi_\beta^*$. Let $v \in \widehat{\mathfrak{U}}_{\mathrm{SSM}}$ and suppose $\jmath_x(v) = -\Lambda_\beta$. In other words, $\widehat\mu_v(F_v) = -\Lambda_\beta$. By the Itô–Krylov formula and Fatou's lemma and since $\breve\psi$ is bounded above, we obtain from (2.76) that

$$\widehat{\mathbb{E}}_x^v\big[\breve\psi(Z_t)\big] - \breve\psi(x) - \widehat{\mathbb{E}}_x^v\bigg[\int_0^t G_v(Z_s)\,\mathrm{d}s\bigg] + \widehat{\mathbb{E}}_x^v\bigg[\int_0^t F_v(Z_s)\,\mathrm{d}s\bigg] \;\ge\; -t\Lambda_\beta\,, \tag{2.77}$$

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

with

$$G_v(z) \;\coloneqq\; \big\langle \big(v(z)+\nabla\breve\psi(z)\big), a(z)\big(v(z)+\nabla\breve\psi(z)\big)\big\rangle\,.$$

Dividing (2.77) by t and taking limits as $t\to\infty$, we obtain $-\widehat\mu_v(G_v)+\jmath_x(v)\ge-\Lambda_\beta$. Therefore, $\widehat\mu_v(G_v)=0$, since G_v is nonnegative. Thus $G_v=0$ a.e. in $\mathbb{R}^d$, and since $\widehat\mu_v$ has a density, this implies that $v=-\nabla\breve\psi=\nabla\psi^*_\beta$ a.e. in $\mathbb{R}^d$.

We now turn to part (c). It is evident that under the control $v=0$, since the diffusion in (2.68) is transient and f vanishes at infinity, we have $\lim_{t\to\infty}\frac1t\,\widehat{\mathbb{E}}^v_x\big[F_0(Z_t)\big]=0$. It is also clear that under any control $v\in\widehat{\mathfrak{U}}_{\rm SM}\setminus\widehat{\mathfrak{U}}_{\rm SSM}$ we have $\lim_{t\to\infty}\frac1t\,\widehat{\mathbb{E}}^v_x\big[F_v(Z_t)\big]\ge0$. Suppose that under some $v\in\widehat{\mathfrak{U}}_{\rm SSM}$, we have

$$\liminf_{t\to\infty}\;\frac1t\,\widehat{\mathbb{E}}^v_x\big[F_v(Z_t)\big]\;=\;\jmath_x(v)\;<\;0\,.$$

Then there exists a solution $\breve\phi$ to the Poisson equation (2.72) which is bounded below in $\mathbb{R}^d$. Thus following the proof of Case 1 in part (a) we obtain by (2.75) that $\jmath_x(v)\ge-\Lambda_\beta$ which is a contradiction. We have therefore shown that $\jmath_x(v)\ge0$ for all $v\in\widehat{\mathfrak{U}}_{\rm SM}$, which implies that 0 is the optimal value in the class of controls $\widehat{\mathfrak{U}}_{\rm SM}$. □

Remark 2.2. The assumption that f is nonnegative can be weakened to $f\in\mathcal{B}_{\rm o}(\mathbb{R}^d)$. From the proof of Theorem 2.2 we note that if $\lambda^*(f+h)<\lambda^*(f)$ for some $h\in\mathcal{B}_{\rm o}(\mathbb{R}^d)$, then the ground state diffusion corresponding to $\lambda^*(f)$ is geometrically ergodic. Moreover, due to [5, Proposition 2.3 (vii)] the function $\beta\mapsto\lambda^*(\beta f)$ is convex for every $f\in\mathcal{B}_{\rm o}(\mathbb{R}^d)$. Instead of the critical value β_c, we can define a critical value λ_c by $\lambda_c\coloneqq\inf_{\beta\in\mathbb{R}}\Lambda_\beta$. Then if we replace the condition $\beta>\beta_c$ by $\Lambda_\beta>\lambda_c$ as done in [18], it is evident that $\lambda^*(\beta f)$ is strictly monotone at βf and the results in Theorem 2.7 (iii) and Theorem 2.8 still hold, provided $\Lambda_\beta\ne0$, and the proofs are the same.

The results in Theorem 2.8 (b) can be also stated for nonstationary controls. Consider the controlled diffusion

$$\mathrm{d}Z_t\;=\;\big(b(Z_t)+2a(Z_t)U_t\big)\,\mathrm{d}t+\boldsymbol{\sigma}(Z_t)\,\mathrm{d}W_t\,. \tag{2.78}$$

Here $U=\{U_t\}$ is an $\mathbb{R}^d$-valued control process which is jointly measurable in $(t,\omega)\in[0,\infty)\times\Omega$, and is *nonanticipative*: for $t>s$, W_t-W_s is independent of

$$\mathfrak{F}_s\;\coloneqq\;\text{the completion of }\cap_{y>s}\,\sigma(X_0,W_r,U_r:r\le y)\text{ relative to }(\mathfrak{F},\mathbb{P})\,.$$

Let $f\in\mathcal{B}_{\rm o}(\mathbb{R}^d)$, not necessarily nonnegative. Assume that $\Lambda_\beta>\lambda_c$, $\Lambda_\beta\le0$, and (2.1) is recurrent (see Remark 2.2). Suppose that under U, the diffusion in (2.78) has a unique weak solution. We claim that

$$\jmath_x(U)\;\coloneqq\;\limsup_{T\to\infty}\;\frac1T\,\widehat{\mathbb{E}}^U_x\left[\int_0^T\big[\langle U_s,a(Z_s)U_s\rangle-\beta f(Z_s)\big]\,\mathrm{d}s\right]\;\ge\;-\Lambda_\beta\,.$$

We can prove this as follows. By (2.59) we obtain

$$\mathcal{L}\breve\psi(z)+2\langle u,a\nabla\breve\psi\rangle+F_u(z)\;\ge\;-\Lambda_\beta$$

for all $u,z\in\mathbb{R}^d$, and we apply the Itô–Krylov formula and Fatou's lemma (using the fact that $\breve\psi$ is bounded above) with $u=U_t$ to obtain analogously to (2.77) that

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

$$\widehat{\mathbb{E}}^U_x\left[\breve{\psi}(Z_t)\right] - \breve{\psi}(x) + \widehat{\mathbb{E}}^U_x\left[\int_0^t F_{U_s}(Z_s)\,\mathrm{d}s\right] \;\geq\; -t\Lambda_\beta\,. \tag{2.79}$$

Dividing (2.79) by t and letting $t \to \infty$, we obtain

$$\liminf_{t\to\infty}\; \frac{1}{t}\,\widehat{\mathbb{E}}^U_x\left[\int_0^t F_{U_s}(Z_s)\,\mathrm{d}s\right] \;\geq\; -\Lambda_\beta\,,$$

thus proving the claim.

2.4.1. Strong duality

The optimality result in Theorem 2.8 can be strengthened. Consider the class of *infinitesimal ergodic occupation measures*, i.e., measures $\boldsymbol{\pi} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ which satisfy

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{A}_u g(x)\,\boldsymbol{\pi}(\mathrm{d}x,\mathrm{d}u) \;=\; 0 \qquad \forall\, g \in C^\infty_{\mathrm{c}}(\mathbb{R}^d)\,, \tag{2.80}$$

with $\mathcal{A}_u \coloneqq \mathcal{L} + \langle 2au, \nabla\rangle$. Disintegrate these as $\boldsymbol{\pi}(\mathrm{d}x,\mathrm{d}u) = \eta_v(\mathrm{d}x)\,v(\mathrm{d}u\,|\,x)$, and denote this disintegration as $\boldsymbol{\pi} = \eta_v \circledast v$. Let $\hat{v}(x) = \int u\,v(\mathrm{d}u\,|\,x)$. Since $\int|u|^2\eta(\mathrm{d}x)\,v(\mathrm{d}u\,|\,x) \geq \int|\hat{v}(x)|^2\eta(\mathrm{d}x)$, and $\eta(\mathrm{d}x)\delta_{\hat{v}(x)}(\mathrm{d}u)$ is also an ergodic occupation measure, it is enough to consider the class of infinitesimal ergodic occupation measures $\boldsymbol{\pi}$ that correspond to a precise control v, i.e., a Borel measurable map from $\mathbb{R}^d$ to $\mathbb{R}^d$. We denote this class by $\mathcal{M}$. Thus for $\boldsymbol{\pi} = \eta_v \circledast v \in \mathcal{M}$, (2.80) takes the form $\int_{\mathbb{R}^d} \mathcal{A}_v g(x)\,\eta_v(\mathrm{d}x) = 0$. Note that v is not necessarily locally bounded, so this class of controls is, in general, larger than $\widehat{\mathfrak{U}}_{\mathrm{SSM}}$.

In Theorem 2.9 below we use the following simple assertions which are stated as remarks.

Remark 2.3. If η_v has density $\rho_v \in L^{d/(d-1)}_{\mathrm{loc}}(\mathbb{R}^d)$, and $v \in L^2(\mathbb{R}^d;\eta_v)$, then

$$\int_{\mathbb{R}^d} \mathcal{A}_v g(x)\,\eta_v(\mathrm{d}x) \;=\; 0 \qquad \forall\, g \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d) \cap C_{\mathrm{c}}(\mathbb{R}^d)\,.$$

This can be proved as follows. We mollify g with a smooth mollifier family $\{\chi_r,\, r>0\}$, so that (2.80) can applied to the function $g * \chi_r$, where '$*$' denotes convolution. Then we separate terms, and applying the Hölder inequality on $\left|\int(\mathcal{L}g - \mathcal{L}(g*\chi_r))\rho_v\right|$, and using the convergence of $\mathcal{L}(g*\chi_r)$ to $\mathcal{L}g$ in $L^d_{\mathrm{loc}}(\mathbb{R}^d)$, we deduce that this term tends to 0 as $r \searrow 0$. Similarly, we apply the Hölder inequality in the form

$$\left|\int \langle 2av, \nabla(g - g*\chi_r)\rangle\,\rho_v\right|^2 \;\leq\; \int|2av|^2\rho_v \int|\nabla(g-g*\chi_r)|^2\rho_v\,.$$

Then the first integral on the right hand side is bounded, and the second integral vanishes as $r \searrow 0$ since $g*\chi_r$ converges to g uniformly on compact sets.

Remark 2.4. Suppose that the drift b in (2.1) has at most affine growth. It is then well known that the map $x \mapsto \mathbb{E}_x[\boldsymbol{\tau}(\mathcal{B}^c)]$ is inf-compact for any open ball $\mathcal{B}$, provided of course that (2.1) is positive recurrent. This fact together with the stochastic representation in (2.18) and Jensen's inequality, imply that if $f \in \mathcal{B}_{\mathrm{o}}(\mathbb{R}^d)$, $\Lambda_\beta < 0$, and (2.1) is recurrent, then the ground state Ψ^*_β is inf-compact, and this of course renders (2.1) positive recurrent. An analogous argument using the ground state diffusion shows that, if $b + a\nabla\Psi^*_\beta$ has at most affine growth, and $\Lambda_\beta > \max\{0,\lambda_c\}$ (see Remark 2.2), then $\widetilde{\Psi} = (\Psi^*_\beta)^{-1}$ is inf-compact.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

The theorem that follows shows that there is no optimality gap between the primal problem which consists of minimizing $\int F_u(x)\boldsymbol{\pi}(\mathrm{d}x,\mathrm{d}u)$ subject to the constraint (2.80), and the dual problem which amounts to a maximization over subsolutions of the HJB equation, as described in Section 1. This theorem is stated for $f \in \mathcal{B}_{\mathrm{o}}(\mathbb{R}^d)$ which is not necessarily nonnegative as discussed in Remark 2.2.

Theorem 2.9. *Assume that $f \in \mathcal{B}_{\mathrm{o}}(\mathbb{R}^d)$, $\Lambda_\beta > \lambda_c$, and that one of the following conditions holds.*

(i) $\Lambda_\beta > 0$, *the coefficients a and b are bounded, and a is uniformly strictly elliptic.*
(ii) $\Lambda_\beta < 0$, (2.1) *is recurrent, and b has at most affine growth.*

Then any $\boldsymbol{\pi} = \eta_v \circledast v \in \mathcal{M}$, such that $\int_{\mathbb{R}^d} F_v \,\mathrm{d}\eta_v < \infty$, satisfies

$$\int_{\mathbb{R}^d} F_v \,\mathrm{d}\eta_v \;=\; -\Lambda_\beta + \int_{\mathbb{R}^d} \|v - \nabla\psi^*_\beta\|^2_a \,\mathrm{d}\eta_v \,. \tag{2.81}$$

In addition, if $\boldsymbol{\pi} = \eta_v \circledast v \in \mathcal{M}$ is optimal, i.e., if it satisfies $\int_{\mathbb{R}^d} F_v \,\mathrm{d}\eta_v = -\Lambda_\beta$, then $v = \nabla\psi^*_\beta$ a.e. in $\mathbb{R}^d$ and $\eta_v = \mu^*_\beta$.

Proof. We first consider case (i). Since a, b, and f are bounded, it follows that $\nabla\psi^*_\beta$ is bounded by [10, Lemma 3.3]. Then $-\psi^*_\beta$ is inf-compact by Remark 2.4. Recall that $\mathcal{A}_v = \mathcal{L} + 2\langle a\, v, \nabla\rangle$. We have

$$-\mathcal{A}_v\psi^*_\beta - \|v - \nabla\psi^*_\beta\|^2_a + F_v \;=\; -\Lambda_\beta \,. \tag{2.82}$$

Let χ be a convex $C^2(\mathbb{R})$ function such that $\chi(x) = x$ for $x \ge 0$, $\chi(x) = -1$ for $x \le -1$, and χ', χ'' are positive on $(-1, 0)$. Define $\chi_R(x) \coloneqq -R + \chi(x + R)$, $R > 0$. Then we have from (2.82) that

$$-\mathcal{A}_v\chi_R(\psi^*_\beta) + \chi''_R(\psi^*_\beta)\|\nabla\psi^*_\beta\|^2_a - \chi'_R(\psi^*_\beta)\|v - \nabla\psi^*_\beta\|^2_a + \chi'_R(\psi^*_\beta)F_v \;=\; -\chi'_R(\psi^*_\beta)\Lambda_\beta \,. \tag{2.83}$$

Since $\int \mathcal{A}_v g \,\mathrm{d}\eta_v =$ for all $g \in C^\infty_{\mathrm{c}}(\mathbb{R}^d)$, an application of [41, Theorem 2.1] shows that η_v has a density $\rho_v \in L^{d/(d-1)}_{\mathrm{loc}}(\mathbb{R}^d)$. Note that this does not require a or b to be bounded. Therefore, since $\chi_R(\psi^*_\beta) + R + 1$ has compact support, we have $\int_{\mathbb{R}^d} \mathcal{A}_v\chi_R(\psi^*_\beta)\,\eta_v(\mathrm{d}x) = 0$ by Remark 2.3. Thus letting $R \to \infty$ in (2.83), using monotone convergence, we obtain (2.81).

We next show uniqueness. Let $\boldsymbol{\pi} = \eta_v \circledast v \in \mathcal{M}$ be optimal, and $\boldsymbol{\pi}_* = \eta_* \circledast v_*$ denote the ergodic occupation measure corresponding to $v_* = \nabla\psi^*_\beta$. Here, $\eta_* = \mu^*_\beta$. Let ρ_* denote the density of η_*. Define $\bar\eta \coloneqq \frac{1}{2}(\eta_v + \eta_*)$ and $\bar v \coloneqq \zeta_v v + \zeta_* v_*$, with ζ_v and ζ_* given by $\zeta_v \coloneqq \frac{\rho_v}{\rho_v + \rho_*}$ and $\zeta_* \coloneqq \frac{\rho_*}{\rho_v + \rho_*}$, respectively. It is straightforward to verify, using the fact that the drift is affine in the control, that $\bar{\boldsymbol{\pi}} = \bar\eta \circledast \bar v$ is in $\mathcal{M}$.

By optimality, we have

$$\begin{aligned}
0 \;&\ge\; \int_{\mathbb{R}^d} F_v(x)\,\eta_v(\mathrm{d}x) + \int_{\mathbb{R}^d} F_{v_*}(x)\,\eta_*(\mathrm{d}x) - 2\int_{\mathbb{R}^d} F_{\bar v}(x)\,\bar\eta(\mathrm{d}x) \\
&=\; 2\int_{\mathbb{R}^d} \Big(\zeta_v(x)\,\|v(x)\|^2_{a(x)} + \zeta_*(x)\,\|v_*(x)\|^2_{a(x)} - \big\|\zeta_v(x)v(x) + \zeta_*(x)v_*(x)\big\|^2_{a(x)}\Big)\,\bar\eta(\mathrm{d}x) \\
&=\; \int_{\mathbb{R}^d} \frac{\rho_v(x)\,\rho_*(x)}{\rho_v(x) + \rho_*(x)}\,\|v(x) - v_*(x)\|^2_{a(x)}\,\mathrm{d}x \,.
\end{aligned} \tag{2.84}$$

Since ρ_* is strictly positive, (2.84) implies that $\rho_v\,|v - v_*| = 0$ a.e. in $\mathbb{R}^d$, and thus $v = v_*$ on the support of η_v. It is clear that if v is modified outside the support of η_v, then the modified $\eta_v \circledast v$ is also an infinitesimal

ergodic occupation measure. Therefore $\eta_v \circledast v_* \in \mathcal{M}$. The uniqueness of the invariant measure of the diffusion with generator $\mathcal{A}_{v_*}$ then implies that $\eta_v = \eta_*$, which in turn implies that $v = \nabla\psi^*_\beta$ a.e. in $\mathbb{R}^d$.

We now turn to case (ii). By Remark 2.4, ψ^*_β is inf-compact. Also, as shown in case (i), η_v has a density $\rho_v \in L^{d/(d-1)}_{\mathrm{loc}}(\mathbb{R}^d)$. We write (2.76) as

$$\mathcal{A}_v\breve{\psi} - \|v + \nabla\breve{\psi}\|^2_a + F_v \;=\; -\Lambda_\beta\,, \tag{2.85}$$

with $\breve{\psi} = -\psi^*_\beta$. Then we have from (2.85) that

$$\mathcal{A}_v\chi_R(\breve{\psi}) - \chi''_R(\breve{\psi})\|\nabla\breve{\psi}\|^2_a - \chi'_R(\breve{\psi})\|v + \nabla\breve{\psi}\|^2_a + \chi'_R(\breve{\psi})F_v \;=\; -\chi'_R(\breve{\psi})\Lambda_\beta\,. \tag{2.86}$$

Using the inequality $\|v+\nabla\breve{\psi}\|^2_a \le 2\|v\|^2_a + 2\|\nabla\breve{\psi}\|^2_a$, then integrating (2.86) with respect to η_v, and rearranging terms we obtain

$$\int \big(\chi''_R(\breve{\psi}) + 2\chi'_R(\breve{\psi})\big)\,\|\nabla\breve{\psi}\|^2_a\,\mathrm{d}\eta_v \;\le\; \int \chi'_R(\breve{\psi})(F_v + 2\beta f)\,\mathrm{d}\eta_v - \Lambda_\beta \int \chi'_R(\breve{\psi})\,\mathrm{d}\eta_v\,. \tag{2.87}$$

Thus letting $R \to \infty$ in (2.87), using monotone convergence, we obtain the energy inequality

$$2\int \|\nabla\breve{\psi}\|^2_a\,\mathrm{d}\eta_v \;\le\; \int (F_v + 2\beta f)\,\mathrm{d}\eta_v - \Lambda_\beta \;<\; \infty\,. \tag{2.88}$$

Then (2.81) follows by letting $R \to \infty$ in (2.87), using again monotone convergence and (2.88). Uniqueness follows as in case (i). This completes the proof. □

Remark 2.5. The proof of Theorem 2.9 provides a general recipe to prove the lack of an optimality gap in ergodic control problems. Note that the model in [16] is such that $\nabla\psi^*_\beta$ is bounded, and a is also bounded. Therefore,

$$\int \chi''_R(\breve{\psi})\|\nabla\breve{\psi}\|^2_a\,\mathrm{d}\eta_v \;\xrightarrow[R\to\infty]{}\; 0\,,$$

and the proof of Theorem 2.9 goes through even for the more general Hamiltonian $H(x,p)$ in [16].

Remark 2.6. If $\Lambda_\beta > \lambda_c$, and under some nonanticipative control U the diffusion (2.78) has a unique weak solution, it was shown in the discussion following Remark 2.2 that $\jmath_x(U) \ge -\Lambda_\beta$, provided $\Lambda_\beta \le 0$ and (2.1) is recurrent. The same conclusion can be drawn if $\Lambda_\beta > 0$ and under the hypotheses of Theorem 2.9. Define the set of *mean empirical measures* $\{\xi^U_{x,t}\,,\ t \ge 0\}$ of (2.78) under the control U by

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} h(x,u)\,\xi^U_{x,t}(\mathrm{d}x,\mathrm{d}u) \;=\; \frac{1}{t}\int_0^t \widehat{\mathbb{E}}^U_x\big[h(Z_t,U_t)\big]\,\mathrm{d}t \qquad \forall h \in C_b(\mathbb{R}^d\times\mathbb{R}^d)\,.$$

If $\Lambda_\beta > 0$, then $F_u(x) - \Lambda_\beta$ is bounded away from zero for all x outside some compact set, and one can follow the arguments in the proof of [34, Lemma 3.4.6] to show that every limit point in $\mathcal{P}(\overline{\mathbb{R}^d\times\mathbb{R}^d})$ (the set of Borel probability measures on the one-point compactification of $\mathbb{R}^d\times\mathbb{R}^d$) of a sequence of mean empirical measures $\{\xi^{U_n}_{x,t_n}\,,\ n \in \mathbb{N}\}$ as $t_n \to \infty$ takes the form $\delta\boldsymbol{\pi} + (1-\delta)\boldsymbol{\pi}_\infty$, where $\boldsymbol{\pi}$ is an infinitesimal ergodic occupation measure and $\boldsymbol{\pi}_\infty(\{\infty\}) = 1$. Using this property, one can show, by following the argument in the proof of [34, Theorem 3.4.7], that if $\jmath_x(U) \le -\Lambda_\beta$, then the mean empirical measures are necessarily tight in $\mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d)$ and $\delta = 1$ in this decomposition. This of course implies that $\jmath_x(U) = -\Lambda_\beta$. This argument establishes optimality over the largest possible class of controls U.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

2.4.2. Differentiability of Λ_β

Differentiability of the map $\beta \mapsto \Lambda_\beta$ for all $\beta > \beta_c$ is established in [18, Proposition 5.4] under the hypothesis that the coefficients a, b, and f are Lipschitz continuous and bounded in $\mathbb{R}^d$, but for a more general class of Hamiltonians (see (A1)–(A3) in [18]). These assumptions are used to show that $\nabla\psi^*$ is bounded in $\mathbb{R}^d$, and this is utilized in the proofs.

In the next theorem we demonstrate this differentiability result for the model in this paper which assumes only measurable b and f, in which case it is not possible, in general, to obtain gradient estimates and follow the approach in [16,18,19]. The first assertion in this theorem should be compared to [18, Proposition 5.4]. Recall the definition $\widetilde{\Psi}_\epsilon = \frac{\Psi^*_{\beta-\epsilon}}{\Psi^*_\beta}$ after (2.65), and let $\widetilde{\psi}_\epsilon = \log \widetilde{\Psi}_\epsilon$.

Theorem 2.10. *Suppose $f \in \mathcal{B}_{\mathrm{o}}^+(\mathbb{R}^d)$, and that $\beta > \beta_c$. Then for all $\epsilon > 0$ such that $\beta - \epsilon > \beta_c$, we have*

$$\epsilon \, \frac{\mu^*_\beta(f\,\widetilde{\Psi}_\epsilon)}{\mu^*_\beta(\widetilde{\Psi}_\epsilon)} \;\le\; \Lambda_\beta - \Lambda_{\beta-\epsilon} \;=\; \mu^*_\beta\bigl(\epsilon f - \|\nabla\widetilde{\psi}_\epsilon\|^2_a\bigr)\,. \tag{2.89}$$

In addition, we have

$$\frac{\mathrm{d}\,\Lambda_\beta}{\mathrm{d}\beta} \;=\; \mu^*_\beta(f)\,. \tag{2.90}$$

Proof. Fix some $\epsilon_1 > 0$ such that $\beta - 2\varepsilon_1 > \beta_c$, and consider (2.66). As argued in the proof of Theorem 2.7, the function $\widetilde{\Psi}_\epsilon$ is bounded away from 0 on $\mathbb{R}^d$ for all $\epsilon \in (0, \varepsilon_1]$. We recall the notation $\widetilde{\mathbb{E}}^{\psi^*_\beta}[\,\cdot\,] = \widetilde{\mathbb{E}}^*[\,\cdot\,]$. Applying the Itô–Krylov formula and Fatou's lemma to (2.66) we obtain

$$\frac{1}{T}\,\widetilde{\mathbb{E}}^*_x\biggl[\int_0^T \bigl(\Lambda_{\beta-\epsilon} - \Lambda_\beta + \epsilon f(Y^*_t)\bigr)\,\widetilde{\Psi}_\epsilon(Y^*_t)\,\mathrm{d}t\biggr] \;\ge\; 0\,,$$

from which the left hand side inequality of (2.89) follows by an application of Birkhoff's ergodic theorem. Also the analogous estimate to (2.54) holds for $\widetilde{\Psi}_\epsilon$, which implies that

$$\lim_{t\to\infty} \frac{1}{t}\,\widetilde{\mathbb{E}}^*_x\bigl[\widetilde{\Psi}_\epsilon(Y^*_t)\bigr] \;=\; 0 \qquad \forall\, \epsilon \in (0, \varepsilon_1]\,. \tag{2.91}$$

The second equality in (2.89) then follows by first using the technique in the proof of Theorem 2.7 and (2.91) to establish (2.56) for $\widetilde{\psi}_\epsilon$, $\epsilon \in (0, \varepsilon_1)$, and then applying the Itô–Krylov formula to the log-transformed equation corresponding to (2.66) as in (2.60), and taking limits at $t \to \infty$.

Using the convexity of $\beta \mapsto \Lambda_\beta$, we write (2.89) as

$$\frac{\mu^*_\beta(f\,\widetilde{\Psi}_\epsilon)}{\mu^*_\beta(\widetilde{\Psi}_\epsilon)} \;\le\; \frac{\Lambda_\beta - \Lambda_{\beta-\epsilon}}{\epsilon} \;\le\; \frac{\Lambda_{\beta+\epsilon} - \Lambda_\beta}{\epsilon} \;\le\; \mu^*_{\beta+\epsilon}(f)\,. \tag{2.92}$$

Fix an open ball $\mathcal{B} \subset \mathbb{R}^d$, such that

$$\Lambda_{\beta-2\epsilon_1} - \Lambda_{\beta-\epsilon} + (2\epsilon_1 - \epsilon) f(x) \;\le\; -\delta \;<\; 0 \qquad \forall\, \epsilon \in [-\epsilon_1, \epsilon_1]\,,\;\; \forall\, x \in \mathcal{B}^c\,. \tag{2.93}$$

This is clearly possible since $\epsilon \mapsto \Lambda_{\beta-\epsilon}$ is nonincreasing, $\Lambda_{\beta-2\epsilon_1} < \Lambda_{\beta-\epsilon_1}$, and f vanishes at infinity. Let $\check{\tau} = \tau(\mathcal{B}^c)$. Since the ground state process corresponding to $\Lambda_{\beta-\epsilon}$ is exponentially ergodic for $\epsilon < \epsilon_1$ by Theorem 2.7, we have

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

$$\Psi^*_{\beta-\epsilon}(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[(\beta-\epsilon)f(X_s)-\Lambda_{\beta-\epsilon}]\,\mathrm{d}s}\,\Psi^*_{\beta-\epsilon}(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<\infty\}}\Big] \qquad \forall\,\epsilon\in[-\epsilon_1,\epsilon_1] \tag{2.94}$$

by Lemma 2.7. Since $\Psi^*_{\beta-\epsilon}$ and its inverse are bounded on $\mathcal{B}$, uniformly in $\epsilon\in[-\epsilon_1,2\epsilon_1]$, it follows from (2.93) and (2.94) that there exists κ such that $\Psi^*_{\beta-\epsilon}\le\kappa\Psi^*_{\beta-2\epsilon_1}$ for all $\epsilon\in[-\epsilon_1,\epsilon_1]$. Therefore, since the collection $\big\{\Psi^*_{\beta-\epsilon}\,,\ \epsilon\in[-\epsilon_1,\epsilon_1]\big\}$, is bounded in $C^{1,\alpha}_{\mathrm{loc}}(\mathcal{B})$, $\alpha>0$, we can use (2.94) and the dominated convergence theorem to conclude that $\widetilde\Psi_\epsilon\to1$ as $\epsilon\searrow0$. Thus, one more application of the dominated convergence theorem shows that $\mu^*_\beta(f\,\widetilde\Psi_\epsilon)\to\mu^*_\beta(f)$ and $\mu^*_\beta(\widetilde\Psi_\epsilon)\to1$ as $\epsilon\searrow0$. This shows that

$$\lim_{\epsilon\searrow0}\frac{\mu^*_\beta(f\,\widetilde\Psi_\epsilon)}{\mu^*_\beta(\widetilde\Psi_\epsilon)} \;=\; \mu^*_\beta(f)\,. \tag{2.95}$$

We next study the term $\mu^*_{\beta+\epsilon}(f)$. Let $\widetilde{\mathbb{E}}^{*,\epsilon}_x$ denote the expectation operator for the ground state diffusion corresponding to $\Lambda_{\beta+\epsilon}$. Since

$$\widetilde{\mathcal{L}}^{\psi^*_{\beta+\epsilon}}\tfrac{\Psi^*_{\beta-2\epsilon_1}}{\Psi^*_{\beta+\epsilon}} \;=\; \big(\Lambda_{\beta-2\epsilon_1}-\Lambda_{\beta+\epsilon}-(2\epsilon_1+\epsilon)f\big)\,\tfrac{\Psi^*_{\beta-2\epsilon_1}}{\Psi^*_{\beta+\epsilon}}\,,$$

it follows by an estimate similar to (2.93) that $\widetilde{\mathbb{E}}^{*,\epsilon}_x[\mathrm{e}^{\kappa\check\tau}]\;\le\;\frac{\Psi^*_{\beta-2\epsilon_1}}{\Psi^*_{\beta+\epsilon}}(x)$ for all $x\in\mathcal{B}^c$ (see also Theorem 3.1 in Section 3).

We claim that

$$\inf_{\epsilon\in[0,\epsilon_1]}\mu^*_{\beta+\epsilon}(\mathcal{B}) \;>\; 0\,. \tag{2.96}$$

Indeed, let $\tilde{\mathcal{B}}$ be a larger ball such that $\bar{\mathcal{B}}\subset\tilde{\mathcal{B}}$. It suffices to exhibit the result for $\tilde{\mathcal{B}}$. For some positive constants δ_i, $i=1,2,3$, we have

$$\sup_{\epsilon\in[0,\epsilon_1]}\sup_{x\in\partial\tilde{\mathcal{B}}}\widetilde{\mathbb{E}}^{*,\epsilon}_x[\check\tau]\;\le\;\kappa^{-1}\sup_{\epsilon\in[0,\epsilon_1]}\sup_{x\in\partial\tilde{\mathcal{B}}}\tfrac{\Psi^*_{\beta-2\epsilon_1}}{\Psi^*_{\beta+\epsilon}}(x)\;=:\;\delta_1<\infty\,,$$

and (see [34, Theorem 2.6.1])

$$0\;<\;\delta_2\;\le\;\inf_{\epsilon\in[0,\epsilon_1]}\sup_{x\in\partial\mathcal{B}}\widetilde{\mathbb{E}}^{*,\epsilon}_x[\uptau(\tilde{\mathcal{B}}^c)]\;\le\;\sup_{\epsilon\in[0,\epsilon_1]}\sup_{x\in\partial\mathcal{B}}\widetilde{\mathbb{E}}^{*,\epsilon}_x[\uptau(\tilde{\mathcal{B}}^c)]\;\le\;\delta_3\;<\;\infty\,.$$

We use the inequality $\mu^*_{\beta+\epsilon}(\tilde{\mathcal{B}})\ge\frac{\delta_2}{\delta_1+\delta_3}$, which follows from the well-known characterization of invariant probability measures due to Has′minskiĭ [34, Theorem 2.6.9], and which establishes the claim.

It follows from (2.96) that the corresponding densities $\eta^*_{\beta+\epsilon}$ are locally bounded and also bounded away from 0 uniformly in $\epsilon\in[0,\epsilon_1]$ by the Harnack inequality (see proof of equation (3.2.6) in [34]). Therefore, standard pde estimates of the Fokker–Planck equation show that this family of densities is locally Hölder equicontinuous [35, Theorem 8.24, p. 202]. Given any $\theta\in(0,1)$ we may enlarge $\mathcal{B}$ so that $\mu^*_\beta(\mathcal{B})\ge1-\theta$ and $|f|\le\theta$ on $\mathcal{B}^c$. Let $\bar\eta_\beta$ be the (uniform) limit of $\eta^*_{\beta+\epsilon_n}$ on $\mathcal{B}$ along some subsequence $\epsilon_n\searrow0$. Since $\nabla\psi^*_{\beta-\epsilon}$ is Hölder equicontinuous on $\mathcal{B}$, uniformly in $\epsilon\in[-\epsilon_1,\epsilon_1]$ as argued earlier, it follows that $\bar\eta_\beta$ is strictly positive on $\bar{\mathcal{B}}$. It is straightforward to show then that $\bar\eta_\beta$ is a positive solution of the Fokker–Planck equation for the (adjoint of the) operator $\mathcal{L}+2\langle a\nabla\psi^*_\beta,\nabla\rangle$. By the uniqueness of the invariant probability measure we have $\bar\eta_\beta=C\eta^*_\beta$ for some positive constant C. Since $\int_{\mathcal{B}}\bar\eta_\beta(x)\,\mathrm{d}x\le1$, we have $C\le(1-\theta)^{-1}$. Thus, since $\sup_{\epsilon\in[0,\epsilon_1]}\|\eta^*_{\beta+\epsilon}\|_\infty<\infty$, and $|f|<\theta$ on $\mathcal{B}^c$, by Fatou's lemma we obtain

$$\limsup_{n\to\infty}\mu^*_{\beta+\epsilon_n}(f)\;\le\;\limsup_{n\to\infty}\int_{\mathcal{B}}f(x)\eta^*_{\beta+\epsilon_n}(x)\,\mathrm{d}x+\theta$$

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

$$\leq \int_{\mathcal{B}} f(x)\bar{\eta}_\beta(x)\,\mathrm{d}x + \theta$$

$$\leq (1-\theta)^{-1} \int_{\mathcal{B}} f(x)\eta^*_\beta\,\mathrm{d}x + \theta$$

$$\leq (1-\theta)^{-1}\,\mu^*_\beta(f) + \theta\,.$$

Since θ can be selected arbitrarily close to 0, we obtain from (2.92) that $\lim_{\epsilon\searrow 0} \frac{\Lambda_{\beta+\epsilon}-\Lambda_\beta}{\epsilon} \leq \mu^*_\beta(f)$. Combining this with (2.94) and (2.95) we obtain (2.90). □

3. Exponential ergodicity and strict monotonicity of principal eigenvalues

In this section we show that exponential ergodicity of (2.1) is a sufficient condition for the strict monotonicity of the principal eigenvalue. In [9,17] exponential ergodicity is used to obtain results similar to Theorem 2.1. In these studies the coefficients a, b, and f are assumed to be C^2, and this assumption seems hard to waive as the technique used relies on a gradient estimate (see [17, Theorem 3.1] and [9, Lemma 2.4]) which is not available for less regular coefficients. Our approach has allowed us to obtain the results in Section 2 under much weaker hypotheses on the coefficients. Under some additional hypotheses, we show in this section that $\lambda^*(f) = \mathcal{E}(f)$. Recall the definition of $\lambda''(f)$ in (2.46). It is straightforward to show that $\lambda''(f) \geq \mathcal{E}(f)$. We present an example where $\lambda^*(f) < \mathcal{E}(f)$, and therefore also $\lambda^*(f) < \lambda''(f)$.

Example 3.1. Let $\phi : \mathbb{R} \to \mathbb{R}_+$ be a smooth function which is strictly positive on $[-1,1]$ and satisfies $\phi(x) = \mathrm{e}^{-\frac{1}{2}|x|}$ for $|x| \geq 1$. Define

$$f(x) \;\coloneqq\; -\frac{1}{\phi(x)}\left(\phi''(x) + \operatorname{sign}(x)\,\phi'(x) - \phi(x)\right).$$

Then $f(x) = \frac{5}{4}$ for $|x| \geq 1$, and

$$\phi''(x) + \operatorname{sign}(x)\,\phi'(x) + f(x)\phi(x) \;=\; \phi(x)\,. \tag{3.1}$$

Consider the one-dimensional controlled diffusion

$$\mathrm{d}X_t \;=\; \operatorname{sign}(X_t)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_t\,. \tag{3.2}$$

From (3.1) and Lemma 2.2 (ii) we have $\lambda^*(f) \leq 1$. It is clear that (3.2) is a transient process. Therefore, for any initial data x

$$\mathcal{E}_x(f) \;\geq\; \limsup_{T\to\infty}\; \frac{1}{T}\;\mathbb{E}_x\left[\int_0^T f(X_t)\,\mathrm{d}t\right] \;=\; \frac{5}{4}\,.$$

Hence $\lambda^*(f) < \mathcal{E}(f)$.

Remark 3.1. Example 3.1 presents a case where the conclusion of [5, Theorem 1.9] fails to hold. Since the operator $\mathcal{L}$ in this example is uniformly elliptic, has bounded coefficients, and $d = 1$, the only aspect that makes it different from the class of operators in part (i) of [5, Theorem 1.9], is that it is not self-adjoint.

Let us start by summarizing some equivalent characterizations of exponential ergodicity.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

Theorem 3.1. *The following are equivalent.*

(a) *For some ball* $\mathcal{B}_\circ$ *there exists* $\delta_\circ > 0$ *and* $x_\circ \in \bar{\mathcal{B}}_\circ^c$ *such that* $\mathbb{E}_{x_\circ}[\mathrm{e}^{\delta_\circ \tau(\mathcal{B}_\circ^c)}] < \infty$.

(b) *For every ball* $\mathcal{B}$ *there exists* $\delta > 0$ *such that* $\mathbb{E}_x[\mathrm{e}^{\delta \tau(\mathcal{B}^c)}] < \infty$ *for all* $x \in \mathcal{B}^c$.

(c) *For every ball* $\mathcal{B}$, *there exists a positive function* $\mathcal{V} \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$, *with* $\inf_{\mathbb{R}^d} \mathcal{V} > 0$, *and positive constants* κ_0 *and* δ *such that*

$$\mathcal{L}\mathcal{V}(x) \;\le\; \kappa_0 \mathbb{1}_{\mathcal{B}}(x) - \delta \mathcal{V}(x) \qquad \forall\, x \in \mathbb{R}^d\,. \tag{3.3}$$

(d) *Equation* (2.1) *is recurrent, and* $\lambda^*(\mathbb{1}_{\mathcal{B}^c}) < 1$ *for every ball* $\mathcal{B}$.

Proof. We first show that (a) $\Rightarrow$ (d). It is clear that (a) implies that (2.1) is positive recurrent, and that it is enough to prove that $\lambda^*(\mathbb{1}_{\mathcal{B}^c}) < 1$ for any $\mathcal{B} \subset \mathcal{B}_\circ$. Let $f = \mathbb{1}_{\mathcal{B}^c}$, and consider the Dirichlet eigensolutions $(\widehat{\Psi}_n, \hat{\lambda}_n)$ in (2.3). It is easy to see that $\hat{\lambda}_n < 1$ for all n. We claim that $\lambda^*(f) < 1$. If not, then $\hat{\lambda}_n \nearrow 1$ as $n \to \infty$, and $\widehat{\Psi}_n$ converges to some $\Psi \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p \ge d$, which satisfies $\mathcal{L}\Psi = \mathbb{1}_{\mathcal{B}}\Psi$ on $\mathbb{R}^d$ and $\Psi(0) = 1$. The same argument used in the proof of Lemma 2.2 then shows that $\Psi(x) = \mathbb{E}_x\big[\Psi(X_{\tau(\mathcal{B}_\circ^c)}\big]$. Therefore, Ψ attains a maximum on $\bar{\mathcal{B}}_\circ$, and by the strong maximum principle it must be constant. Thus $\mathcal{L}\Psi = 0$ which contradicts the fact that $\Psi(0) = 1$.

Next we show that (d) $\Rightarrow$ (c). If $\lambda^*(f) < 1$, for $f = \mathbb{1}_{\mathcal{B}^c}$, then any limit point Ψ of the Dirichlet eigenfunctions $\widehat{\Psi}_n$ as $n \to \infty$ satisfies

$$\mathcal{L}\Psi \;=\; \mathbb{1}_{\bar{\mathcal{B}}}\Psi - \big(1 - \lambda^*(f)\big)\,\Psi \;\le\; \Big(\sup_{\mathcal{B}} \Psi\Big)\mathbb{1}_{\bar{\mathcal{B}}} - \big(1 - \lambda^*(f)\big)\,\Psi\,.$$

Also by [10, Lemma 2.1 (c)], we have $\inf_{\mathbb{R}^d} \Psi = \min_{\bar{\mathcal{B}}} \Psi > 0$. Thus (c) holds with $\delta = 1 - \lambda^*(f)$.

That (c) $\Rightarrow$ (b) is well known, and can be shown by a standard application of the Itô–Krylov formula to (3.3), by which we obtain

$$\begin{aligned}\Big(\inf_{\mathbb{R}^d} \mathcal{V}\Big)\,\mathbb{E}_x\Big[\mathrm{e}^{\delta(\tau(\mathcal{B}^c)\wedge\tau_R)}\Big] - \mathcal{V}(x) \;&\le\; \mathbb{E}_x\Big[\mathrm{e}^{\delta(\tau(\mathcal{B}^c)\wedge\tau_R)}\mathcal{V}(X_{\tau(\mathcal{B}^c)\wedge\tau_R})\Big] - \mathcal{V}(x)\\ &\le\; \mathbb{E}_x\bigg[\int_0^{\tau(\mathcal{B}^c)\wedge\tau_R} \Big(\delta \mathrm{e}^{\delta t}\,\mathcal{V}(X_t) + \mathrm{e}^{\delta t}\mathcal{L}\mathcal{V}(X_t)\Big)\,\mathrm{d}t\bigg] \;\le\; 0\,.\end{aligned}$$

The result then follows by letting $R \to \infty$, and this completes the proof. □

We introduce the following hypothesis.

(H2) There exists a lower-semicontinuous, inf-compact function $\ell : \mathbb{R}^d \to [0, \infty)$ such that $\mathcal{E}(\ell) < \infty$, where $\mathcal{E}(\cdot)$ is as defined in (1.3).

Lemma 3.1. *Under* (H2), *we have* $\mathcal{E}(\ell) = \lambda^*(\ell)$, *and there exists a positive* $V \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p \ge d$, *with* $\inf_{\mathbb{R}^d} V > 0$, *and* $V(0) = 1$, *satisfying*

$$\mathcal{L}V + \ell V \;=\; \lambda^*(\ell)\,V \quad \text{a.e. in } \mathbb{R}^d\,. \tag{3.4}$$

In particular, the unique strong solution of (2.1) *is exponentially ergodic.*

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

Proof. By (2.4) we have

$$\mathcal{E}_x(\ell) \;\geq\; \limsup_{T\to\infty} \frac{1}{T}\, \mathbb{E}_x\biggl[\int_0^T \ell(X_s)\,\mathrm{d}s\biggr]. \tag{3.5}$$

Since $\mathcal{E}(\ell) = \inf_{\mathbb{R}^d} \mathcal{E}_x(\ell)$, (3.5) implies that

$$\limsup_{T\to\infty} \frac{1}{T}\, \mathbb{E}_x\biggl[\int_0^T \ell(X_s)\,\mathrm{d}s\biggr] \;<\; \infty$$

for some $x \in \mathbb{R}^d$. The inf-compactness of ℓ then implies that the unique strong solution of (2.1) is positive recurrent. That $\mathcal{E}(\ell) = \lambda^*(\ell)$, and the existence of a solution V then follow by Theorem 1.4 and Lemma 2.1 in [10], respectively. Exponential ergodicity then follows from (3.4), using Theorem 3.1. □

An application of the Itô–Krylov formula to (3.4), followed by Fatou's lemma, shows that

$$\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\breve{\tau}_r}[\ell(X_t)-\lambda^*(\ell)]\,\mathrm{d}t}\, V(X_{\breve{\tau}})\Bigr] \;\leq\; V(x) \qquad \forall\, x \in B_r^c\,,\ \forall\, r > 0\,, \tag{3.6}$$

where $\breve{\tau}_r$, as defined earlier, denotes the first hitting time of the ball B_r.

The next result shows that (H2) implies (P1).

Theorem 3.2. *Assume* (H2)*, and suppose that f is a potential such that $\ell - f$ is inf-compact. Then for any continuous $h \in C_{\mathrm{o}}^+(\mathbb{R}^d)$ we have*

$$\lambda^*(f-h) \;<\; \lambda^*(f) \;=\; \mathcal{E}_x(f) \quad \forall\, x \in \mathbb{R}^d\,.$$

Proof. Let $h \in C_{\mathrm{o}}^+(\mathbb{R}^d)$, and $\tilde{f} \coloneqq f - h$. It is easy to see that $\mathcal{E}(f)$ and $\mathcal{E}(\tilde{f})$ are both finite. It is shown in [10,28] that the Dirichlet eigensolutions $(\widehat{\Psi}_r, \hat{\lambda}_r)$ in (2.3) converge, along some subsequence as $r \to \infty$, to $\bigl(\Psi^*, \lambda^*(f)\bigr)$ which satisfies

$$\mathcal{L}\Psi^* + f\,\Psi^* \;=\; \lambda^*(f)\,\Psi^* \quad \text{on } \mathbb{R}^d\,, \qquad \text{and} \quad \lambda^*(f) \;\leq\; \mathcal{E}_x(f) \quad \forall\, x \in \mathbb{R}^d\,. \tag{3.7}$$

It is also clear that Lemma 2.2 (i) holds for $(\widehat{\Psi}_n, \hat{\lambda}_n)$. Now choose a bounded ball $\mathcal{B}$ such that

$$\bigl(|f(x)| + |h(x)|\bigr) + \sup_n\bigl(|\hat{\lambda}_n(f)| + |\hat{\lambda}_n(\tilde{f})|\bigr) + \lambda^*(\ell) + 1 \;<\; \ell(x) \quad \forall\, x \in \mathcal{B}^c\,.$$

This is possible since $\ell - f$ is inf-compact. In view of (3.6) we note that (2.12) holds with $f - h - \lambda^*(f-h)$ replaced by $\ell - \lambda^*(\ell)$. Thus with the above choice of $\mathcal{B}$, we can justify the passing to the limit in (2.14), and therefore, we obtain

$$\Psi^*(x) \;=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\breve{\tau}}[f(X_t)-\lambda^*(f)]\,\mathrm{d}t}\, \Psi^*(X_{\breve{\tau}})\Bigr] \qquad \forall\, x \in \mathcal{B}^c\,, \tag{3.8}$$

with $\breve{\tau} = \tau(\mathcal{B}^c)$. Recall the definition of $\bigl(\tilde{\Psi}^*, \lambda^*(\tilde{f})\bigr)$ in (2.31). A similar argument also gives

$$\tilde{\Psi}^*(x) \;=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\breve{\tau}}[\tilde{f}(X_t)-\lambda^*(\tilde{f})]\,\mathrm{d}t}\, \tilde{\Psi}^*(X_{\breve{\tau}})\Bigr] \qquad \forall\, x \in \mathcal{B}^c\,. \tag{3.9}$$

In fact, the above relations hold for any bounded domain $D \supset \mathcal{B}$ with $\breve{\tau} = \tau(D^c)$.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

Suppose that $\lambda^*(f) = \lambda^*(\tilde f)$. Then

$$f(x) - h(x) - \lambda^*(\tilde f) \;\le\; f(x) - \lambda^*(f)\,.$$

Thus if we multiply Ψ^* with a suitable positive constant such that $\Psi^* - \tilde\Psi^*$ is nonnegative in $\mathcal{B}$ and attains a minimum of 0 in $\mathcal{B}$, it follows from (3.8) and (3.9) that $\Psi^* - \tilde\Psi^*$ is nonnegative in $\mathbb{R}^d$. Since (2.34) holds, and we conclude exactly as in the proof of Theorem 2.2 that $\lambda^*(\tilde f) < \lambda^*(f)$.

Next we show that $\lambda^*(f) = \mathcal{E}_x(f)$ for all $x \in \mathbb{R}^d$. We have already established the strict monotonicity of $\lambda^*(f)$ at f, and therefore, Theorem 2.1 applies. Hence for any continuous g with compact support we have from [40, Theorem 1.3.10] that

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^T [f(X_t)-\lambda^*(f)]\,\mathrm{d}t}\, g(X_T)\Big] \;=\; \Psi^*(x)\,\widetilde{\mathbb{E}}_x^{\psi^*}\left[\frac{g(Y^*_T)}{\Psi^*(Y^*_T)}\right] \;\xrightarrow[T\to\infty]{}\; \Psi^*(x)\,\mu^*\Big(\frac{g}{\Psi^*}\Big) \;>\; 0\,, \tag{3.10}$$

where μ^* denotes the invariant measure of the twisted process Y^* satisfying (2.25). Let $\tilde{\mathcal{B}}$ be a ball such that $f(x) - \lambda^*(f) < \ell(x) - \lambda^*(\ell)$ for $x \in \tilde{\mathcal{B}}^c$. Thus from (3.4) we obtain

$$\mathcal{L}V + \big(f - \lambda^*(f)\big)V \;\le\; \kappa \mathbb{1}_{\tilde{\mathcal{B}}}\,, \tag{3.11}$$

with $\kappa = \max_{\tilde{\mathcal{B}}}\big(|f| + |\ell| + |\lambda^*| + \lambda^*(\ell)\big)\,V$. Applying the Itô–Krylov formula to (3.11) followed by Fatou's lemma we obtain

$$\begin{aligned}
\Big(\min_{\mathbb{R}^d} V\Big)\,\mathbb{E}_x\big[\mathrm{e}^{\int_0^T [f(X_t)-\lambda^*(f)]\,\mathrm{d}t}\big] \;&\le\; \mathbb{E}_x\big[\mathrm{e}^{\int_0^T [f(X_t)-\lambda^*(f)]\,\mathrm{d}t}\, V(X_T)\big] \\
&\le\; \kappa \int_0^T \mathbb{E}_x\big[\mathrm{e}^{\int_0^t [f(X_t)-\lambda^*(f)]\,\mathrm{d}t}\, \mathbb{1}_{\tilde{\mathcal{B}}}(X_T)\big]\,\mathrm{d}t + V(x) \\
&\le\; \kappa'\,T + V(x)\,,
\end{aligned}$$

for some constant κ', where in the last inequality we have used (3.10). Taking logarithms on both sides of the preceding inequality, then dividing by T, and letting $T \to \infty$, we obtain $\lambda^*(f) \ge \mathcal{E}_x(f)$ for all $x \in \mathbb{R}^d$. Combining this with (3.7) results in equality. □

Remark 3.2. Continuity of h is superfluous in Theorem 3.2. The result holds if h is a non-trivial, nonnegative measurable function, vanishing at infinity.

Corollary 3.1. *Under the assumptions of Theorem 3.2, for any potential $\tilde f \lneqq f$, we have $\lambda^*(\tilde f) < \lambda^*(f)$.*

Proof. Note that for any cut-off function χ we have $\lambda^*(\tilde f) \le \lambda^*(\chi\tilde f + (1-\chi)f)$. Then the result follows from Theorem 3.2 and Remark 3.2. □

Remark 3.3. In Theorem 3.2 we can replace the assumption that f is bounded from below in $\mathbb{R}^d$ by the hypothesis that $\ell - |f|$ is inf-compact.

Let us now discuss the exponential ergodicity and show that this implies (H2).

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

Proposition 3.1. *Let $\ell : \mathbb{R}^d \to \mathbb{R}$ be inf-compact, and suppose $\phi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ is bounded below in $\mathbb{R}^d$ and satisfies*

$$\mathcal{L}\phi + \langle \nabla\phi, a\nabla\phi\rangle \;=\; -\ell\,, \tag{3.12}$$

then $\mathcal{E}_x(\ell) < \infty$ for all $x \in \mathbb{R}^d$.

Proof. Let $\Phi(x) = \exp(\phi(x))$. Then $\inf_{\mathbb{R}^d} \Phi > 0$, and (3.12) gives

$$\mathcal{L}\Phi + \ell\Phi \;=\; 0\,. \tag{3.13}$$

Now apply the Itô–Krylov formula to (3.13) followed by Fatou's lemma to obtain

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^T \ell(X_s)\,\mathrm{d}s}\,\Phi(X_T)\Big] \;\le\; \Phi(x)\,.$$

Taking logarithm on both sides, diving by T and letting $T \to \infty$, we obtain $\mathcal{E}_x(\ell) < \infty$. □

Example 3.2. Let $a = \frac{1}{2}I$ and $b(x) = b_1(x) + B(x)$ where B is bounded and

$$\langle b_1(x), x\rangle \;\le\; -\kappa|x|^\alpha, \quad \text{for some } \alpha \in (1,2]\,.$$

Then we take $\phi(x) = \theta\,|x|^\alpha$ for $|x| \ge 1\,, \theta \in (0,1)$. It is easy to check that for a suitable choice of $\theta \in (0,1)$, (3.12) holds for $\ell(x) \sim |x|^{2\alpha-2}$.

Remark 3.4. Equation (3.12) is a stronger condition than strict monotonicity of $\lambda^*(f)$ at f. In fact, (3.12) might not hold in many important situations. For instance, if a and b are both bounded, and a is uniformly elliptic, then it is not possible to find inf-compact ℓ satisfying (3.12). Otherwise, we can find a finite principal eigenvalue for the operator $\mathcal{L}^\ell$, by a same method as in (3.7), which would contradict [5, Proposition 2.6].

Even though (3.12) does not hold for bounded a and b, strict monotonicity of $\lambda^*(f)$ at f can be asserted under suitable hypotheses. This is the subject of the following theorem.

Theorem 3.3. *Let $\mathcal{V} \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ such that $\inf_{\mathbb{R}^d} \mathcal{V} > 0$, satisfying*

$$\mathcal{L}\mathcal{V} \;\le\; \kappa_0 \mathbb{1}_{\mathcal{K}} - \gamma\mathcal{V} \quad \text{on } \mathbb{R}^d\,, \tag{3.14}$$

for some compact set $\mathcal{K}$ and positive constants κ_0 and γ. Let f be a nonnegative bounded measurable function with $\limsup_{x\to\infty} f(x) < \gamma$. Then for any $h \in C_{\mathrm{o}}^+(\mathbb{R}^d)$, we have $\lambda^*(f-h) < \lambda^*(f) = \mathcal{E}_x(f)$ for all $x \in \mathbb{R}^d$.

Proof. Let $\tilde f = f - h$. Suppose $\lambda^*(\tilde f) = \lambda^*(f)$. Applying an argument similar to (3.7) we can find Ψ^* and $\tilde\Psi^*$ that satisfy

$$\begin{aligned} \mathcal{L}\Psi^* + f\,\Psi^* \;&=\; \lambda^*(f)\,\Psi^*, \\ \mathcal{L}\tilde\Psi^* + (f-h)\tilde\Psi^* \;&=\; \lambda^*(\tilde f)\tilde\Psi^*. \end{aligned}$$

Let $\mathcal{K}_0 \supset \mathcal{K}$ be any compact set such that $f < \gamma$ on $\mathcal{K}_0^c$. If $\breve\tau$ denotes the first hitting time to the compact set $\mathcal{K}_0$, then by an application of the Itô–Krylov formula to (3.14) we obtain

$$\mathbb{E}_x\big[\mathrm{e}^{\gamma\breve\tau}\big] < \infty\,, \quad x \in \mathcal{K}_0^c\,.$$

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

We next use the fact that if $\mathcal{L}$ corresponds to a recurrent diffusion and f is nonnegative then $\lambda^*(f) \ge 0$. Indeed, in this case we have $\mathcal{L}\Psi^* \le \lambda^*(f)\Psi^*$. If $\lambda^*(f) \le 0$, this implies that $\Psi^*(X_t)$ is a nonnegative supermartingale and since it is integrable, it converges a.s. Since the process is recurrent, this implies that Ψ^* must equal to a constant, which, in turn, necessitates that $\lambda^*(f) = 0$ (and $f = 0$). Thus, since $\lambda^*(f) \ge 0$, an argument similar to the proof of Lemma 2.2 (ii) shows that

$$\Psi^*(x) \;=\; \mathbb{E}_x\left[\mathrm{e}^{\int_0^{\check\tau}[f(X_t)-\lambda^*(f)]\,\mathrm{d}t}\,\Psi^*(X_{\check\tau})\right],$$

$$\tilde\Psi^*(x) \;=\; \mathbb{E}_x\left[\mathrm{e}^{\int_0^{\check\tau}[\tilde f(X_t)-\tilde\lambda^*(\tilde f)]\,\mathrm{d}t}\,\tilde\Psi^*(X_{\check\tau})\right],$$

for $x \in \mathcal{K}_0^{\mathrm{c}}$. Therefore, applying the strong maximum principle as in Theorem 3.2, we obtain $h\,\tilde\Psi^* = 0$ which is a contradiction since $h \ne 0$ and $\tilde\Psi^* > 0$. Thus we have $\lambda^*(f-h) < \lambda^*(f)$. That $\lambda^*(f) = \mathcal{E}_x(f)$ for all $x \in \mathbb{R}^d$ follows by an argument similar to the one used in the proof Theorem 3.2. □

Example 3.3. Suppose $a = \frac{1}{2}I$, where I denote the identity matrix, and

$$\langle b(x), x\rangle \;\le\; -|x|, \quad \text{outside a compact set } \mathcal{K}_1\,.$$

With $\mathcal{V}(x) = \exp(|x|)$ for $|x| \ge 1$, we have

$$\mathcal{L}\mathcal{V} \;=\; \left(\frac{d-1}{2|x|} + \frac{1}{2} - \frac{\langle b(x), x\rangle}{|x|}\right)\mathcal{V} \;\le\; \left(\frac{d-1}{2|x|} - \frac{1}{2}\right)\mathcal{V} \quad \text{for } |x| \ge 1\,.$$

4. Risk-sensitive control

In this section we apply the results developed in the previous sections to the risk-sensitive control problem. As mentioned earlier, we establish the existence and uniqueness of solutions to the risk-sensitive HJB equation, and use this to completely characterize the optimal Markov controls (see Theorems 4.1 and 4.2). Another interesting result is the continuity of the controlled principal eigenvalue with respect to the stationary Markov controls. This is done in Theorem 4.3. We first introduce the control problem.

4.1. The controlled diffusion model

Consider a controlled diffusion process $X = \{X_t,\, t \ge 0\}$ which takes values in the d-dimensional Euclidean space $\mathbb{R}^d$, and is governed by the Itô equation

$$\mathrm{d}X_t \;=\; b(X_t, U_t)\,\mathrm{d}t + \boldsymbol{\sigma}(X_t)\,\mathrm{d}W_t\,. \tag{4.1}$$

All random processes in (4.1) live in a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. The process W is a d-dimensional standard Wiener process independent of the initial condition X_0. The control process U takes values in a compact, metrizable set $\mathbb{U}$, and $U_t(\omega)$ is jointly measurable in $(t,\omega) \in [0,\infty) \times \Omega$. The set $\mathfrak{U}$ of *admissible controls* consists of the control processes U that are *non-anticipative*: for $s < t$, $W_t - W_s$ is independent of

$$\mathfrak{F}_s \;\coloneqq\; \text{the completion of } \cap_{y>s}\,\sigma\{X_0, U_r, W_r,\; r \le y\} \text{ relative to } (\mathfrak{F}, \mathbb{P})\,.$$

We impose the following standard assumptions on the drift b and the diffusion matrix $\boldsymbol{\sigma}$ to guarantee existence and uniqueness of solutions.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

(B1) *Local Lipschitz continuity:* The functions $b\colon \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are continuous, and satisfy

$$|b(x,u)-b(y,u)| + \|\sigma(x)-\sigma(y)\| \;\le\; C_R\,|x-y| \qquad \forall\, x,y \in B_R\,,\ \forall\, u \in \mathbb{U}\,,$$

for some constant $C_R > 0$ depending on $R > 0$.

(B2) *Affine growth condition:* For some $C_0 > 0$, we have

$$\sup_{u\in\mathbb{U}} \langle b(x,u),x\rangle^+ + \|\sigma(x)\|^2 \;\le\; C_0\big(1+|x|^2\big) \qquad \forall\, x \in \mathbb{R}^d\,.$$

(B3) *Nondegeneracy:* Assumption (A3) in Subsection 1.1 holds.

It is well known that under (B1)–(B3), for any admissible control there exists a unique solution of (4.1) [34, Theorem 2.2.4]. We define the family of operators $\mathcal{L}_u\colon C^2(\mathbb{R}^d) \mapsto C(\mathbb{R}^d)$, where $u \in \mathbb{U}$ plays the role of a parameter, by

$$\mathcal{L}_u f(x) \;\coloneqq\; a^{ij}(x)\,\partial_{ij} f(x) + b^i(x,u)\,\partial_i f(x)\,, \quad u \in \mathbb{U}\,.$$

The risk-sensitive criterion Let $\mathfrak{C}$ denote the class of functions $c(x,u)$ in $C(\mathbb{R}^d \times \mathbb{U}, \mathbb{R}_+)$ that are locally Lipschitz in x uniformly with respect to $u \in \mathbb{U}$. We let $c \in \mathfrak{C}$ denote the *running cost* function, and for any admissible control $U \in \mathfrak{U}$, we define the risk-sensitive objective function $\mathcal{E}_x^U(c)$ by

$$\mathcal{E}_x^U(c) \;\coloneqq\; \limsup_{T\to\infty}\; \frac{1}{T}\,\log \mathbb{E}_x\Big[\mathrm{e}^{\int_0^T c(X_s,U_s)\,\mathrm{d}s}\Big]\,. \tag{4.2}$$

We also define $\Lambda_x^* \coloneqq \inf_{U\in\mathfrak{U}} \mathcal{E}_x^U(c)$.

4.2. Relaxed controls

We adopt the well-known *relaxed control* framework [34]. According to this relaxation, a stationary Markov control is a measurable map from $\mathbb{R}^d$ to $\mathcal{P}(\mathbb{U})$, the latter denoting the set of probability measures on $\mathbb{U}$ under the Prokhorov topology. Let $\mathfrak{U}_{\mathrm{SM}}$ denote the class of all such stationary Markov controls. A control $v \in \mathfrak{U}_{\mathrm{SM}}$ may be viewed as a kernel on $\mathcal{P}(\mathbb{U}) \times \mathbb{R}^d$, which we write as $v(\mathrm{d}u\,|\,x)$. We say that a control $v \in \mathfrak{U}_{\mathrm{SM}}$ is precise if it is a measurable map from $\mathbb{R}^d$ to $\mathbb{U}$. We extend the definition of b and c as follows. For $v \in \mathfrak{U}_{\mathrm{SM}}$ we let

$$b_v(x) \;\coloneqq\; \int_{\mathbb{U}} b(x,u)\,v(\mathrm{d}u\,|\,x)\,, \quad \text{and} \quad c_v(x) \;\coloneqq\; \int_{\mathbb{U}} c(x,u)\,v(\mathrm{d}u\,|\,x) \quad \text{for } v \in \mathcal{P}(\mathbb{U})\,.$$

It is easy to see from (B2) and Jensen's inequality that

$$\sup_{v\in\mathfrak{U}_{\mathrm{SM}}} \langle b_v(x),x\rangle^+ \;\le\; C_0(1+|x|^2) \qquad \forall\, x \in \mathbb{R}^d\,.$$

For $v \in \mathfrak{U}_{\mathrm{SM}}$, consider the relaxed diffusion

$$\mathrm{d}X_t \;=\; b_v(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\,. \tag{4.3}$$

It is well known that under $v \in \mathfrak{U}_{\mathrm{SM}}$ (4.3) has a unique strong solution [42], which is also a strong Markov process. It also follows from the work in [41] that under $v \in \mathfrak{U}_{\mathrm{SM}}$, the transition probabilities of X have densities which are locally Hölder continuous. Thus $\mathcal{L}_v$ defined by

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

$$\mathcal{L}_v f(x) \;\coloneqq\; a^{ij}(x)\,\partial_{ij} f(x) + b_v^i(x)\,\partial_i f(x)\,, \quad v \in \mathfrak{U}_{\rm SM}\,,$$

for $f \in C^2(\mathbb{R}^d)$, is the generator of a strongly-continuous semigroup on $C_b(\mathbb{R}^d)$, which is strong Feller. We let $\mathbb{P}_x^v$ denote the probability measure and $\mathbb{E}_x^v$ the expectation operator on the canonical space of the process under the control $v \in \mathfrak{U}_{\rm SM}$, conditioned on the process X starting from $x \in \mathbb{R}^d$ at $t = 0$. We denote by $\mathfrak{U}_{\rm SSM}$ the subset of $\mathfrak{U}_{\rm SM}$ that consists of *stable controls*, i.e., under which the controlled process is positive recurrent, and by μ_v the invariant probability measure of the process under the control $v \in \mathfrak{U}_{\rm SSM}$.

Definition 4.1. For $v \in \mathfrak{U}_{\rm SM}$ and a locally bounded measurable function $f\colon \mathbb{R}^d \to \mathbb{R}$, we let $\lambda_v^*(f)$ denote the principal eigenvalue of the operator $\mathcal{L}_v^f \coloneqq \mathcal{L}_v + f$ on $\mathbb{R}^d$ (see Definition 2.1).

We also adapt the notation in (2.4) to the control setting, and define

$$\mathcal{E}_x^v(f) \;\coloneqq\; \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}_x^v\Big[\mathrm{e}^{\int_0^T f(X_s)\,\mathrm{d}s}\Big]\,, \quad \text{and} \quad \mathcal{E}^v(f) \;\coloneqq\; \inf_{x\in\mathbb{R}^d} \mathcal{E}_x^v(f)\,, \qquad v \in \mathfrak{U}_{\rm SM}\,.$$

We refer to $\mathcal{E}^v(f)$ as the *risk-sensitive* average of f under the control v.

Recall the risk-sensitive objective function $\mathcal{E}_x^U$ defined in (4.2) and the optimal value Λ^*. We say that a stationary Markov control $v \in \mathfrak{U}_{\rm SM}$ is optimal (for the risk-sensitive criterion) if $\mathcal{E}_x^v(c_v) = \Lambda_x^*$ for all $x \in \mathbb{R}^d$, and we let $\mathfrak{U}_{\rm SM}^*$ denote the class of these controls.

4.3. Optimal Markov controls and the risk-sensitive HJB

We start with the following assumption.

Assumption 4.1 *(uniform exponential ergodicity).* There exists an inf-compact function $\ell \in C(\mathbb{R}^d)$ and a positive function $\mathcal{V} \in \mathcal{W}_{\rm loc}^{2,d}(\mathbb{R}^d)$, satisfying $\inf_{\mathbb{R}^d} \mathcal{V} > 0$, such that

$$\sup_{u\in\mathbb{U}} \mathcal{L}_u \mathcal{V} \;\le\; \bar\kappa\, \mathbb{1}_{\mathcal{K}} - \ell\mathcal{V} \quad \text{a.e. on } \mathbb{R}^d\,, \tag{4.4}$$

for some constant $\bar\kappa$, and a compact set $\mathcal{K}$.

It is easy to see that for $\bar\kappa_\circ \coloneqq \frac{\bar\kappa}{\min_{\mathbb{R}^d} \mathcal{V}}$ we obtain from (4.4) that

$$\sup_{u\in\mathbb{U}} \mathcal{L}_u \mathcal{V} + (\ell - \bar\kappa_\circ)\mathcal{V} \;\le\; 0\,,$$

and therefore, applying the Itô–Krylov formula, we have $\mathcal{E}_x^v(\ell) \le \bar\kappa_\circ$ for any stationary Markov control $v \in \mathfrak{U}_{\rm SM}$, and all $x \in \mathbb{R}^d$.

Example 4.1. Let $\boldsymbol{\sigma}$ be bounded and $b : \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}^d$ be such that

$$\langle b(x,u) - b(0,u), x\rangle \;\le\; -\kappa\,|x|^\alpha, \quad \text{for some } \alpha \in (1,2]\,, \quad (x,u) \in \mathbb{R}^d \times \mathbb{U}\,.$$

Then as seen in Example 3.2, $\mathcal{V}(x) = \exp(\theta\,|x|^\alpha)$, for $|x| \ge 1$, satisfies (4.4) for sufficiently small $\theta > 0$, and $\ell(x) \sim |x|^{2\alpha-2}$. Note that $\alpha = 2$ and $\boldsymbol{\sigma} = I$ is considered in [23].

We introduce the class of running costs $\mathcal{C}_\ell$ defined by

$$\mathcal{C}_\ell \;\coloneqq\; \Big\{ c \in \mathfrak{C} :\; \ell(\cdot) - \max_{u\in\mathbb{U}} c(\cdot,u) \text{ is inf-compact} \Big\}\,.$$

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

The first important result of this section is the following.

Theorem 4.1. *Suppose Assumption 4.1 holds, and $c \in \mathcal{C}_\ell$. Then $\Lambda^* = \Lambda^*_x$ does not depend on x, and there exists a positive solution $V \in C^2(\mathbb{R}^d)$ satisfying*

$$\min_{u\in\mathbb{U}} [\mathcal{L}_u V + c(\cdot,u)V] \;=\; \Lambda^* V \quad \text{on } \mathbb{R}^d\,, \quad \text{and } V(0) = 1\,. \tag{4.5}$$

In addition, if $\overline{\mathfrak{U}}_{\rm SM} \subset \mathfrak{U}_{\rm SM}$ denotes the class of Markov controls v which satisfy

$$\mathcal{L}_v V + c_v V \;=\; \min_{u\in\mathbb{U}} [\mathcal{L}_u V + c(\cdot,u)V] \qquad \text{a.e. in } \mathbb{R}^d\,,$$

then the following hold.

(a) $\overline{\mathfrak{U}}_{\rm SM} \subset \mathfrak{U}^*_{\rm SM}$, *and it holds that $\lambda^*_v(c_v) = \Lambda^*$ for all $v \in \overline{\mathfrak{U}}_{\rm SM}$;*
(b) $\mathfrak{U}^*_{\rm SM} \subset \overline{\mathfrak{U}}_{\rm SM}$ *;*
(c) *Equation (4.5) has a unique positive solution in $C^2(\mathbb{R}^d)$ (up to a multiplicative constant).*

Proof. Using a standard argument (see [10,26,28]) we can find a pair $(V,\hat\lambda) \in C^2(\mathbb{R}^d)\times\mathbb{R}$, with $V>0$ on $\mathbb{R}^d$, and $V(0)=1$, that satisfies

$$\min_{u\in\mathbb{U}} [\mathcal{L}_u V + c(\cdot,u)V] \;=\; \hat\lambda V\,, \quad \hat\lambda \;\le\; \inf_{x\in\mathbb{R}^d} \Lambda^*_x\,. \tag{4.6}$$

This is obtained as a limit of Dirichlet eigensolutions $(\widehat V_n, \hat\lambda_n) \in \big(\mathcal{W}^{2,p}_{\rm loc}(B_n)\cap C(\bar B_n)\big)\times\mathbb{R}$, for any $p>d$, satisfying $\widehat V_n > 0$ on B_n, $\widehat V_n = 0$ on ∂B_n, $\widehat V_n(0)=1$, and

$$\min_{u\in\mathbb{U}} \big[\mathcal{L}_u \widehat V_n(x) + c(x,u)\,\widehat V_n(x)\big] \;=\; \hat\lambda_n\,\widehat V_n(x) \qquad \text{a.e. } x\in B_n\,.$$

For $v \in \overline{\mathfrak{U}}_{\rm SM}$ we have

$$\mathcal{L}_v V + c_v V \;=\; a^{ij}\partial_{ij} V + \langle b_v, \nabla V\rangle + c_v V \;=\; \hat\lambda V \quad \text{on } \mathbb{R}^d\,. \tag{4.7}$$

By Corollary 2.1 we obtain $\hat\lambda \ge \lambda^*_v(c_v)$. Also by Theorem 3.2 we have $\lambda^*_v(c_v) = \mathcal{E}^v_x(c_v)$ for all $x\in\mathbb{R}^d$. Combining these estimates with (4.6) we obtain

$$\Lambda^*_x \;\le\; \mathcal{E}^v_x(c_v) \;=\; \lambda^*_v(c_v) \;\le\; \hat\lambda \;\le\; \inf_{z\in\mathbb{R}^d} \Lambda^*_z \qquad \forall\, x\in\mathbb{R}^d\,.$$

This of course shows that $\hat\lambda = \lambda^*_v(c_v) = \Lambda^*_x$ for all $x\in\mathbb{R}^d$, and also proves part (a).

We continue with part (b). By Theorem 3.2 we have

$$\lambda^*_v(c_v - h) \;<\; \lambda^*_v(c_v) \qquad \forall\, h\in C^+_{\rm o}(\mathbb{R}^d)\,, \quad \forall\, v\in\mathfrak{U}_{\rm SM}\,. \tag{4.8}$$

In turn, by Lemma 2.4 there exists a unique eigenfunction $\Psi_v \in \mathcal{W}^{2,d}_{\rm loc}(\mathbb{R}^d)$ which is associated with the principal eigenvalue $\lambda^*_v(c_v)$ of the operator $\mathcal{L}^{c_v}_v = \mathcal{L}_v + c_v$. Since $\hat\lambda = \lambda^*_v(c_v)$ for all $v\in\overline{\mathfrak{U}}_{\rm SM}$ by part (a), it follows by (4.7) that

$$V \;=\; \Psi_v \quad \forall\, v\in\overline{\mathfrak{U}}_{\rm SM}\,. \tag{4.9}$$

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

By (4.8) and Lemma 2.2 (ii), and since (4.3) is recurrent, we have

$$\Psi_v(x) \;=\; \mathbb{E}^v_x\Big[e^{\int_0^{\check\tau}[c_v(X_s)-\Lambda^*]\,\mathrm{d}s}\,\Psi_v(X_{\check\tau})\Big] \qquad \forall\, x\in\mathcal{B}^c\,, \quad \forall\, v\in\mathfrak{U}^*_{\mathrm{SM}}\,, \tag{4.10}$$

and all sufficiently large balls $\mathcal{B}$ centered at 0, where $\check\tau=\tau(\mathcal{B}^c)$, as usual.

Since the Dirichlet eigenvalues satisfy $\hat\lambda_n<\hat\lambda=\Lambda^*$ for all $n\in\mathbb{N}$, the Dirichlet problem

$$\min_{u\in\mathbb{U}}\big[\mathcal{L}_u\varphi_n(x)+\big(c(x,u)-\Lambda^*\big)\,\varphi_n(x)\big] \;=\; -\alpha_n\,\mathbb{1}_{\mathcal{B}}(x) \qquad \text{a.e. } x\in B_n\,, \qquad \varphi_n=0 \;\text{ on }\; \partial B_n\,, \tag{4.11}$$

with $\alpha_n>0$, has a unique solution $\varphi_n\in\mathcal{W}^{2,p}_{\mathrm{loc}}(B_n)\cap C(\bar B_n)$, for any $p\ge1$ [4, Theorem 1.9] (see also [43, Theorem 1.1 (ii)]). We choose α_n as follows: first select $\tilde\alpha_n>0$ such that the solution φ_n of (4.11) with $\alpha_n=\tilde\alpha_n$ satisfies $\varphi_n(0)=1$, and then set $\alpha_n=\min(1,\tilde\alpha_n)$. Passing to the limit in (4.11) as $n\to\infty$ along a subsequence, we obtain a nonnegative solution $\Phi\in\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$ of

$$\min_{u\in\mathbb{U}}\big[\mathcal{L}_u\Phi(x)+\big(c(x,u)-\Lambda^*\big)\,\Phi(x)\big] \;=\; -\alpha\,\mathbb{1}_{\mathcal{B}}(x)\,, \qquad x\in\mathbb{R}^d\,. \tag{4.12}$$

It is evident from the construction that if $\alpha=0$ then $\Phi(0)=1$. On the other hand, if $\alpha>0$, then necessarily Φ is positive on $\mathbb{R}^d$. Let $\hat v\in\mathfrak{U}_{\mathrm{SM}}$ be a selector from the minimizer of (4.12). If $\alpha>0$, then (4.12) implies that there exists $h\in C^+_{\mathrm{o}}(\mathbb{R}^d)$ such that $\lambda^*_{\hat v}(c_{\hat v}+h)\le\Lambda^*$. Since $\lambda^*_{\hat v}(c_{\hat v})=\mathcal{E}^{\hat v}_x(c_{\hat v})$ for all $x\in\mathbb{R}^d$ by Theorem 3.2, and $\mathcal{E}^{\hat v}_x(c_{\hat v})\ge\Lambda^*$, then, in view of Corollary 2.1, this contradicts (4.8) and the convexity of $\lambda^*_{\hat v}$. Therefore, we must have $\alpha=0$. Let $\bar v\in\mathfrak{U}^*_{\mathrm{SM}}$. Applying the Itô–Krylov formula to (4.11) we obtain

$$\varphi_n(x) \;\le\; \mathbb{E}^{\bar v}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[c_{\bar v}(X_s)-\Lambda^*]\,\mathrm{d}s}\,\varphi_n(X_{\check\tau})\,\mathbb{1}_{\{\check\tau<T\wedge\tau_n\}}\Big]+\mathbb{E}^{\bar v}_x\Big[\mathrm{e}^{\int_0^{T}[c_{\bar v}(X_s)-\Lambda^*]\,\mathrm{d}s}\,\varphi_n(X_T)\,\mathbb{1}_{\{T<\check\tau\wedge\tau_n\}}\Big} \qquad \forall\, T>0\,,$$

and for all $x\in B_n\setminus\mathcal{B}$, where $\check\tau=\tau(\mathcal{B}^c)$. Using the argument in the proof of [10, Lemma 2.11], we obtain

$$\Phi(x) \;\le\; \mathbb{E}^{\bar v}_x\Big[\mathrm{e}^{\int_0^{\check\tau}[c_{\bar v}(X_s)-\Lambda^*]\,\mathrm{d}s}\,\Phi(X_{\check\tau})\Big] \qquad \forall\, x\in\mathcal{B}^c\,, \quad \forall\,\bar v\in\mathfrak{U}^*_{\mathrm{SM}}\,. \tag{4.13}$$

Comparing (4.10) and (4.13), it follows that, given any $\bar v\in\mathfrak{U}^*_{\mathrm{SM}}$, we can scale $\Psi_{\bar v}$ by a positive constant so that it touches Φ from above at some point in $\bar{\mathcal{B}}$. However, $\bar v$ satisfies

$$\mathcal{L}_{\bar v}\Phi+c_{\bar v}\,\Phi \;\ge\; \Lambda^*\,\Phi \qquad \text{a.e. in } \mathbb{R}^d$$

by (4.12). Thus we have

$$\mathcal{L}_{\bar v}(\Psi_{\bar v}-\Phi)-(c_{\bar v}-\Lambda^*)^-\,(\Psi_{\bar v}-\Phi) \;\le\; 0 \qquad \text{a.e. in } \mathbb{R}^d\,,$$

and it follows by the strong maximum principle that $\Phi=\Psi_{\bar v}$ for all $\bar v\in\mathfrak{U}^*_{\mathrm{SM}}$. Since $\overline{\mathfrak{U}}_{\mathrm{SM}}\subset\mathfrak{U}^*_{\mathrm{SM}}$ by part (a), it then follows by (4.9) that $V=\Psi_{\bar v}$ for all $\bar v\in\mathfrak{U}^*_{\mathrm{SM}}$. Thus we have

$$\mathcal{L}_{\bar v}V+c_{\bar v}V \;=\; \mathcal{L}_{\bar v}\Psi_{\bar v}+c_{\bar v}\Psi_{\bar v} \;=\; \lambda^*_{\bar v}(c_{\bar v})\Psi_{\bar v} \;=\; \Lambda^*V \;=\; \min_{u\in\mathbb{U}}\left[\mathcal{L}_uV+c(\cdot,u)V\right].$$

This proves the verification of optimality result in part (b).

Suppose now that $\tilde V\in C^2(\mathbb{R}^d)$ is a positive solution of

$$\min_{u\in\mathbb{U}}\left[\mathcal{L}_u\tilde V+c(\cdot,u)\tilde V\right] \;=\; \Lambda^*\,\tilde V \quad \text{on } \mathbb{R}^d\,. \tag{4.14}$$

Let $\tilde v\in\mathfrak{U}_{\mathrm{SM}}$ be a selector from the minimizer of (4.14). We have $\lambda^*_{\tilde v}(c_{\tilde v})=\mathcal{E}^{\tilde v}_x(c_{\tilde v})\ge\Lambda^*$ for all $x\in\mathbb{R}^d$ by Theorem 3.2 and the definition of Λ^*, and $\lambda^*_{\tilde v}(c_{\tilde v})\le\Lambda^*$ by Corollary 2.1. Thus $\mathcal{E}^{\tilde v}_x(c_{\tilde v})=\Lambda^*$ for all $x\in\mathbb{R}^d$,

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

which implies that $\tilde{v} \in \mathfrak{U}_{\rm SM}^*$. Then $\tilde{V} = \Psi_{\tilde{v}}$ by the uniqueness of the latter. Therefore, $\tilde{V} = \Psi_{\tilde{v}} = V$ by part (b). This completes the proof. □

As mentioned in Remark 3.4 the existence of an inf-compact ℓ in Assumption 4.1 is not possible when a and b are bounded. So we consider the following alternative assumption.

Assumption 4.2. There exists a function $\mathcal{V} \in \mathcal{W}^{2,d}_{\rm loc}(\mathbb{R}^d)$, such that $\inf_{\mathbb{R}^d} \mathcal{V} > 0$, a compact set $\mathcal{K}$, and positive constants κ_0 and γ, satisfying

$$\max_{u\in\mathbb{U}} \mathcal{L}_u \mathcal{V}(x) \;\le\; \kappa_0 \mathbb{1}_{\mathcal{K}}(x) - \gamma \mathcal{V}(x), \quad x \in \mathbb{R}^d,$$

$$\limsup_{|x|\to\infty} \sup_{u\in\mathbb{U}} c(x,u) \;<\; \gamma.$$

A similar assumption is used in [26] where the author has obtained only the existence of the solution V to the HJB, and an optimal control. Also it is shown in [26] that there exists a constant γ_1, depending on γ, such that if $\|c\|_\infty < \gamma_1$, then (4.15) below has a solution. We improve these results substantially by proving uniqueness of the solution V, and verification of optimality.

Theorem 4.2. *Under Assumption 4.2, there exists a positive solution $V \in C^2(\mathbb{R}^d)$ satisfying*

$$\min_{u\in\mathbb{U}} [\mathcal{L}_u V + c(\cdot,u)V] \;=\; \Lambda^* V. \tag{4.15}$$

Let $\overline{\mathfrak{U}}_{\rm SM} \subset \mathfrak{U}_{\rm SM}$ be as in Theorem 4.1. Then (a) and (b) of Theorem 4.1 hold, and (4.15) has a unique positive solution in $C^2(\mathbb{R}^d)$ up to a multiplicative constant.

Proof. Part (a) follows exactly as in the proof of Theorem 4.1.

By Theorem 3.3 and Lemma 2.7 for any $v \in \mathfrak{U}_{\rm SM}$ there exists a unique eigenpair (Ψ_v, λ_v^*) for $\mathcal{L}_v^{c_v}$. In addition,

$$\Psi_v(x) \;=\; \mathbb{E}_x^v \Big[\mathrm{e}^{\int_0^{\check{\tau}_r} [c_v(X_s) - \lambda_v^*]\,\mathrm{d}s}\, \Psi_v(X_{\check{\tau}_r}) \Big], \quad x \in \bar{B}_r^c.$$

The rest follows as in Theorem 4.1. □

4.4. Continuity of the risk-sensitive value

It is known from [34] that the set of relaxed stationary Markov controls $\mathfrak{U}_{\rm SM}$ is compactly metrizable (see also [44] for a detailed construction of this topology). In particular $v_n \to v$ in $\mathfrak{U}_{\rm SM}$ if and only if

$$\int_{\mathbb{R}^d} f(x) \int_{\mathbb{U}} g(x,u)\, v_n(\mathrm{d}u|x)\,\mathrm{d}x \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^d} f(x) \int_{\mathbb{U}} g(x,u)\, v(\mathrm{d}u|x)\,\mathrm{d}x$$

for all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $g \in C_b(\mathbb{R}^d \times \mathbb{U})$. For $v \in \mathfrak{U}_{\rm SM}$ we denote by $(\Psi_v, \lambda_v^*(f))$ the principal eigenpair of the operator $\mathcal{L}_v^f$, i.e.,

$$\mathcal{L}_v \Psi_v(x) + f(x)\,\Psi_v(x) \;=\; \lambda_v^*(f)\,\Psi_v(x), \quad \Psi_v(x) > 0, \quad x \in \mathbb{R}^d.$$

When $f = c_v$, we occasionally drop the dependence on c_v and denote the eigenvalue as $\lambda_v^* = \lambda_v^*(c_v)$. The next result concerns the continuity of λ_v^* with respect to stationary Markov controls, and extends the

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

result in [5, Proposition 9.2]. The continuity result in [5, Proposition 9.2] is established with respect to the L^∞ norm convergence of the coefficients, whereas Theorem 4.3 that follows asserts continuity under a much weaker topology.

Theorem 4.3. *Assume one of the following.*

(i) *Assumption 4.1 holds, and $c \in \mathcal{C}_{\beta\ell}$ for some $\beta \in (0,1)$.*
(ii) *Assumption 4.2 holds.*

Then the map $v \mapsto \lambda^*_v$ is continuous.

Proof. We demonstrate the result under (i). For case (ii) the proof is analogous. Let $v_n \to v$ in the topology of Markov controls. Let (Ψ_n, λ^*_n) be the principal eigenpair which satisfies

$$\mathcal{L}_{v_n}\Psi_n(x) + c_{v_n}(x)\,\Psi_n(x) \;=\; \lambda^*_n\,\Psi_n(x), \quad x \in \mathbb{R}^d, \quad \text{and} \quad \lambda^*_n = \mathcal{E}^{v_n}(c_{v_n})\,, \tag{4.16}$$

where the equality $\lambda^*_n = \mathcal{E}^{v_n}(c_{v_n})$ is a consequence of Theorem 3.2 and Proposition 3.1. It is obvious that $\lambda^*_n \ge 0$ for all n.

Since $\ell(\cdot) - \max_{u\in\mathbb{U}} c(\cdot,u)$ is inf-compact, we can find a constant κ_1 such that $\max_{u\in\mathbb{U}} c(x,u) \le \kappa_1 + \ell(x)$. Recall that $\mathcal{E}^v(\ell) < \bar\kappa_\circ$ for all $v \in \mathfrak{U}_{\mathrm{SM}}$ (as shown in the paragraph after Assumption 4.1), and this implies that $\lambda^*_n \le \kappa_1 + \bar\kappa_\circ$ for all n. Thus $\{\lambda^*_n : n \ge 1\}$ is bounded. Therefore, passing to a subsequence we may assume that $\lambda^*_n \to \lambda^*$ as $n \to \infty$. To complete the proof we only need to show that $\lambda^* = \lambda^*_v$. Since $\Psi_n(0) = 1$ for all n, and the coefficients b_{v_n}, and c_{v_n} are uniformly locally bounded, applying Harnack's inequality and Sobolev's estimate we can find $\Psi \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p \ge 1$, such that $\Psi_n \to \Psi$ weakly in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$. Therefore, by [34, Lemma 2.4.3] and (4.16), we obtain

$$\mathcal{L}\Psi(x) + c_v(x)\,\Psi(x) \;=\; \lambda^*\,\Psi(x), \quad x \in \mathbb{R}^d, \quad \Psi > 0\,. \tag{4.17}$$

By Corollary 2.1 we have $\lambda^* \ge \lambda^*_v$.

Let $\mathcal{B} \supset \mathcal{K}$ be an open ball such that $|c(x,u) - \lambda^*| \le \beta\ell(x)$ for all $(x,u) \in \mathcal{B}^c \times \mathbb{U}$, and $R > 0$ be large enough so that $\mathcal{B} \subset B_R$. Let $\check\tau = \tau(\mathcal{B}^c)$. Applying the Itô–Krylov formula to (4.17), we obtain

$$\Psi(x) \;=\; \mathbb{E}^v_x\Big[\mathrm{e}^{\int_0^{\check\tau\wedge\tau_R\wedge T}[c_v(X_s)-\lambda^*]\,\mathrm{d}s}\,\Psi(X_{\check\tau\wedge\tau_R\wedge T})\Big]\,, \quad x \in \mathcal{B}^c \cap B_R\,, \tag{4.18}$$

for any $T > 0$. Since

$$\begin{aligned}\mathbb{E}^v_x\Big[\mathrm{e}^{\int_0^{\check\tau}[c_v(X_s)-\lambda^*]\,\mathrm{d}s}\Big] \;&\le\; \mathbb{E}^v_x\Big[\mathrm{e}^{\int_0^{\check\tau}\beta\ell(X_s)\,\mathrm{d}s}\Big]\\ &\le\; \Big(\mathbb{E}^v_x\Big[\mathrm{e}^{\int_0^{\check\tau}\ell(X_s)\,\mathrm{d}s}\Big]\Big)^\beta \;<\; \infty \quad \text{for } x \in \mathcal{B}^c,\end{aligned} \tag{4.19}$$

and Ψ in bounded in $\mathcal{B}^c \cap B_R$, for every fixed R, letting $T \to \infty$ in (4.18) we have

$$\Psi(x) \;=\; \mathbb{E}^v_x\Big[\mathrm{e}^{\int_0^{\check\tau\wedge\tau_R}[c_v(X_s)-\lambda^*]\,\mathrm{d}s}\,\Psi(X_{\check\tau\wedge\tau_R})\Big]\,, \quad x \in \mathcal{B}^c \cap B_R\,. \tag{4.20}$$

Equation (4.19) which also holds, possibly for a larger ball $\mathcal{B}$, if we replace v and λ^* with v_n and λ^*_n, respectively, shows that, for some constant $\tilde\kappa$, we have $\Psi_n(x) \le \tilde\kappa\big(\mathcal{V}(x)\big)^\beta$ for all $n \in \mathbb{N}$, and $x \in \mathcal{B}^c$. Therefore, $\Psi(x) \le \tilde\kappa\big(\mathcal{V}(x)\big)^\beta$ for all $x \in \mathcal{B}^c$.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

We write

$$\mathbb{E}_x^v\Big[\mathrm{e}^{\int_0^{\check\tau\wedge\tau_R}\ell(X_s)\,\mathrm{d}s}\Big] \;=\; \mathbb{E}_x^v\Big[\mathrm{e}^{\int_0^{\check\tau}\ell(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\{\check\tau<\tau_R\}}\Big]+\mathbb{E}_x^v\Big[\mathrm{e}^{\int_0^{\tau_R}\ell(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\{\tau_R<\check\tau\}}\Big]\,. \tag{4.21}$$

The left hand side of (4.21) and the first term on the right hand side both converge to $\mathbb{E}_x^v\big[\mathrm{e}^{\int_0^{\check\tau}\ell(X_s)\,\mathrm{d}s}\big]$ as $R\to\infty$, by monotone convergence. Thus we have

$$\mathbb{E}_x^v\Big[\mathrm{e}^{\int_0^{\tau_R}\ell(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\{\tau_R<\check\tau\}}\Big] \;\xrightarrow[R\to\infty]{}\; 0\,. \tag{4.22}$$

On the other hand Assumption 4.1 implies that

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\tau_R}\ell(X_s)\,\mathrm{d}s}\,\mathcal{V}(X_{\tau_R})\,\mathbb{1}_{\{\tau_R<\check\tau\}}\Big] \;\le\; \mathcal{V}(x) \qquad \forall\, x\in B_R\setminus\mathcal{B}^c\,,\ \forall\, R>0\,.$$

We proceed as in the proof of Theorem 2.7. Let $\Gamma(R,m)\coloneqq\{x\in\partial B_R\colon \Psi(x)\ge m\}$ for $m\ge 1$. Since $\Psi\le\tilde\kappa\mathcal{V}^\beta$ on $\mathcal{B}^c$, we have $\mathcal{V}^{\beta-1}\le\big(\frac{\Psi}{\tilde\kappa}\big)^{1-\frac1\beta}$ on $\mathcal{B}^c$, and, therefore,

$$\Psi\,\mathbb{1}_{\Gamma(R,m)} \;\le\; \tilde\kappa\,\mathcal{V}\,\mathcal{V}^{\beta-1}\,\mathbb{1}_{\Gamma(R,m)} \;\le\; \tilde\kappa^{\frac1\beta}\,m^{1-\frac1\beta}\,\mathcal{V} \quad \text{on } \partial B_R\,. \tag{4.23}$$

Thus, using (4.23), we obtain

$$\begin{aligned}
\mathbb{E}_x^v\Big[\mathrm{e}^{\int_0^{\tau_R}[c_v(X_s)-\lambda^*]\,\mathrm{d}s}\,\Psi(X_{\tau_R})\,\mathbb{1}_{\{\tau_R<\check\tau\}}\Big] &\le\; m\,\mathbb{E}_x^v\Big[\mathrm{e}^{\int_0^{\tau_R}\ell(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\{\tau_R<\check\tau\}}\Big]\\
&\quad+\mathbb{E}_x^v\Big[\mathrm{e}^{\int_0^{\tau_R}\ell(X_s)\,\mathrm{d}s}\,\Psi(X_{\tau_R})\,\mathbb{1}_{\Gamma(R,m)}(X_{\tau_R})\,\mathbb{1}_{\{\tau_R<\check\tau\}}\Big]\\
&\le\; m\,\mathbb{E}_x^v\Big[\mathrm{e}^{\int_0^{\tau_R}\ell(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\{\tau_R<\check\tau\}}\Big]\\
&\quad+\tilde\kappa^{\frac1\beta}\,m^{1-\frac1\beta}\;\mathbb{E}_x^v\Big[\mathrm{e}^{\int_0^{\tau_R}\ell(X_s)\,\mathrm{d}s}\,\mathcal{V}(X_{\tau_R})\mathbb{1}_{\{\tau_R<\check\tau\}}\Big]\\
&\le\; m\,\mathbb{E}_x^v\Big[\mathrm{e}^{\int_0^{\tau_R}\ell(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\{\tau_R<\check\tau\}}\Big]+\tilde\kappa^{\frac1\beta}\,m^{1-\frac1\beta}\,\mathcal{V}(x)\,,
\end{aligned} \tag{4.24}$$

and by first letting $R\to\infty$, using (4.22), and then $m\to\infty$, it follows that the left hand side of (4.24) vanishes as $R\to\infty$. Therefore, letting $R\to\infty$ in (4.20), we obtain

$$\Psi(x) \;=\; \mathbb{E}_x^v\Big[e^{\int_0^{\check\tau}[c_v(X_s)-\lambda^*]\,\mathrm{d}s}\,\Psi(X_{\check\tau})\Big]\,,\quad x\in\mathcal{B}^c\,.$$

It then follows by Corollary 2.3 that $\lambda^*=\lambda_v^*$, and this completes the proof. □

Remark 4.1. Following the proof of Theorem 4.3 we can obtain the following continuity result which should be compared with [5, Proposition 9.2 (ii)]. Consider a sequence of operators $\mathcal{L}_n^{f_n}$ with coefficients (a_n,b_n,f_n), where b_n, f_n are locally bounded uniformly in n, and $\inf_n(\inf_{\mathbb{R}^d}f_n)>-\infty$. The coefficients a_n and b_n are assumed to satisfy (A1)–(A3) uniformly in n. Assume that $a_n\to a$ in $C_{\mathrm{loc}}(\mathbb{R}^d)$, and $b_n\to b$ and $f_n\to f$ weakly in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Moreover we suppose that one of the following hold.

(a) There exists an inf-compact function $\ell\in C(\mathbb{R}^d)$ and $\mathcal{V}\in\mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$, with $\inf_{\mathbb{R}^d}\mathcal{V}>0$, such that $\mathcal{L}_n\mathcal{V}\le\bar\kappa\,\mathbb{1}_{\mathcal{K}}-\ell\mathcal{V}$ a.e. on $\mathbb{R}^d$ for some constant $\bar\kappa$, and a compact set $\mathcal{K}$. In addition, $\beta\ell-\sup_n f_n$ is inf-compact for some $\beta\in(0,1)$.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

(b) The sequence $\mathcal{L}_n$ satisfies (3.14) for all n, $\lim_{n\to\infty}\|f_n^-\|_\infty = \|f^-\|_\infty$, and

$$\limsup_{|x|\to\infty} \sup_n f_n(x) + \|f_n^-\|_\infty \;<\; \gamma\,.$$

Then the principal eigenvalue $\lambda^*(f_n)$ converges to $\lambda^*(f)$ as $n\to\infty$.

As an application of Theorem 4.3 we have the following existence result for the risk-sensitive control problem under (Markovian) risk-sensitive type constraints.

Theorem 4.4. *Assume one of the following.*

(i) *Assumption 4.1 holds, and $c, r_1, \dots, r_m \in \mathcal{C}_{\beta\ell}$ for some $\beta\in(0,1)$.*
(ii) *Assumption 4.2 holds, and $r_1,\dots,r_m \in \mathfrak{C}$ satisfy*

$$\max_{i=1,\dots,m}\Big\{\limsup_{|x|\to\infty}\max_{u\in\mathbb{U}} r_i(x,u)\Big\} \;<\; \gamma\,.$$

In addition, suppose that K_i, $i=1,\dots,m$, are closed subsets of $\mathbb{R}$, and that there exists $\hat v\in\mathfrak{U}_{\rm SM}$ such that $\mathcal{E}^{\hat v}(r_{i,\hat v})\in K_i$ for all i, where we use the usual notation $r_{i,v}(x) \coloneqq r_i\big(x,v(x)\big)$.

Then the following constrained minimization problem admits an optimal control in $\mathfrak{U}_{\rm SM}$

$$\text{minimize over } v\in\mathfrak{U}_{\rm SM}: \quad \mathcal{E}^v(c_v), \quad \text{subject to} \quad \mathcal{E}^v(r_{i,v})\in K_i\,,\ i=1,\dots,m\,.$$

Proof. Let $v_n\in\mathfrak{U}_{\rm SM}$ be a sequence of controls along which the constraints are met, and $\mathcal{E}^{v_n}(c_{v_n})$ converges to its infimum. Since $\mathfrak{U}_{\rm SM}$ is compact under the topology of Markov controls, we may assume, without loss of generality, that v_n converges to some $\bar v\in\mathfrak{U}_{\rm SM}$ as $n\to\infty$. By Theorem 4.3 we know that $v\mapsto\lambda^*_v(c_v)$, and $v\mapsto\lambda^*_v(r_{i,v})$, $i=1,\dots,m$, are continuous maps, and that $\mathcal{E}^v(c_v)=\lambda^*_v(c_v)$, and $\mathcal{E}^v(r_{i,v})=\lambda^*_v(r_{i,v})$ for $i=1,\dots,m$. It follows that the constraints are met at $\bar v$. Therefore, $\bar v$ is an optimal Markov control for the constrained problem. □

Another application of Theorem 4.3 is a following characterization of λ^* which provides a positive answer to [5, Conjecture 1.8] for a certain class of a, b and f. In Theorem 4.5 below, we consider the uncontrolled generator $\mathcal{L}$ in Section 3. Let us introduce the following definition from [5]

$$\lambda'(f) \;=\; \sup\big\{\lambda:\ \exists\,\varphi\in\mathcal{W}^{2,d}_{\rm loc}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d), \varphi>0,\ \mathcal{L}\varphi+(f-\lambda)\varphi\ \ge\ 0 \text{ a.e. in } \mathbb{R}^d\big\}\,.$$

Recall the definition of λ'' in (2.46). From [5, Theorem 1.7], under (A1)–(A2), we have $\lambda^*(f)\le\lambda'(f)\le\lambda''(f)$ whenever f is bounded above. It is conjectured in [5, Conjecture 1.8] that for bounded a, b, and f, one has $\lambda'(f)=\lambda''(f)$. It should be noted from Example 3.1 that $\lambda^*(f)$ could be strictly smaller than $\lambda''(f)$. The following result complements those in [5, Theorems 1.7 and 1.9].

Theorem 4.5. *For a potential f the following are true.*

(i) *Suppose that $\mathcal{E}_x(f)<\infty$. Then under (A1)–(A3) we have*

$$\lambda^*(f)\;\le\;\lambda'(f)\;\le\;\mathcal{E}_x(f)\;\le\;\lambda''(f)\,.$$

(ii) *Let $\mathcal{L}$, $\mathcal{V}$ and γ satisfy (3.14), and suppose that $\sup_{\mathbb{R}^d}(f+\|f^-\|_\infty)<\gamma$. Then $\lambda^*(f)\;=\;\lambda''(f)$.*
(iii) *Let $\mathcal{L}$, $\mathcal{V}$ and ℓ satisfy (4.4), and suppose that $\beta\ell-f$ is inf-compact for some $\beta\in(0,1)$. Then $\lambda^*(f)=\lambda''(f)$.*

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

Proof. We first show (i). By [5, Theorem 1.7 (ii)] we have $\lambda^*(f) \le \lambda'(f)$. Let $\varphi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $\varphi > 0$, be such that

$$\mathcal{L}\varphi + (f - \lambda)\varphi \;\ge\; 0\,.$$

Recall that τ_n is the exit time from the open ball $B_n(0)$. Therefore, applying the Itô–Krylov formula, we obtain

$$\varphi(x) \;\le\; \mathbb{E}_x\Big[e^{\int_0^{\tau_n\wedge T}[f(X_s)-\lambda]\,\mathrm{d}s}\,\varphi(X_{\tau_n\wedge T})\Big] \;\le\; \Big(\sup_{\mathbb{R}^d}\varphi\Big)\,\mathbb{E}_x\Big[e^{\int_0^{\tau_n\wedge T}[f(X_s)-\lambda]\,\mathrm{d}s}\Big]\,,\quad T \ge 0\,. \tag{4.25}$$

Since $\mathcal{E}_x(f)$ is finite, letting $n \to \infty$ in (4.25), taking logarithms on both sides, dividing by T and then letting $T \to \infty$ we obtain $\lambda \le \mathcal{E}_x(f)$. This implies $\lambda'(f) \le \mathcal{E}_x(f)$. Now suppose $\varphi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$, with $\inf_{\mathbb{R}^d}\varphi > 0$, satisfies

$$\mathcal{L}\varphi + (f - \lambda)\varphi \;\le\; 0\,.$$

Repeating the analogous calculation as above, we obtain $\lambda \ge \mathcal{E}_x(f)$, which implies that $\mathcal{E}_x(f) \le \lambda''(f)$.

Next we prove (ii). Since $\lambda^*(f + c) = \lambda^*(f) + c$ for any constant c, we may replace f by $f + \|f^-\|_\infty$. Therefore, f is non-negative and $\|f\|_\infty < \gamma$. By (i) we have $\lambda^*(f) \le \lambda''(f)$. Let $\chi_n\colon \mathbb{R}^d \to [0,1]$ be a cut-off function such that $\chi_n(x) = 1$ for $|x| \le n$, and $\chi_n(x) = 0$ for $|x| \ge n+1$. Define $f_n := \chi_n f + (1-\chi_n)\|f\|_\infty$. Let $\big(\Psi^*_n, \lambda^*(f_n)\big)$ denote the principal eigenpair of $\mathcal{L}^{f_n}$. By Remark 4.1 we have $\lambda^*(f_n) \to \lambda^*(f)$ as $n \to \infty$. Thus to complete the proof it is enough to show that $\inf_{\mathbb{R}^d}\Psi_n > 0$, which implies that $\lambda^*(f_n) = \lambda''(f_n) \ge \lambda''(f)$ for all n, and thus $\lambda^*(f) \ge \lambda''(f)$. Note that $\lambda^*(f_n) \le \mathcal{E}(f_n) \le \|f\|_\infty$ for all n. Now fix n and let $\check\tau_n$ be the first hitting time to the ball B_n. Then applying the Itô–Krylov formula to

$$\mathcal{L}\Psi^*_n + \big(f_n - \lambda^*(f_n)\big)\,\Psi^*_n \;=\; 0$$

together with Fatou's lemma, we have

$$\min_{z\in B_{n+1}} \Psi^*_n(z) \;\le\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\check\tau_n}[f_n(X_s)-\lambda^*(f_n)]\,\mathrm{d}s}\,\Psi^*_n(X_{\check\tau_n})\Big] \;\le\; \Psi^*_n(x)$$

for all $x \in \overline{B}^c_{n+1}(0)$. Hence $\inf_{\mathbb{R}^d}\Psi^*_n > 0$ which completes the proof.

The proof of (iii) is completely analogous to the proof of part (ii). Since $\beta\ell - f^+$ is inf-compact, we can find $g\colon \mathbb{R}^d \to \mathbb{R}_+$, such that $\lim_{|x|\to\infty} g(x) = \infty$, and $\beta\ell - f^+ - g$ is inf-compact. We let $f_n = \chi_n f + (1-\chi_n)(g + f^+)$. Note that

$$\inf_n\,(\beta\ell - f_n) \;=\; \inf_{n\in\mathbb{N}}\big(\chi_n(\beta\ell - f) + (1-\chi_n)(\beta\ell - f^+ - g)\big)$$

is inf-compact. On the other hand, $f_n \ge f$ for all n. The rest follows as part (ii). □

Acknowledgements

The research of Ari Arapostathis was supported in part by the Army Research Office through grant W911NF-17-1-001, in part by the National Science Foundation through grant DMS-1715210, and in part by the Office of Naval Research through grant N00014-16-1-2956. The research of Anup Biswas was supported in part by an INSPIRE faculty fellowship, and a DST-SERB grant EMR/2016/004810.

References

[1] M.G. Kreĭn, M.A. Rutman, Linear Operators Leaving Invariant a Cone in a Banach Space, Amer. Math. Soc. Translation, vol. 26, 1950, p. 128.

[2] R.G. Pinsky, Positive Harmonic Functions and Diffusion, Cambridge Studies in Advanced Mathematics, vol. 45, Cambridge University Press, Cambridge, 1995.
[3] H. Berestycki, L. Nirenberg, S.R.S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators in general domains, Commun. Pure Appl. Math. 47 (1) (1994) 47–92, https://doi.org/10.1002/cpa.3160470105.
[4] A. Quaas, B. Sirakov, Principal eigenvalues and the Dirichlet problem for fully nonlinear elliptic operators, Adv. Math. 218 (1) (2008) 105–135, https://doi.org/10.1016/j.aim.2007.12.002.
[5] H. Berestycki, L. Rossi, Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains, Commun. Pure Appl. Math. 68 (6) (2015) 1014–1065, https://doi.org/10.1002/cpa.21536.
[6] Y. Furusho, Y. Ogura, On the existence of bounded positive solutions of semilinear elliptic equations in exterior domains, Duke Math. J. 48 (3) (1981) 497–521, https://doi.org/10.1215/S0012-7094-81-04828-6.
[7] Y. Pinchover, On positive solutions of second-order elliptic equations, stability results, and classification, Duke Math. J. 57 (3) (1988) 955–980, https://doi.org/10.1215/S0012-7094-88-05743-2.
[8] H. Berestycki, L. Rossi, On the principal eigenvalue of elliptic operators in $\mathbb{R}^N$ and applications, J. Eur. Math. Soc. 8 (2) (2006) 195–215, https://doi.org/10.4171/JEMS/47.
[9] H. Kaise, S.-J. Sheu, On the structure of solutions of ergodic type Bellman equation related to risk-sensitive control, Ann. Probab. 34 (1) (2006) 284–320, https://doi.org/10.1214/009117905000000431.
[10] A. Arapostathis, A. Biswas, Infinite horizon risk-sensitive control of diffusions without any blanket stability assumptions, Stoch. Process. Appl. 128 (5) (2018) 1485–1524, https://doi.org/10.1016/j.spa.2017.08.001.
[11] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000.
[12] M.D. Donsker, S.R.S. Varadhan, On a variational formula for the principal eigenvalue for operators with maximum principle, Proc. Natl. Acad. Sci. USA 72 (1975) 780–783.
[13] M.D. Donsker, S.R.S. Varadhan, Asymptotic evaluation of certain Markov process expectations for large time. III, Commun. Pure Appl. Math. 29 (4) (1976) 389–461, https://doi.org/10.1002/cpa.3160290405.
[14] I. Kontoyiannis, S.P. Meyn, Spectral theory and limit theorems for geometrically ergodic Markov processes, Ann. Appl. Probab. 13 (1) (2003) 304–362, https://doi.org/10.1214/aoap/1042765670.
[15] L. Wu, Feynman–Kac semigroups, ground state diffusions, and large deviations, J. Funct. Anal. 123 (1) (1994) 202–231, https://doi.org/10.1006/jfan.1994.1087.
[16] N. Ichihara, Criticality of viscous Hamilton–Jacobi equations and stochastic ergodic control, J. Math. Pures Appl. (9) 100 (3) (2013) 368–390, https://doi.org/10.1016/j.matpur.2013.01.005.
[17] N. Ichihara, Recurrence and transience of optimal feedback processes associated with Bellman equations of ergodic type, SIAM J. Control Optim. 49 (5) (2011) 1938–1960, https://doi.org/10.1137/090772678.
[18] N. Ichihara, The generalized principal eigenvalue for Hamilton–Jacobi–Bellman equations of ergodic type, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 32 (3) (2015) 623–650, https://doi.org/10.1016/j.anihpc.2014.02.003.
[19] G. Barles, J. Meireles, On unbounded solutions of ergodic problems in $\mathbb{R}^m$ for viscous Hamilton–Jacobi equations, Commun. Partial Differ. Equ. 41 (12) (2016) 1985–2003, https://doi.org/10.1080/03605302.2016.1244208.
[20] A.G. Bhatt, V.S. Borkar, Occupation measures for controlled Markov processes: characterization and optimality, Ann. Probab. 24 (3) (1996) 1531–1562, https://doi.org/10.1214/aop/1065725192.
[21] R.H. Stockbridge, Time-average control of martingale problems: a linear programming formulation, Ann. Probab. 18 (1) (1990) 206–217, https://doi.org/10.1214/aop/1176990945.
[22] H. Kaise, S.-J. Sheu, Evaluation of large time expectations for diffusion processes, preprint, 2004.
[23] W.H. Fleming, W.M. McEneaney, Risk-sensitive control on an infinite time horizon, SIAM J. Control Optim. 33 (6) (1995) 1881–1915, https://doi.org/10.1137/S0363012993258720.
[24] H. Nagai, Bellman equations of risk-sensitive control, SIAM J. Control Optim. 34 (1) (1996) 74–101, https://doi.org/10.1137/S0363012993255302.
[25] J.-L. Menaldi, M. Robin, Remarks on risk-sensitive control problems, Appl. Math. Optim. 52 (3) (2005) 297–310, https://doi.org/10.1007/s00245-005-0829-y.
[26] A. Biswas, An eigenvalue approach to the risk sensitive control problem in near monotone case, Syst. Control Lett. 60 (3) (2011) 181–184, https://doi.org/10.1016/j.sysconle.2010.12.002.
[27] A. Biswas, V.S. Borkar, K. Suresh Kumar, Risk-sensitive control with near monotone cost, Appl. Math. Optim. 62 (2) (2010) 145–163, https://doi.org/10.1007/s00245-009-9096-7.
[28] A. Biswas, Risk sensitive control of diffusions with small running cost, Appl. Math. Optim. 64 (1) (2011) 1–12, https://doi.org/10.1007/s00245-010-9127-4.
[29] A. Basu, M.K. Ghosh, Zero-sum risk-sensitive stochastic differential games, Math. Oper. Res. 37 (3) (2012) 437–449, https://doi.org/10.1287/moor.1120.0542.
[30] A. Biswas, S. Saha, Zero-sum stochastic differential game with risk-sensitive cost, Appl. Math. Optim. (2018), https://doi.org/10.1007/s00245-018-9479-8, https://arxiv.org/abs/1704.02689.
[31] M.K. Ghosh, K.S. Kumar, C. Pal, Zero-sum risk-sensitive stochastic games for continuous time Markov chains, Stoch. Anal. Appl. 34 (5) (2016) 835–851, https://doi.org/10.1080/07362994.2016.1180995.
[32] X. Zhang, Strong solutions of SDES with singular drift and Sobolev diffusion coefficients, Stoch. Process. Appl. 115 (11) (2005) 1805–1818, https://doi.org/10.1016/j.spa.2005.06.003.
[33] N.V. Krylov, Controlled Diffusion Processes, Applications of Mathematics, vol. 14, Springer-Verlag, New York, 1980.
[34] A. Arapostathis, V.S. Borkar, M.K. Ghosh, Ergodic Control of Diffusion Processes, Encyclopedia of Mathematics and its Applications, vol. 143, Cambridge University Press, Cambridge, 2012.
[35] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd edition, Grundlehren der Mathematischen Wissenschaften, vol. 224, Springer-Verlag, Berlin, 1983.
[36] R.S. Liptser, A.N. Shiryayev, Statistics of Random Processes. I. General Theory, Applications of Mathematics, vol. 5, Springer-Verlag, New York, 1977, Translated by A. B. Aries.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008

[37] A. Arapostathis, M.K. Ghosh, S.I. Marcus, Harnack's inequality for cooperative weakly coupled elliptic systems, Commun. Partial Differ. Equ. 24 (9–10) (1999) 1555–1571, https://doi.org/10.1080/03605309908821475.
[38] R.Z. Has′minskiĭ, Stochastic Stability of Differential Equations, Sijthoff & Noordhoff, The Netherlands, 1980.
[39] A. Arapostathis, A. Biswas, D. Ganguly, Some Liouville-type results for eigenfunctions of elliptic operators, arXiv e-prints, arXiv:1708.09640, https://arxiv.org/abs/1708.09640.
[40] H. Kunita, Stochastic Flows and Stochastic Differential Equations, Cambridge Studies in Advanced Mathematics, vol. 24, Cambridge University Press, Cambridge, 1990.
[41] V.I. Bogachev, N.V. Krylov, M. Röckner, On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions, Commun. Partial Differ. Equ. 26 (11–12) (2001) 2037–2080, https://doi.org/10.1081/PDE-100107815.
[42] I. Gyöngy, N. Krylov, Existence of strong solutions for Itô's stochastic equations via approximations, Probab. Theory Relat. Fields 105 (2) (1996) 143–158, https://doi.org/10.1007/BF01203833.
[43] Y. Yoshimura, A note on demi-eigenvalues for uniformly elliptic Isaacs operators, in: Viscosity Solution Theory of Differential Equations and Its Developments, 2006, pp. 106–114.
[44] V.S. Borkar, A topology for Markov controls, Appl. Math. Optim. 20 (1) (1989) 55–62, https://doi.org/10.1007/BF01447645.

Please cite this article in press as: A. Arapostathis et al., Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.05.008