Spatial Array Processing

Signal and Image Processing Seminar

Murat Torlak

Telecommunications & Information Sys. Eng.

The University of Texas at Austin

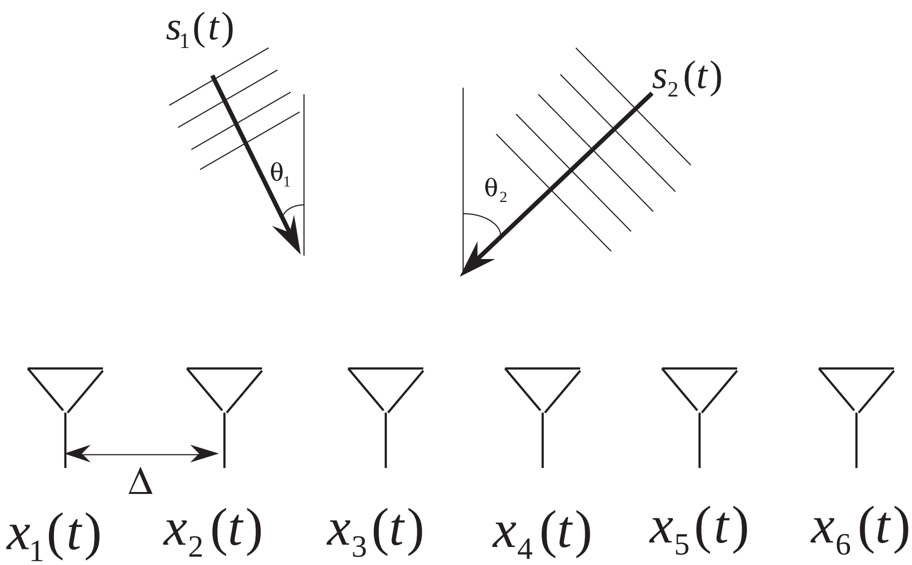
Introduction

- A sensor array is a group of sensors located at spatially separated points

- Sensor array processing focuses on data collected at the sensors to carry out a given estimation task

- Application Areas
  - Radar
  - Sonar
  - Seismic exploration
  - Anti-jamming communications
  - YES! Wireless communications

Problem Statement



Find

1. Number of sources

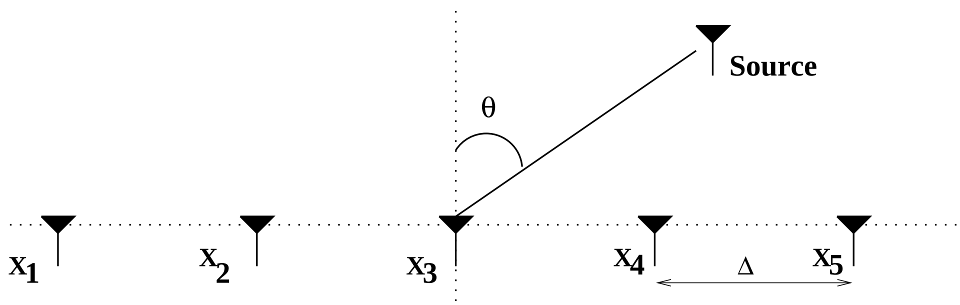
2. Their direction-of-arrivals (DOAs)

3. Signal Waveforms

Assumptions

- Isotropic and nondispersive medium
  - Uniform propagation in all directions
- Far-Field
  - Radius of propogation $>>$ size of array
  - Plane wave propogation
- Zero mean white noise and signal, uncorrelated
- No coupling and perfect calibration

Antenna Array



- Array Response Vector–Far-Field Assumption

  - Delay $\underset{\text{Assumption}}{\overset{\text{Narrowband}}{\Longrightarrow}}$ Phase Shift

$$\mathbf{a}(\theta) = [1, e^{j2\pi f_c \triangle \sin\theta / c}, \ldots, e^{j2\pi f_c 4\triangle \sin\theta / c}]^T$$

- Single Source Case $\Longrightarrow \mathbf{x}(t)$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_M(t) \end{bmatrix} = \begin{bmatrix} s_1(t) \\ s_1(t-\tau) \\ \vdots \\ s_1(t-(M-1)\tau) \end{bmatrix} \approx \begin{bmatrix} 1 \\ e^{-j2\pi f_c \tau} \\ \vdots \\ e^{-j2\pi f_c (M-1)\tau} \end{bmatrix} s_1(t) = \mathbf{a}(\theta_1) s_1(t)$$

where $\tau = \triangle \sin\theta_1 / c$.

General Model

- By superposition, for d signals,

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{a}(\theta_1)s_1(t) + \cdots + \mathbf{a}(\theta_d)s_d(t) \\ &= \sum_{k=1}^{d} \mathbf{a}(\theta_k)s_k(t)\end{aligned}$$

- Noise

$$\begin{aligned}\mathbf{x}(t) &= \sum_{k=1}^{d} \mathbf{a}(\theta_k)s_k(t) + \mathbf{n}(t) \\ &= \mathbf{AS}(t) + \mathbf{n}(t)\end{aligned}$$

where

$$\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_d)]$$

and

$$\mathbf{S}(t) = [s_1(t), \ldots, s_d(t)]^T.$$

Low-Resolution Approach:Beamforming

- Basic Idea

$$x_i(t) = \sum_{k=1}^{d} = e^{(i-1)(j2\pi f_c \triangle \sin\theta_k / c)} s_k(t) = \sum_{k=1}^{d} s_k(t) e^{j w_k (i-1)}$$

where $w_k = 2\pi \triangle \sin(\theta_k)/c$ and $i = 1, \ldots, M$.

- Use DFT (or FFT) to find the frequencies $\{w_k\}$

$$\mathbf{F} = [\mathbf{F}(w_1) \cdots \mathbf{F}(w_M)] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{jw_1} & e^{jw_2} & \cdots & e^{jw_M} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j(M-1)w_1} & e^{j(M-1)w_2} & \cdots & e^{j(M-1)w_M} \end{bmatrix}$$

- Look for the peaks in

$$|\mathcal{F}(x_i(t))| = |\mathbf{F}^* \mathbf{x}(t)|^2$$

- To smooth out noise

$$B(w_i) = \frac{1}{N} \sum_{t=1}^{N} |\mathbf{F}^* \mathbf{x}(t)|^2$$

Beamforming Algorithm

- Algorithm

  1. Estimate $\mathbf{R}_x = \frac{1}{N}\sum_{t=1}^{N}\mathbf{x}(t)\mathbf{x}^*(t)$
  2. Calculate $B(w_i) = \mathbf{F}^*(w_i)\mathbf{R}_x\mathbf{F}(w_i)$
  3. Find peaks of $B(w_i)$ for all possible w_i's.
  4. Calculate θ_k, $i = 1, \ldots, d$.

- Advantage

  - Simple and easy to understand

- Disadvantage

  - Low resolution

Number of Sources

- Detection of number of signals for $d < M$,

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t)$$

$$\begin{aligned}\mathbf{R}_x &= E\{\mathbf{x}(t)\mathbf{x}^*(t)\} = \mathbf{A}\underbrace{E\{\mathbf{s}(t)\mathbf{s}^*(t)\}}_{\mathbf{R}_s}\mathbf{A}^* + \underbrace{E\{\mathbf{n}(t)\mathbf{n}^*(t)\}}_{\sigma_n^2\mathbf{I}} \\ &= \underbrace{\mathbf{A}}_{M\times d}\underbrace{\mathbf{R}_s}_{d\times d}\underbrace{\mathbf{A}^*}_{d\times M} + \sigma_n^2\mathbf{I}\end{aligned}$$

  where σ_n^2 is the noise power.

- No noise and rank of $\mathbf{R}_s$ is d
  - Eigenvalues of $\mathbf{R}_x = \mathbf{A}\mathbf{R}_s\mathbf{A}^*$ will be

$$\{\lambda_1, \ldots, \lambda_d, 0, \ldots, 0\}.$$

  - Real positive eigenvalues because $\mathbf{R}_x$ is real, Hermition-symmetric
  - rank d

- Check the rank of $\mathbf{R}_x$ or its nonzero eigenvalues to detect the number of signals

- Noise eigenvalues are shifted by σ_n^2

$$\{\lambda_1 + \sigma_n^2, \ldots, \lambda_d + \sigma_n^2, \sigma_n^2, \ldots, \sigma_n^2\}.$$

  where $\lambda_1 > \ldots > \lambda_d$ and $\lambda >> 0$

- Detect the number of principal (distinct) eigenvalues

MUSIC

- Subspace decomposition by performing eigenvalue decomposition

$$\mathbf{R}_x = \mathbf{A}\mathbf{R}_s\mathbf{A}^* + \sigma_n^2\mathbf{I} = \sum_{k=1}^{M} \lambda_k \mathbf{e}_k \mathbf{e}_k^*$$

  where $\mathbf{e}_k$ is the eigenvector of the λ_k eigenvalue

- $span\{\mathbf{A}\} = span\{\mathbf{e}_1, \ldots, \mathbf{e}_d\} = span\{\mathbf{E}_s\}$

- Check which $\mathbf{a}(\theta) \, \epsilon \, span\{\mathbf{E}_s\}$ or $\mathbf{P}_\mathbf{A}\mathbf{a}(\theta)$ or $\mathbf{P}_\mathbf{A}^\perp \mathbf{a}(\theta)$, where $\mathbf{P}_\mathbf{A}$ is a projection matrix

- Search for all possible θ such that

$$|\mathbf{P}_\mathbf{A}^\perp \mathbf{a}(\theta)|^2 = 0 \text{ or } M(\theta) = \frac{1}{\mathbf{P}_\mathbf{A}\mathbf{a}(\theta)} = \infty$$

- After EVD of $\mathbf{R}_x$

$$\mathbf{P}_\mathbf{A}^\perp = \mathbf{I} - \mathbf{E}_s\mathbf{E}_s^* = \mathbf{E}_n\mathbf{E}_n^*$$

  where the noise eigenvector matrix $\mathbf{E}_n = [\mathbf{e}_{d+1}, \ldots, \mathbf{e}_M]$

Root-MUSIC

- For a true θ, $e^{j2\pi f_c \triangle \sin\theta / c}$ is a root of

$$P(z) = \sum_{k=d+1}^{M} [1, z, \ldots, z^{M-1}]^T \mathbf{e}_k \mathbf{e}_k^* [1, z^{-1}, \ldots, z^{-(M-1)}].$$

- After eigenvalue decomposition,
  - Obtain $\{\mathbf{e}_k\}_{k=1}^{d}$
  - Form $p(z)$
  - Obtain $2M - 2$ roots by rooting $p(z)$
  - Pick d roots lying on the unit circle
  - Solve for $\{\theta_k\}$

Estimation of Signal Parameters via Rotationally Invariant Techniques (ESPRIT)

- Decompose a uniform linear array of M sensors into two subarrays with $M-1$ sensors
- Note the shift invariance property

$$\mathbf{a}^{(2)}(\theta) = \begin{bmatrix} e^{jw} \\ e^{j2w} \\ \vdots \\ e^{j(M-1)w} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{jw} \\ \vdots \\ e^{j(M-1)w} \end{bmatrix} e^{jw} = \mathbf{a}^{(1)} e^{jw}$$

- General form relating subarray (1) to subarray (2)

$$\mathbf{A}^{(2)} = \mathbf{A}^{(1)} \begin{bmatrix} e^{jw_1} & & \\ & \ddots & \\ & & e^{jw_d} \end{bmatrix} = \mathbf{A}(1)\Phi.$$

- Φ contains sufficient information of $\{\theta_k\}$

ESPRIT

- $span\{\mathbf{E}_s\} = span\{\mathbf{A}\}$ and $\mathbf{E}_s = \mathbf{A}\mathbf{T}$
  - $\mathbf{T}$ is a $d \times d$ nonsingular unitary matrix
  - $\mathbf{T}$ comes from a Grahm-Schmit orthogonalization of Ab in

$$\begin{aligned} \mathbf{R}_x &= \mathbf{E}_s \Lambda_s \mathbf{E}_s^* + \mathbf{E}_n \Lambda \mathbf{E}_n^* \\ & \mathbf{A}^H \mathbf{R}_s \mathbf{A} + \sigma_n^2 \mathbf{I} \end{aligned}$$

- $\mathbf{E}_s^{(2)} = \mathbf{A}^{(2)}\mathbf{T}$ and $\mathbf{E}_s^{(1)} = \mathbf{A}^{(1)}\mathbf{T}$

$$\mathbf{E}_s(2) = \mathbf{A}^{(2)}\mathbf{T} = \mathbf{A}^{(1)}\mathbf{\Phi}\mathbf{T} = \mathbf{E}_s(1)\mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T}$$

- Multiply both sides by the pseudo inverse of $\mathbf{E}_s^{(1)}$

$$\mathbf{E}_s^{(1)\#}\mathbf{E}_s(2) = (\mathbf{E}^{(1)*}\mathbf{E}^{(1)})^{-1}\mathbf{E}^{(1)*}\mathbf{E}^{(1)}\mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T} = \mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T}$$

where $\#$ means the pseudo-inverse

$$\mathbf{A}^{\#} = (\mathbf{A}^{sH}\mathbf{A})^{-1}\mathbf{A}^{sH}$$

- Eigenvalues of $\mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T}$ are those of Φ.

Superresolution Algorithms

1. Calculate $\mathbf{R}_x = \frac{1}{N}\sum_{k=1}^{N}\mathbf{x}(k)\mathbf{x}^*(k)$

2. Perform eigenvalue decomposition

3. Based on the distribution of $\{\lambda_k\}$, determine d

4. Use your favorite diraction-of-arrival estimation algorithm:

   (a) MUSIC: Find the peaks of $M(\theta)$ for θ from 0 to 180°

   - Find $\{\hat{\theta}_k\}_{k=1}^{d}$ corresponding the d peaks of $M(\cdot)$.

   (b) Root-MUSIC: Root the polynomial $p(z)$

   - Pick the d roots that are closest to the unit circle $\{r_k\}_{k=1}^{d}$ and $\hat{\theta}_k = \sin^{-1}\frac{r_k c}{2\pi f_c \Delta}$.

   (c) ESPRIT: Find the eigenvalues of $\mathbf{E}_s^{(1)\#}\mathbf{E}_s^{(2)}$, $\{\phi_k\}$

   - $\hat{\theta}_k = \sin^{-1}\frac{\phi_k c}{2\pi f_c \triangle}$

Signal Waveform Estimation

- Given $\mathbf{A}$, recover $\mathbf{s(t)}$ from $\mathbf{x}(t)$.
- Deterministic Method
  - No noise case: find $\mathbf{w}_k$ such that

$$\mathbf{w}_k \perp \mathbf{a}(\theta_i), i \neq k, \mathbf{w}_k \not\perp \mathbf{a}(\theta_k)$$

- $\mathbf{A}^{\#}$ can do the job

$$\mathbf{A}^{\#}\mathbf{x}(t) = \mathbf{A}^{\#}\mathbf{A}\mathbf{s(t)} = \mathbf{s(t)}$$

- With noise, $\mathbf{n}(t)$

$$\mathbf{A}^{\#}\mathbf{x}(t) = \mathbf{s(t)} + \mathbf{A}^{\#}\mathbf{n(t)}$$

  - Disadvantage $\Longrightarrow$ increased noise

Stocastic Approach

- Find $\mathbf{w}_k$ to minimize

$$\min_{\mathbf{a}^*(\theta_k)\mathbf{w}_k=1} E\{|\mathbf{w}_k\mathbf{x}(t)|^2\} = \min_{\mathbf{a}^*(\theta_k)\mathbf{w}_k=1} \mathbf{w}_k^*\mathbf{R}_k\mathbf{w}_k$$

- Use the Langrange method

$$\min_{\mathbf{a}^*(\theta_k)\mathbf{w}_k=1} E\{|\mathbf{w}_k\mathbf{x}(t)|^2\} \Leftrightarrow \min_{\mu,\mathbf{w}_k} \mathbf{w}_k^*\mathbf{R}_k\mathbf{w}_k + 2\mu(\mathbf{a}^*(\theta_k)\mathbf{w}_k \Leftrightarrow 1)$$

- Differentiating it, we obtain

$$\mathbf{R}_x\mathbf{w}_k = \mu\mathbf{a}(\theta_k), or \mathbf{w}_k = \mu\mathbf{R}_x^{-1}\mathbf{a}(\theta_k)$$

.

- Since $\mathbf{a}^*(\theta_k)\mathbf{w}_k = \mu\mathbf{a}^*(\theta_k)\mathbf{R}_x^{-1}\mathbf{a}(\theta_k) = 1$,
- Then

$$\mu = \mathbf{a}^*(\theta_k)\mathbf{R}_x^{-1}\mathbf{a}(\theta_k)$$

- Capon's Beamformer

$$\mathbf{w}_k = \mathbf{R}_x^{-1}\mathbf{a}(\theta_k)/(\mathbf{a}^*(\theta_k)\mathbf{R}_x^{-1}\mathbf{a}(\theta_k))$$

Subspace Framework for Sinusoid Detection

- $x(t) = \sum_{k=1}^{d} \beta_k e^{(\alpha_k + j\omega_k)t}$
- Let us select a window of M, *i.e.,*

$$\mathbf{x}(t) = [x(t), \ldots, x(t-M+1)]^T$$

- Then

$$\begin{aligned}
\mathbf{x}(t) &= \begin{bmatrix} x(t) \\ x(t-1) \\ \vdots \\ x(t-M+1) \end{bmatrix} = \sum_{k=1}^{d} \begin{bmatrix} \beta_k e^{(\alpha_k + j\omega_k)t} \\ \beta_k e^{(\alpha_k + j\omega_k)(t-1)} \\ \vdots \\ \beta_k e^{(\alpha_k + j\omega_k)(t-M+1)} \end{bmatrix} \\
&= \sum_{k=1}^{d} \underbrace{\begin{bmatrix} 1 \\ e^{-(\alpha_k + j\omega_k)} \\ \vdots \\ e^{(\alpha_k + j\omega_k)(-M+1)} \end{bmatrix}}_{\mathbf{a}(\rho_k)} \underbrace{\beta_k e^{(\alpha_k + j\omega_k)t}}_{s_k(t)} \\
&= \sum_{k=1}^{d} \mathbf{a}(\rho_k) s_k(t) = \mathbf{A}\mathbf{s}(t),
\end{aligned}$$

where M is the window size, d the number of sinusoids, and $\rho_k = e^{\alpha_k + j\omega_k}$.

Subspace Framework for Sinusoid Detection

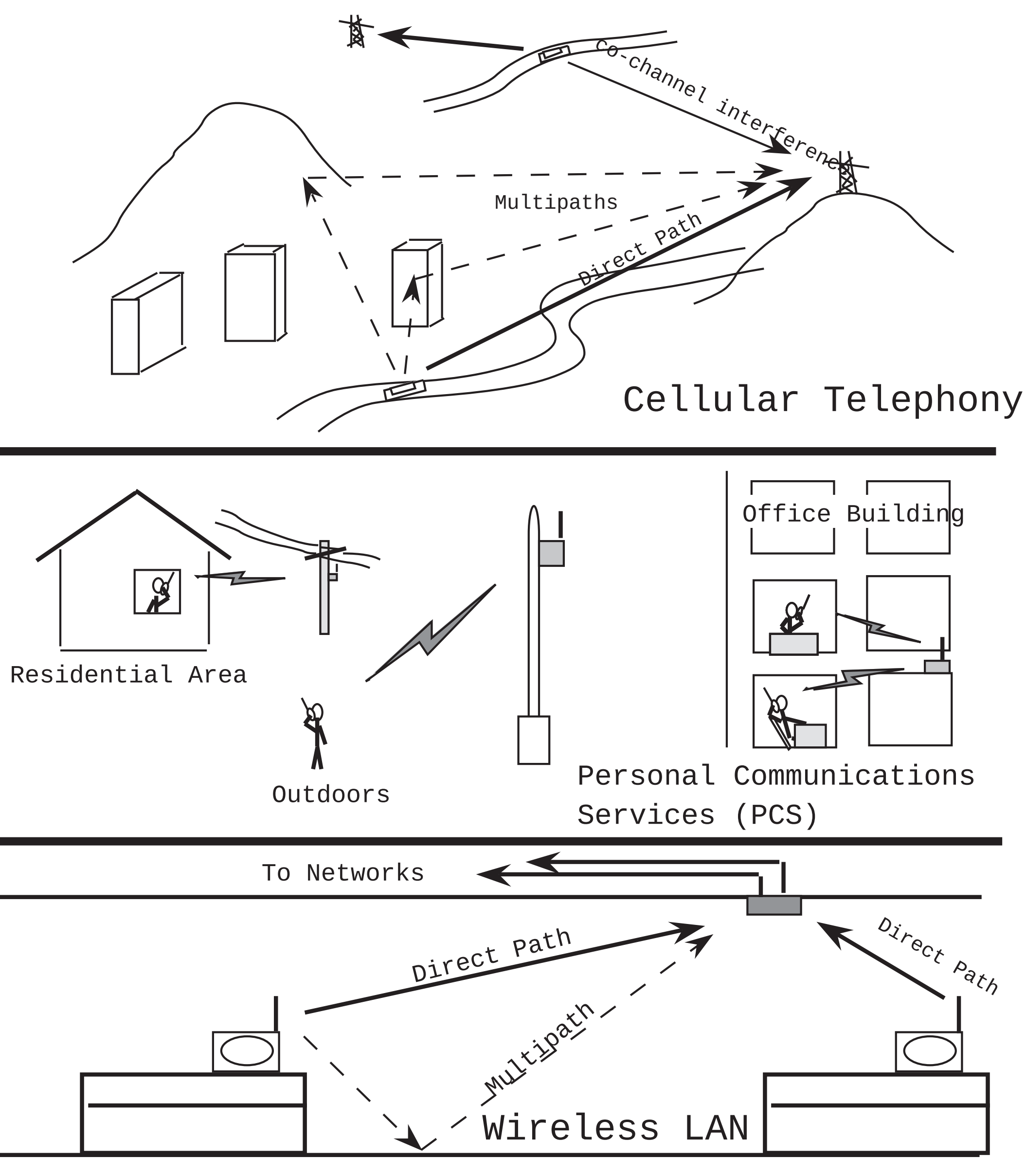
- Therefore, the subspace methods can be applied to find $\{\alpha_k + j\omega_k\}$

- Recall

$$x(t) = \sum_{k=1}^{d} \beta_k e^{(\alpha_k + j\omega_k)t}$$

- Then finding $\{\beta_k\}$ is a simple least squares problem.

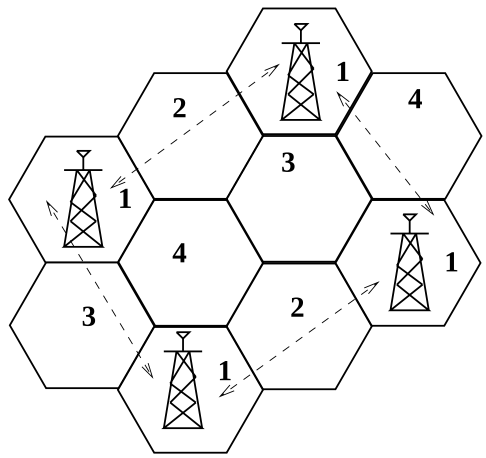
Wireless Communications



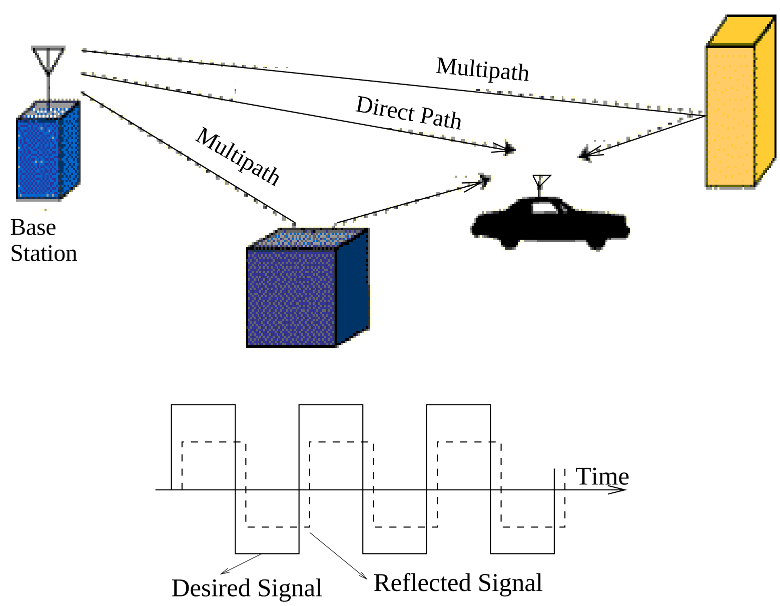
- Increasing Demand for Wireless Services
- Unique Problems compared to Wired communications

Problems in Wireless Communications

- Scarce Radio Spectrum and Co-channel Interference



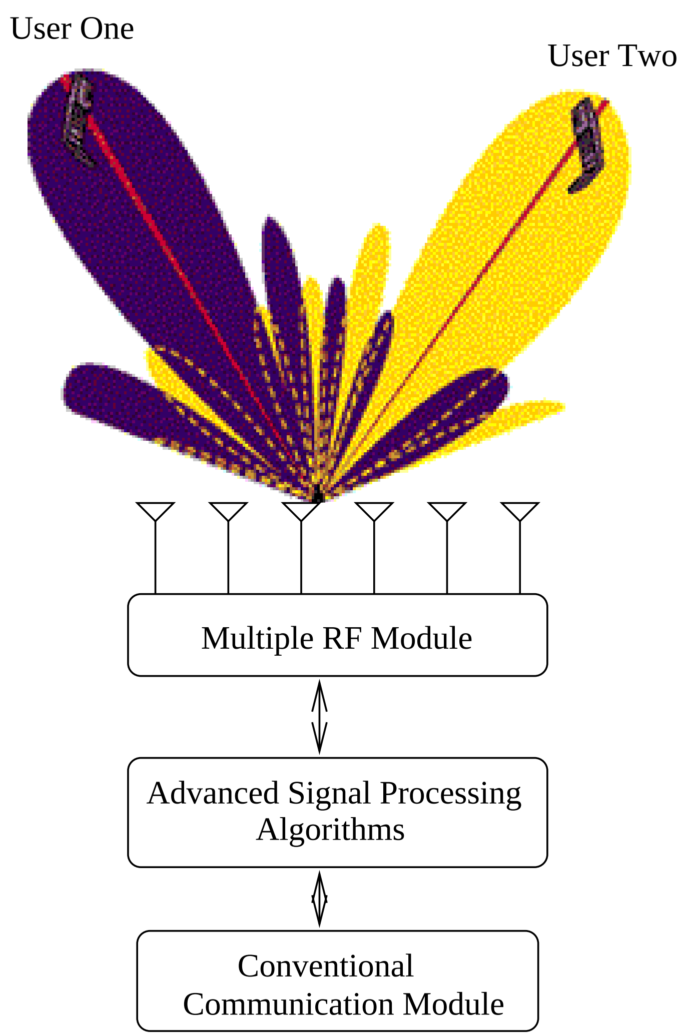
- Multipath



- Coverage/Range

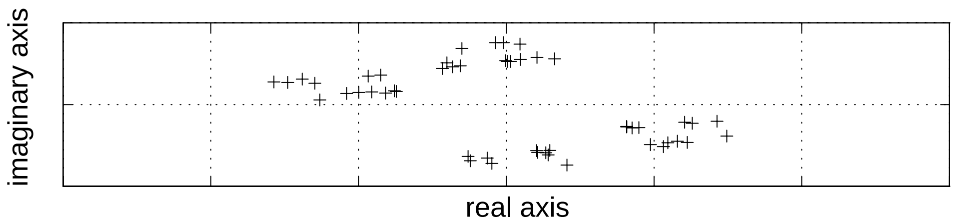
Smart Antenna Systems

- Employ more than one antenna element and exploit the spatial dimension in signal processing to improve some system operating parameter(s):
  - Capacity, Quality, Coverage, and Cost.

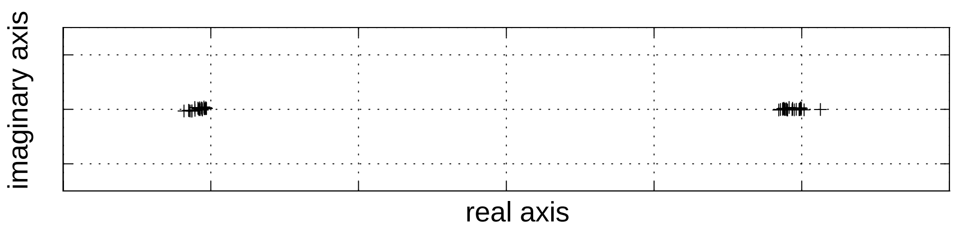


Experimental Validation of Smart Uplink Algorithm

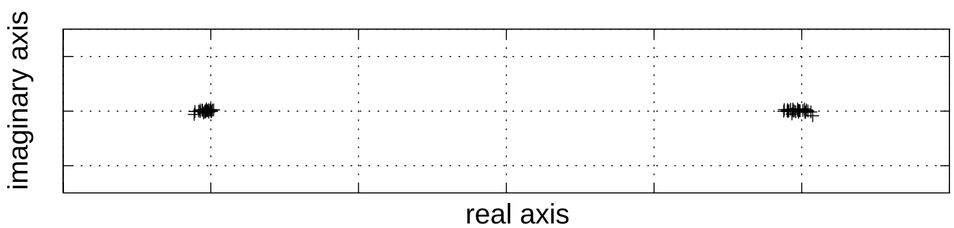
- Comparison of constellation before (upper) and after smart uplink processing (middle and lower)



Antenna Output



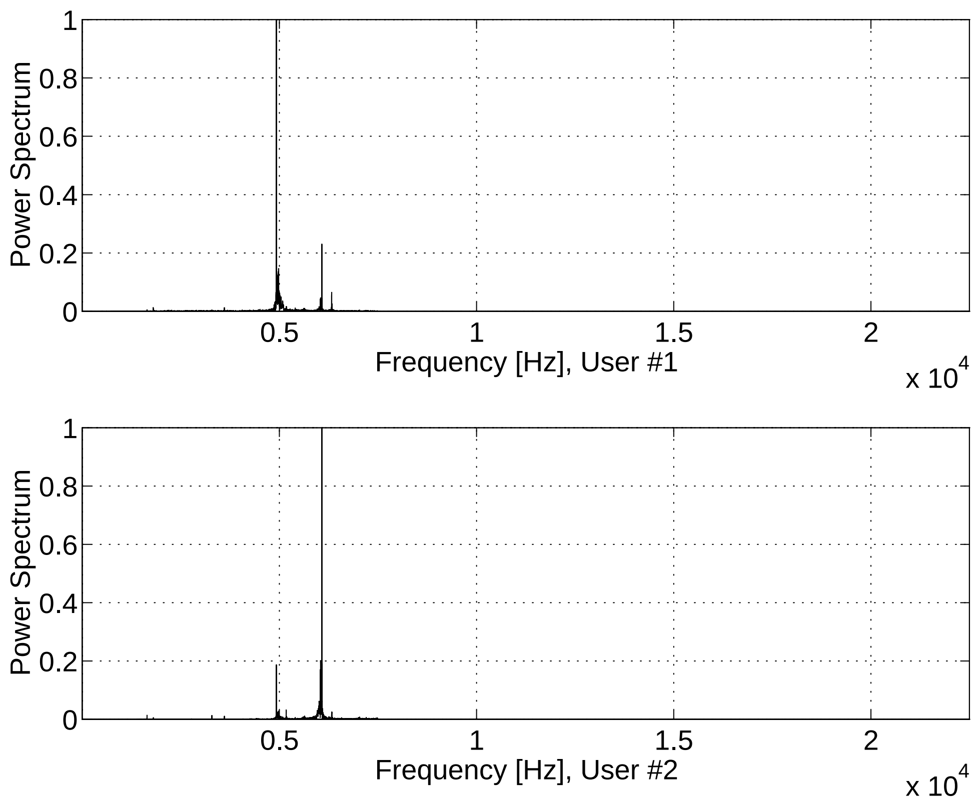
Equalized Signal 1



Equalized Signal 2

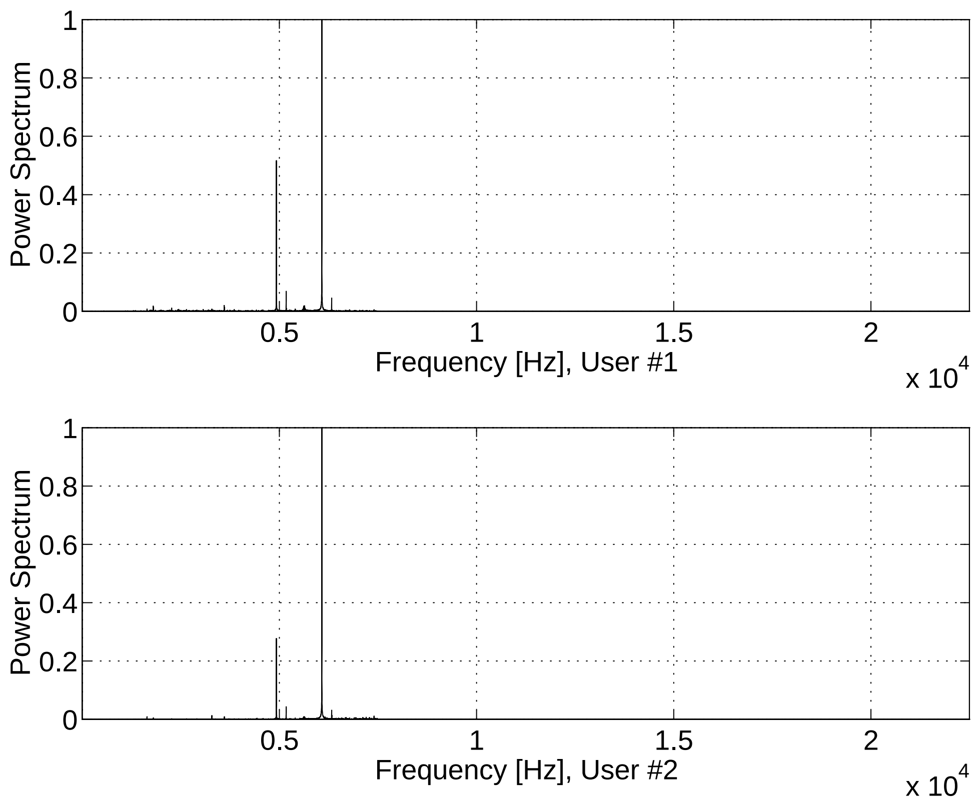
Selective Transmission Using DOAs

- Beamforming results for two sources separated by 20°



Selective Transmission Using DOAs

- Beamforming results for two sources separated by 3°



Future Directions

- Adapt the theoretical methods to fit the particular demands in specific applications
  - Smart Antennas
  - Synthetic aperture radar
  - Underwater acoustic imaging
  - Chemical sensor arrays
- Bridge the gap between theoretical methods and real-time applications