

HARNACK'S INEQUALITY FOR COOPERATIVE  
WEAKLY COUPLED ELLIPTIC SYSTEMS

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1. INTRODUCTION

There is considerable literature on the Harnack inequality for uniformly elliptic partial differential equations [2], [3], [5]. Harnack's inequality, apart from being interesting in its own right, plays a very important role in the theory of partial differential equations. For example, it is applied to derive the interior estimates of the gradients of the solutions. Let us first state this result in the simplest situation. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^d$ ,  $\Gamma$

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a closed subset of  $\Omega$  and  $u : \Omega \rightarrow \mathbb{R}$  a nonnegative harmonic function, i.e.,  $\Delta u = 0$  in  $\Omega$ . Then there exists a constant  $C$  which depends only on the dimension  $d$ , on the diameter of  $\Omega$ , and on the distance between  $\Gamma$  and  $\partial\Omega$ , such that

$$u(x) \leq C u(y), \quad \forall x, y \in \Gamma.$$

The Harnack inequality is also valid for both weak and strong solutions of second-order, uniformly elliptic operators with bounded coefficients [2], [3]. Extensions to unbounded coefficients have also been established [9].

Consider a system of equations in  $\mathbf{u}(x) = (u_1(x), \dots, u_n(x))$  of the form

$$(L\mathbf{u})_k(x) := L_k u_k(x) + \sum_{\substack{j=1 \\ j \neq k}}^n c_{kj}(x) u_j(x) = 0, \quad 1 \leq k \leq n, \quad (1.1)$$

where  $n$  is a positive integer and each  $L_k$  is a second-order, uniformly elliptic operator given by

$$L_k := \sum_{i,j=1}^d a_{ij}^k(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i^k(x) \frac{\partial}{\partial x_i} + c_{kk}(x). \quad (1.2)$$

The operator  $L$  is called *cooperative* if the coupling coefficients  $c_{kj}$  are nonnegative for  $k \neq j$ .

**Definition 1.1.** We denote by  $\mathfrak{L}(\lambda, d, n)$  the class of all cooperative operators  $L$  of the form (1.1)–(1.2), with coefficients  $a_{ij}^k \in C^{0,1}(\mathbb{R}^d)$ ,  $b_i^k, c_{kj} \in L^\infty(\mathbb{R}^d)$ , bounded in  $L^\infty$ -norm by a constant  $\lambda \geq 1$ , and satisfying the uniform ellipticity condition

$$\lambda^{-1} \|\zeta\|^2 \leq \sum_{i,j=1}^d a_{ij}^k(x) \zeta_i \zeta_j \leq \lambda \|\zeta\|^2, \quad \text{for all } x, \zeta \in \mathbb{R}^d, \quad 1 \leq k \leq n.$$

A function  $\mathbf{u}$  is called *L-harmonic in a domain*  $\Omega \subset \mathbb{R}^d$  provided  $\mathbf{u}$  is a strong solution of  $L\mathbf{u} = 0$  in the Sobolev space  $W_{loc}^{2,p}(\Omega; \mathbb{R}^n)$ , for some  $p \in [1, \infty)$ .

Systems of the above form appear in the study of jump diffusion processes with a discrete component [1]. In this paper, we obtain analogues of Harnack's inequality for  $L$ -harmonic functions of operators in the class  $\mathfrak{L}(\lambda, d, n)$ . We use the technique introduced by Krylov for estimating the

oscillation of a harmonic function on bounded sets [3]. The main results are given in Section 2. Section 3 is devoted to proofs and auxiliary results.

After this work was submitted for publication, similar results were reported in [10]. Our work and [10] differ both in methodology and results. In [10] the authors assume Hölder continuous coefficients, and the proofs are based on estimates of the Green function in small balls, while this paper, motivated from a stochastic control problem, assumes only measurable coefficients, and the proofs are based on the approach of Krylov. Also, in our work an averaged coupling matrix (see Definition 2.1) appears explicitly in the Harnack constant. This enables us to provide a rather general version of the maximum principle and some further Harnack inequalities valid for a certain class of supersolutions.

## 2. MAIN RESULTS

Throughout the paper,  $\Omega$  denotes a bounded domain in  $\mathbb{R}^d$ . We first establish a weak version of Harnack's inequality, under general conditions.

**Theorem 2.1.** *Let  $\Gamma \subset \Omega$  be a closed set. There exists a constant  $K_1 > 0$ , depending only on  $d, n$ , the diameter of  $\Omega$ , the distance between  $\Gamma$  and  $\partial\Omega$ , and the bound  $\lambda$ , such that for any nonnegative  $L$ -harmonic function  $\mathbf{u}$  in  $\Omega$ , with  $L \in \mathfrak{L}(\lambda, d, n)$ ,*

$$\sup_{x \in \Gamma} \{u_i(x)\} \leq K_1 \max_{1 \leq k \leq n} \inf_{x \in \Gamma} \{u_k(x)\}, \quad \forall i \in \{1, \dots, n\}. \quad (2.1)$$

An inequality stronger than (2.1) is obtained under an irreducibility condition on the coupling coefficients. We need to introduce some additional notation.

For a measurable set  $A \subset \mathbb{R}^d$ ,  $|A|$  denotes the Lebesgue measure of  $A$ , while  $\|\cdot\|_{p,A}$  denotes the norm of  $L^p(A)$ ,  $1 \leq p \leq \infty$ . Also, for  $A \subset \Omega$ ,  $\|\cdot\|_{k,p,A}$  denotes the restriction to  $A$  of the standard norm of  $W^{k,p}(\Omega)$ . These norms are extended to vector valued functions  $\mathbf{u}$  using the convention  $\|\mathbf{u}\| = \sum_{i=1}^n \|u_i\|$ .

**Definition 2.1.** For  $\Omega \subset \mathbb{R}^d$  and  $L \in \mathfrak{L}(\lambda, d, n)$ , let  $\mathbf{C}_L(\Omega) \in \mathbb{R}^{n \times n}$  denote the matrix defined by

$$[C_L(\Omega)]_{ij} := \frac{\|c_{ij}\|_{1,\Omega}}{|\Omega|}, \quad \text{for } i \neq j, \quad i, j \in \{1, \dots, n\},$$

with diagonal entries equal to 0. Given a nonnegative matrix  $M \in \mathbb{R}^{n \times n}$  and a pair  $i, j \in \{1, \dots, n\}$ , we say that  $j$  is *reachable from*  $i$  provided that the  $ij$ 'th element of  $(I + M)^{n-1}$  is positive, and we denote this by  $i \xrightarrow{M} j$ . Furthermore, the matrix  $M$  is called *irreducible* if  $i \xrightarrow{M} j$  for all  $i, j \in \{1, \dots, n\}$ ; otherwise, it is called *reducible*. We say that  $L \in \mathfrak{L}(\lambda, d, n)$  is  $\mu_\Omega$ -*irreducible in*  $\Omega$  if there exists an irreducible matrix  $S \in \mathbb{R}^{n \times n}$ , with elements in  $\{0, 1\}$  and  $\mu_\Omega \in \mathbb{R}$ , such that  $\mu_\Omega C_L(\Omega) \geq S$  (here, the inequality is meant elementwise). The class of all  $\mu_\Omega$ -irreducible operators whose coefficients  $a_{ij}^k$  have a uniform Lipschitz constant  $\gamma$  is denoted by  $\mathfrak{L}(\lambda, d, n, \gamma, \mu_\Omega)$ .

**Theorem 2.2.** *Let  $\Gamma \subset \Omega$  be a closed set. There exists a positive constant  $K_2 = K_2(\Omega, \Gamma, \lambda, d, n, \gamma, \mu_\Omega)$  such that for any nonnegative  $L$ -harmonic function  $u$  in  $\Omega$ , with  $L \in \mathfrak{L}(\lambda, d, n, \gamma, \mu_\Omega)$ ,*

$$u_i(x) \leq K_2 u_j(y), \quad \forall x, y \in \Gamma, \quad i, j \in \{1, \dots, n\}. \quad (2.2)$$

*More generally, if  $L \in \mathfrak{L}(\lambda, d, n)$ , and  $\bar{c}_\Omega$  denotes the smallest positive element of  $C_L(\Omega)$ , then*

$$u_i(x) \leq K'_2 u_j(y), \quad \forall x, y \in \Gamma, \quad \text{if } j \xrightarrow{C_L(\Omega)} i, \quad (2.3)$$

where  $K'_2 = K_2(\Omega, \Gamma, \lambda, d, n, \gamma, \frac{1}{\bar{c}_\Omega})$ .

*Remark 2.1.* Let  $\Gamma \subset \Omega$  and  $L \in \mathfrak{L}(\lambda, d, n)$  be given. Then, for the existence of a constant  $K_2 > 0$  satisfying (2.2) for all nonnegative  $L$ -harmonic functions  $u$  in  $\Omega$ , it is necessary that  $L$  be  $\mu_\Omega$ -irreducible in  $\Omega$ . Otherwise, there exists a nontrivial partition  $\{\mathcal{I}_1, \mathcal{I}_2\}$  of  $\{1, \dots, n\}$  such that  $c_{ij} = 0$  a.e. in  $\Omega$ , for all  $(i, j) \in \mathcal{I}_1 \times \mathcal{I}_2$ ; therefore, any nonzero  $L$ -harmonic function  $u$  satisfying  $u_k = 0$ , for  $k \in \mathcal{I}_1$ , violates (2.2).

There is a fair amount of work in the literature on maximum principles for cooperative, weakly-coupled systems [6], [7]. In [6], it is assumed that the coupling coefficients are positive. Note that the notion of irreducibility in Definition 2.1 is in an ‘average’ sense only and that  $C_L(\Omega)$  may be irreducible even if  $[c_{ij}(x)]$  is reducible at every  $x \in \Omega$ . We state the following version of

the strong maximum principle, which follows immediately from Theorem 2.2, and does not seem to be available in the existing literature.

**Corollary 2.1.** *Let  $L \in \mathfrak{L}(\lambda, d, n)$  be such that  $C_L(\Omega)$  is irreducible. Then any nonnegative  $L$ -harmonic function  $u$  in  $\Omega$  is either positive in  $\Omega$  or identically zero.*

It is well known that, in general, there is no Harnack inequality for nonnegative  $L$ -superharmonic functions, i.e., functions  $u$  satisfying  $Lu \leq 0$  in  $\Omega$ , for an elliptic operator  $L$ . Serrin [8] has utilized the maximum principle to provide a growth estimate in terms of the Harnack constant of a comparison function and the value of  $\|Lu\|_\infty$ , but this estimate does not result in a Harnack inequality. Theorem 2.2 can be employed to provide a Harnack constant for all superharmonic functions  $u$  for which  $-Lu$  lies in a convex positive cone of  $L^\infty$ . We introduce the following definition.

**Definition 2.2.** For a measurable set  $A \subset \mathbb{R}^d$  having finite, nonzero measure and for a constant  $\theta \geq 1$ , we define the positive convex cone  $\mathcal{K}(\theta, A) \subset L^\infty(A; \mathbb{R}^n)$  by

$$\mathcal{K}(\theta, A) := \left\{ f \in L^\infty(A; \mathbb{R}^n) : f \geq 0, \min_{1 \leq k \leq n} \frac{\|f_k\|_{1;A}}{|A|\|f_k\|_{\infty;A}} \geq \frac{1}{\theta} \right\}.$$

Suppose, for the moment, that  $n = 1$  and  $u$  is a nonnegative function satisfying  $Lu = -f$  in  $\Omega$ , with  $L \in \mathfrak{L}(\lambda, d, 1)$  and  $f \in \mathcal{K}(\theta, \Omega)$ . We form the cooperative system

$$\begin{aligned} Lv_1 + \frac{\theta f(x)}{\|f\|_{\infty; \Omega}} v_2 &= 0 \\ \Delta v_2 &= 0. \end{aligned}$$

Clearly,  $(v_1, v_2) = (u, \theta^{-1}\|f\|_{\infty; \Omega})$  is a nonnegative solution and  $\tilde{c}_\Omega \geq 1$ . Therefore, from (2.3), we deduce Harnack's inequality for  $u$  by setting  $\lambda = \max\{\lambda, \theta\}$  and  $\mu_\Omega = 1$  in the Harnack constant  $K_2$ .

For the elliptic system in (1.1)–(1.2), this procedure leads to the following:

**Corollary 2.2.** *Let  $\Gamma \subset \Omega$  be a closed set and  $u$  a nonnegative function satisfying  $-Lu \in \mathcal{K}(\theta, \Omega)$ . The following are true:*

- (i) *If  $L \in \mathfrak{L}(\lambda, d, n, \gamma)$  then (2.1) holds, with a Harnack constant*

$$K_1 K_2(\Omega, \Gamma, \max\{\lambda, \theta\}, d, 2n, \gamma, 1).$$


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(ii) If  $L \in \mathfrak{L}(\lambda, d, n, \gamma, \mu_\Omega)$ , then (2.2) holds with a constant

$$K_2(\Omega, \Gamma, \max\{\lambda, \frac{\theta}{\mu_\Omega}\}, d, 2n, \gamma, \mu_\Omega).$$

### 3. PROOFS OF THE RESULTS

If  $\mathbf{u} \in W_{loc}^{2,p}(\Omega; \mathbb{R}^n)$ , for some  $p \in [1, \infty)$ , is a solution of  $L\mathbf{u} = \mathbf{f}$  in  $\Omega$  and  $\mathbf{f} \in L^\infty(\Omega; \mathbb{R}^n)$ , then  $\mathbf{u} \in W_{loc}^{2,p}(\Omega; \mathbb{R}^n)$ , for all  $p \in [1, \infty)$ . This fact follows from the interior  $L^p$  estimates for second derivatives of uniformly elliptic equations and the well known Sobolev inequalities. However, the natural space for some considerations is  $W^{2,d}$ . This is the case, for example, for the Aleksandroff estimate (Lemma 3.2) and the comparison principle [2] which states that if  $\varphi, \psi \in W_{loc}^{2,d}(\Omega; \mathbb{R}^n) \cap C^0(\overline{\Omega}; \mathbb{R}^n)$  satisfy  $L\varphi \leq L\psi$  in  $\Omega$  and  $\varphi \geq \psi$  on  $\partial\Omega$ , then  $\varphi \geq \psi$  in  $\overline{\Omega}$ .

Let  $\mathbf{u} \in W_{loc}^{2,d}(\Omega; \mathbb{R}^n)$  be a nonnegative solution of  $L\mathbf{u} = 0$  in  $\Omega$ , with  $L \in \mathfrak{L}(\lambda, d, n)$ . Augmenting the dimension of the domain, let  $I \subset \mathbb{R}$  be a bounded open interval and define the function  $\mathbf{v} : \Omega \times I \rightarrow \mathbb{R}^d$  by  $\mathbf{v}(x, x_{d+1}) := \mathbf{u}(x) \exp(\sqrt{n\lambda}x_{d+1})$ , and the operator  $\tilde{L} \in \mathfrak{L}((n+1)\lambda, d+1, n)$  by

$$\tilde{L}_k := L_k + \frac{\partial^2}{\partial x_{d+1}^2} - n\lambda + c_{kk}.$$

Then  $\tilde{L}\mathbf{v} = 0$ , and any Harnack estimates obtained for  $\tilde{L}$ -harmonic functions clearly hold for  $\mathbf{u}$ . Observe that the coefficients  $\tilde{c}_{kj}$  of the operator  $\tilde{L}$  form a *sub-stochastic* matrix, i.e., they satisfy  $\sum_{j=1}^n \tilde{c}_{kj} \leq 0$ , for all  $k = 1, \dots, n$ . Hence, without loss of generality, we restrict the proofs to operators in  $\mathfrak{L}(\lambda, d, n)$  and  $\mathfrak{L}(\lambda, d, n, \gamma, \mu_\Omega)$  whose coupling coefficients form a sub-stochastic matrix, and we denote the corresponding classes by  $\mathfrak{L}_0(\lambda, d, n)$  and  $\mathfrak{L}_0(\lambda, d, n, \gamma, \mu_\Omega)$ , respectively.

Let  $\mathfrak{U}_\Omega$  ( $\mathfrak{U}_\Omega^-$ ) denote the space of all nonnegative functions  $\mathbf{u} \in W_{loc}^{2,d}(\Omega; \mathbb{R}^n) \cap C^0(\overline{\Omega}; \mathbb{R}^n)$ , satisfying  $L\mathbf{u} = 0$  ( $L\mathbf{u} \leq 0$ ) in  $\Omega$ , for some  $L \in \mathfrak{L}_0(\lambda, d, n)$ . If  $\xi \in \mathbb{R}$ , then  $\mathbf{u} \geq \xi$  is to be interpreted as  $u_i \geq \xi$ , for all  $i \in \{1, \dots, n\}$ , and if  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ , then  $\mathbf{u} \geq \boldsymbol{\xi} \iff u_i \geq \xi_i$ , for all  $i \in \{1, \dots, n\}$ . In general, all scalar operations on  $\mathbb{R}^n$ -valued functions

are meant to be componentwise. For greater clarity, we denote all  $\mathbb{R}^n$ -valued quantities by a bold letter. If  $\Gamma$  is a closed subset of  $\Omega$ ,  $x \in \Omega$ , and  $\boldsymbol{\xi} \in \mathbb{R}_+^n$ , we define

$$\Psi_x(\boldsymbol{\mathcal{M}}_\Omega, \Gamma; \boldsymbol{\xi}) := \inf_{\boldsymbol{u} \in \boldsymbol{\mathcal{M}}_\Omega} \{ \boldsymbol{u}(x) : \boldsymbol{u} \geq \boldsymbol{\xi} \text{ on } \Gamma \}.$$

Lastly, deviating from the usual vector space notation, if  $D$  is a cube in  $\mathbb{R}^d$  and  $\delta > 0$ ,  $\delta D$  denotes the cube which is *concentric* to  $D$  and whose edges are  $\delta$  times as long.

We start with a measure theoretic result, announced in [4]. For a proof see [2].

**Lemma 3.1.** *Let  $K \subset \mathbb{R}^d$  be a cube,  $\Gamma \subset K$  a closed subset, and  $0 < \alpha < 1$ . Define*

$$\begin{aligned} \mathcal{Q} &:= \{ Q : Q \text{ is a subcube of } K \text{ and } |Q \cap \Gamma| \geq \alpha |Q| \} \\ \tilde{\Gamma} &:= \bigcup_{Q \in \mathcal{Q}} (3Q \cap K). \end{aligned}$$

*Then either  $\tilde{\Gamma} = K$  or  $|\tilde{\Gamma}| \geq \frac{1}{\alpha} |\Gamma|$ .*

Next we state a variant of the weak maximum principle of A. D. Alexandroff. This particular form of the estimate is derived by first using a transformation to remove the first-order terms and then dominating the  $L^d$  norm with the  $L^\infty$  norm. The steps of the proof are quite standard and are therefore omitted.

**Lemma 3.2.** *There exist constants  $C_1 > 0$  and  $\kappa_0 \in (0, 1]$  such that, if  $D \subset \mathbb{R}^d$  is any cube of volume  $|D| \leq \kappa_0$ , and  $\varphi \in W_{loc}^{2,d}(D) \cap C^0(\bar{D})$  satisfies  $L_k \varphi \geq f$  in  $D$ , and  $\varphi = 0$  on  $\partial D$ , with  $f \in L^d(D)$  and  $L \in \mathfrak{L}(\lambda, d, n)$ , then*

$$\sup_{x \in D} \{ \varphi(x) \} \leq C_1 |D|^{1/d} \|f\|_{d;D}.$$

For the remainder of this section,  $D$  denotes an open cube in  $\mathbb{R}^d$  of volume not exceeding the constant  $\kappa_0$  in Lemma 3.2.

**Lemma 3.3.** *There exist constants  $\beta_0 > 0$  and  $\alpha_0 < 1$  such that, if  $\Gamma$  is a closed subset of some cube  $D \subset \mathbb{R}^d$  satisfying  $|\Gamma| \geq \alpha_0 |D|$ , then*

$$\inf_{x \in \frac{1}{3}D} \Psi_x(\boldsymbol{\mathcal{M}}_D^-, \Gamma; \boldsymbol{\xi}) \geq \beta_0 \boldsymbol{\xi}, \quad \forall \boldsymbol{\xi} \in \mathbb{R}_+^n.$$

*Proof.* Observe that if  $\mathbf{u} \in \mathfrak{U}_{\overline{D}}$ , then each component  $u_k$  satisfies  $L_k u_k \leq 0$  on  $D$ . Define  $\varphi', \varphi'' \in W_{loc}^{2,d}(D) \cap C^0(\overline{D})$  by

$$L_k \varphi' = -I_{\Gamma}, \quad L_k \varphi'' = -I_{\Gamma^c}, \quad \text{in } D$$

$$\text{and } \varphi' = \varphi'' = 0, \quad \text{on } \partial D.$$

Then  $\varphi := \varphi' + \varphi''$  satisfies  $L_k \varphi = -1$  in  $D$  and  $\varphi = 0$  on  $\partial D$ . Without loss of generality, suppose  $D$  is centered at the origin and consider the function

$$\psi(x) := \prod_{i=1}^d (|D|^{2/d} - 4x_i^2).$$

Clearly,  $\psi = 0$  on  $\partial D$  and  $\psi > 0$  in  $D$ ; moreover, there exists a positive constant  $C_2$  such that

$$\inf_{x \in \frac{1}{3}D} \{\psi(x)\} \geq C_2 |D|^{2/d} \|L_k \psi\|_{\infty; D}, \quad \forall L \in \mathfrak{L}_0(\lambda, d, n).$$

Therefore, by the comparison principle

$$\varphi(x) \geq \frac{\psi(x)}{\|L_k \psi\|_{\infty; D}} \geq C_2 |D|^{2/d}, \quad \forall x \in \frac{1}{3}D. \quad (3.1)$$

Using Lemma 3.2, we obtain

$$\begin{aligned} \varphi' &\leq C_1 |D|^{1/d} |\Gamma|^{1/d} = C_1 |D|^{2/d} \left(\frac{|\Gamma|}{|D|}\right)^{1/d} \\ \varphi'' &\leq C_1 |D|^{1/d} |\Gamma^c|^{1/d} = C_1 |D|^{2/d} \left(1 - \frac{|\Gamma|}{|D|}\right)^{1/d}. \end{aligned} \quad (3.2)$$

By (3.1) and (3.2),

$$\varphi'(x) \geq C_2 |D|^{2/d} - C_1 |D|^{2/d} \left(1 - \frac{|\Gamma|}{|D|}\right)^{1/d}, \quad \forall x \in \frac{1}{3}D.$$

On the other hand, since  $L_k \varphi' = 0$  in  $D \setminus \Gamma$  and  $\varphi' = 0$  on  $\partial D$ , the comparison principle yields

$$\inf_{x \in \frac{1}{3}D} \{u_k(x)\} \geq \xi_k \frac{C_2 - C_1 \left(1 - \frac{|\Gamma|}{|D|}\right)^{1/d}}{C_1 \left(\frac{|\Gamma|}{|D|}\right)^{1/d}}. \quad (3.3)$$



Selecting  $\alpha_0$  to satisfy

$$\alpha_0 \geq 1 - \left( \frac{C_2}{2C_1} \right)^d,$$

inequality (3.3) yields

$$\inf_{x \in \frac{1}{3}D} u_k(x) \geq \frac{C_2 \xi_k}{2C_1}.$$

Hence, the claim follows with  $\beta_0 = \frac{C_2}{2C_1}$ .  $\square$

**Lemma 3.4.** *For each  $\delta > 0$ , there exists  $k'_\delta > 0$  such that if  $Q \subset (1 - \delta)D$  is a subcube of an open cube  $D \subset \mathbb{R}^d$ , then*

$$\Psi_x(\mathfrak{M}_D^-, \frac{1}{3}Q; \xi) \geq k'_\delta \xi, \quad \forall x \in 3Q \cap (1 - \delta)D, \quad \forall \xi \in \mathbb{R}_+^n.$$

*Proof.* Let  $B(r) \subset \mathbb{R}^d$  denote the ball of radius  $r$  centered at the origin. We claim that there exists a constant  $m_0 > 0$  such that if  $r \leq 1$ , then

$$\inf_{x \in B(\frac{3r}{4})} \Psi_x(\mathfrak{M}_{B(r)}^-, B(\frac{r}{4}); \xi) \geq m_0 \xi, \quad \forall \xi \in \mathbb{R}_+^n. \quad (3.4)$$

In order to establish (3.4) we use the function

$$\varphi(x) := \exp \left\{ 16\lambda^2(d+1) \left( 1 - \frac{\|x\|^2}{r^2} \right) \right\} - 1, \quad x \in B(r),$$

which satisfies  $L_k \varphi(x) \geq 0$ , for all  $L \in \mathfrak{L}_0(\lambda, d, n)$ , provided  $\|x\| \geq \frac{r}{4}$  and  $r \leq 1$ . By the comparison principle, (3.4) holds with

$$m_0 = \frac{e^{7\lambda^2(d+1)} - 1}{e^{15\lambda^2(d+1)} - 1}.$$

It follows that if  $B(r)$  is centered at  $y$ , and  $x$  is a point in  $D$  such that the distance between  $\partial D$  and the line segment joining  $x$  and  $y$  is at least  $r$ , then

$$\Psi_x(\mathfrak{M}_D^-, B(\frac{r}{4}); \xi) \geq (m_0)^\ell \xi, \quad \text{with } \ell = \left\lceil \frac{4\|x-y\| - r}{2r} \right\rceil, \quad (3.5)$$

for all  $\xi \in \mathbb{R}_+^n$ . If we define

$$k'_\delta := m_0^{\ell(\delta)}, \quad \ell(\delta) := \left\lceil \frac{6\sqrt{d}}{\min\{1, \delta\}} \right\rceil,$$

then an easy calculation, using (3.5) with  $r = \min\{\frac{2}{3}, \frac{\delta}{2}\} |Q|^{1/d}$ , establishes the result.  $\square$

**Lemma 3.5.** *Suppose there exist constants  $\varepsilon$  and  $\theta$  such that if  $\Gamma \subset (1-\delta)D$  is a closed subset of some cube  $D$  and  $\xi \in \mathbb{R}_+^n$ , then*

$$\inf_{x \in \frac{1}{3}D} \Psi_x(\mathfrak{M}_D^-, \Gamma; \xi) \geq \varepsilon \xi, \quad \text{whenever } |\Gamma| \geq \theta|D|.$$

*Then there exists a constant  $k_\delta > 0$  such that*

$$\inf_{x \in \frac{1}{3}D} \Psi_x(\mathfrak{M}_D^-, \Gamma; \xi) \geq \varepsilon k_\delta \xi, \quad \text{whenever } |\Gamma| \geq \alpha_0 \theta |D|,$$

*where  $\alpha_0$  is the constant in Lemma 3.3.*

*Proof.* Suppose  $|\Gamma| \geq \alpha_0 \theta |D|$  and let  $y \in \tilde{\Gamma}$ , with  $\tilde{\Gamma}$  as defined in Lemma 3.1 corresponding to  $\alpha = \alpha_0$  and  $K = (1-\delta)D$ . Then there exists a subcube  $Q \subset K$  such that  $|\Gamma \cap Q| \geq \alpha_0 |Q|$  and  $y \in 3Q \cap K$ . We use the identities,

$$\Psi_x(\mathfrak{M}_D^-, \Gamma; \xi) \geq \Psi_x(\mathfrak{M}_D^-, \tilde{\Gamma}; \inf_{y \in \tilde{\Gamma}} \Psi_x(\mathfrak{M}_D^-, \Gamma; \xi)) \tag{3.6}$$

and

$$\begin{aligned} \Psi_y(\mathfrak{M}_D^-, \Gamma; \xi) &\geq \Psi_y(\mathfrak{M}_D^-, \frac{1}{3}Q; \inf_{z \in \frac{1}{3}Q} \Psi_z(\mathfrak{M}_D^-, \Gamma; \xi)) \\ &\geq \Psi_y(\mathfrak{M}_D^-, \frac{1}{3}Q; \inf_{z \in \frac{1}{3}Q} \Psi_z(\mathfrak{M}_Q^-, \Gamma \cap Q; \xi)). \end{aligned} \tag{3.7}$$

By Lemma 3.3,

$$\inf_{z \in \frac{1}{3}Q} \Psi_z(\mathfrak{M}_Q^-, \Gamma \cap Q; \xi) \geq \beta_0 \xi, \tag{3.8}$$

while from Lemma 3.4, we obtain  $\Psi_y(\mathfrak{M}_D^-, \frac{1}{3}Q; \beta_0 \xi) \geq \beta_0 k'_\delta \xi$ , for all  $y \in 3Q \cap K$ . Hence, combining (3.7) and (3.8), we obtain

$$\inf_{y \in \tilde{\Gamma}} \Psi_y(\mathfrak{M}_D^-, \Gamma; \xi) \geq k_\delta \xi, \quad \text{with } k_\delta := \beta_0 k'_\delta. \tag{3.9}$$

By Lemma 3.1,  $|\tilde{\Gamma}| \geq \frac{1}{\alpha_0} |\Gamma| \geq \theta |D|$ . Therefore, by hypothesis,

$$\inf_{x \in \frac{1}{3}D} \Psi_x(\mathfrak{M}_D^-, \tilde{\Gamma}; k_\delta \xi) \geq \varepsilon k_\delta \xi,$$

which along with (3.6) and (3.9) yield the desired result.  $\square$

**Proposition 3.1.** *The following estimates hold:*

- (i) *Let  $D$  be a cube and  $\Gamma \subset (1 - \delta)D$  a closed subset. Then for all  $\xi \in \mathbb{R}_+^n$ ,*

$$\inf_{x \in \frac{1}{3}D} \Psi_x(\mathfrak{U}_D^-, \Gamma; \xi) \geq \beta_0 \left(\frac{|\Gamma|}{|D|}\right)^{\rho(\delta)} \xi, \quad \rho(\delta) := \frac{\log k_\delta}{\log \alpha_0}, \quad (3.10)$$

where the constants  $\alpha_0$ ,  $\beta_0$  and  $k_\delta$  are as in Lemma 3.3 and Lemma 3.5.

- (ii) *There exists a real function  $F$  defined on  $[0, 1]$ , satisfying  $F(\theta) > 0$  for  $\theta > 0$ , such that if  $\Gamma \subset D$  is a closed subset of a cube  $D$ , then*

$$\inf_{x \in \frac{1}{3}D} \Psi_x(\mathfrak{U}_D^-, \Gamma; \xi) \geq F\left(\frac{|\Gamma|}{|D|}\right) \xi, \quad \forall \xi \in \mathbb{R}_+^n. \quad (3.11)$$

*Proof.* Part (i) is direct consequence of Lemmas 3.3 and 3.5. For part (ii), choose  $\delta = \frac{|\Gamma|}{4|D|}$ . Then,

$$\frac{|\Gamma \cap (1 - \delta)D|}{|D|} \geq \frac{|\Gamma|}{|D|} - (1 - (1 - \delta)^d) \geq \frac{|\Gamma|}{|D|} - d\delta \geq \frac{3|\Gamma|}{4|D|}. \quad (3.12)$$

Since

$$\Psi_x(\mathfrak{U}_D^-, \Gamma; \xi) \geq \Psi_x(\mathfrak{U}_D^-, \Gamma \cap (1 - \delta)D; \xi),$$

then if we let

$$F(\theta) := \beta_0 \left(\frac{3\theta}{4}\right)^{\rho(\frac{\theta}{4d})},$$

the bound in (3.11) follows from (3.10) and (3.12).  $\square$

**Proposition 3.2.** *If  $D$  is a cube,  $u \in \mathfrak{U}_D$  and  $q = F(\frac{1}{2})$ , with  $F(\cdot)$  as defined in Proposition 3.1 (ii), then*

$$\text{osc}(u_k; \frac{1}{3}D) \leq \left(1 - \frac{q}{2}\right) \max_{1 \leq k \leq n} \sup_{x \in D} \{u_k(x)\}, \quad \forall k \in \{1, \dots, n\},$$

where  $\text{osc}(f; A)$  denotes the oscillation of a function  $f$  over a set  $A$ .

*Proof.* Let

$$\begin{aligned} M_k^a &:= \sup_{x \in \frac{1}{3}D} \{u_k(x)\}, & M^a &:= \max_{1 \leq k \leq n} M_k^a \\ m_k^a &:= \inf_{x \in \frac{1}{3}D} \{u_k(x)\}, & m^a &:= \min_{1 \leq k \leq n} m_k^a \end{aligned}$$

and  $M^b, m^b$  be the corresponding bounds relative to  $D$ . Consider the sets

$$\begin{aligned}\Gamma_1^{(k)} &:= \{x \in D : u_k(x) \leq \frac{M^b + m^b}{2}\} \\ \Gamma_2^{(k)} &:= \{x \in D : u_k(x) \geq \frac{M^b + m^b}{2}\}.\end{aligned}$$

Suppose  $|\Gamma_1^{(k)}| \geq \frac{1}{2}|D|$ . Since  $M^b - \mathbf{u}$  is nonnegative and  $M^b - u_k \geq \frac{M^b - m^b}{2}$  in  $\Gamma_1^{(k)}$ , applying Proposition 3.1 (ii), we get

$$M^b - u_k(x) \geq q \frac{M^b - m^b}{2}, \quad \forall x \in \frac{1}{3}D.$$

Consequently,  $M_k^a \leq M^b - q \frac{M^b - m^b}{2}$ , and since  $m^a \geq m^b$ , we obtain

$$M_k^a - m^a \leq M^b - m^b - q \frac{M^b - m^b}{2} \leq (1 - \frac{q}{2})M^b. \quad (3.13)$$

On the other hand, if  $|\Gamma_2^{(k)}| \geq \frac{1}{2}|D|$ , the analogous argument relative to the nonnegative function  $\mathbf{u}$ , yields

$$M^a - m_k^a \leq (1 - \frac{q}{2})M^b, \quad (3.14)$$

and the result follows by (3.13)–(3.14).  $\square$

**Proposition 3.3.** *There exists a constant  $M_1 > 0$  such that, for any  $\mathbf{u} \in \mathfrak{U}_D$*

$$\sup_{x \in \frac{1}{3}D} \{u_i(x)\} \leq M_1 \max_{1 \leq k \leq n} \inf_{x \in \frac{1}{3}D} \{u_k(x)\}, \quad \forall i \in \{1, \dots, n\}.$$

*Proof.* Let  $\beta_0$  be as in Lemma 3.3 and  $\rho(\cdot)$  and  $q$  as in (3.10) and Proposition 3.2, respectively. Define

$$\rho := \frac{1}{d\rho(\frac{2}{3})} \quad \text{and} \quad q_0 := \frac{(1 - \frac{q}{4})}{(1 - \frac{q}{2})}. \quad (3.15)$$

We claim that the value of the constant  $M_1$  may be chosen as

$$M_1 := \frac{4q_0}{q\beta_0} \left[ \frac{27n^{1/d}}{2(q_0^\rho - 1)} \right]^{1/\rho}. \quad (3.16)$$

We argue by contradiction. Suppose  $\mathbf{u}$  violates this bound and let

$\{x^{(1)}, \dots, x^{(n)}\}$  denote the points in  $\frac{1}{9}\bar{D}$  where the minima of  $\mathbf{u}$  are attained, i.e.,

$$\inf_{x \in \frac{1}{9}D} \{u_k(x)\} = u_k(x^{(k)}), \quad 1 \leq k \leq n.$$

Without loss of generality, suppose that  $\max_{1 \leq k \leq n} \{u_k(x^{(k)})\} = 1$  and that for some  $y_0 \in \frac{1}{9}D$  and  $k_0 \in \{1, \dots, n\}$ ,  $u_{k_0}(y_0) = M > aM_1$ , with  $a > 1$ . Using the estimate for the growth of the oscillation of  $\mathbf{u}$  in Proposition 3.2, we then show that  $\mathbf{u}$  has to be unbounded in  $\frac{1}{3}D$ . By hypothesis,  $\frac{M}{a}$  exceeds  $M_1$  in (3.16) and, in order to facilitate the construction which follows, we choose to express this as

$$\frac{1}{9} + 3n^{1/d} \left( \frac{4a}{q\beta_0 M} \right)^\rho \sum_{i=0}^{\infty} \left( \frac{1}{q_0} \right)^{i\rho} < \frac{1}{3}. \quad (3.17)$$

For  $\xi > 0$ , define

$$\mathcal{D}_k^{(\xi)} := \left\{ x \in \frac{1}{3}\bar{D} : u_k(x) \geq \xi \right\}, \quad \mathcal{D}^{(\xi)} := \bigcup_{1 \leq k \leq n} \mathcal{D}_k^{(\xi)}.$$

If  $\mathbf{1}_k \in \mathbb{R}_+^n$  stands for the vector whose  $k$ -th component is equal to 1 and the others 0, then

$$\mathbf{u}(x^{(k)}) \geq \Psi_{x^{(k)}}(\mathbf{u}_D, \mathcal{D}_k^{(\xi)}; \xi \mathbf{1}_k), \quad \forall k \in \{1, \dots, n\}, \quad (3.18)$$

while, on the other hand, Proposition 3.1 yields

$$\Psi_{x^{(k)}}(\mathbf{u}_D, \mathcal{D}_k^{(\xi)}; \xi \mathbf{1}_k) \geq \beta_0 \left( \frac{|\mathcal{D}_k^{(\xi)}|}{|D|} \right)^{\rho(\frac{2}{3})} \xi \mathbf{1}_k, \quad \forall k \in \{1, \dots, n\}. \quad (3.19)$$

By (3.18)–(3.19) and using (3.15), we obtain the estimate

$$|\mathcal{D}^{(\xi)}| \leq \sum_{1 \leq k \leq n} |\mathcal{D}_k^{(\xi)}| \leq \sum_{1 \leq k \leq n} \left( \frac{u_k(x^{(k)})}{\xi \beta_0} \right)^{\rho d} |D| \leq n \left( \frac{1}{\xi \beta_0} \right)^{\rho d} |D|, \quad (3.20)$$

for all  $\xi > 0$ . Choosing  $\xi = \frac{qM}{4}$ , we have by (3.20)

$$\left| \left\{ x \in \frac{1}{3}D : \max_{1 \leq k \leq n} \{u_k(x)\} \geq \frac{qM}{4} \right\} \right| \leq n \left( \frac{4}{q\beta_0 M} \right)^{\rho d} |D|.$$

Hence, if  $Q_0$  is a cube of volume  $|Q_0| = n\left(\frac{4a}{q\beta_0 M}\right)^{\rho d}|D|$  centered at  $y_0$ , then

$$\text{osc}(u_{k_0}; Q_0) > \left(1 - \frac{q}{4}\right)M. \quad (3.21)$$

By Proposition 3.2 and (3.21), there exists  $y^{(1)} \in 3Q_0$  and  $k_1 \in \{1, \dots, n\}$  such that

$$u_{k_1}(y^{(1)}) > \frac{\left(1 - \frac{q}{4}\right)}{\left(1 - \frac{q}{2}\right)}M = q_0 M.$$

Note that (3.17) implies that  $3Q_0 \subset \frac{1}{3}D$ . This allows us to repeat the argument above, this time choosing  $\xi = q_0 \frac{qM}{4}$  in (3.20) and a cube  $Q_1$  of volume  $n\left(\frac{4a}{q_0 q \beta_0 M}\right)^{\rho d}|D|$  centered at  $y^{(1)}$ , to conclude that there exists  $y^{(2)} \in 3Q_1$  and  $k_2 \in \{1, \dots, n\}$  such that  $u_{k_2}(y^{(2)}) \geq q_0^2 M$ . Inductively, we construct a sequence  $\{y^{(i)}, k_i, Q_i\}_{i=0}^{\infty}$  satisfying, for all  $i = 0, 1, \dots$ ,

$$\begin{aligned} y^{(0)} = y_0 &\in \frac{1}{9}\overline{D} \cap Q_0, & y^{(i)} &\in Q_i \cap 3Q_{i-1}, \\ |Q_i|^{1/d} &= n^{1/d} \left(\frac{1}{q_0}\right)^{i\rho} \left(\frac{4a}{q\beta_0 M}\right)^{\rho} |D|^{1/d}, & & (3.22) \\ u_{k_i}(y^{(i)}) &\geq q_0^i M. \end{aligned}$$

The inequality in (3.17) guarantees that  $y^{(i)} \in \frac{1}{3}D$ , for all  $i$ . Hence, (3.22) implies that  $u$  is unbounded in  $\frac{1}{3}D$ , and we reach a contradiction.  $\square$

Theorem 2.1 now follows via the standard technique of selecting an appropriate cover of  $\Gamma$  consisting of congruent cubes and applying the estimates in Proposition 3.1 and Proposition 3.3.

We next proceed to prove Theorem 2.2. We need the following lemma.

**Lemma 3.6.** *Let  $D \subset \mathbb{R}^d$  be a cube,  $L \in \mathfrak{L}_0(\lambda, d, 1, \gamma)$ , and  $f \in \mathcal{K}(\theta, D)$ . There exists a constant  $C' = C'(|D|, \lambda, d, \gamma, \theta) > 0$  such that if  $\varphi$  is a solution to the Dirichlet problem  $L\varphi = -f$  on  $D$ , with  $\varphi = 0$  on  $\partial D$ , then*

$$\inf_{x \in \frac{1}{3}D} \{\varphi(x)\} \geq C' \|f\|_{\infty, D}.$$

*Proof.* First note that the Dirichlet problem has a unique strong solution  $\varphi \in W_{loc}^{2,p}(D) \cap C^0(\overline{D})$ , for all  $p \in [d, \infty)$ . Then, arguing by contradiction, suppose there exists a sequence of operators  $\{L^{(m)}\}_{m=1}^{\infty} \subset \mathfrak{L}_0(\lambda, d, 1, \gamma)$ , with coefficients  $\{a_{ij}^{(m)}, b_i^{(m)}, c^{(m)}\}$ , and a sequence of functions

$\{f^{(m)}\}_{m=1}^\infty \subset \mathcal{K}(\theta, D)$ , with  $\|f^{(m)}\|_{\infty;D} = 1$ , such that the corresponding solutions  $\{\varphi^{(m)}\}_{m=1}^\infty$  satisfy

$$\inf_{x \in \frac{1}{3}D} \{\varphi^{(m)}(x)\} < \frac{1}{m^2}, \quad m = 1, 2, \dots$$

By Proposition 3.1,

$$\left| \left\{ x \in (1 - \delta)D : \varphi^{(m)}(x) \geq \frac{1}{m} \right\} \right| \leq \left( \frac{1}{\beta_0 m} \right)^{1/\rho(\delta)} |D|, \quad \forall \delta > 0. \quad (3.23)$$

Since the sequence  $\varphi^{(m)}$  is bounded in  $L^\infty(D)$  (by Lemma 3.2), it follows from (3.23) that  $\varphi^{(m)} \rightarrow 0$  in  $L^p(D)$ , as  $m \rightarrow \infty$ , for all  $p \in [1, \infty)$ . For any subcube  $D' = \delta D$ , with  $\delta < 1$ , and  $p \in [1, \infty)$ , we use the well known estimate

$$\|\varphi^{(m)}\|_{2,p;D'} \leq C'' (\|\varphi^{(m)}\|_{p;D} + \|f^{(m)}\|_{p;D}),$$

for some constant  $C'' = C''(|D|, p, \delta, \lambda, d, \gamma)$ , to conclude that the first and second derivatives of  $\varphi^{(m)}$  converge weakly to 0 in  $L^p(D')$ , for all  $p \in [1, \infty)$ . In turn, since  $W_0^{2,p}(D') \hookrightarrow W_0^{1,p}(D')$  is compact for  $p > d$ , using the standard approximation argument, we deduce that  $\frac{\partial \varphi^{(m)}}{\partial x_i}$  converges in  $L^p(D')$  strongly, for all  $i = 1, \dots, d$ . Also, since the sequence  $\{a_{ij}^{(m)}\}$  is uniformly Lipschitz, we can extract a subsequence which converges uniformly. The previous arguments combined imply that  $\{L^{(m)}\varphi^{(m)}\}$  converges weakly to 0 in  $L^p(D')$ ,  $p \in [1, \infty)$ . On the other hand, if we choose  $\delta \geq (1 - \frac{1}{2\theta})^{1/d}$ , an easy calculation yields,

$$\int_{D'} f^{(m)}(x) dx \geq \frac{|D|}{2\theta}, \quad m = 1, 2, \dots,$$

resulting in a contradiction.  $\square$

*Proof of Theorem 2.2.* Let  $L \in \mathfrak{L}_0(\lambda, d, n, \gamma, \mu_\Omega)$  and  $\mathbf{S} = [s_{ij}]$  as in Definition 2.1. Select a collection  $\{D_\ell, \ell = 1, \dots, \ell_0\}$  of disjoint, congruent open cubes, whose closures form a cover of  $\Gamma$ , in such a manner that  $3\bar{D}_\ell \subset \Omega$ ,  $1 \leq \ell \leq \ell_0$ , and  $\mathcal{D} := \bigcup_{\ell=1}^{\ell_0} \bar{D}_\ell$  is a connected set satisfying

$|\mathcal{D}| \leq \left(1 - \frac{1}{2\lambda\mu_\Omega}\right) |\Omega|$ . It follows that  $2\mu_\Omega \mathbf{C}_L(\mathcal{D}) \geq \mathbf{S}$ . Therefore, for each pair  $i, j \in \{1, \dots, n\}$ ,  $i \neq j$ , there exists  $\ell(i, j) \in \{1, \dots, \ell_0\}$  such that

$$\|c_{ij}\|_{\infty; D_{\ell(i,j)}} \geq \frac{\|c_{ij}\|_{1; D_{\ell(i,j)}}}{|D_{\ell(i,j)}|} \geq \frac{s_{ij}}{2\mu_{\Omega}}. \quad (3.24)$$

Define a collection  $\{\varphi_{ij}\}_{i \neq j} \subset W_{loc}^{2,d}(D_{\ell(i,j)}) \cap C^0(\overline{D}_{\ell(i,j)})$ , by

$$L_i \varphi_{ij} = -c_{ij} \quad \text{in } D_{\ell(i,j)}, \quad \text{and} \quad \varphi_{ij} = 0 \quad \text{on } \partial D_{\ell(i,j)}.$$

Then, by Lemma 3.6 and (3.24), there exists a constant  $C' > 0$ , such that

$$\varphi_{ij}(x) \geq \frac{C'}{2\mu_{\Omega}} s_{ij}, \quad \forall x \in \frac{1}{3} D_{\ell(i,j)}, \quad i \neq j. \quad (3.25)$$

By the comparison principle,

$$u_i(x) \geq \varphi_{ij}(x) \inf_{z \in D_{\ell(i,j)}} \{u_j(z)\}, \quad \forall x \in D_{\ell(i,j)}, \quad i \neq j. \quad (3.26)$$

By Proposition 3.1, (3.25) and (3.26),

$$\begin{aligned} u_i(y) &\geq F\left(\frac{1}{9d}\right) \inf_{x \in \frac{1}{3} D_{\ell(i,j)}} \{u_i(x)\} \\ &\geq \varepsilon' F\left(\frac{1}{9d}\right) s_{ij} \inf_{z \in D_{\ell(i,j)}} \{u_j(z)\}, \quad \forall y \in D_{\ell(i,j)}, \quad i \neq j. \end{aligned} \quad (3.27)$$

Moreover, provided  $3D_k \supset D_{k'}$ ,  $1 \leq k, k' \leq \ell_0$ , Proposition 3.1 also asserts that

$$\inf_{x \in D_k} \{u(x)\} \geq F\left(\frac{1}{3d}\right) \inf_{x \in D_{k'}} \{u(x)\},$$

from which we deduce that

$$\inf_{x \in D_k} \{u(x)\} \geq \left(F\left(\frac{1}{3d}\right)\right)^{\ell_0} \inf_{x \in D_{\ell}} \{u(x)\}, \quad \forall k, \ell \in \{1, \dots, \ell_0\}. \quad (3.28)$$

Therefore, by (3.27) and (3.28), for all  $i \neq j$ ,

$$\begin{aligned} \inf_{x \in \mathcal{D}} \{u_i(x)\} &\geq \left(F\left(\frac{1}{3d}\right)\right)^{\ell_0} \inf_{y \in D_{\ell(i,j)}} \{u_i(y)\} \\ &\geq \frac{C'}{2\mu_{\Omega}} F\left(\frac{1}{9d}\right) \left(F\left(\frac{1}{3d}\right)\right)^{\ell_0} s_{ij} \inf_{z \in D_{\ell(i,j)}} \{u_j(z)\} \\ &\geq \frac{C'}{2\mu_{\Omega}} F\left(\frac{1}{9d}\right) \left(F\left(\frac{1}{3d}\right)\right)^{2\ell_0} s_{ij} \inf_{x \in \mathcal{D}} \{u_j(x)\}, \end{aligned}$$

and in turn, the irreducibility of  $S$  implies that, for all  $i, j \in \{1, \dots, n\}$ ,

$$\inf_{x \in \mathcal{D}} \{u_i(x)\} \geq \left(\frac{C'}{2\mu_{\Omega}} F\left(\frac{1}{9d}\right) \left(F\left(\frac{1}{3d}\right)\right)^{2\ell_0}\right)^{n-1} \inf_{x \in \mathcal{D}} \{u_j(x)\}. \quad (3.29)$$



The result follows by combining (3.29) and the estimate in Theorem 2.1 relative to the closed set  $\mathcal{D} \subset \Omega$ .  $\square$

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