Spatial Array Processing

Signal and Image Processing Seminar

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Introduction

- A sensor array is a group of sensors located at spatially separated points.

- Sensor array processing focuses on data collected at the sensors to carry out a given estimation task.

- Application Areas
  - Radar
  - Sonar
  - Seismic exploration
  - Anti-jamming communications
  - YES! Wireless communications
Problem Statement

Find

1. Number of sources

2. Their direction-of-arrivals (DOAs)

3. Signal Waveforms
Assumptions

- Isotropic and nondispersive medium
  - Uniform propagation in all directions

- Far-Field
  - Radius of propagation $>>$ size of array
  - Plane wave propagation

- Zero mean white noise and signal, uncorrelated

- No coupling and perfect calibration
Antenna Array

- **Array Response Vector—Far-Field Assumption**

  **Narrowband**
  - **Delay**  \[\Rightarrow\]  **Phase Shift**
  **Assumption**

  \[
a(\theta) = [1, e^{j2\pi f_c \triangle \sin \theta/c}, \ldots, e^{j2\pi f_c 4 \triangle \sin \theta/c}]^T
\]

- **Single Source Case**  \[\Rightarrow\]  **x(t)**

  \[
  \begin{bmatrix}
  x_1(t) \\
  x_2(t) \\
  \vdots \\
  x_M(t)
  \end{bmatrix} = \begin{bmatrix}
  s_1(t) \\
  s_1(t - \tau) \\
  \vdots \\
  s_1(t - (M - 1)\tau)
  \end{bmatrix} \approx \begin{bmatrix}
  1 \\
  e^{-j2\pi f_c \tau} \\
  \vdots \\
  e^{-j2\pi f_c (M-1)\tau}
  \end{bmatrix} s_1(t) = a(\theta_1)s_1(t)
  \]

  where  \[\tau = \triangle \sin \theta_1/c.\]
General Model

- By superposition, for $d$ signals,

$$x(t) = a(\theta_1)s_1(t) + \cdots + a(\theta_d)s_d(t)$$

$$= \sum_{k=1}^{d} a(\theta_k)s_k(t)$$

- Noise

$$x(t) = \sum_{k=1}^{d} a(\theta_k)s_k(t) + n(t)$$

$$= AS(t) + n(t)$$

where

$$A = [a(\theta_1), \ldots, a(\theta_d)]$$

and

$$S(t) = [s_1(t), \ldots, s_d(t)]^T.$$
Low-Resolution Approach: Beamforming

- Basic Idea

\[ x_i(t) = \sum_{k=1}^{d} e^{(i-1)(j2\pi f_c \Delta \sin \theta_k / c)} s_k(t) = \sum_{k=1}^{d} s_k(t)e^{jw_k(i-1)} \]

where \( w_k = 2\pi \Delta \sin(\theta_k) / c \) and \( i = 1, \ldots, M \).

- Use DFT (or FFT) to find the frequencies \( \{w_k\} \)

\[
F = [F(w_1) \cdots F(w_M)] = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
e^{jw_1} & e^{jw_2} & \cdots & e^{jw_M} \\
\vdots & \vdots & \ddots & \vdots \\
e^{j(M-1)w_1} & e^{j(M-1)w_2} & \cdots & e^{j(M-1)w_M}
\end{bmatrix}
\]

- Look for the peaks in

\[ |F(x_i(t))| = |F^*x(t)|^2 \]

- To smooth out noise

\[ B(w_i) = \frac{1}{N} \sum_{t=1}^{N} |F^*x(t)|^2 \]
Beamforming Algorithm

• Algorithm

1. Estimate $\mathbf{R}_x = \frac{1}{N} \sum_{t=1}^{N} \mathbf{x}(t)\mathbf{x}^*(t)$
2. Calculate $B(w_i) = \mathbf{F}^*(w_i)\mathbf{R}_x\mathbf{F}(w_i)$
3. Find peaks of $B(w_i)$ for all possible $w_i$'s.
4. Calculate $\theta_k, i = 1, \ldots, d$.

• Advantage
  - Simple and easy to understand

• Disadvantage
  - Low resolution
Number of Sources

- Detection of number of signals for $d < M$,

$$x(t) = As(t) + n(t)$$

$$R_x = E\{x(t)x^*(t)\} = A E\{s(t)s^*(t)\} A^* + E\{n(t)n^*(t)\}$$

$$= \underbrace{A}^{M \times d} \underbrace{R_s}_{d \times d} \underbrace{A^*}_{d \times M} + \sigma_n^2 I$$

where $\sigma_n^2$ is the noise power.

- No noise and rank of $R_s$ is $d$
  - Eigenvectors of $R_x = AR_s A^*$ will be
    $$\{\lambda_1, \ldots, \lambda_d, 0, \ldots, 0\}.$$  
  - Real positive eigenvalues because $R_x$ is real, Hermition-symmetric
  - rank $d$

- Check the rank of $R_x$ or its nonzero eigenvalues to detect the number of signals

- Noise eigenvalues are shifted by $\sigma_n^2$

  $$\{\lambda_1 + \sigma_n^2, \ldots, \lambda_d + \sigma_n^2, \sigma_n^2, \ldots, \sigma_n^2\}.$$  

  where $\lambda_1 > \ldots > \lambda_d$ and $\lambda > 0$

- Detect the number of principal (distinct) eigenvalues
MUSIC

- Subspace decomposition by performing eigenvalue decomposition

\[ R_x = AR_s A^* + \sigma_n^2 I = \sum_{k=1}^{M} \lambda_k e_k e_k^* \]

where \( e_k \) is the eigenvector of the \( \lambda_k \) eigenvalue

- \( \text{span}\{A\} = \text{span}\{e_1, \ldots, e_d\} = \text{span}\{E_s\} \)

- Check which \( a(\theta) \in \text{span}\{E_s\} \) or \( P_A a(\theta) \) or \( P_A^\perp a(\theta) \), where \( P_A \) is a projection matrix

- Search for all possible \( \theta \) such that

\[ |P_A^\perp a(\theta)|^2 = 0 \text{ or } M(\theta) = \frac{1}{P_A a(\theta)} = \infty \]

- After EVD of \( R_x \)

\[ P_A^\perp = I - E_s E_s^* = E_n E_n^* \]

where the noise eigenvector matrix

\[ E_n = [e_{d+1}, \ldots, e_M] \]
Root-MUSIC

• For a true $\theta$, $e^{j2\pi f_c} \triangleq \sin \theta/c$ is a root of

$$P(z) = \sum_{k=d+1}^{M} [1, z, \ldots, z^{M-1}]^T e_k^* e_k^* [1, z^{-1}, \ldots, z^{-(M-1)}].$$

• After eigenvalue decomposition,

- Obtain $\{e_k\}_{k=1}^{d}$
- Form $p(z)$
- Obtain $2M - 2$ roots by rooting $p(z)$
- Pick $d$ roots lying on the unit circle
- Solve for $\{\theta_k\}$
Estimation of Signal Parameters via Rotationally Invariant Techniques (ESPRIT)

- Decompose a uniform linear array of $M$ sensors into two subarrays with $M - 1$ sensors
- Note the shift invariance property

\[
a^{(2)}(\theta) = \begin{bmatrix}
e^{jw} \\
e^{j2w} \\
\vdots \\
e^{j(M-1)w}
\end{bmatrix} = \begin{bmatrix}
1 \\
e^{jw} \\
\vdots \\
e^{j(M-1)w}
\end{bmatrix} e^{jw} = a^{(1)} e^{jw}
\]

- General form relating subarray (1) to subarray (2)

\[
A^{(2)} = A^{(1)} \begin{bmatrix}
e^{jw_1} \\
\ddots \\
e^{jw_d}
\end{bmatrix} = A(1)\Phi.
\]

- $\Phi$ contains sufficient information of $\{\theta_k\}$
ESPRIT

- \( \text{span}\{E_s\} = \text{span}\{A\} \) and \( E_s = AT \)
  - \( T \) is a \( d \times d \) nonsingular unitary matrix
  - \( T \) comes from a Graham-Schmit orthogonalization of \( Ab \) in
    \[
    R_x = E_s \Lambda_s E^*_s + E_n \Lambda E^*_n
    \]
    \[
    A^H R_s A + \sigma^2_n I
    \]

- \( E_s^{(2)} = A^{(2)} T \) and \( E_s^{(1)} = A^{(1)} T \)
  \[
  E_s(2) = A^{(2)} T = A^{(1)} \Phi T = E_s(1) T^{-1} \Phi T
  \]

- Multiply both sides by the pseudo inverse of \( E_s^{(1)} \)
  \[
  E_s^{(1)} \# E_s(2) = (E^{(1)*} E^{(1)})^{-1} E^{(1)*} E^{(1)} T^{-1} \Phi T = T^{-1} \Phi T
  \]
  where \( \# \) means the pseudo-inverse
  \[
  A^\# = (A^{sH} A)^{-1} A^{sH}
  \]

- Eigenvalues of \( T^{-1} \Phi T \) are those of \( \Phi \).
Superresolution Algorithms

1. Calculate $R_x = \frac{1}{N} \sum_{k=1}^{N} x(k)x^*(k)$

2. Perform eigenvalue decomposition

3. Based on the distribution of $\{\lambda_k\}$, determine $d$

4. Use your favorite direction-of-arrival estimation algorithm:

   (a) MUSIC: Find the peaks of $M(\theta)$ for $\theta$ from 0 to $180^\circ$
       - Find $\{\hat{\theta}_k\}_{k=1}^{d}$ corresponding the $d$ peaks of $M(\cdot)$.

   (b) Root-MUSIC: Root the polynomial $p(z)$
       - Pick the $d$ roots that are closest to the unit circle $\{r_k\}_{k=1}^{d}$ and
         $\hat{\theta}_k = \sin^{-1} \frac{r_k c}{2\pi f_c \Delta}$.

   (c) ESPRIT: Find the eigenvalues of $E_s^{(1)} \# E_s^{(2)}$,

       $\{\phi_k\}$
       - $\hat{\theta}_k = \sin^{-1} \frac{\phi_k c}{2\pi f_c \Delta}$
Signal Waveform Estimation

• Given $A$, recover $s(t)$ from $x(t)$.

• Deterministic Method
  – No noise case: find $w_k$ such that
    \[ w_k \perp a(\theta_i), i \neq k, w_k \not\perp a(\theta_k) \]

• $A^\#$ can do the job
  \[ A^\#x(t) = A^\#As(t) = s(t) \]

• With noise, $n(t)$
  \[ A^\#x(t) = s(t) + A^\#n(t) \]
  – Disadvantage $\implies$ increased noise
Stochastic Approach

- Find $w_k$ to minimize

$$\min_{a^*(\theta_k)w_k = 1} E\{|w_k x(t)|^2\} = \min_{a^*(\theta_k)w_k = 1} w_k^* R_k w_k$$

- Use the Langrange method

$$\min_{a^*(\theta_k)w_k = 1} E\{|w_k x(t)|^2\} \Leftrightarrow \min_{\mu, w_k} w_k^* R_k w_k + 2\mu (a^*(\theta_k)w_k \Leftrightarrow 1)$$

- Differentiating it, we obtain

$$R_x w_k = \mu a(\theta_k), \text{or } w_k = \mu R_x^{-1} a(\theta_k)$$

- Since $a^*(\theta_k)w_k = \mu a^*(\theta_k)R_x^{-1} a(\theta_k) = 1$,

- Then

$$\mu = a^*(\theta_k)R_x^{-1} a(\theta_k)$$

- Capon’s Beamformer

$$w_k = R_x^{-1} a(\theta_k)/(a^*(\theta_k)R_x^{-1} a(\theta_k))$$
Subspace Framework for Sinusoid Detection

- \( x(t) = \sum_{k=1}^{d} \beta_k e^{(\alpha_k + j \omega_k) t} \)

- Let us select a window of \( M \), i.e.,
  \[
  \mathbf{x}(t) = [x(t), \ldots, x(t - M + 1)]^T
  \]

- Then
  \[
  \mathbf{x}(t) = \begin{bmatrix}
  x(t) \\
  x(t - 1) \\
  \vdots \\
  x(t - M + 1)
  \end{bmatrix}
  = \sum_{k=1}^{d}
  \begin{bmatrix}
  e^{(\alpha_k + j \omega_k)} \\
  e^{(\alpha_k + j \omega_k)(-1)} \\
  \vdots \\
  e^{(\alpha_k + j \omega_k)(-M+1)}
  \end{bmatrix}
  \begin{bmatrix}
  \beta_k e^{(\alpha_k + j \omega_k) t} \\
  \beta_k e^{(\alpha_k + j \omega_k)(t-1)} \\
  \vdots \\
  \beta_k e^{(\alpha_k + j \omega_k)(t-M+1)}
  \end{bmatrix}
  = \sum_{k=1}^{d} a(\rho_k) s_k(t)
  \]

  where \( M \) is the window size, \( d \) the number of sinusoids, and
  \[
  \rho_k = e^{\alpha_k + j \omega_k}.
  \]
Subspace Framework for Sinusoid Detection

- Therefore, the subspace methods can be applied to find \( \{\alpha_k + j\omega_k\} \)

- Recall

\[
x(t) = \sum_{k=1}^{d} \beta_k e^{(\alpha_k + j\omega_k)t}
\]

- Then finding \( \{\beta_k\} \) is a simple least squares problem.
• Increasing Demand for Wireless Services

• Unique Problems compared to Wired communications
Problems in Wireless Communications

- Scarce Radio Spectrum and Co-channel Interference

- Multipath

- Coverage/Range
Smart Antenna Systems

- Employ more than one antenna element and exploit the spatial dimension in signal processing to improve some system operating parameter(s):
  - Capacity, Quality, Coverage, and Cost.

![Smart Antenna System Diagram]

User One

Multiple RF Module

Advanced Signal Processing Algorithms

Conventional Communication Module

User Two
Experimental Validation of Smart Uplink Algorithm

- Comparison of constellation before (upper) and after smart uplink processing (middle and lower)
Selective Transmission Using DOAs

- Beamforming results for two sources separated by 20°
Selective Transmission Using DOAs

- Beamforming results for two sources separated by $3^\circ$

![Power Spectrum](Frequency [Hz], User #1)

![Power Spectrum](Frequency [Hz], User #2)
Future Directions

- Adapt the theoretical methods to fit the particular demands in specific applications
  - Smart Antennas
  - Synthetic aperture radar
  - Underwater acoustic imaging
  - Chemical sensor arrays
- Bridge the gap between theoretical methods and real-time applications