THE UNIVERSITY OF TEXAS AT AUSTIN
Dept. of Electrical and Computer Engineering

EE381K-14 Multidimensional Digital Signal Processing
Problem Set #3: Processing Sampled Signals

Date assigned: February 2, 2008
Date due: February 8, 2008

Reading: D&M, Sections 2.1–2.2; resampling slides; vector-valued signals handout

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You may use any computer program to help you solve these problems, check answers, etc.

Homework is due on Friday, February 8th, by 11:00 AM. You may turn in the homework to Prof. Evans' office in ENS 433B, by fax to 471-6512, or by e-mail. If you submit the homework by fax, please let me know by e-mail.

Regularly scheduled office hours for Prof. Evans are Wednesdays 10–11 AM, Thursdays 12:30–1:30 PM, and Fridays 9:00–10:00 AM in ENS 433B. Feel free to send questions by e-mail to bevans@ece.utexas.edu.

Be sure to submit your own independent homework solutions.

Problem 3.1 Vector-Valued Convolution

This problem asks you to derive the vector-valued convolution theorem.

(a) Let $\boldsymbol{x}[n_1, n_2]$ be the input to a linear, shift-invariant system with impulse response matrix $\tilde{\boldsymbol{h}}[n_1, n_2]$ and output $\boldsymbol{y}[n_1, n_2]$. Show that

$$\boldsymbol{Y}(\omega_1, \omega_2) = \tilde{\boldsymbol{H}}(\omega_1, \omega_2)\boldsymbol{X}(\omega_1, \omega_2).$$

(b) Show that if two LSI vector-valued systems with impulse response matrices $\tilde{\boldsymbol{g}}[n_1, n_2]$ and $\tilde{\boldsymbol{h}}[n_1, n_2]$ are connected in cascade, then the overall frequency response matrix is

$$\tilde{\boldsymbol{H}}_{eq}(\omega_1, \omega_2) = \tilde{\boldsymbol{H}}(\omega_1, \omega_2)\tilde{\boldsymbol{G}}(\omega_1, \omega_2).$$

Problem 3.2 Sampling of Bandlimited Signals

Consider the bandlimited (analog) signal whose Fourier transform occupies the band shown in Figure 1 What is the minimum number of samples per square meter that is required to sample this signal without aliasing?

Problem 3.3 Convolution of a Hexagonally Sampled Signal

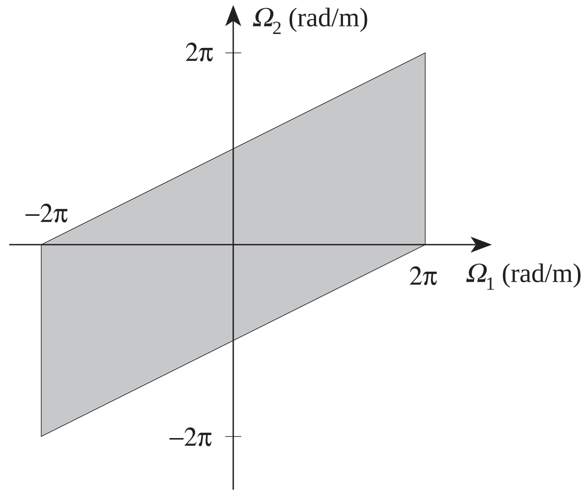

Figure 1:

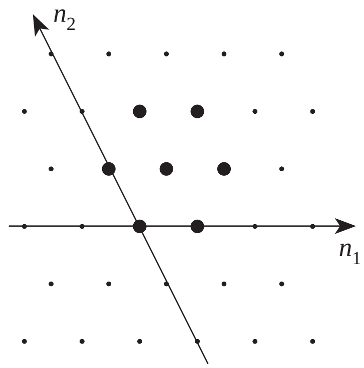

Figure 2:

Determine the convolution of the hexagonally sampled signal shown in Figure 2 with itself. The sequence values are 1 on the heavy dots and 0 on the light ones.

Problem 3.4 Circular and Linear Convolution

(a) Compute the discrete-space circular convolution of the two arrays

$$x_1(n_1, n_2) = \delta(n_1), 0 \le n_1 < N_1; 0 \le n_2 < N_2$$

$$x_2(n_1, n_2) = \delta(n_2), 0 \le n_1 < N_1; 0 \le n_2 < N_2$$

(b) Compute the discrete-space linear convolution of these arrays.

(c) Repeat parts (a) and (b), but replace $x_2(n_1, n_2)$ by

$$x_3(n_1, n_2) = \delta(n_1 - n_2)$$

and assume that $N_1 = N_2 = N$.

Problem 3.5 One-Dimensional Resampling

Prove that in one dimension, a cascade of an upsampler by L and a downsampler by M is commutative if and only if L and M are relatively prime.

An upsampler by L with input $x[n]$ and output $x_\mathrm{u}[n]$ is defined in the discrete-time domain as

$$x_\mathrm{u}[n] = \begin{cases} x[L^{-1}n] & \text{if } L^{-1}n \in \mathcal{I} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and is denoted by $\uparrow L$ or $\uparrow_n L$. An upsampler maps the input signal, $x[n]$, defined on $\mathcal{I}$, to the signal $x_\mathrm{u}[n]$ which is also defined on $\mathcal{I}$. The values of $x[n]$ are mapped to locations in $x_\mathrm{u}[n]$ which are on the sublattice(L). All the values of samples of $x_\mathrm{u}[n]$ not on the sublattice(L) are set to zero. The discrete-time Fourier transform relationship for an upsampler is

$$X_\mathrm{u}(\omega) = X(L\omega) \tag{2}$$

where ω is the discrete-time frequency variables.

A downsampler by M with input $x[n]$ and output $x_d[n]$ is defined in the discrete-time domain as

$$x_d[n] = x[Mn] \tag{3}$$

and is denoted by $\downarrow M$ or $\downarrow_n M$. A downsampler maps the input signal $x[n]$ to the output signal $x_d[n]$ by discarding samples of $x[n]$ that do not lie on sublattice(M) and then compacts sublattice(M) onto $\mathcal{I}$. Downsampling introduces aliasing and increases the sampling density (i.e. decreases the sampling rate) by a factor of M called the downsampling factor.

The equivalent input/output Fourier transform relationship is

$$X_d(\omega) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(M^{-1}(\omega - 2\pi k_i)\right) \tag{4}$$

where ω is the discrete-time frequency variables and k_i is a distinct coset of M such that $k_i = i$ for $i = 0, 1, \ldots, M-1$, so k_0 is zero. The term $2\pi M^{-1}k_i$ (for $i \neq 0$) is an *aliasing term.* Aliasing terms are periodic with a period of 2π in each dimension. A *normalized* aliasing term is an aliasing term with the 2π term removed so it is periodic with period 1.

Note that this theorem was proved by two separate groups of authors at the same IEEE conference in the 1980s. One group proved it by hand, and the other stumbled on the proof by using an expert system to rearrange operations in resampling structures.