

Course notes for EE394V

Restructured Electricity Markets: Locational Marginal Pricing

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Economic dispatch

- (i) Formulation,
- (ii) Changes in demand and capacity,
- (iii) Problem characteristics,
- (iv) Optimality conditions,
- (v) Algorithms,
- (vi) Discussion,
- (vii) Examples,

- (viii) Linear programming approximation,
- (ix) Homework exercise: Due Wednesday, October 8.

3.1 Formulation

3.1.1 Variables

- Suppose there are n generators.
- We consider their electric energy production over a period of time.
- The length T of this period of time is assumed to be short enough so that the power from each generator can be well approximated by a constant over the period of time.
- We will deal separately with variations of power production and demand that occur over shorter time scales through:
 - economic dispatch defined over shorter time scales, and
 - other **ancillary services**.
- Define $x_k \in \mathbb{R}$ to be the (average) power level of generator k during the time period.
- We collect the production decisions of generators $k = 1, \dots, n$, into a

vector $x \in \mathbb{R}^n$, so that $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

3.1.2 Generator constraints

- We assume that generator k has:
 - a maximum production capacity, say \bar{x}_k , and
 - a minimum production capacity, $\underline{x}_k \geq 0$.
- That is, x_k must satisfy:

$$\underline{x}_k \leq x_k \leq \bar{x}_k. \quad (3.1)$$

- Equivalently, the feasible operating set for generator k is:

$$\mathbb{S}_k = [\underline{x}_k, \bar{x}_k].$$

3.1.3 Production costs

- We suppose that for $k = 1, \dots, n$ there are functions $f_k : \mathbb{S}_k \rightarrow \mathbb{R}$ such that $f_k(x_k)$ is the cost for generator k to produce at power level x_k for the time period T .
- We will consider the properties of f_k by first considering the **average cost per unit of production** $f_k(x_k)/x_k$.

Average production costs

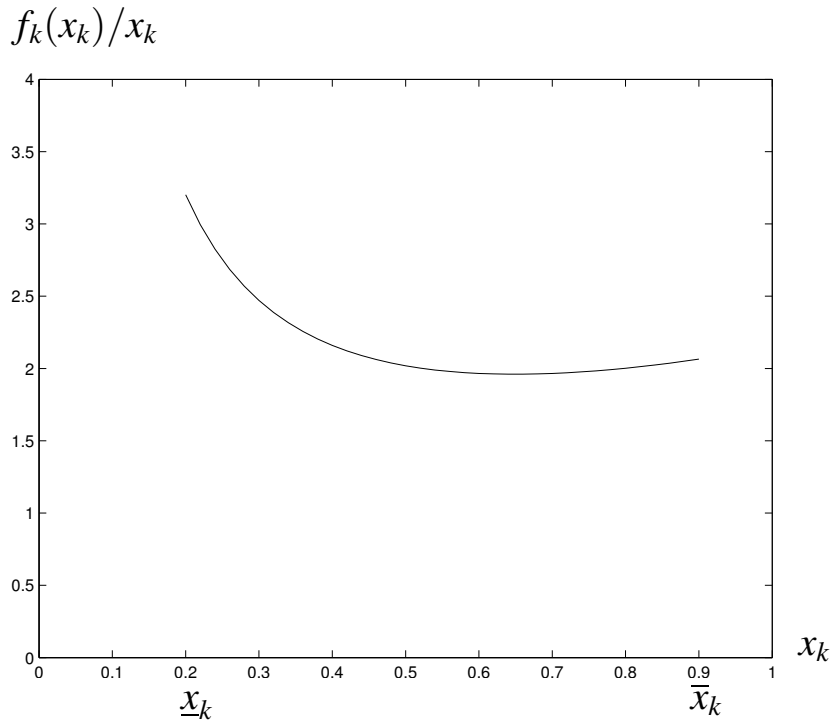


Fig. 3.1. The average production cost $f_k(x_k)/x_k$ versus production x_k for a typical generator for $\underline{x}_k \leq x_k \leq \bar{x}_k$.

Average production costs, continued

- At low levels of production, we would expect the average production cost to be relatively high.
- This is because there are usually “auxiliary” costs that must be incurred whenever the plant is in-service and producing non-zero levels of output.
- As x_k increases from low levels, the average production costs typically decrease because the costs of operating the auxiliary equipment are averaged over a greater amount of production.
- For some x_k , the average costs $f_k(x_k)/x_k$ reach a minimum and then may begin to increase again for larger values of x_k .
- The point where $f_k(x_k)/x_k$ is at a minimum is the point of maximum efficiency of the generator.

Production costs

- If we multiply the values of $f_k(x_k)/x_k$ in Figure 3.1 by x_k , we obtain the production costs $f_k(x_k)$ as illustrated in Figure 3.2.

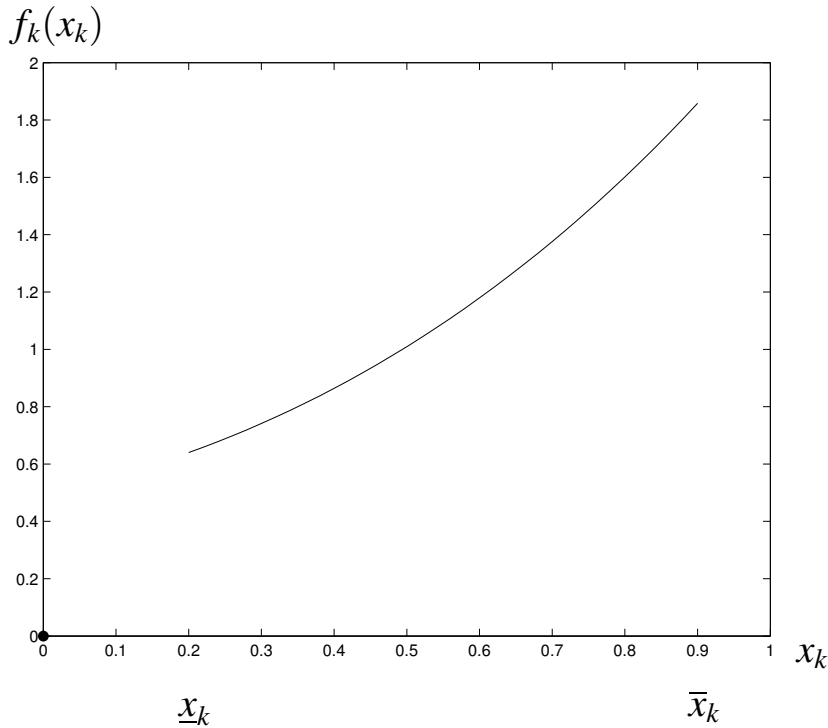


Fig. 3.2. Production cost $f_k(x_k)$ versus production x_k for a typical generator.

Production costs, continued

- Extrapolating the shape of f_k from \underline{x}_k to values $x_k < \underline{x}_k$ we find that at $x_k = 0$ the extrapolated value of the production cost function would be greater than zero due to the auxiliary operating costs.

Convexity

- Figure 3.2 suggests that f_k is convex on \mathbb{S}_k .
- It is often reasonable to assume that $f_k : \mathbb{S}_k \rightarrow \mathbb{R}$ is quadratic:

$$\forall x_k \in \mathbb{S}_k, f_k(x_k) = \frac{1}{2}Q_{kk}(x_k)^2 + c_k x_k + d_k. \quad (3.2)$$

- We will assume that the cost function of each generator has been extrapolated to a function that is convex on the *whole* of \mathbb{R} .
- For convex costs, $Q_{kk} \geq 0$.
- With non-zero auxiliary costs, $d_k > 0$.
- We also usually expect that $c_k > 0$.
- Note that the **marginal costs**, $\nabla f_k(x_k) = Q_{kk}x_k + c_k$, increase with x_k .

3.1.4 Objective

- We must consider the production costs of all generators combined.
- We want to minimize the objective $f : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by:

$$\forall x \in \mathbb{R}^n, f(x) = \sum_{k=1}^n f_k(x_k). \quad (3.3)$$

- Adding together the cost functions for all generators, we obtain:

$$\forall x \in \mathbb{R}^n, f(x) = \frac{1}{2}x^\dagger Qx + c^\dagger x + d,$$

- where $Q \in \mathbb{R}^{n \times n}$ is a diagonal matrix with k -th diagonal entry equal to Q_{kk} ,
- $c \in \mathbb{R}^n$ has k -th entry equal to c_k , and
- $d = \sum_{k=1}^n d_k \in \mathbb{R}$.

$$\begin{aligned} \forall x \in \mathbb{R}^n, \nabla f(x) &= \begin{bmatrix} \nabla f_1(x_1) \\ \vdots \\ \nabla f_n(x_n) \end{bmatrix}, \\ &= Qx + c. \end{aligned}$$

3.1.5 Power balance constraint

- Let us assume that during the time period of interest we face (an average) power demand of D .
- To meet demand, we must satisfy the constraint:

$$D = \sum_{k=1}^n x_k. \quad (3.4)$$

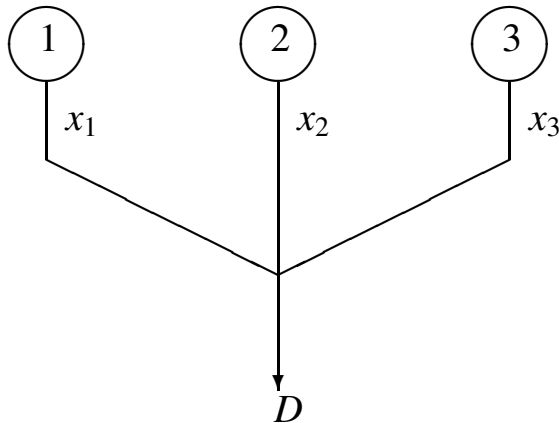


Fig. 3.3. Production from three generators.

3.1.6 Power balance constraint, continued

- We can write the power balance constraint in the form $Ax = b$ with either of the following two choices for $A \in \mathbb{R}^{1 \times n}$ and $b \in \mathbb{R}$:
 $A = \mathbf{1}^\dagger, b = [D]$, or
 $A = -\mathbf{1}^\dagger, b = [-D]$.
- For reasons that will be made clear when we discuss an economic interpretation of the problem, we prefer to use the second choice for A and b .

3.1.7 Generator and power balance constraints combined

- The feasible operating set for all the generators is: $(\prod_{k=1}^n S_k) \subset \mathbb{R}^n$, where the symbol \prod means the **Cartesian product**, so that the feasible set for the problem is:

$$\begin{aligned} S &= \left(\prod_{k=1}^n S_k \right) \cap \{x \in \mathbb{R}^n | Ax = b\}, \\ &= \{x \in \mathbb{R}^n | Ax = b, \underline{x} \leq x \leq \bar{x}\}, \end{aligned}$$

- where $\underline{x} \in \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$ are constant vectors with k -th entries \underline{x}_k and \bar{x}_k , respectively.

3.1.8 Problem

$$\min_{x \in \mathbb{R}^n} \{f(x) | Ax = b, \underline{x} \leq x \leq \bar{x}\} = \min_{\forall k, x_k \in S_k} \{f(x) | Ax = b\}. \quad (3.5)$$

3.2 Changes in demand and capacity

- We may want to estimate the change in the costs due to a change in demand from D to $D + \Delta D$, say.
- If the capacity of a generator k changes or it fails then the corresponding entries \bar{x}_k and \underline{x}_k of \bar{x} and \underline{x} will change.

3.3 Problem characteristics

3.3.1 Objective

- For typical cost functions, f_k is convex on $[\underline{x}_k, \bar{x}_k]$.
- Therefore, f is convex.

3.3.2 Equality constraints

- The equality constraint $D = \sum_{k=1}^n x_k$ is linear.

3.3.3 Inequality constraints and the feasible region

- The intersection of the box with the equality constraint restricts the feasible region to being a planar slice through the box.
- This is illustrated in Figure 3.4 for $n = 3$, $D = 10$, and:

$$\underline{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \bar{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

Inequality constraints and the feasible region, continued

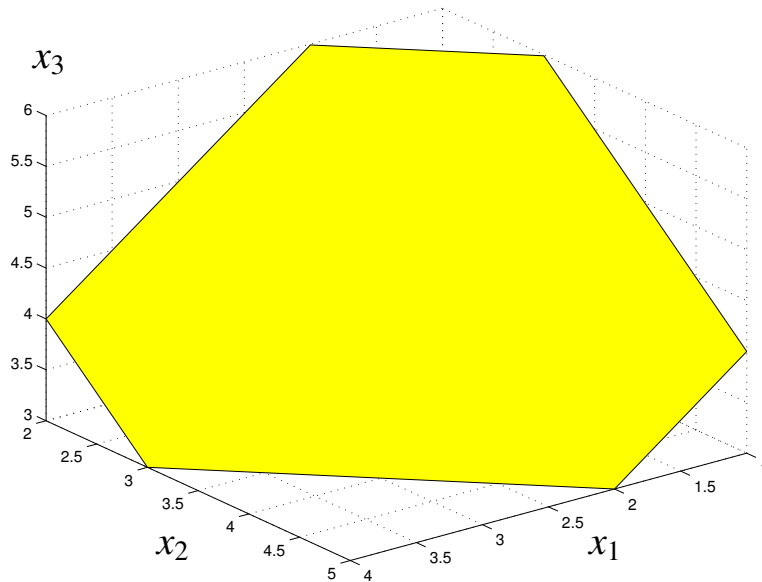


Fig. 3.4. Feasible set for economic dispatch example.

3.3.4 Solvability

- Problem (3.5) is convex.
- It is possible for there to be no feasible points for Problem (3.5).

3.4 Optimality conditions

3.4.1 First-order necessary conditions

- Assuming that there is a minimizer $x^* \in \mathbb{R}^n$, the first-order necessary conditions are that:

$$\begin{aligned}\exists \lambda^* \in \mathbb{R}, \exists \underline{\mu}^*, \bar{\mu}^* \in \mathbb{R}^n \text{ such that: } & \nabla f(x^*) - \mathbf{1}\lambda^* - \underline{\mu}^* + \bar{\mu}^* = \mathbf{0}; \\ & \underline{M}^*(\underline{x} - x^*) = \mathbf{0}; \\ & \bar{M}^*(x^* - \bar{x}) = \mathbf{0}; \\ & -\mathbf{1}^\dagger x^* = [-D]; \\ & x^* \geq \underline{x}; \\ & x^* \leq \bar{x}; \\ & \underline{\mu}^* \geq \mathbf{0}; \text{ and} \\ & \bar{\mu}^* \geq \mathbf{0},\end{aligned}$$

- where $\underline{M}^* = \text{diag}\{\underline{\mu}^*\} \in \mathbb{R}^{n \times n}$ and $\bar{M}^* = \text{diag}\{\bar{\mu}^*\} \in \mathbb{R}^{n \times n}$ are diagonal matrices with entries specified by the entries of $\underline{\mu}^*$ and $\bar{\mu}^*$, respectively.
- These first-order necessary conditions involve the marginal costs ∇f_k .

First-order necessary conditions, continued

- If the generator capacity constraints are not binding then:

$\underline{\mu}^* = \bar{\mu}^* = \mathbf{0}$ and the first and fourth lines of the first-order necessary conditions become:

$$\begin{aligned}\exists \lambda^* \in \mathbb{R}, \text{ such that: } \nabla f(x^*) - \mathbf{1}\lambda^* &= \mathbf{0}; \\ -\mathbf{1}^\dagger x^* &= [-D].\end{aligned}$$

That is, under economic dispatch, the marginal costs for each generator are equalized (and all equal to λ^*) and total generation equals demand.

To interpret, note that if $\nabla f_k(x_k) \neq \nabla f_\ell(x_\ell)$, we could improve dispatch by “backing off” the generator with higher marginal cost and increasing generation at the generator with lower marginal cost.

- If a generator maximum production capacity constraint is binding then its marginal cost is less than λ^* .
- If a generator minimum production capacity constraint is binding then its marginal cost is greater than λ^* .

3.4.2 Sensitivity

- By the sensitivity theorem, the Lagrange multiplier λ^* equals the sensitivity of the total costs to changes in demand.
- Each Lagrange multiplier $\underline{\mu}_k^*$ equals the sensitivity of the total costs to changes in the corresponding minimum capacity of generator k .
- Each Lagrange multiplier $\overline{\mu}_k^*$ equals the sensitivity of the total costs to changes in the corresponding maximum capacity of generator k .

3.4.3 Algorithms

3.4.3.1 Lagrange multipliers

- The first-order necessary conditions are:

$$\forall k = 1, \dots, n, \frac{df_k}{dx_k}(x_k^*) - \lambda^* = 0,$$

$$D - \sum_{k=1}^n x_k^* = 0.$$

- If f_k is quadratic then the marginal costs are linear and these equations are linear.
- If the marginal costs are non-linear then these equations are non-linear and can be solved using the Newton–Raphson update.
- If each f_k is strictly convex then there will be a unique minimizer.

3.4.3.2 Dual maximization

$$\begin{aligned}\forall k = 1, \dots, n, x_k^{(v)} &\in \operatorname{argmin}_{x_k \in \mathbb{R}} \{f_k(x_k) - \lambda^{(v)} x_k\}, & (3.6) \\ \Delta \lambda^{(v)} &= Ax^{(v)} - b, \\ &= D - \sum_{k=1}^n x_k^{(v)}, \\ \lambda^{(v+1)} &= \lambda^{(v)} + \alpha^{(v)} \Delta \lambda^{(v)}.\end{aligned}$$

- If f_k is quadratic then, at each iteration v , the k -th sub-problem on the right-hand side of (3.6) can be solved directly in one step by solving the linear necessary conditions.
- If f_k is not quadratic then (3.6) can be solved by applying the Newton–Raphson update until a value of $x_k^{(v)}$ is obtained that satisfies the necessary conditions to within a tolerance.
- That is, if f_k is non-quadratic, then at each outer iteration v and for each k we must perform several inner iterations to solve the necessary conditions of (3.6).

3.4.4 Discussion

- Maximizing the dual has a suggestive economic interpretation if we think of λ as the price paid for producing the commodity.
- The values $\lambda^{(v)}$ are tentative prices that are suggested at each iteration by a central agent.
- The goal of the central agent is to pick prices such that supply matches demand.
- The Lagrange multiplier λ^* is the final price that matches supply to demand.
- Each cost function f_k is associated with a decision-making agent that makes decisions based on:
 - its own cost function, and
 - the tentative prices.

Discussion, continued

- Each decision-making agent sells a quantity of product x_k to maximize its profits, which is equivalent to minimizing the difference between:
 - the *cost* of production $f_k(x_k)$ for the quantity x_k , minus
 - the *revenues* $x_k\lambda^{(v)}$, based on the current value of the dual variable, $\lambda^{(v)}$.
- The solution of (3.6) maximizes the agent's *profit*, that is, revenues minus costs, for the given value of the dual variable.
- At each iteration, the central agent adjusts the tentative prices based on comparing the sum of offered productions by the agents to the target value D :
 - price is raised or lowered to encourage or discourage production.
- At the optimum, the “marginal cost of production” for each agent, that is, the derivative of its cost function, is the same for all agents.
- The value of the Lagrange multiplier is sometimes called the **shadow price**.

3.5 Examples

3.5.1 Capacity constraints not binding

- Suppose that $n = 3$, with quadratic costs:

$$\forall x_1 \in \mathbb{R}_+, f_1(x_1) = (x_1)^2 \times 0.5 \text{ \$/(\text{MW})}^2\text{h},$$

$$\forall x_2 \in \mathbb{R}_+, f_2(x_2) = (x_2)^2 \times 1 \text{ \$/(\text{MW})}^2\text{h},$$

$$\forall x_3 \in \mathbb{R}_+, f_3(x_3) = (x_3)^2 \times 1.5 \text{ \$/(\text{MW})}^2\text{h}.$$

- That is, the marginal costs are assumed to be linear:

$$\forall x_1 \in [0, 1500], \nabla f_1(x_1) = x_1 \times 1 \text{ \$/(\text{MW})}^2\text{h},$$

$$\forall x_2 \in [0, 1000], \nabla f_2(x_2) = x_2 \times 2 \text{ \$/(\text{MW})}^2\text{h},$$

$$\forall x_3 \in [0, 1500], \nabla f_3(x_3) = x_3 \times 3 \text{ \$/(\text{MW})}^2\text{h}.$$

- Let $D = 11$ MW.
- We claim that the minimizer of this economic dispatch problem is $x_1^* = 6$, $x_2^* = 3$, and $x_3^* = 2$.

Capacity constraints not binding, continued

- The optimality conditions are:

$$\begin{aligned}\exists \lambda^* \in \mathbb{R}, \exists \underline{\mu}^*, \bar{\mu}^* \in \mathbb{R}^n \text{ such that: } \nabla f(x^*) - \mathbf{1}\lambda^* - \underline{\mu}^* + \bar{\mu}^* &= \mathbf{0}; \\ \underline{M}^*(\underline{x} - x^*) &= \mathbf{0}; \\ \bar{M}^*(x^* - \bar{x}) &= \mathbf{0}; \\ -\mathbf{1}^\dagger x^* &= [-D]; \\ x^* &\geq \underline{x}; \\ x^* &\leq \bar{x}; \\ \underline{\mu}^* &\geq \mathbf{0}; \text{ and} \\ \bar{\mu}^* &\geq \mathbf{0},\end{aligned}$$

- We can find the Lagrange multipliers by observing that none of the generators are at their minimum or maximum limits at the claimed solution.

Capacity constraints not binding, continued

- We claim that:

$$\lambda^* = 6\$/MWh,$$

- and all other Lagrange multipliers have value zero.
- Substituting into the first line of the first-order necessary conditions:

$$\begin{aligned}\nabla f_1(x_1^*) - \lambda^* &= 6 \times 1 - 6, \\ &= 0,\end{aligned}$$

$$\begin{aligned}\nabla f_2(x_2^*) - \lambda^* &= 3 \times 2 - 6, \\ &= 0,\end{aligned}$$

$$\begin{aligned}\nabla f_3(x_3^*) - \lambda^* &= 2 \times 3 - 6, \\ &= 0.\end{aligned}$$

- The other lines of the first-order necessary conditions are also satisfied.

Capacity constraints not binding, continued

- The sensitivity of total costs to changes in demand is $\lambda^* = 6\$/MWh$.
- The sensitivity of total costs to changes in the capacities is zero.

3.5.2 Capacity constraints binding

- Suppose that $n = 3$, with:

$$\forall x_1 \in [0, 1500], f_1(x_1) = x_1 \times 40\$/\text{MWh},$$

$$\forall x_2 \in [0, 1000], f_2(x_2) = x_2 \times 20\$/\text{MWh},$$

$$\forall x_3 \in [0, 1500], f_3(x_3) = x_3 \times 50\$/\text{MWh}.$$

- That is, the marginal costs are assumed constant for each machine over their feasible production sets:

$$\forall x_1 \in [0, 1500], \nabla f_1(x_1) = 40\$/\text{MWh},$$

$$\forall x_2 \in [0, 1000], \nabla f_2(x_2) = 20\$/\text{MWh},$$

$$\forall x_3 \in [0, 1500], \nabla f_3(x_3) = 50\$/\text{MWh}.$$

- Let $D = 3000$ MW.
- We claim that the minimizer of this economic dispatch problem is $x_1^* = 1500$, $x_2^* = 1000$, and $x_3^* = 500$.

Capacity constraints binding, continued

- The optimality conditions are:

$$\begin{aligned}\exists \lambda^* \in \mathbb{R}, \exists \underline{\mu}^*, \bar{\mu}^* \in \mathbb{R}^n \text{ such that: } \nabla f(x^*) - \mathbf{1}\lambda^* - \underline{\mu}^* + \bar{\mu}^* &= \mathbf{0}; \\ \underline{M}^*(\underline{x} - x^*) &= \mathbf{0}; \\ \bar{M}^*(x^* - \bar{x}) &= \mathbf{0}; \\ -\mathbf{1}^\dagger x^* &= [-D]; \\ x^* &\geq \underline{x}; \\ x^* &\leq \bar{x}; \\ \underline{\mu}^* &\geq \mathbf{0}; \text{ and} \\ \bar{\mu}^* &\geq \mathbf{0},\end{aligned}$$

- We can find the Lagrange multipliers by observing that only generator 3 is not at its minimum or maximum limits at the claimed solution.

Capacity constraints binding, continued

- We claim that:

$$\lambda^* = 50\$/MWh,$$

$$\underline{\mu}_1^* = 50 - 40 = 10\$/MWh,$$

$$\underline{\mu}_2^* = 50 - 20 = 30\$/MWh,$$

- and all other Lagrange multipliers have value zero.
- Substituting into the first line of the first-order necessary conditions:

$$\begin{aligned}\nabla f_1(x_1^*) - \lambda^* - \underline{\mu}_1^* + \bar{\mu}_1^* &= 40 - 50 - 0 + 10, \\ &= 0,\end{aligned}$$

$$\begin{aligned}\nabla f_2(x_2^*) - \lambda^* - \underline{\mu}_2^* + \bar{\mu}_2^* &= 20 - 50 - 0 + 30, \\ &= 0,\end{aligned}$$

$$\begin{aligned}\nabla f_3(x_3^*) - \lambda^* - \underline{\mu}_3^* + \bar{\mu}_3^* &= 50 - 50 - 0 + 0, \\ &= 0.\end{aligned}$$

- The other lines of the first-order necessary conditions are also satisfied.

Capacity constraints binding, continued

- The sensitivity of total costs to changes in demand is $\lambda^* = 50\$/\text{MWh}$.
- The sensitivity of total costs to changes in the maximum capacity of generator 1 is $\bar{\mu}_1^* = 10(\$/\text{h})/\text{MW}$.
- The sensitivity of total costs to changes in the maximum capacity of generator 2 is $\bar{\mu}_2^* = 30(\$/\text{h})/\text{MW}$.
- The sensitivity of total costs to changes in other capacities is zero.

3.6 Linear programming approximation

- Typical generator cost functions are non-linear, for example, quadratic.
- To use linear programming software to solve economic dispatch, we need to approximate the generator costs function.
- A typical approximation is to **piece-wise linearize** the function.
- This approximates the marginal costs as being piece-wise constant.

3.6.1 Piece-wise linearization

- For a function $f : [0, 1] \rightarrow \mathbb{R}$ we might:
 - define subsidiary variables ξ_1, \dots, ξ_5 ,
 - include constraints:

$$x = \sum_{j=1}^5 \xi_j,$$
$$0 \leq \xi_j \leq 0.2,$$

- define parameters:

$$d = f(0),$$
$$c_j = \frac{1}{0.2} [f(0.2 \times j) - f(0.2 \times (j - 1))], j = 1, \dots, 5,$$

and

- replace the objective f by the piece-wise linearized objective $\phi : \mathbb{R}^5 \rightarrow \mathbb{R}$ defined by:

$$\forall \xi \in \mathbb{R}^5, \phi(\xi) = c^\dagger \xi + d.$$

3.6.2 Quadratic example function

$$\forall x \in [0, 1], f(x) = (x)^2.$$

$f(x), \phi(\xi)$

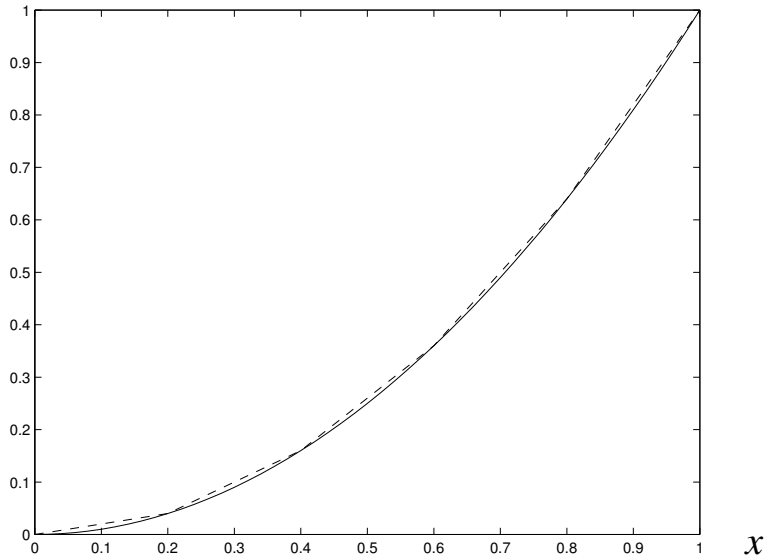


Fig. 3.5. Piece-wise linearization (shown dashed) of a function (shown solid).

Quadratic example function, continued

- For the function f illustrated in Figure 3.5:

$$\begin{aligned}d &= f(0), \\ &= 0, \\ c_j &= \frac{1}{0.2} (f(0.2 \times j) - f(0.2 \times (j - 1))), \\ &= (0.4 \times j) - 0.2, j = 1, \dots, 5.\end{aligned}$$

- To piece-wise linearize f in an optimization problem, we use ϕ as the objective instead of f , augment the decision vector to include ξ , and include the constraints that link ξ and x .
- Similarly, non-linear constraints can also be piece-wise linearized.

3.7 Summary

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- (v) Algorithms,
- (vi) Discussion,
- (vii) Examples,
- (viii) Linear programming approximation.

This chapter is based on:

- Sections 12.1, 13.5, and 15.1 of *Applied Optimization: Formulation and Algorithms for Engineering Systems*, Cambridge University Press 2006,
- Daniel S. Kirschen and Goran Strbac, *Power System Economics*, Wiley, 2004.

Homework exercise: Due Wednesday, October 8

- Consider economic dispatch Problem (3.5) in the case that $n = 3$, $D = 5$,

$$\underline{x} = \mathbf{0}, \bar{x} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}, \text{ and the } f_k \text{ are of the form:}$$

$$\forall x_1 \in \mathbb{R}, f_1(x_1) = \frac{1}{2}(x_1)^2 + x_1,$$

$$\forall x_2 \in \mathbb{R}, f_2(x_2) = \frac{1}{2} \times 1.1(x_2)^2 + 0.9x_2,$$

$$\forall x_3 \in \mathbb{R}, f_3(x_3) = \frac{1}{2} \times 1.2(x_3)^2 + 0.8x_3.$$

- Solve the economic dispatch problem by solving the first-order necessary conditions in terms of the minimizer x^* and the Lagrange multipliers λ^* , $\underline{\mu}^*$, and $\bar{\mu}^*$.
- (Hint: Because the minimum capacities are low enough and because the maximum capacities are large enough, none of the minimum and maximum capacity constraints will be binding. By complementary slackness, what can you say about $\underline{\mu}^*$ and $\bar{\mu}^*$?)