Course notes for EE394V
Restructured Electricity Markets:
Locational Marginal Pricing

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Linearized power flow

(i) Discussion of Newton–Raphson update,
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1.1 Discussion of Newton–Raphson update

- In principle, the Newton–Raphson update is repeated until a suitable **stopping criterion** is satisfied.
- Approximations and variations have been developed due to:
  - the computational effort of performing multiple iterations, and
  - the potential that the iterates fail to form a convergent sequence.
- One variation is to perform just *one* Newton–Raphson update starting from a suitable initial guess to obtain an approximate answer.
- We will develop this variation because it:
  - is used in many market models, and
  - sheds light on decomposition approaches even when the non-linear equations are being solved more accurately.

1.2 Re-formulation of power flow problem

- We formulated the power flow problem as the solution of non-linear simultaneous equations:

$$g(x) = \mathbf{0},$$

- where the entries of g were functions $p_\ell, q_\ell : \mathbb{R}^n \rightarrow \mathbb{R}$ of the form:

$$\begin{aligned}
\forall x \in \mathbb{R}^n, p_\ell(x) &= \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} u_\ell u_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)] - P_\ell, \\
\forall x \in \mathbb{R}^n, q_\ell(x) &= \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} u_\ell u_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)] - Q_\ell.
\end{aligned}$$

- The vector g includes p_ℓ and q_ℓ for each bus except the reference bus.
- If $x^\star$ can be found that satisfies $g(x^\star) = \mathbf{0}$ then the real and reactive power generation at the reference bus can be calculated according to:

$$\begin{aligned}
P_1 &= \tilde{p}_1(x^\star), \\
Q_1 &= \tilde{q}_1(x^\star),
\end{aligned}$$

- where we have assumed that bus 1 is the reference bus.

Re-formulation of power flow problem, continued

- For example, for a three bus system with buses $\ell = 1, 2, 3$ and bus 1 the reference bus, the entries of $x \in \mathbb{R}^4$ and $g : \mathbb{R}^4 \to \mathbb{R}^4$ would be:

$$\begin{aligned} x &= \begin{bmatrix} \theta_2 \\ \theta_3 \\ u_2 \\ u_3 \end{bmatrix}, \\ \forall x \in \mathbb{R}^4, g(x) &= \begin{bmatrix} p_2(x) \\ p_3(x) \\ q_2(x) \\ q_3(x) \end{bmatrix}, \end{aligned}$$

- where P_2 is the net generation (generation minus demand) at bus 2, and similarly for other buses and for the reactive power at the buses.
- If we solve $g(x) = \mathbf{0}$, we can then use the resulting value of x to evaluate the real power and reactive power that must be produced at the reference bus to satisfy Kirchhoff's laws.

Re-formulation of power flow problem, continued

- To explore linearization, we will re-formulate the equations to make the dependence on the real and reactive power generation more explicit.
- Define new functions by omitting the values of the net real and reactive generation, P_ℓ and Q_ℓ, from the definitions of the functions p_ℓ and q_ℓ, respectively.
- That is, define $\tilde{p}_\ell : \mathbb{R}^n \to \mathbb{R}$ and $\tilde{q}_\ell : \mathbb{R}^n \to \mathbb{R}$ to be:

$$\begin{aligned}
\forall x \in \mathbb{R}^n, \tilde{p}_\ell(x) &= \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} u_\ell u_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)], \\
\forall x \in \mathbb{R}^n, \tilde{q}_\ell(x) &= \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} u_\ell u_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)].
\end{aligned}$$

Re-formulation of power flow problem, continued

- Collect the entries $\tilde{p}_\ell$ for all buses together into a vector function $\tilde{p} : \mathbb{R}^n \rightarrow \mathbb{R}^{n_{PQ}+1}$ and collect the entries $\tilde{q}_\ell$ for all buses together into a vector function $\tilde{q} : \mathbb{R}^n \rightarrow \mathbb{R}^{n_{PQ}+1}$.
- Let $\hat{p}$ and $\hat{q}$ be the sub-vectors of $\tilde{p}$ and $\tilde{q}$, respectively, that omit the reference bus.
- For example, if bus 1 is the reference bus then:

$$\begin{aligned} \tilde{p} &= \begin{bmatrix} \tilde{p}_1 \\ \hat{p} \end{bmatrix}, \\ \tilde{q} &= \begin{bmatrix} \tilde{q}_1 \\ \hat{q} \end{bmatrix}. \end{aligned}$$

- Collect the real power generation P_ℓ for all the buses together into a vector $P \in \mathbb{R}^{n_{PQ}+1}$ and collect the reactive power generation Q_ℓ for all buses together into a vector $Q \in \mathbb{R}^{n_{PQ}+1}$.
- Let $\hat{P}$ and $\hat{Q}$ be the sub-vectors of P and Q, respectively, that omit the reference bus.

Re-formulation of power flow problem, continued

- We now re-express $g(x) = \mathbf{0}$ in the equivalent form:

$$\begin{aligned} \hat{p}(x) &= \hat{P}, \\ \hat{q}(x) &= \hat{Q}. \end{aligned}$$

- For the three bus example, these vector equations are:

$$\begin{aligned} \begin{bmatrix} \tilde{p}_2(x) \\ \tilde{p}_3(x) \end{bmatrix} &= \begin{bmatrix} P_2 \\ P_3 \end{bmatrix}, \\ \begin{bmatrix} \tilde{q}_2(x) \\ \tilde{q}_3(x) \end{bmatrix} &= \begin{bmatrix} Q_2 \\ Q_3 \end{bmatrix}. \end{aligned}$$

- For future reference, note that if we find x that satisfies $\hat{p}(x) = \hat{P}$ then the real power generation at the reference bus can be evaluated as $P_1 = \tilde{p}_1(x)$, so x also satisfies $\tilde{p}(x) = P$ and, moreover:

$$\mathbf{1}^\dagger P = \mathbf{1}^\dagger \tilde{p}(x).$$

- This expression evaluates the total losses in the system, since it sums the total net real power injected into the transmission lines.

1.3 Linearized power flow

1.3.1 Base-case

- Suppose that we are given values of real and reactive generation $P^\star \in \mathbb{R}^{n_{PQ}}$ and $Q^\star \in \mathbb{R}^{n_{PQ}}$ that specify a **base-case**.
  - For example, the base-case real and reactive generations could be the current operating conditions.
  - As another example, $P^\star = P^{(0)} = \mathbf{0}$ is the (unrealistic) condition of zero net real power injection.
- Also suppose that we have a solution $x^\star$ to the base-case equations.
- That is, $g(x^\star) = \mathbf{0}$, or in our re-formulation:

$$\begin{aligned} \hat{p}(x^\star) &= \hat{P}^\star, \\ \hat{q}(x^\star) &= \hat{Q}^\star, \end{aligned}$$

- where $\hat{P}^\star$ and $\hat{Q}^\star$ are the sub-vectors of $Q^\star$ and $Q^\star$, respectively, that omit the reference bus.

1.3.2 *Change-case*

- Now suppose that the real and reactive power generations change:
  - from $P^\star$ and $Q^\star$,
  - to $P^\star + \Delta P$ and $Q^\star + \Delta Q$, respectively.
- Similarly, we suppose that the value of x changes from $x^\star$ to $x^\star + \Delta x$ to re-establish satisfaction of the power flow equations $g(x) = \mathbf{0}$.
- That is, the **change-case** power flow equations are given by:

$$\begin{aligned} \hat{p}(x^\star + \Delta x) &= \hat{P}^\star + \Delta \hat{P}, \\ \hat{q}(x^\star + \Delta x) &= \hat{Q}^\star + \Delta \hat{Q}, \end{aligned}$$

- where $\Delta \hat{P}$ and $\Delta \hat{Q}$ are the sub-vectors of ΔP and ΔQ, respectively, that omit the reference bus.
- The equations are non-linear equations in Δx.
- Note the change in net generation at the reference bus is required to be consistent with this change, so that we also have (assuming bus 1 is the reference bus):

$$\begin{aligned} \tilde{p}_1(x^\star + \Delta x) &= P_1^\star + \Delta P_1, \\ \tilde{q}_1(x^\star + \Delta x) &= Q_1^\star + \Delta Q_1. \end{aligned}$$

1.3.3 First-order Taylor approximation

- To find an approximate solution to the change-case equations, we form **first-order Taylor approximations** to $\hat{p}$ and $\hat{q}$:

$$\begin{aligned}
\hat{p}(x^\star + \Delta x) &\approx \hat{p}(x^\star) + \frac{\partial \hat{p}}{\partial x}(x^\star)\Delta x, \\
\hat{q}(x^\star + \Delta x) &\approx \hat{q}(x^\star) + \frac{\partial \hat{q}}{\partial x}(x^\star)\Delta x.
\end{aligned}$$

- For future reference, note that the matrices $\frac{\partial \hat{p}}{\partial x}(x^\star)$ and $\frac{\partial \hat{q}}{\partial x}(x^\star)$ form the Jacobian, $\widehat{J} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, of the system of equations $\hat{p}(x) = \hat{P}, \hat{q}(x) = \hat{Q}$, where $\widehat{J}$ is defined by:

$$\widehat{J} = \begin{bmatrix} \frac{\partial \hat{p}}{\partial x} \\ \frac{\partial \hat{q}}{\partial x} \end{bmatrix}.$$

1.3.4 Linearization of change-case equations

- Substituting the first-order Taylor approximations into the change-case equations, we obtain:

$$\begin{aligned}\hat{p}(x^\star)+\frac{\partial\hat{p}}{\partial x}(x^\star)\Delta x &\approx P^\star+\Delta P,\\ \hat{q}(x^\star)+\frac{\partial\hat{q}}{\partial x}(x^\star)\Delta x &\approx Q^\star+\Delta Q.\end{aligned}$$

- From the base-case solution, we have $\hat{p}(x^\star)=\hat{P}^\star$ and $\hat{q}(x^\star)=\hat{Q}^\star$.
- Ignoring the error in the first-order Taylor approximation, we have:

$$\begin{aligned}\frac{\partial\hat{p}}{\partial x}(x^\star)\Delta x &= \Delta\hat{P},\\ \frac{\partial\hat{q}}{\partial x}(x^\star)\Delta x &= \Delta\hat{Q}.\end{aligned}$$

Linearization of change-case equations, continued

- Typically, the Jacobian $\widehat{J}(x^{\star}) = \begin{bmatrix} \dfrac{\partial \hat{p}}{\partial x}(x^{\star}) \\ \dfrac{\partial \hat{q}}{\partial x}(x^{\star}) \end{bmatrix}$ is non-singular.
- That is, we can solve

$$\widehat{J}(x^{\star})\Delta x = \begin{bmatrix} \Delta \hat{P} \\ \Delta \hat{Q} \end{bmatrix},$$

- for Δx.
- These are **sparse** linear equations, which can be solved efficiently for Δx.
- This approximation to the solution of the change-case power flow equations is equivalent to performing one iteration of the Newton–Raphson method, starting at the base-case.

Linearization of change-case equations, continued

- Moreover, the change in real and reactive power at the reference bus will approximately satisfy:

$$\begin{aligned} \Delta P_1 &= \frac{\partial \tilde{p}_1}{\partial x}(x^\star)\Delta x, \\ \Delta Q_1 &= \frac{\partial \tilde{q}_1}{\partial x}(x^\star)\Delta x, \end{aligned}$$

- where we have assumed that bus 1 is the reference bus.

1.3.5 Jacobian

1.3.5.1 Terms

- The entries in $\tilde{p} : \mathbb{R}^n \rightarrow \mathbb{R}^{n_{PQ}+1}$ are defined by:

$$\forall x \in \mathbb{R}^n, \tilde{p}_\ell(x) = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} u_\ell u_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)].$$

- The entries in $\tilde{q} : \mathbb{R}^n \rightarrow \mathbb{R}^{n_{PQ}+1}$ are defined by: $q_\ell : \mathbb{R}^n \rightarrow \mathbb{R}$:

$$\forall x \in \mathbb{R}^n, \tilde{q}_\ell(x) = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} u_\ell u_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)].$$

- The entries in the vector x are either of the form θ_k or of the form u_k.
- To examine the terms in the Jacobian, partition x so that all the voltage angles appear first in a sub-vector θ followed by all the voltage magnitudes in a sub-vector u.

Terms, continued

- There are four qualitative types of partial derivative terms corresponding to each combination:

$$\forall x \in \mathbb{R}^n, \frac{\partial \tilde{p}_\ell}{\partial \theta_k}(x) = \begin{cases} \sum_{j \in \mathbb{J}(\ell)} u_\ell u_j [-G_{\ell j} \sin(\theta_\ell - \theta_j) + B_{\ell j} \cos(\theta_\ell - \theta_j)], & \text{if } k = \ell, \\ u_\ell u_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)], & \text{if } k \in \mathbb{J}(\ell), \\ 0, & \text{otherwise}, \end{cases}$$

$$\forall x \in \mathbb{R}^n, \frac{\partial \tilde{p}_\ell}{\partial u_k}(x) = \begin{cases} 2u_\ell G_{\ell\ell} + \sum_{j \in \mathbb{J}(\ell)} u_j [G_{\ell j} \cos(\theta_\ell - \theta_j) + B_{\ell j} \sin(\theta_\ell - \theta_j)], & \text{if } k = \ell, \\ u_\ell [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)], & \text{if } k \in \mathbb{J}(\ell), \\ 0, & \text{otherwise}, \end{cases}$$

Terms, continued

$$\forall x \in \mathbb{R}^n, \frac{\partial \tilde{q}_\ell}{\partial \theta_k}(x)$$

$$= \begin{cases} \sum_{j \in \mathbb{J}(\ell)} u_\ell u_j [G_{\ell j} \cos(\theta_\ell - \theta_j) + B_{\ell j} \sin(\theta_\ell - \theta_j)], & \text{if } k = \ell, \\ u_\ell u_k [-G_{\ell k} \cos(\theta_\ell - \theta_k) - B_{\ell k} \sin(\theta_\ell - \theta_k)], & \text{if } k \in \mathbb{J}(\ell), \\ 0, & \text{otherwise,} \end{cases}$$

$$\forall x \in \mathbb{R}^n, \frac{\partial \tilde{q}_\ell}{\partial u_k}(x)$$

$$= \begin{cases} -2u_\ell B_{\ell\ell} + \sum_{j \in \mathbb{J}(\ell)} u_j [G_{\ell j} \sin(\theta_\ell - \theta_j) - B_{\ell j} \cos(\theta_\ell - \theta_j)], & \text{if } k = \ell, \\ u_\ell [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)], & \text{if } k \in \mathbb{J}(\ell), \\ 0, & \text{otherwise.} \end{cases}$$

1.3.5.2 Partitioning by types of terms

- Based on the partitioning of x, we can partition the Jacobian into four blocks:

$$
\begin{aligned}
\forall x \in \mathbb{R}^n, \widehat{J}(x) &= \begin{bmatrix} \widehat{J}_{p\theta}(x) & \widehat{J}_{pu}(x) \\ \widehat{J}_{q\theta}(x) & \widehat{J}_{qu}(x) \end{bmatrix}, \\
\forall x \in \mathbb{R}^n, \widehat{J}_{p\theta}(x) &= \frac{\partial \tilde{p}}{\partial \theta}(x), \\
\forall x \in \mathbb{R}^n, \widehat{J}_{pu}(x) &= \frac{\partial \tilde{p}}{\partial u}(x), \\
\forall x \in \mathbb{R}^n, \widehat{J}_{q\theta}(x) &= \frac{\partial \tilde{q}}{\partial \theta}(x), \\
\forall x \in \mathbb{R}^n, \widehat{J}_{qu}(x) &= \frac{\partial \tilde{q}}{\partial u}(x).
\end{aligned}
$$

1.3.6 Decoupled equations

- Recall that for typical lines $\forall \ell, \forall k \in \mathbb{J}(\ell) \cup \{\ell\}, |G_{\ell k}| \ll |B_{\ell k}|$.
- This implies that the terms in the matrices $\widehat{J}_{pu}$ and $\widehat{J}_{q\theta}$ are small compared to the terms in the matrices $\widehat{J}_{p\theta}$ and $\widehat{J}_{qu}$.
- If we neglect all the terms in $\widehat{J}_{pu}$ and $\widehat{J}_{q\theta}$, then we can then approximate the Jacobian by $\widehat{J}(x) \approx \begin{bmatrix} \widehat{J}_{p\theta}(x) & \mathbf{0} \\ \mathbf{0} & \widehat{J}_{qu}(x) \end{bmatrix}$.
- Letting $\Delta x = \begin{bmatrix} \Delta\theta \\ \Delta u \end{bmatrix}$, this allows decoupling of the linearized equations into:

$$\begin{aligned} \frac{\partial \hat{p}}{\partial \theta}(x^\star)\Delta\theta &= \Delta\hat{P}, \\ \frac{\partial \hat{q}}{\partial u}(x^\star)\Delta u &= \Delta\hat{Q}, \end{aligned}$$

- These decoupled equations require less computation than solving the full system.

1.4 Fixed voltage schedule

- If **real power** generations and flows are our main concern and there is adequate **voltage support** in the form of controllable reactive sources then we may be justified in assuming that the voltage magnitudes can be held fixed by controlling reactive power:
  - A typical assumption is that all voltage magnitudes are 1 per unit, $u = \mathbf{1}$.
  - More generally, any fixed voltage schedule $u^{(0)}$ can be used.
- In this case, we can assume that all buses, except the reference bus, are *PV* buses:
  - The unknowns are voltage angles and reactive power generations.
  - We first solve $\hat{p}\left(\begin{bmatrix} \theta^\star \\ u^{(0)} \end{bmatrix}\right) = \hat{P}^\star$ for $\theta^\star$, given the fixed voltage schedule $u^{(0)}$.
  - To complete the solution, reactive generations are chosen to satisfy $Q^\star = \tilde{q}\left(\begin{bmatrix} \theta^\star \\ u^{(0)} \end{bmatrix}\right)$, specifying the reactive power at all the buses, including the reference bus, in order to achieve the voltage schedule $u^{(0)}$.

1.5 DC power flow

- We combine the ideas of fixed voltage profile and linearization.

1.5.1 Fixed voltage schedule

- We again assume that there are controllable voltage sources available to provide a fixed voltage schedule $u^{(0)}$.
- Based on the analysis in the previous section, we must in principle solve $\tilde{p}\left(\begin{bmatrix}\theta \\ u^{(0)}\end{bmatrix}\right) = P^{\star}$, which we can do by first solving $\hat{p}\left(\begin{bmatrix}\theta^{\star} \\ u^{(0)}\end{bmatrix}\right) = \hat{P}^{\star}$ for $\theta^{\star}$ and then evaluating $P_1 = \tilde{p}_1\left(\begin{bmatrix}\theta^{\star} \\ u^{(0)}\end{bmatrix}\right)$.
- This again enables us to focus on real power generation and angles.
- However, instead of solving $\hat{p}\left(\begin{bmatrix}\theta^{\star} \\ u^{(0)}\end{bmatrix}\right) = \hat{P}^{\star}$ exactly for $\theta^{\star}$, we solve a linearized version.

1.5.2 Linearization

- In DC power flow, we linearize about a *fixed* base-case solution, $x^{(0)} = \begin{bmatrix} \theta^{(0)} \\ u^{(0)} \end{bmatrix}$.
- The base-case power generations $P^{(0)}$ that determine the base-case solution are chosen to be convenient for calculations.
- A typical base-case involves:
  - zero net real power generation at all buses, so that $P^{(0)} = \mathbf{0}$, and
  - all voltage magnitudes 1 per unit, so that $u^{(0)} = \mathbf{1}$.
- If the transmission lines have zero real values for their shunt elements then $\theta^{(0)} = \mathbf{0}$ solves the base-case.
- $x^{(0)} = \begin{bmatrix} \theta^{(0)} \\ u^{(0)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ is called a **flat start**.

Linearization, continued

- Linearizing about the flat start condition, we approximate the solution $\theta = \theta^{(0)} + \Delta\theta = \mathbf{0} + \Delta\theta = \Delta\theta$ corresponding to injections $P = P^{(0)} + \Delta P = \mathbf{0} + \Delta P = \Delta P$.
- We now interpret:

  $P^{(0)} + \Delta P = \Delta P = P$ to be the power generation for the change-case we are trying to solve, and

  $\theta^{(0)} + \Delta\theta = \Delta\theta = \theta$ to be the solution for the angles for the change-case we are trying to solve.
- We solve the linearized power flow equations: $\frac{\partial \hat{p}}{\partial \theta}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)\theta = \hat{P}$,
- where $\frac{\partial \hat{p}}{\partial \theta}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)$ is a constant matrix,
- θ is the vector of unknown angles, and
- where $\hat{P}$ is the sub-vector of P that omits the reference bus.

Linearization, continued

- These equations are in the form $\widehat{J}_{p\theta}^{(0)}\theta = \hat{P}$, where:

$$\widehat{J}_{p\theta}^{(0)} = \widehat{J}_{p\theta}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right) = \frac{\partial \hat{p}}{\partial \theta}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right).$$

- These are again sparse linear equations, which can be solved efficiently for θ.
- Paralleling the earlier observation, this approximation to the solution of the power flow equations for power generation $\hat{P}$ is equivalent to performing one iteration of the Newton–Raphson method, starting at a flat start.

Linearization, continued

- Moreover, the real power at the reference bus for the change-case can be estimated by:

$$P_1 = \frac{\partial \tilde{p}_1}{\partial \theta}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)\theta.$$

- We will see that the reference bus real power estimation can be simplified under certain assumptions on the base-case system.

1.5.3 Interpretation

- We have interpreted the approximation as equivalent to performing one iteration of the Newton–Raphson method, starting at a flat start.
- This differs from the “traditional” interpretation of DC power flow that emphasizes:
  - the small angle approximations for cos and sin, and
  - the solution of DC power flow being the same as the solution of an analogous DC circuit with current sources specified by the power injections and voltages specified by the angles.
- Our interpretation provides a clearer and more general perspective on the conditions when the DC power flow provides a good approximation:
  - see homework.
- It also provides a connection to **decomposition** algorithms.

1.5.4 Terms in Jacobian

- The entries for the sub-matrix $\widehat{J}^{(0)}_{p\theta} = \frac{\partial \hat{p}}{\partial \theta}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)$ of the Jacobian are:

$$\frac{\partial \tilde{p}_\ell}{\partial \theta_k}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right) = \begin{cases} \sum_{j \in \mathbb{J}(\ell)} B_{\ell j}, & \text{if } k = \ell, \\ -B_{\ell k}, & \text{if } k \in \mathbb{J}(\ell), \\ 0, & \text{otherwise,} \end{cases}$$

- Note that these entries correspond to the imaginary part of the admittance matrix, *not* to the inverse of the line inductive reactances.

1.5.5 *Losses*

- If the transmission lines have zero real part for their shunt elements then:

$$
\begin{aligned}
\forall k, \sum_{\ell} \frac{\partial p_\ell}{\partial \theta_k}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right) &= \frac{\partial p_k}{\partial \theta_k}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right) + \sum_{\ell \neq k} \frac{\partial p_\ell}{\partial \theta_k}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right), \\
& \quad \text{where the summation includes the reference bus,} \\
&= \sum_{j \in \mathbb{J}(k)} B_{kj} - \sum_{\ell \in \mathbb{J}(k)} B_{\ell k}, \\
&= \sum_{j \in \mathbb{J}(k)} B_{kj} - \sum_{\ell \in \mathbb{J}(k)} B_{k\ell}, \text{ since } B_{\ell k} = B_{k\ell}, \\
&= 0.
\end{aligned}
$$

- That is, each column of $\frac{\partial \tilde{p}}{\partial \theta}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$ sums to zero.
- Equivalently, $\mathbf{1}^\dagger \frac{\partial \tilde{p}}{\partial \theta}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right) = \mathbf{0}$.

Losses, continued

- Moreover, the losses in the system are given by:

$$\begin{aligned}
\mathbf{1}^{\dagger}P &= \mathbf{1}^{\dagger}\begin{bmatrix} P_1 \\ \hat{P} \end{bmatrix}, \\
&= \mathbf{1}^{\dagger}\begin{bmatrix} \frac{\partial \tilde{p}_1}{\partial \theta}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)\theta \\ \\ \frac{\partial \hat{p}}{\partial \theta}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)\theta \end{bmatrix}, \\
&= \mathbf{1}^{\dagger}\frac{\partial \tilde{p}}{\partial \theta}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)\theta, \\
&= \mathbf{0}\theta, \text{ from the previous page,} \\
&= 0,
\end{aligned}$$

- so that the linearized representation is lossless, given the assumption of the flat start as base-case and that the transmission lines have zero real part for their shunt elements.

1.5.6 Inverting the power flow equations

- In the usual case that the matrix $\widehat{J}^{(0)}_{p\theta}$ is non-singular, we can write:

$$\theta = \left[\widehat{J}^{(0)}_{p\theta}\right]^{-1} \hat{P},$$

- allowing us to eliminate θ.
- (If $\widehat{J}^{(0)}_{p\theta}$ is singular, then the system is at a steady-state stability limit.)
- Given a specification of $\hat{P}$, we can calculate the net generation at the reference bus through:

$$\begin{aligned} 0 &= -\mathbf{1}^{\dagger} P, \\ &= -P_1 - \mathbf{1}^{\dagger} \hat{P}. \end{aligned}$$

- Equivalently, $P_1 = -\mathbf{1}^{\dagger}\hat{P}$, so that the reference bus exactly compensates for the net demand or withdrawal summed across all other buses, since the approximation is lossless.

1.5.7 Demand

- So far, the vector P has represented the vector of *net* injections at the buses.
- In some formulations, we want to consider demand and generation separately.
- For example, if the net injection is $P - D$, where:

  P is now the vector of generations, and
  D is the vector of demands,

- then the power flow equations become:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}D, \\ \theta &= \left[\widehat{J}_{p\theta}^{(0)}\right]^{-1}(\hat{P} - \hat{D}), \end{aligned}$$

- where $\hat{P}$ and $\hat{D}$ are the sub-vectors of P and D, respectively, that omit the reference bus.

1.5.8 Line flow

- We typically use the results of power flow to evaluate whether the flow along a line is within limits.
- For a line joining bus ℓ to bus k, if we ignore shunt elements in the models, then the real and reactive power flows $p_{\ell k}$ and $q_{\ell k}$ along the line are given by:

$$\begin{aligned} \forall x \in \mathbb{R}^n, p_{\ell k}(x) &= u_\ell u_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)] - (u_\ell)^2 G_{\ell k}, \\ \forall x \in \mathbb{R}^n, q_{\ell k}(x) &= u_\ell u_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)] + (u_\ell)^2 B_{\ell k}. \end{aligned}$$

- We will approximate these expressions by again linearizing about a base-case.

1.5.9 Linearized Line flow

- Linearizing the expressions for $p_{\ell k}$ and $q_{\ell k}$ about $\theta^{(0)}$, we obtain:

$$\begin{aligned}
& p_{\ell k}\left(\begin{bmatrix}\theta^{(0)}+\Delta\theta \\ u^{(0)}\end{bmatrix}\right) \\
\approx \; & p_{\ell k}\left(\begin{bmatrix}\theta^{(0)} \\ u^{(0)}\end{bmatrix}\right)+\frac{\partial p_{\ell k}}{\partial\theta}\left(\begin{bmatrix}\theta^{(0)} \\ u^{(0)}\end{bmatrix}\right)\Delta\theta, \\
= \; & p_{\ell k}\left(\begin{bmatrix}\theta^{(0)} \\ u^{(0)}\end{bmatrix}\right)+u_\ell^{(0)}u_k^{(0)}\begin{bmatrix}-G_{\ell k}\sin(\theta_\ell^{(0)}-\theta_k^{(0)}) \\ +B_{\ell k}\cos(\theta_\ell^{(0)}-\theta_k^{(0)})\end{bmatrix}(\Delta\theta_\ell-\Delta\theta_k), \\
& q_{\ell k}\left(\begin{bmatrix}\theta^{(0)}+\Delta\theta \\ u^{(0)}\end{bmatrix}\right) \\
\approx \; & q_{\ell k}\left(\begin{bmatrix}\theta^{(0)} \\ u^{(0)}\end{bmatrix}\right)+\frac{\partial q_{\ell k}}{\partial\theta}\left(\begin{bmatrix}\theta^{(0)} \\ u^{(0)}\end{bmatrix}\right)\Delta\theta, \\
= \; & q_{\ell k}\left(\begin{bmatrix}\theta^{(0)} \\ u^{(0)}\end{bmatrix}\right)+u_\ell^{(0)}u_k^{(0)}\begin{bmatrix}G_{\ell k}\cos(\theta_\ell^{(0)}-\theta_k^{(0)}) \\ +B_{\ell k}\sin(\theta_\ell^{(0)}-\theta_k^{(0)})\end{bmatrix}(\Delta\theta_\ell-\Delta\theta_k).
\end{aligned}$$

Linearized Line flow, continued

- We focus on the real power $p_{\ell k}$.
- Define the row vector $K_{(\ell k)}$ by:

$$\forall j, K_{(\ell k)j} = \begin{cases} u_\ell^{(0)} u_k^{(0)} [-G_{\ell k} \sin(\theta_\ell^{(0)} - \theta_k^{(0)}) + B_{\ell k} \cos(\theta_\ell^{(0)} - \theta_k^{(0)})], \\ \hfill \text{if } j = \ell, \\ -u_\ell^{(0)} u_k^{(0)} [-G_{\ell k} \sin(\theta_\ell^{(0)} - \theta_k^{(0)}) + B_{\ell k} \cos(\theta_\ell^{(0)} - \theta_k^{(0)})], \\ \hfill \text{if } j = k, \\ \hfill 0, \text{ otherwise,} \end{cases}$$

- That is, $K_{(\ell k)j}$ is the j-th entry in the row vector $K_{(\ell k)}$.
- Then the linear approximation to $p_{\ell k}$ is given by:

$$p_{\ell k}\left(\begin{bmatrix} \theta^{(0)} + \Delta\theta \\ u^{(0)} \end{bmatrix}\right) \approx p_{\ell k}\left(\begin{bmatrix} \theta^{(0)} \\ u^{(0)} \end{bmatrix}\right) + K_{(\ell k)} \Delta\theta.$$

1.5.10 *Line flow constraints*

- Suppose that we have line flow constraints of the form:

$$p_{\ell k}(x) \leq \overline{p}_{\ell k}.$$

- Using the linear approximation, we obtain:

$$p_{\ell k}\left(\begin{bmatrix}\theta^{(0)}\\ u^{(0)}\end{bmatrix}\right) + K_{(\ell k)}\Delta\theta \leq \overline{p}_{\ell k}.$$

- We now consider the case that there are line flow constraints on each line (ℓk).
- By defining a matrix K with rows $K_{(\ell k)}$ and a vector d with entries $d_{(\ell k)}$ of the form:

$$d_{(\ell k)} = \overline{p}_{\ell k} - p_{\ell k}\left(\begin{bmatrix}\theta^{(0)}\\ u^{(0)}\end{bmatrix}\right),$$

- we can approximate the collection of line flow constraints in the form $K\Delta\theta \leq d$.

1.5.11 DC power flow approximation to line flow constraints

- Using a flat start $x^{(0)} = \begin{bmatrix} \theta^{(0)} \\ u^{(0)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ as the base-case for the linearization, we find:

$$\begin{aligned}
p_{\ell k}\left(\begin{bmatrix} \mathbf{0} \\ u^{(0)} \end{bmatrix}\right) &= u_\ell^{(0)}(u_k^{(0)} - u_\ell^{(0)})G_{\ell k}, \\
p_{\ell k}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right) &= 0, \\
K_{(\ell k)j} = \frac{\partial p_{\ell k}}{\partial \theta_j}\left(\begin{bmatrix} \mathbf{0} \\ u^{(0)} \end{bmatrix}\right) &= \begin{cases} u_\ell u_k B_{\ell k}, & \text{if } j = \ell, \\ -u_\ell u_k B_{\ell k}, & \text{if } j = k, \\ 0, & \text{otherwise}, \end{cases} \\
K_{(\ell k)j} = \frac{\partial p_{\ell k}}{\partial \theta_j}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right) &= \begin{cases} B_{\ell k}, & \text{if } j = \ell, \\ -B_{\ell k}, & \text{if } j = k, \\ 0, & \text{otherwise}, \end{cases} \\
d_{(\ell k)} &= \overline{p}_{\ell k} - p_{\ell k}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right), \\
&= \overline{p}_{\ell k}.
\end{aligned}$$

DC power flow approximation to line flow constraints, continued

- Summarizing, we can approximate the flows at the angle $\theta = \theta^{(0)} + \Delta\theta = \Delta\theta$ using the linearized equations $K\theta \leq d$, where:

$$
\begin{aligned}
\forall(\ell k), \forall j, K_{(\ell k)j} &= \begin{cases} B_{\ell k}, & \text{if } j = \ell, \\ -B_{\ell k}, & \text{if } j = k, \\ 0, & \text{otherwise}, \end{cases} \\
\forall(\ell k), d_{(\ell k)} &= \overline{p}_{\ell k}.
\end{aligned}
$$

1.5.12 Eliminating the angles

- We previously found that the power flow equations could be expressed as:

$$\begin{aligned} -\mathbf{1}^\dagger P &= -\mathbf{1}^\dagger D, \\ \theta &= \left[\widehat{J_{p\theta}^{(0)}}\right]^{-1}(\hat{P}-\hat{D}). \end{aligned}$$

- We use the second equation to substitute into $K\theta \leq d$ to obtain the equality and inequality constraints with the angles eliminated:

$$\begin{aligned} -\mathbf{1}^\dagger P &= -\mathbf{1}^\dagger D, \\ K\left[\widehat{J_{p\theta}^{(0)}}\right]^{-1}\hat{P} &\leq K\left[\widehat{J_{p\theta}^{(0)}}\right]^{-1}\hat{D}+d. \end{aligned}$$

1.5.13 Shift factor matrix

- The matrix $K\left[\widehat{J_{p\theta}^{(0)}}\right]^{-1}$ is the matrix of DC **shift factors**.
- That is, entries in the matrix represent the fraction of flow along each line for:

  injection at the buses represented in the vector $\hat{P}$, and
  withdrawal at the reference bus.

- We occasionally want to express line flows in terms of the vector P of all net injections.
- Define the augmented shift factor matrix $\hat{C} = \left[\mathbf{0} \;\; K\left[\widehat{J_{p\theta}^{(0)}}\right]^{-1}\right]$.
- That is, $\hat{C}$ consists of the columns of $K\left[\widehat{J_{p\theta}^{(0)}}\right]^{-1}$ augmented by an additional zero column corresponding to the entry that of P that was deleted to form $\hat{P}$.
- Consider the column $\hat{C}_k$ of $\hat{C}$ corresponding to a generator at bus k.
- Each entry of $\hat{C}_k$ represents the fraction of the generation from generator at bus k that flows on the corresponding line.
- The flows are given by $\hat{C}(P-D)$.

1.6 Example

- Consider the following system with MW capacities and per unit impedances (on a 1 MVA base) as shown.
- Bus 0 is the angle reference bus, so the unknown angles are $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$.

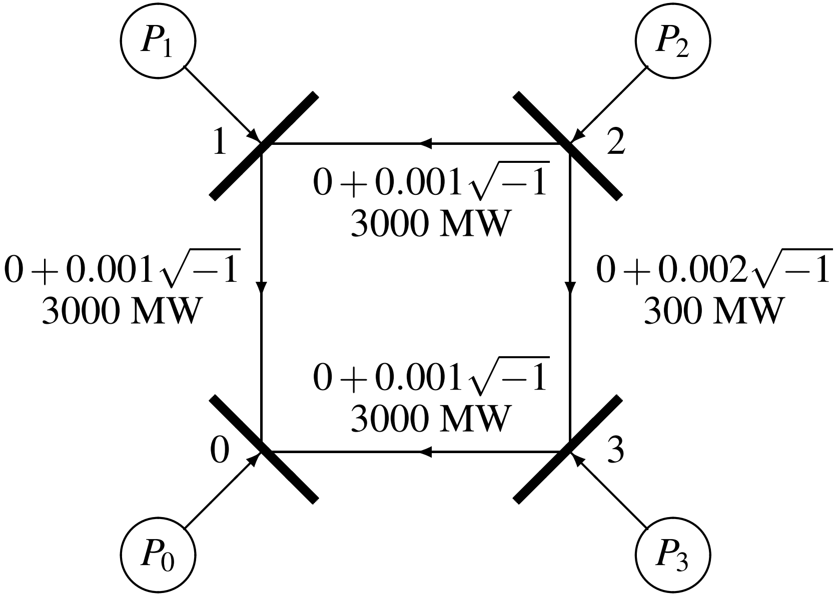


Fig. 1.1. Four-line four-bus network.

1.6.1 Admittance matrix

- The line admittances are:

$$\begin{aligned} Y_{01} = Y_{12} = Y_{03} &= \frac{1}{0 + 0.001\sqrt{-1}} = -1000\sqrt{-1}, \\ Y_{23} &= \frac{1}{0 + 0.002\sqrt{-1}} = -500\sqrt{-1}. \end{aligned}$$

- The bus admittance matrix is:

$$\begin{aligned} \begin{bmatrix} -Y_{01} & 0 & -Y_{03} \\ Y_{01} + Y_{12} & -Y_{12} & 0 \\ -Y_{12} & Y_{12} + Y_{23} & -Y_{23} \\ 0 & -Y_{23} & Y_{03} + Y_{23} \end{bmatrix} &= \begin{bmatrix} 1000\sqrt{-1} & 0 & 1000\sqrt{-1} \\ -2000\sqrt{-1} & 1000\sqrt{-1} & 0 \\ 1000\sqrt{-1} & -1500\sqrt{-1} & 500\sqrt{-1} \\ 0 & 500\sqrt{-1} & -1500\sqrt{-1} \end{bmatrix}, \\ &= \begin{bmatrix} B_{01}\sqrt{-1} & 0 & B_{03}\sqrt{-1} \\ B_{11}\sqrt{-1} & B_{12}\sqrt{-1} & 0 \\ B_{21}\sqrt{-1} & B_{22}\sqrt{-1} & B_{23}\sqrt{-1} \\ 0 & B_{32}\sqrt{-1} & B_{33}\sqrt{-1} \end{bmatrix}. \end{aligned}$$

1.6.2 Jacobian

- Evaluating the sub-matrix of the Jacobian corresponding to real power and angles at the condition of flat start:

$$
\begin{aligned}
\frac{\partial \tilde{p}}{\partial \theta}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right) &= \begin{bmatrix} -B_{01} & 0 & -B_{03} \\ B_{12}+B_{01} & -B_{12} & 0 \\ -B_{21} & B_{21}+B_{23} & -B_{23} \\ 0 & -B_{23} & B_{23}+B_{03} \end{bmatrix}, \\
&= \begin{bmatrix} -1000 & 0 & -1000 \\ 2000 & -1000 & 0 \\ -1000 & 1500 & -500 \\ 0 & -500 & 1500 \end{bmatrix}.
\end{aligned}
$$

1.6.3 DC power flow

- We invert the power flow equations to eliminate θ to obtain the following form:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}D, \\ \theta &= \left[\widehat{J}^{(0)}_{p\theta}\right]^{-1}(\hat{P}-\hat{D}). \end{aligned}$$

- where:

$$\begin{aligned} \widehat{J}^{(0)}_{p\theta} &= \begin{bmatrix} 2000 & -1000 & 0 \\ -1000 & 1500 & -500 \\ 0 & -500 & 1500 \end{bmatrix}, \\ \left[\widehat{J}^{(0)}_{p\theta}\right]^{-1} &= \begin{bmatrix} 0.0008 & 0.0006 & 0.0002 \\ 0.0006 & 0.0012 & 0.0004 \\ 0.0002 & 0.0004 & 0.0008 \end{bmatrix}. \end{aligned}$$

DC power flow, continued

- If there is only demand at bus 0 then $\hat{D} = \mathbf{0}$ and the power flow equations become:

$$
\begin{aligned}
-P_1 - P_2 - P_3 &= -D_0, \\
\theta &= \begin{bmatrix} 0.0008 & 0.0006 & 0.0002 \\ 0.0006 & 0.0012 & 0.0004 \\ 0.0002 & 0.0004 & 0.0008 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}.
\end{aligned}
$$

1.6.4 DC power flow approximation to line flow constraints

- The line flow constraints are specified by $K\theta \leq d$, where:

$$\begin{aligned} d &= \begin{bmatrix} \overline{p}_{10} \\ \overline{p}_{21} \\ \overline{p}_{23} \\ \overline{p}_{30} \end{bmatrix}, \\ &= \begin{bmatrix} 3000 \\ 3000 \\ 300 \\ 3000 \end{bmatrix}, \\ K &= \begin{bmatrix} B_{01} & 0 & 0 \\ -B_{12} & B_{12} & 0 \\ 0 & B_{23} & -B_{23} \\ 0 & 0 & B_{03} \end{bmatrix}, \\ &= \begin{bmatrix} 1000 & 0 & 0 \\ -1000 & 1000 & 0 \\ 0 & 500 & -500 \\ 0 & 0 & 1000 \end{bmatrix}. \end{aligned}$$

1.6.5 DC shift factors

- The matrix of DC shift factors is:

$$K\left[\widehat{J}^{(0)}_{p\theta}\right]^{-1} = \begin{bmatrix} 1000 & 0 & 0 \\ -1000 & 1000 & 0 \\ 0 & 500 & -500 \\ 0 & 0 & 1000 \end{bmatrix} \begin{bmatrix} 0.0008 & 0.0006 & 0.0002 \\ 0.0006 & 0.0012 & 0.0004 \\ 0.0002 & 0.0004 & 0.0008 \end{bmatrix},$$

$$= \begin{bmatrix} 0.8 & 0.6 & 0.2 \\ -0.2 & 0.6 & 0.2 \\ 0.2 & 0.4 & -0.2 \\ 0.2 & 0.4 & 0.8 \end{bmatrix}.$$

- The augmented shift factor matrix is:

$$\hat{C} = \begin{bmatrix} \mathbf{0} & K\left[\widehat{J}^{(0)}_{p\theta}\right]^{-1} \end{bmatrix},$$

$$= \begin{bmatrix} 0.0 & 0.8 & 0.6 & 0.2 \\ 0.0 & -0.2 & 0.6 & 0.2 \\ 0.0 & 0.2 & 0.4 & -0.2 \\ 0.0 & 0.2 & 0.4 & 0.8 \end{bmatrix}.$$

1.6.6 Line flow constraints in terms of shift factors

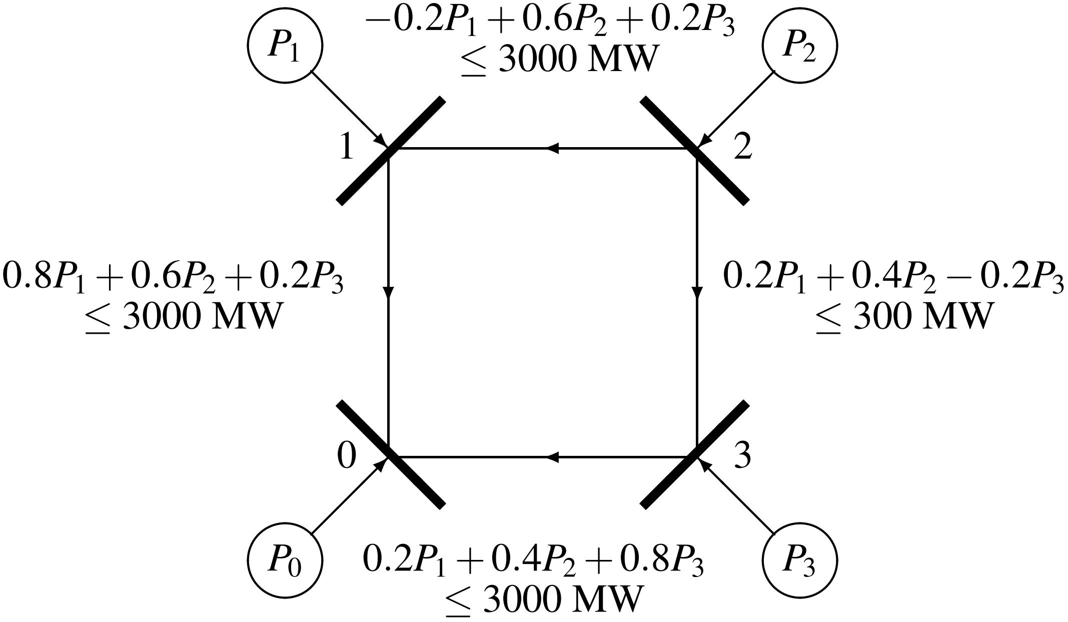
- The flows on the lines are given by $\hat{C}(P-D)$ or, equivalently, $K\left[\widehat{J_{p\theta}^{(0)}}\right]^{-1}(\hat{P}-\hat{D})$.
- The equality and inequality constraints with angles eliminated are:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}D, \\ K\left[\widehat{J_{p\theta}^{(0)}}\right]^{-1}\hat{P} &\leq K\left[\widehat{J_{p\theta}^{(0)}}\right]^{-1}\hat{D}+d. \end{aligned}$$

- Again note that $\hat{D}=\mathbf{0}$ for this particular example.
- Also, $d=\begin{bmatrix}3000\\3000\\300\\3000\end{bmatrix}$, so these constraints become:

$$\begin{aligned} -P_1-P_2-P_3 &= -D_0, \\ \begin{bmatrix} 0.8 & 0.6 & 0.2 \\ -0.2 & 0.6 & 0.2 \\ 0.2 & 0.4 & -0.2 \\ 0.2 & 0.4 & 0.8 \end{bmatrix}\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix} &\leq \begin{bmatrix}3000\\3000\\300\\3000\end{bmatrix}. \end{aligned}$$

Line flow constraints in terms of shift factors, continued



$-0.2P_1 + 0.6P_2 + 0.2P_3$
≤ 3000 MW

$0.8P_1 + 0.6P_2 + 0.2P_3$
≤ 3000 MW

$0.2P_1 + 0.4P_2 - 0.2P_3$
≤ 300 MW

$0.2P_1 + 0.4P_2 + 0.8P_3$
≤ 3000 MW

Fig. 1.2. DC power flow approximation to line flow constraints for four-line four-bus network.

1.6.7 Line flow constraints at other operating points

- The derivation so far used the flat start condition as the base-case for evaluating the shift factors and the line flow constraints.
- Other base-cases could be used, such as:
  - another assumed operating point, or
  - a measured or estimated operating point from a **state estimator**.
- The lossless assumption will typically not hold at other base-cases.

1.7 Summary

- In this chapter we re-formulated the power flow problem.
- We considered a linearization of power flow.
- We considered fixed voltage profiles.
- We considered the DC power flow.

This chapter is based on Sections 8.2 and 9.2 of *Applied Optimization: Formulation and Algorithms for Engineering Systems,* Cambridge University Press 2006.

Homework exercise: Due Wednesday, September 17

Consider a power system consisting of just two buses and one transmission line:

- bus 1 (the angle reference bus), where there is a generator, and
- bus 2, where there is load.

Suppose that the reference bus voltage is specified to be $V_1 = 1\angle 0^\circ$ and that net real power flow out of bus 2 is given by:

$$\forall u_2 \in \mathbb{R}_+, \forall \theta_2 \in \mathbb{R}, p_2(u_2, \theta_2) = u_2 \sin\theta_2 + (-P_2).$$

(That is, we assume that $G_{22} = G_{12} = B_{22} = 0$ and $B_{12} = 1$.) Suppose $u_2 = 1.0$.

(i) What is the largest value of demand $(-P_2)$ for which there is a solution to the equation $p_2(1.0, \theta_2) = 0$? What is the corresponding value of θ_2? We will write $\underline{\theta}_2$ for this value of θ_2.

(ii) What happens if θ_2 is smaller than $\underline{\theta}_2$?

(iii) Show that there are two solutions to the equation $p_2(1.0, \theta_2) = 0$ with $0 \geq \theta_2 > -2\pi$ if $(-P_2) = 0.5$. What are the corresponding values of θ_2?

(iv) Use the DC power flow to approximate the relationship between θ_2 and $(-P_2)$.

(v) When do you expect the DC power flow to be a poor approximation to the exact solution?