Infinite Impulse Response Filters

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Introduction

• Background

• Bounded Input, Bounded Output (BIBO) stability of quarter-plane IIR filters

• Not all output masks are recursively computable: Figure 1.

\[
\begin{array}{cccccccc}
\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\
\cdot \cdot \cdot \circ \cdot \cdot \cdot \cdot \cdot \\
\cdot \cdot \cdot \cdot \cdot \cdot \circ \cdot \\
\end{array}
\]

Figure 1: Two examples of not recursive computable masks
Background

- 2-D difference equations

\[ y(n_1, n_2) = \sum_{r_1} \sum_{r_2} a(n_1 - r_1, n_2 - r_2) \cdot x(r_1, r_2) \]
\[ - \sum_{k_1 \neq n_1} \sum_{k_2 \neq n_2} b(n_1 - k_1, n_2 - k_2) \cdot y(k_1, k_2) \]

- Input and output masks

  - \( a(n_1, n_2) \) is called the support of the input mask
  - \( b(n_1, n_2) \) is called the support of the output mask
  - Angle of support \( \beta \): minimum angle enclosing support of mask
  - \( y(n_1, n_2) \) is computed as in Figure 2

\[ n_2 \]
\[ n_1 \]
\[ \text{Weight and Sum} \]
\[ + \]
\[ \text{Weight and Sum} \]

Figure 2: Recursive computation.

- What degrees of freedom exist for moving the output mask?
- What must the boundary conditions be?
Recursive Computation

- Consider a four-tap all-pole IIR filter to be passed over an $N \times N$ image in a raster scan fashion, e.g. in Floyd-Steinberg error diffusion halftoning.
- What are the boundary conditions?
- How many rows of the image do we need to keep in memory at one time?
- What parallelism exists if any?
- What is the tradeoff between the amount of parallelism exploited and memory?
Recursive Computability

- Recursive computability: Computing the difference equation using known values of the shifted output samples and initial conditions.
  - Support of output mask
  - Initial conditions
    * Initial conditions must be 0, and must lie outside the support of the output sequence, for the filter to be Linear Shift Invariant (LSI).
  - Order of recursion

- Types of recursively computable masks
  - Quarter-plane masks: supported in a quadrant in $\mathbb{R}^2$
  - Non-Symmetric Half Plane (NSHP): supported in a half-plane in $\mathbb{R}^2$

- Not all possible orderings are equivalent
  - Amount of storage
  - Degree of parallelism
Boundary Conditions

- For a recursive system, how do you choose the boundary conditions.

  - If the system is to be LTI, the initial conditions must be zero outside the support of the filter.

- Example: \( y[n_1, n_2] = y[n_1 - 1, n_2] + y[n_1 + 1, n_2 - 1] + x[n_1, n_2] \)

  - Now assume a different set of boundary conditions: Figure 3.

  \[
  \begin{array}{cccccc}
  0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
  \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\
  \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
  \cdot & \cdot & \cdot & 0 & 0 & 0 \\
  \cdot & \cdot & \cdot & \cdot & 0 & 0 \\
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  \end{array}
  \quad
  \begin{array}{cccccc}
  0 & 1 & 4 & 10 & 20 & 35 \\
  0 & 1 & 3 & 6 & 10 & 15 \\
  0 & 1 & 2 & 3 & 4 & 5 \\
  \cdot & \cdot & 0 & 1 & 1 & 1 \\
  \cdot & \cdot & 0 & 0 & 0 & 0 \\
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  \end{array}
  \]

Figure 3: Boundary conditions (left) Response of \( y[n_1, n_2] \) to \( \delta[n_1, n_2] \) (right)

- Response to \( \delta[n_1, n_2] \): Figure 3

- Response to \( \delta[n_1 - 1, n_2 - 1] \): Figure 4

- This is a shifted version of the above: consistent with support of the impulse response

- Let's begin guessing at some boundary conditions: Figure 4

- Calculated response to \( \delta[n_1, n_2] \): Figure 5
Figure 4: Response of $y[n_1, n_2]$ to $\delta[n_1 - 1, n_2 - 1]$ (left) Another boundary condition (right)

Figure 5: Response of $y[n_1, n_2]$ to $\delta[n_1, n_2]$ (left) Response of $y[n_1, n_2]$ to $\delta[n_1 - 1, n_2 - 1]$ (right)

- Response to $\delta[n_1 - 1, n_2 - 1]$: Figure 5
- Note: This is not a shifted version of the response $\delta[n_1, n_2]$: These boundary conditions do not lead to an LTI system.
The Transfer Function

\[ H(z_1, z_2) = \sum_{n_1} \sum_{n_2} h[n_1, n_2] z_1^{-n_1} z_2^{-n_2} \]

- The \( z \)-transform will not converge for all values of \((z_1, z_2)\)

- If it converges for \( z_1 = e^{j\omega_1} \), \( z_2 = e^{j\omega_0} \), then the Discrete-Time Fourier Transform exists and the system is stable.

\[ |z_1| = 1 \text{ and } |z_2| = 1 \Rightarrow \text{Unit Bicircle} \]

Properties of the \( z \)-transform

- Separable Signals: \( v[n_1] w[n_2] \iff V(z_1) W(z_2) \)

- Linearity

- Shift

- Convolution

- Linear mapping (look familiar?)

\[
x[n_1, n_2] = \begin{cases} 
  w[m_1, m_2] & n_1 = Im_1 + Jm_2, n_2 = Km_1 + Lm_2 \\
  0 & \text{otherwise} 
\end{cases}
\]

\( IL - JK \neq 0 \)

\[ X(z_1, z_2) = W(z_1^I z_2^K, z_1^J z_2^L) ; \]

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Z-transform of Linear Mappings

- Linear mapping

\[ x[n_1, n_2] = \begin{cases} 
  w[m_1, m_2] & n_1 = Im_1 + Jm_2, n_2 = Km_1 + Lm_2 \\
  0 & \text{otherwise}
\end{cases} \]

\[ IL - JK \neq 0 \]

\[ X(z_1, z_2) = W(z_1^J z_2^K, z_1^J z_2^L) \]

- Notice the regular insertion of zeros

- This is upsampling by upsampling matrix

\[ R = \begin{bmatrix}
  I & J \\
  K & L
\end{bmatrix} \]

where \( \det R = IL - JK \neq 0 \)

- Frequency response \( X(\omega) \) of an upsampler to input \( W(\omega) \)

\[ X_u(\omega) = X(R^t \omega) \]

- Let \( z_1 = e^{j\omega_1} \) and \( z_2 = e^{j\omega_2} \)

\[ X(z_1, z_2) = W(z_1^J z_2^K, z_1^J z_2^L) \]

\[ X(e^{j\omega_1}, e^{j\omega_2}) = W(e^{jL\omega_1} e^{jK\omega_2}, e^{jL\omega_1} e^{jL\omega_2}) \]

\[ R^t = \begin{bmatrix}
  I & K \\
  J & L
\end{bmatrix} \]
Inverse $z$-transform

\[ x[n_1, n_2] = \frac{1}{(2\pi j)^2} \oint_{c_2} \oint_{c_1} X(z_1, z_2) z_1^{n_1-1} z_2^{n_2-1} dz_1 dz_2 \]

\[ X(z_1, z_2) = \frac{1}{1 - az_1^{-1} - bz_2^{-1}} \quad |a| + |b| \leq 1 \]

- Choose ROC that includes the unit bicircle.

\[ x[n_1, n_2] = \frac{1}{(2\pi j)^2} \oint_{c_2} \oint_{c_1} \frac{z_1^{n_1} z_2^{n_2}}{z_1 z_2 - a z_2 - b z_1} dz_1 dz_2 \]

\[ = \frac{1}{(2\pi j)^2} \oint_{c_2} \oint_{c_1} \frac{1}{z_2 - b} \frac{z_1^{n_1}}{z_1 - \frac{az_2}{z_2 - b}} dz_1 dz_2 \]

\[ = \frac{1}{(2\pi j)^2} \oint_{c_2} \frac{1}{z_2 - b} \oint_{c_1} \frac{z_1^{n_1}}{z_1 - \frac{az_2}{z_2 - b}} dz_1 dz_2 \]

- The inner integral (with respect to $z_1$) is the inverse $z$-transform of a first-order system with a pole at $\frac{az_2}{z_2 - b}$

\[ x[n_1, n_2] = \frac{1}{2\pi j} \oint_{c_2} \frac{z_2^{n_2}}{z_2 - b} \left( \frac{az_2}{z_2 - b} \right)^n u(n_1) dz_2 \]

\[ = a^{n_1} u[n_1] \frac{1}{2\pi j} \oint_{c_2} \frac{z_2^{n_1+n_2}}{(z_2 - b)^{n_1+1}} dz_2 \]

\[ = \frac{(n_1 + n_2)!}{n_1! n_2!} a^{n_1} b^{n_2} u[n_1, n_2] \]

- What does this tell us about other examples? very little

- Consider an example:

\[ H(z_1, z_2) = \frac{1}{1 - az_1^{-1} - bz_2^{-1} - c z_1^{-1} z_2^{-1}} \]

\[ B(e^{j\omega_1}, z_2) = 1 - ae^{-j\omega_1} - bz_2^{-1} - ce^{-j\omega_1} z_2^{-1} \]
Setting $B(e^{j\omega_1}, z_2) = 0$ and solving gives

$$z_2 = \frac{b + ce^{-j\omega_1}}{1 - ae^{-j\omega_1}}$$

This is a bilinear transformation which must map circles into circles.

![Figure 6: Bilinear transform](image)

Point A $\omega_1 = \pi \quad z_2 = \frac{b - c}{1 + a}$

Point B $\omega_1 = 0 \quad z_2 = \frac{b + c}{1 - a}$

$$\left| \frac{b - c}{1 + a} \right| < 1 \quad \left| \frac{b + c}{1 - a} \right| < 1$$

From the other part of the locus

$$\left| \frac{a - c}{1 + b} \right| < 1 \quad \left| \frac{a + b}{1 - b} \right| < 1$$