Solving three-player games by the matrix approach with application to an electric power market

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***Abstract*--In models of imperfect competition of deregulated electricity markets, the key task is to find the Nash equilibrium (NE). The approaches for finding the NE have had two major bottlenecks: computation of mixed strategy equilibrium and treatment of multi-player games. This paper proposes a payoff matrix approach that resolves these bottlenecks. The proposed method can efficiently find a mixed strategy equilibrium in a multi-player game. The formulation of the NE condition for a three-player game is introduced and a basic computation scheme of solving nonlinear equalities and checking inequalities is proposed. In order to relieve the inevitable burden of searching the subspace of payoffs, several techniques are adopted in this paper. Two example application problems arising from electricity markets and involving a Cournot and a Bertrand model, respectively, are investigated for verifying the proposed method. The proposed method outperforms a publicly available game theory software for the application problems.**

***Index Terms*— Bimatrix Game, Complementarity Problem, Deregulation, Dominated Strategy, Electric Power Market, Game Theory, Mixed Strategy, Nash Equilibrium, Payoff Matrix**

I. Introduction

Competition among electric generation companies is a major goal of restructuring in the electricity industry. It is expected that the more competitive the market for selling power, the lower is the price. However, strategic behavior of generation companies for maximizing their profits has appeared as an undesirable situation resulting in higher market prices than expected.

Such phenomenon can be analyzed with game theory. The individual behavior and the market clearing mechanism are represented as a bilevel optimization problem and solved to find a Nash equilibrium [1]-[3]. Particular characteristics of electricity markets, such as continuous strategy spaces and transmission constraints that affect the market clearing mechanism, severely complicate the search for a Nash equilibrium [4][5].

Many efforts have been made to develop solution methods for finding the NE of games representing transmission-constrained electricity markets. There are at least five categories of solution methods: the mathematical programming approach [1][6][7], the payoff matrix approach [8][9], co-evolutionary programming [10], the exhaustive search approach [11], and analytical derivation [12]. However, there have been some bottlenecks for developing a rigorous tool applicable to electric power markets. One of them is the consideration of transmission system constraints, since the constraints complicate the market clearing mechanism and cause the payoff functions to be non-differentiable and non-concave. Another difficulty is to deal with a multi-player game where three or more players participate. In many electricity markets there are somewhat more than two but less than ten major players.

The mathematical programming approach uses a numerical framework such as: the linear complementarity problem [7]; mathematical programming with equilibrium constraints [1]; or a conventional optimization technique [6]. This approach can solve for the NE of a multi-player game that has differentiable and concave payoffs. When the problem includes transmission constraints or generation capacities, however, this method has difficulties in determining equilibria because the payoff can be non-differentiable and non-convex, unless simplifying assumptions are made. In some cases, "false equilibria" are identified that satisfy local optimality conditions for a Nash equilibrium but which are not Nash equilibria.

The co-evolutionary programming and exhaustive search approaches require neither differentiability nor concavity of the payoffs, so they can deal with transmission constrained problems. In some cases, it may be possible to interpret the distribution of strategies from a co-evolutionary programming approach as representing the distribution of a mixed equilibrium. However, neither the co-evolutionary programming nor the exhaustive search approaches are designed to find mixed equilibria.

On the other hand, the payoff matrix approach in a bimatrix game representing two players can find a global solution for the given payoff matrices and can represent mixed strategy equilibria. This approach is based on Lemke's algorithm, which uses a linear complementarity pivot [13]. Unfortunately, the optimality conditions for a NE are nonlinear in a multi-player game. So the payoff matrix approach has been restricted to bimatrix games.

In this paper, a new formulation and its solution method for multi-player games are introduced. Since it is based on the payoff matrix approach, mixed strategy equilibria can be sought. Since it is developed for dealing with multi-player games, it can resolve the bottleneck of computing multi-player equilibria.

The payoffs in this method depend on the choices of three or more participants, so that a higher dimensional payoff than

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a two dimensional matrix is required and introduced in this paper. The notation and results are developed for a three-player game. However, the formulation and the solution method can, in principle, be applied to a multi-player game. (We comment briefly in the conclusion about potential difficulties for multi-player games.)

The main scheme for the solution technique is different from the algorithm for the nonlinear complementarity problem [14] in that we discard dominated strategies and use heuristics for reducing the search space. The heuristics result from numerical studies of the relation between the model of imperfect competition in a transmission-constrained electricity market and its mixed strategy equilibria. Cournot and Bertrand models of market interaction are simulated, and the computation time is compared with GAMBIT [15], which is a public domain software for game theory.

The outline of this paper is as follows. Section II formulates the three-player game, while section III outlines a solution method. Sections IV and V apply the solution method to Cournot and price competition models of electricity markets. We conclude in section VI.

II. Formulation of a Three-player game

A. Notation for two-players' payoffs

A bimatrix game is a two-player, nonzero-sum matrix game. In a bimatrix game, two matrices, A and B, are used for denoting the payoff of each player, respectively. A game with a continuous strategy space can be converted into matrix game using discretization of the strategy space. The strategy pair (x^*, y^*) of the players is said to be an equilibrium pair if:

- $x^{*t}e_{N_1}=1$, $x^*\geq 0$, $y^{*t}e_{N_2}=1$, $y^*\geq 0$, where e_{N_1} and e_{N_2} denote the column vectors in R^{N_1} and R^{N_2}, respectively, in which all the elements are equal to 1 and
- no player benefits by unilaterally changing his own strategy while the other player's strategy is fixed.

Thus the NE is characterized as follows [13]:

$x^{*t}Ay^*\geq x^tAy^*$, for all $x\in R^{N_1}$, such that $x^te_{N_1}=1$, $x\geq 0$, (1)

and $x^{*t}By^*\geq x^{*t}By$, for all $y\in R^{N_2}$, such that $y^te_{N_2}=1$, $y\geq 0$. (2)

B. Notation for three-players' payoffs

In a multi-player game the NE conditions appear in a similar way to (1) and (2). However, the notation for payoffs must be modified to represent information about multiple players. In this paper, the formulation and case studies are performed for three-player games, but the development is applicable to a general multi-player game. The three players are designated P_1, P_2, and P_3. Player P_1 chooses amongst N_1 pure strategies, P_2 chooses amongst N_2, and P_3 chooses amongst N_3. In a play, if P_1 picks the pure strategy i, P_2 picks j, and P_3 picks k, then the payoff to P_1 is a_{ijk}, the payoff to P_2 is b_{ijk}, and the payoff to P_3 is c_{ijk}. Define the $N_1\times N_2\times N_3$ payoff "cubes" A, B, and C by:

$A=[a_{ijk}]\in R^{N_1\times N_2\times N_3}$, $B=[b_{ijk}]\in R^{N_1\times N_2\times N_3}$, $C=[c_{ijk}]\in R^{N_1\times N_2\times N_3}$.

More generally, in an n-player game, each player would have an associated hypercube of dimension n to represent its payoff.

In order to express the NE condition analogously to (1) and (2) for the case of a three-player game, the multiplication between a cube and a vector is required. A notation is proposed for describing the multiplicative operation of the cube. When a cube is multiplied by a vector, the result is a matrix. This operation can be represented by the following notation:

$$A\otimes[x,\cdot,\cdot]=[a_{jk}]\in R^{N_2\times N_3},\quad where\ x\in R^{N_1},\ and\ a_{jk}=\sum_{i=1}^{N_1}a_{ijk}x_i$$

$$A\otimes[\cdot,y,\cdot]=[a_{ik}]\in R^{N_1\times N_3},\quad where\ y\in R^{N_2},\ and\ a_{ik}=\sum_{j=1}^{N_2}a_{ijk}y_j$$

$$A\otimes[\cdot,\cdot,z]=[a_{ij}]\in R^{N_1\times N_2},\quad where\ z\in R^{N_3},\ and\ a_{ij}=\sum_{k=1}^{N_3}a_{ijk}z_k$$

When a cube is multiplied by two vectors successively, it becomes a conventional column vector. This operation can be represented by:

$$A\otimes[\cdot,y,z]=(A\otimes[\cdot,y,\cdot])z=(A\otimes[\cdot,\cdot,z])y\in R^{N_1},\quad where\ y\in R^{N_2},\ z\in R^{N_3}$$

$$A\otimes[x,\cdot,z]=(A\otimes[x,\cdot,\cdot])z=(A\otimes[\cdot,\cdot,z])^t x\in R^{N_2},\quad where\ x\in R^{N_1},\ z\in R^{N_3}$$

$$A\otimes[x,y,\cdot]=(A\otimes[x,\cdot,\cdot])^t y=(A\otimes[\cdot,y,\cdot])^t x\in R^{N_3},\quad where\ x\in R^{N_1},\ y\in R^{N_2}$$

When three vectors are multiplied to a cube successively, it becomes a scalar. The following notation represents this operation:

$$A\otimes[x,y,z]=z^t(A\otimes[x,y,\cdot])=y^t(A\otimes[x,\cdot,z])=x^t(A\otimes[\cdot,y,z])\in R,\quad where\ x\in R^{N_1},\ y\in R^{N_2},\ z\in R^{N_3}$$

C. Nash equilibrium

The inequalities (1) and (2) for a two-player game can be extended to those for a three-player game using the notation of multiplication between a cube and vectors. Strategies (x^*, y^*, z^*) satisfying $x^{*t}e_{N_1}=1$, $x^*\geq 0$, $y^{*t}e_{N_2}=1$, $y^*\geq 0$, $z^{*t}e_{N_3}=1$, $z^*\geq 0$, constitute a NE if they satisfy following inequalities:

$$\begin{aligned}&A\otimes[x^*,y^*,z^*]\geq A\otimes[x,y^*,z^*],\quad for\ all\ x\in R^{N_1},\ such\ that\ x^te_{N_1}=1,\ x\geq 0\\&B\otimes[x^*,y^*,z^*]\geq B\otimes[x^*,y,z^*],\quad for\ all\ y\in R^{N_2},\ such\ that\ y^te_{N_2}=1,\ y\geq 0\\&C\otimes[x^*,y^*,z^*]\geq C\otimes[x^*,y^*,z],\quad for\ all\ z\in R^{N_3},\ such\ that\ z^te_{N_3}=1,\ z\geq 0\end{aligned}\quad(3)$$

In the bimatrix game approach, the NE conditions (1) and (2) are converted to a set of equations, and can solved by a linear complementarity algorithm[13]. But this approach cannot be applied directly to a multi-player game, since the condition (3) involves nonlinear inequalities. In this paper, (3) is transformed into a set of nonlinear equations, and solved numerically.

D. Necessary condition of the Nash equilibrium

In a bimatrix game, each solution has associated with it a square sub-matrix of the payoff matrix corresponding to the non-zero entries in x^* and y^* of the equilibrium strategy [16][17]. Therefore x^* and y^* have the same number of nonzero elements. This is called the "equal number of non-zeros property" [17]. In a similar way, each solution in a three-player game has associated with it a certain sub-cube of the payoff cube corresponding to the non-zero entries in (x^*, y^*, z^*). We call this sub-cube the "kernel."

For a kernel of (x^*, y^*, z^*), let the number of non-zero entries in x^*, y^*, z^* be K_1, K_2, and K_3, respectively. Then the kernels for the payoffs A, B, and C are represented as $\hat{A}$, $\hat{B}$, and $\hat{C}$ respectively:

$$\hat{A} \in R^{K_1 \times K_2 \times K_3},\ \hat{B} \in R^{K_1 \times K_2 \times K_3},\ \hat{C} \in R^{K_1 \times K_2 \times K_3}.$$

Since under the equilibrium strategies of (x^*, y^*, z^*), each player will not change its strategy unilaterally, there exist $\lambda_1 \in R$, $\lambda_2 \in R$, $\lambda_3 \in R$, such that the following conditions are satisfied by the equilibrium strategies:

$$\begin{aligned} &\hat{A} \otimes [\,\cdot\,, \hat{y}^*, \hat{z}^*] = \lambda_1 e_{K_1}, \hat{B} \otimes [\,\hat{x}^*, \cdot\,, \hat{z}^*] = \lambda_2 e_{K_2}, \\ &\hat{C} \otimes [\hat{x}^*, \hat{y}^*, \cdot\,] = \lambda_3 e_{K_3}, \end{aligned} \quad (4)$$

where $\hat{x}^* \in R^{K_1}$, $\hat{y}^* \in R^{K_2}$, $\hat{z}^* \in R^{K_3}$ are vectors corresponding to the non-zero entries of x^*, y^*, and z^* respectively, and e_{K_1}, e_{K_2}, e_{K_3} denote the column vectors in R^{K_1}, R^{K_2}, R^{K_3} respectively, in which all the elements are equal to 1.

The equations (4) are necessary conditions for the NE. All of the equations in (4) are quadratic. For example, the pth equation among the K_1 equations corresponding to $\hat{A}$ is as follows:

$$\sum_{q=1}^{K_2} \hat{y}_q^* \sum_{r=1}^{K_3} \hat{a}_{pqr} \hat{z}_r^* = \lambda_1,$$

where $\hat{A}$=[$\hat{a}_{pqr}$]. Equation (4) has K_1+K_2+K_3 equations.

There are 3 more to represent the normalizing equations ($\hat{x}^t e_{K_1} = 1$, $\hat{y}^t e_{K_2} = 1$, $\hat{z}^t e_{K_3} = 1$) and there are also non-negativity constraints. The equalities (4) together with the normalizing equations can be solved by Newton's method for the values of $\hat{x}$, $\hat{y}$, $\hat{z}$, λ_1, λ_2, and λ_3.

E. Inequalities for checking the equilibrium

Not all of the solutions of (4) are NEs. Some of the solutions do not satisfy non-negativity constraints, others do not meet (3). Thus each solution of (4) must be checked to see whether it is non-negative and meets (3). The check for being a NE is performed easily by the non-negativeness and by checking the following inequalities:

$$\begin{aligned} &A \otimes [\,\cdot\,, y^*, z^*] \le \lambda_1 e_{N_1}, B \otimes [\,x^*, \cdot\,, z^*] \le \lambda_2 e_{N_2}, \\ &C \otimes [x^*, y^*, \cdot\,] \le \lambda_3 e_{N_3}, \end{aligned} \quad (5)$$

where x^*, y^*, and z^* have non-zero elements corresponding to the solution of (4).

The vectors in the left hand side of (5) evaluate the expected payoff to each player for each of its strategies, when the distribution of the strategies of the other players is fixed. For example, the pth element of $A \otimes [\,\cdot\,, y^*, z^*]$ evaluates the expected payoff to P_1, when P_1 picks a pure strategy p among N_1, P_2 picks the mixed strategy specified by y^*, and P_3 picks the mixed strategy specified by z^*. Thus (5) verifies that the profits are the highest for each player at the candidate equilibrium, given that the other players choose the candidate equilibrium. The values $\lambda_1 \in R$, $\lambda_2 \in R$, $\lambda_3 \in R$, are the payoffs given to the players when they pick the set of strategies (x^*, y^*, z^*).

A set of strategies satisfying the normalizing equations, the non-negativity constraints, and both (4) and (5) is a NE.

III. Solution Method of a Three-Player Game

A. Search space for finding a kernel

The equations of the necessary condition (4) are computed for a given kernel. However it is not simple to guess a kernel of the NE, while the computation of (4) and the check of (5) are rather simple. So the problem of finding a kernel of the NE is key to the overall solution method for the multi-player game. In a bimatrix game, the kernel has a useful property of "equal number of non-zeros property" [17]. Unfortunately that property does not hold in a multi-player game.

The space to be searched for finding a kernel of the NE is huge. For example, the number of sub-cubes in a three-player game of 10×10×10 dimensions is more than 10^9, which is calculated by: $\sum_{i=1}^{10}\sum_{j=1}^{10}\sum_{k=1}^{10} {}_{10}C_i \cdot {}_{10}C_j \cdot {}_{10}C_k = 1.07*10^9$. Another approach to solve (3) was proposed using the nonlinear complementarity problem (NLCP) [14]. As a result of tests on electric power market problems, the approach seems not to be useful for problems that result from discretizing a continuous strategy space. The comparison between a NLCP method and the proposed method is provided in section IV.

In order to reduce the search space, several techniques are used in this paper. Some are based on game theory, and others are heuristics based on properties of power markets. The major techniques used in the proposed method are introduced in the following sections.

B. Discarding the dominated strategies

In a game, a player may have some strategies that will not be picked by the "rational" player, no matter how other players play. Those strategies are called "dominated" [18][19]. Those strategies do not affect the game, so can be eliminated without any changes to the NE.

The proposed method uses this property in reducing the search space. In an initial step, the payoffs are reduced to a sub-cube that consists of only the undominated strategies. The search for finding the kernel of the NE is then performed only on the reduced payoffs. Reduction of dominated strategies is also executed prior to each solution of (4).

Typical problems that arise in power market application have continuous strategy spaces. The problems are converted into discrete problems to apply the matrix approach by discretizing the continuous strategy space. The number of strategies in the discretized problem increases as the discretization becomes finer. Moreover, the number of undominated strategies increases as the grid becomes finer. So the searching problem becomes harder as the discretization becomes finer. For this reason, several heuristics are also necessary. These are described in the next three sections.

C. Variable grids

Through numerical studies, we found the following tendency: If the non-zero elements appear in an interval [a,b], with no other non-zero elements outside the interval, then after using a finer discretization, the refined solution of the non-zero elements still appears only near the interval [a,b]. This tendency can be utilized for reducing the number of effective payoffs.

Even though the number of payoffs becomes larger with finer discretization, the non-zero elements in the NE tend to concentrate in a relatively stable interval, independent of the fineness of the grid. In other words, the distribution of the mixed strategy equilibrium does not change significantly with discretization fineness. Therefore partial information about the discretized strategies can be enough to capture the shape of the solution. Such information is extracted from the payoffs with a coarsely discretized grid. This extraction is different from discarding the dominated strategies, as described in section III.B, in that the strategies not selected by the grid are not thrown away forever.

Using coarse grids, the reduced strategies are selected, and the reduced payoff cubes are calculated with the coarse grids. After the serial process of elimination, extraction by grid, search, computation, and check, it can be estimated where the non-zero entries in the solution are concentrated, and what shape the distribution of the NE has. In the next step, the discretization grids can be refined to obtain a finer solution using the estimated information. This refining process is executed iteratively until the grid reaches the desired discretization level. Performance of this heuristic depends on the shape of the payoffs. If the original payoff is continuous in continuous strategies, this coarse grid method shows good results. If the payoff has many abrupt changes, the adjustment of the variable grids is less efficient, so additional techniques are applied.

D. Consecutive non-zeros strategy

In a Cournot model with price-sensitive demand at each node, the nodal prices vary continuously with the bid quantities, even if a constraint becomes binding. Hence, the payoffs of the Cournot model are continuous in the quantity strategy space. However, in a Bertrand model with strategies of bid prices, the quantity of generation at each firm changes abruptly depending on the difference of the bid prices.

Empirical studies on the mixed strategy equilibrium result in some heuristics. One of them is that the NE is often a mixed strategy consisting of a series of consecutive non-zero elements if the payoff has abrupt changes as a function of the strategic variables, such as in the Bertrand model. Therefore it is recommended to search only over kernels corresponding to consecutive non-zero entries in the strategy vector when the payoffs are discontinuous. If the search space for finding a kernel of the NE is restricted only to the consecutive strategies, the amount of the search space reduces dramatically.

On the other hand, a mixed strategy where the non-zero elements are split into two parts generally appears if the payoff is a continuous function of the strategic variables, such as in the Cournot model and the supply function equilibrium model. In this type of payoff, a series of many consecutive non-zero elements does not usually appear. Each region of non-zero elements has only one or a few non-zero elements. This heuristic is also useful in adjusting the fineness of the grid.

These two typical patterns of mixed strategies are simulated and introduced in sections IV and V.

E. Using the bimatrix game

The continuous payoffs also show another tendency that one of the players, say P_1, chooses a pure strategy in equilibrium, while the others choose mixed strategies. Taking advantage of this observation allows conversion of the three-player game into a bimatrix game between players P_2 and P_3. As is well known, the solution of a bimatrix game can be computed rapidly using the linear complementarity algorithm of Lemke and Howson [13].

If this tendency is observed in a three-player game after coarse grid process, then the strategy of the player that is apparently choosing a pure strategy is tentatively fixed as a pure strategy. This heuristic is also quite effective in reducing the search space. Lemke's algorithm [13] can then be used instead of the search and computation process explained previously. This bimatrix game computation produces a solution very rapidly. This heuristic is also valuable when there are more than three players; however, in this case if one player adopts a pure strategy, the other players are still involved in a multi-player game.

IV. Application to the Cournot Model

A. System and problem description

The test system for the Cournot model is shown in Fig.1, and the marginal cost function of generators and the inverse demand function at each node are provided in Table 1. This system and the market data are quoted from [11]. The power flows are approximated using a DC power flow solution, and the transmission lines are assumed to be lossless and have equal reactance [20].

The three generation firms bid generation quantities in the Cournot model. The clearing prices are determined by benefit

Table I. Benefit and Cost data. (Source: This data is quoted from [11, table I].)

		Utility A	Utility B	Utility C
Inverse Demand $P_i = \beta_i - \alpha_i q_i$	$\beta_i =$	108.4096	103.8238	105.6709
	$\alpha_i =$	0.055500	0.066909	0.063703
Total Cost $C_i = 1/2\phi_i q_i^2 + \gamma_i q_i + \eta_i$	$\phi_i =$	0.015718	0.021052	0.012956
	$\gamma_i =$	1.360575	-2.07807	8.105354
	$\eta_i =$	9490.366	11128.95	6821.482

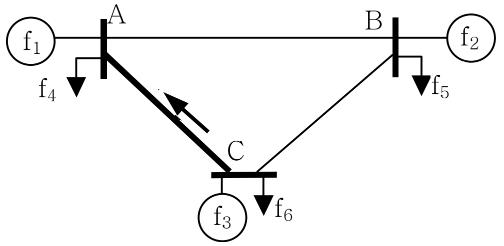

Fig.1 Sample system for the Cournot model. (Source: This figure is based on [11, Fig.1].)

maximization of demands located at each node and by considering the system constraints. The payoffs of the firms are computed by the product of the bid quantity and the clearing price at each node less the cost. By this mechanism and discretization, the payoff cubes of the firms are produced. The minimum size of the bid quantity is assumed 1MW in this simulation.

B. Solution of the Cournot model

In this simulation, two cases are considered:

1. no constraints and
2. a transmission line limit on the line joining node A and node C.

For the given market data, the result of the perfect competition shows that the generation levels are between 1200~1500MW. Therefore the initial range of the generation bid is guessed as 600~1500MW, since imperfect competition such as in the Cournot model typically results in less generation than under perfect competition.

1. No constraints

The coarse grid with discretization of 100MW in the range 600~1500MW is applied initially to the case of no constraints. The process of discarding the dominated strategies reduces the original dimension of 10×10×10 to 2×3×2. The search and computation process results in the pure strategies of 1100, 1000, and 1000MW for players A, B, and C, respectively. So, using the variable grid heuristic of section III.*C*, the range of the strategies are shrunk to 1000~1200 for A, 900~1100 for B, and 900~1100 for C. After several iterations, the NE solution is found as 1106, 1045, and 995 MW respectively.

Figure 2 illustrates the pure strategies and the expected payoff distribution showing that each pure strategy computed is the best to each player. In figure 2 and subsequently, the vertical lines indicate the value and probability mass of the NE strategy, while the curves show, for each player, the payoff versus strategy for that player, given the equilibrium strategy of the other players. The resulting flow along the line from node C to node A is 29MW. This result is consistent with that presented in [11].

2. Transmission constraints

For the transmission constrained problem a transmission limit of 40MW was imposed on the line between A and C in a bi-directional way. The same initial grid as for the case of no constraints is applied for the constrained case. The initial undominated payoffs result in the reduced payoffs of 5×4×5.

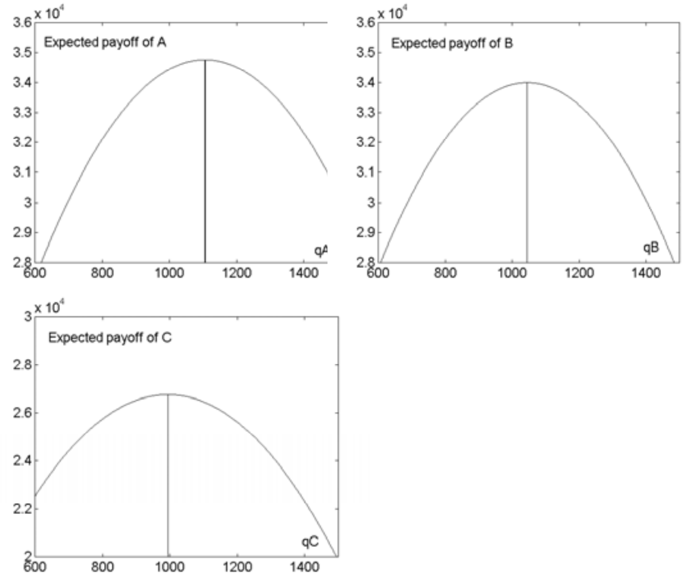

Fig.2 Strategies and expected payoffs of unconstrained Cournot NE.

The search and computation process produces the following mixed strategies: A picks quantities [800, 900, 1100] with probabilities of [0.11, 0.27, 0.61], respectively, B picks quantity 1100 as a pure strategy, and C picks quantities [700, 800, 900] with probabilities of [0.12, 0.18, 0.70], respectively.

Because B picks a pure strategy, the bimatrix game heuristic of section III.*E* is applied to the next step. The computation is executed like a bimatrix game with varying the pure strategies of B from 1000 to 1200. This heuristic computes rapidly. The final solution is as follows: A picks quantities [829, 1073, 1074] with probabilities of [0.21, 0.2, 0.59], respectively, B picks quantity 1115 as a pure strategy, and C picks quantities [699, 873, 874] with probabilities of [0.08, 0.22, 0.7], respectively. Figure 3 shows the mixed strategies as vertical lines and the payoffs. Because the equilibrium generations are random, the resulting flow along the line from node C to node A is also random. The maximum

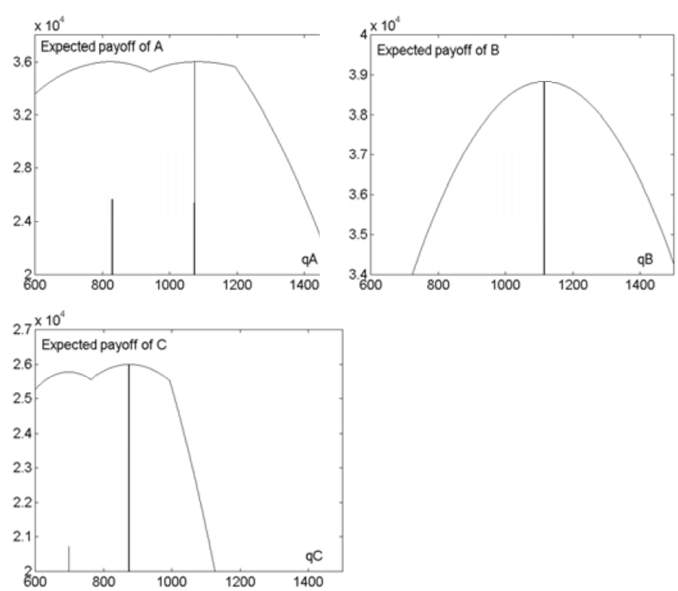

Fig.3 Strategies and expected payoffs of constrained Cournot NE.

flow from C to A is 40MW and the expected flow is 4.07MW.

The reason why B maintains a pure strategy while A and C have mixed strategies is that the transmission constraint between A and C primarily affects generation at nodes A and C, making the payoff of both A and C non-concave and non-differentiable. However, B's payoff remains concave and differentiable and a pure strategy remains best for B.

In practice, for any given demand level and supply configuration, transmission constraints in a large interconnected electricity system will tend to affect some players strongly while not significantly affecting others. Consequently, the heuristic of section III.*E* is likely to be applicable in practice.

It is interesting to note that the transmission constraint of 40 MW is great than the flow on the line from C to A in the unconstrained case solved in section IV.B.1, yet this constraint disrupts the unconstrained equilibrium. This somewhat surprising result is consistent with [11,12] and is due to the multiple local maxima of the profit functions in the constrained case.

C. Computational effort and comparison to GAMBIT

The initial estimated range of the generation bid is large, involving a grid of size 900×900×900. However, the regions of the non-zero elements can be narrowed gradually as the variable grid method is applied. At each step, the cubes extracted by the grids are reduced again to lower dimension by eliminating the dominated strategies. The characteristic that one player chooses a pure strategy contributes much to saving the overall computation time and to reducing the manual effort to adjust the grids. As explained in section III.*D*, this Cournot model has continuous payoffs even in the constrained case, and the mixed equilibrium shows the pattern of being split into two regions. The bimatrix game heuristic also contributes to reducing the search space by neglecting other regions and focusing on the two regions where the strategies have non-zero probability of being played.

The overall computation time is hard to estimate since, between iterations, manual adjustment was included in the prototype implementation, and the performance is highly dependent on the heuristic decision. Hence, the computation time is reported for the extracted payoffs using several particular choices of fixed grids rather than using the variable grids. We ran each case under two assumptions: first, with the heuristics in sections III.*B*-III.*E* applied and second with only the heuristics in sections III.*B*-III.*D* applied.

We compared the results to that of the game theory software, GAMBIT. As an option for the solution method in GAMBIT, the 'Simpdiv' option is more stable and faster than other options in the software, so it was used throughout for the numerical studies with GAMBIT. The Simpdiv method uses the NLCP [14].

Table 2 shows the comparison of computation time between GAMBIT [15] and the proposed method for the Cournot model. All computations were performed on a Pentium III processor of 233MHz. Several grids of different dimension are used.

The test cases are extracted by the variable grids heuristic of section III.*C*. The simple case of 3×3×3 shows fast computation for the proposed method. In the proposed method, the elimination of the dominated strategies leads to a pure strategy directly. Since this undominated strategy is the NE, no further heuristics are required and the NE is found quickly.

In GAMBIT, the 'Simpdiv' method can be run to seek a mixed strategy or to seek only pure strategies. If the former option is used, GAMBIT takes 2 seconds to find the pure strategy for the 3×3×3 case; however, if it is run with the option for seeking only the pure strategy then it takes 0.05 sec for this case (as indicated by the 0.05* in parenthesis in the table), which is comparable to the computation time for the proposed method. In all subsequent cases, GAMBIT was run to seek only one Nash equilibrium (rather than seeking all possible NEs.)

In Table 2, the cases using the heuristic of sections III.*B*-III.*E* are compared with the cases where the heuristics of sections III.*B*-III.*D* only are used. In the row of 5×5×5, the computation (0.03sec) using the bimatrix game heuristic of section III.*E* in addition to the other heuristics improves the performance by a factor of 15 times (from 0.5sec). The improvement due to using the heuristic of section III.*E* increases for the largest case considered.

As the dimension goes up, the computation time of GAMBIT increases extremely rapidly. While the computation of the GAMBIT depends on the total size of the cubes, the proposed method depends on the initial undominated sub-cube. Since this comparison is performed only for a sample of electricity market problems, it is hard to compare the overall performance of both methods for general game theory problems. However, in the game of the power market, the proposed heuristics are applicable, and can speed up the computation.

TABLE 2. COMPARISONS OF COMPUTATION TIME BETWEEN THE PROPOSED METHOD AND GAMBIT

Test cases		Proposed method			GAMBIT	Solutions
Dimension	Grids of generation quantities	Initial undominated sub-cube	Computation time [sec]: Heuristics in III.*B*-III.*E* used	Computation time [sec]: Heuristics in III.*B*-III.*D* used	Computation time [sec]	Dimension of Kernel
3×3×3	[800~1200] by 200	1×1×1	0.01	0.01	2 (0.05*)	1×1×1
5×5×5	[800~1200] by 100	5×3×3	0.03	0.50	38	2×1×2
7×7×7	[700~1300] by 100	5×3×4	0.06	0.71	240	3×1×3
9×9×9	[800~1200] by 50	8×4×5	0.11	5.22	2028	3×1×3

V. Application to the Price competition model

A. Market description

Another model for analyzing imperfect competition is the Bertrand price competition model. The market rules used in this simulation by the Bertrand model with constraints are defined as follows: the participants submit the prices with quantity they want to sell, and suppliers are ordered by price to meet the demand [21]. Prices are set using a pay-as-bid model. For example, if there were two bidders P_1 and P_2 and they bid \$3/MWh for 5MW and \$5/MWh for 10MW, respectively, and 10MW of demand is cleared. In the pay-as-bid model, prices of P_1 and P_2 are \$3 and \$5 respectively. (In comparison, a clearing price of \$5 would be awarded to both in a uniform-price model.)

The cost data, demand function, and the power system in this simulation are the same as those in section IV. Only the market rules are replaced by the price competition model. Under perfect competition, the nodal price is 24.4 at all the nodes. The price under Cournot competition without constraints is 41.47 as shown in Fig.2. Since the price of Bertrand equilibrium generally lies between that of the perfect competition and that of the Cournot equilibrium, the range of the bid prices is set between 20~40 in this simulation.

B. Solution of the Bertrand model

The discretization for bidding prices is assumed to be 1, so the initial cubes have a dimension of 21×21×21. Since the transaction is determined by the difference among the bid prices, the profit of each player is discontinuous in the continuous price space given a pure strategy for the other players. Thus in this simulation, the consecutive non-zero heuristic of section III.*D* is applied directly without the variable grid heuristic of section III.*C*.

The first step is an elimination of the dominated strategies, which reduces the cubes to 8×7×7 sub-cubes. The solution is obtained directly by search and computation using the consecutive non-zero heuristic of section III.C. Figure 4 illustrates the distributions of the mixed equilibrium and the expected profits.

This result shows that the solution has a kernel of 6×2×6. While the proposed solution method took 19 seconds to compute this solution, the computation time of GAMBIT could not be checked, since it stalled. The authors of GAMBIT apparently primarily intended GAMBIT for problems where each player had only a small number of strategies, so that problems that arise from continuous strategy spaces, such as in electricity markets, are not well-suited to GAMBIT.

In this simulation, another pattern of the mixed equilibrium in power market problem is investigated, in that the mixed equilibrium of the discontinuous payoffs has consecutive non-zero elements. We confirm that the heuristics can speed up the solution procedure and can be applied to problems such as the Bertrand model.

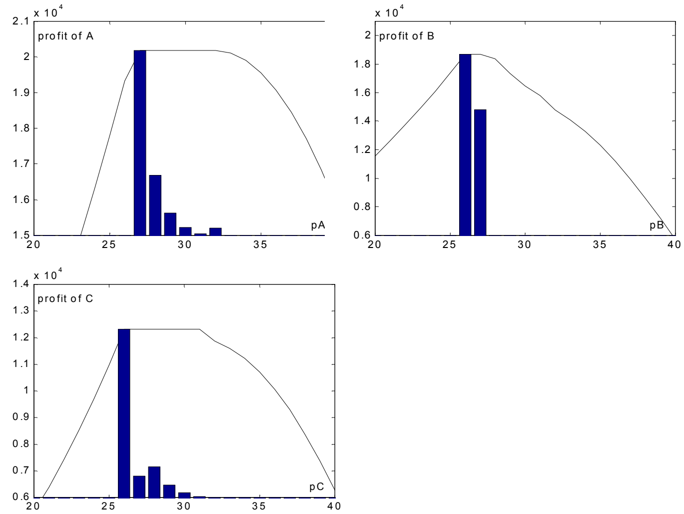

Fig.4 Strategies and expected payoffs of the Bertrand NE

VI. Conclusion

In the approaches for finding the NE, there have been two major bottlenecks: computation of the mixed strategy equilibrium and treatment of the multi-player game. This paper proposes a payoff matrix approach that resolves these bottlenecks.

In order to deal with the payoffs of a three-player game, a "cube" of payoff is proposed and the NE condition is expressed by the payoffs. By solving the nonlinear equations of the necessary NE condition and checking inequalities, the equilibrium solution is obtained. However, a difficult problem arises in finding the kernel of the solution because it involves searching the large number of candidate kernel sub-cubes.

Several techniques are used for decreasing the search space. First, dominated strategies that do not affect the solutions are eliminated before the search process. The variable grid method is applied to extract the key features of the distribution of mixed strategies by partial information of the payoffs. The LCP method is executed when one of the players is estimated to maintain a pure strategy. The shape of payoffs is considered heuristically to confine the region of the non-zero elements of strategies.

Two example payoff models in power market problem are used to test the proposed method. It is verified that the proposed method is effective in decreasing the search space and the overall computation time is relatively modest.

The proposed method for three-player game may not be practical to apply directly to a realistic many players game. However, the heuristics are a starting point in solving multi-player games.

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