Stabilization and Stability Testing of Multidimensional Recursive Digital Filters

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Introduction

- Stability of general IIR filters
- Stability tests
  - Graphical root locus techniques
  - FFT based cepstral methods
- Stabilization of unstable filters based on
  - Double Planar Least Squares Inversion (DPLSI)
  - Discrete Hilbert Transform (DHT)

The Transfer Function

$$H(z_1, z_2) \;=\; \sum_{n_1} \sum_{n_2} h(n_1, n_2)\, z_1^{-n_1} z_2^{-n_2}$$

- The z-transform will not converge for all values of (z_1, z_2)
- If it converges for $z_1 = e^{j\omega_1}$, $z_2 = e^{j\omega_2}$, then the Discrete-Time Fourier Transform exists and the system is stable.

$$|z_1| = 1 \text{ and } |z_2| = 1 \Rightarrow \text{Unit Bicircle}$$

Stability of 2-D LSI Systems

- Bound-Input, Bounded-Output (BIBO) criterion

$$\text{if } |x(n_1, n_2)| < P < \infty \;\; \text{then} \;\; |y(n_1, n_2)| < Q < \infty$$

- Spatial domain necessary and sufficient condition for BIBO stability

$$\sum_{n_1} \sum_{n_2} |h(n_1, n_2)| = S < \infty$$

Implies $H(z_1, z_2)$ is analytic on the unit bicircle

- For a rational transfer function

$$T(z) = \frac{A(z)}{B(z)}$$

of a causal system, recall that the stability condition is that all roots of $B(z)$ should be inside unit circle

Effect of Numerator Polynomial on Stability:

[*Goodman*]

- No effect in 1-D case: Use factorization theorem

$$T(z) = \frac{a(z^{-1} - c_1)(z^{-1} - c_2)\cdots(z^{-1} - c_m)}{b(z^{-1} - d_1)(z^{-1} - d_2)\cdots(z^{-1} - d_n)}$$

- Situation is not so simple in 2-D

$$G_1(z) = \frac{(1 - z_1^{-1})^8(1 - z_2^{-1})^8}{1 - 0.5z_1^{-1} - 0.5z_2^{-1}}$$

$$G_2(z) = \frac{(1 - z_1^{-1})(1 - z_2^{-1})}{1 - 0.5z_1^{-1} - 0.5z_2^{-1}}$$

- What happens when $z_1 = z_2 = 1$?
  - Note the indeterminate forms
  - Goodman established that $G_1(z)$ is unstable while $G_2(z)$ is stable
  - Such singularities are called non-essential singularities of the second kind.
  - There is no known method to test for stability in the presence of such singularities

Necessary and Sufficient Conditions for Stability

- Let $\vec{z} = \{z_1, z_2, \ldots, z_N\}$
- (*Shanks and Justice*) Let $B(\vec{z})$ be the denominator polynomial of a first quadrant multidimensional recursive digital filter. The filter is stable if and only if $B(\vec{z}) \neq 0$ whenever
  $|z_k| \geq 1, k = 1, 2, \ldots, N$ simultaneously.
  Disadvantage: Whole exterior of the unit bicircle must be searched for points of singularity.
- (*DeCarlo-Strintzis)* $B(\vec{z})$ is stable if and only if
  1. $B(\vec{z}) \neq 0$ for $\vec{z} \in T^n$ where $T^n = \{|z_1| = 1, |z_2| = 1, \ldots |z_N| = 1\}$ and
  2. $B(z, z, \ldots z) \neq 0, \ |z| \geq 1$
     This second condition is equivalent to
     $$B(1, 1, \ldots, z_k, \ldots, 1) \neq 0, \ |z_k| \geq 1, \ k = 1, 2 \ldots N$$
- The DeCarlo-Strintzis Theorem suggests a stability test that consists of N 1-D stability tests plus a search for roots of $B(z)$ over the N-dimensional surface $|z_1| = |z_2| = \cdots = |z_N| = 1$.

The O'Connor-Huang Mapping Theorem

- How do we test the stability of an NSHP filter?

  Consider two sectors $S_1[(M_1, N_1), (M_2, N_2)]$ and $S_2[(1,0),(0,1)]$, with $D = M_1N_2 - M_2N_1 \neq 0$.

  The following is an injective linear map from S_1 into S_2

  $$m = k_1m' + k_2n' \quad n = k_3m' + k_4n'$$

  with k_1, k_2, k_3, k_4 defined as:

  $$\begin{aligned} k_1 &= \operatorname{sgn}(D)N_2 & k_2 &= -\operatorname{sgn}(D)M_2 \\ k_3 &= -\operatorname{sgn}(D)N_1 & k_4 &= \operatorname{sgn}(D)M_1 \end{aligned}$$

- Let $b(m,n)$ be a recursive array with angle of support β. Then $b(m,n)$ is stable if and only if with $K = k_1k_4 - k_2k_3 \neq 0$, the recursive array $g(m,n) = b(m',n')$ is stable, $(m',n') \in \beta$

- We need $D \neq 0$ to ensure that we have support in a sector and the two rays (line segments) that define the sector are not colinear.

- Example:$B(z_1,z_2) = 0.5z_1^{-1}z_2 + 1 + 0.85z_1 + 0.1z_1z_2 + 0.5z_1^2z_2^{-1}$

  $$\begin{aligned} (0,0) &\rightarrow (0,0) & (-1,0) &\rightarrow (1,1) \\ (-1,-1) &\rightarrow (2,3) & (1,-1) &\rightarrow (0,1) \end{aligned}$$

  So the mapped polynomial to be tested is

  $$M(z_1,z_2) = 1 + 0.5z_1^{-1} + 0.5z_2^{-1} + 0.85z_1^{-1}z_2^{-1} + 0.1z_1^{-2}z_2^{-3}$$

Root-Locus Techniques

- Consider:

$$B(z_1, z_2) = 1 - 1.5z_1 - 0.6z_1^2 - 1.2z_2 + 1.8z_1z_2 \\ -0.72z_1^2z_2 + 0.5z_2^2 - 0.75z_1z_2^2 + 0.25z_1^2z_2^2$$

- We can hold z_1 constant

$$B([z_1], z_2) = (1 - 1.5[z_1] + 0.6[z_1]^2) \\ +(-1.2 + 1.8[z_1] - 0.72[z_1]^2)\ z_2 \\ +(0.5 - 0.75[z_1] + 0.25[z_1]^2)\ z_2^2$$

- Roots of $B([z_1], z_2)$ with respect to z_2 are functions of z_1

- Plot roots in z_1 plane. Rootlets must lie completely inside the unit hyperdisk for the filter to be stable.

Cepstrum/2-D cepstral stability tests

- 2-D complex cepstrum of $b(m,n)$

$$\hat{b}(m,n) = Z^{-1}[\log[Z[b(m,n)]]$$

- $\hat{b}(m,n)$ is real for a real sequence
- It is called "complex" due to the use of the complex logarithm.

$$\log z = \log|z| + \ j \ \arg(z) \text{ if } z \in C$$

- (*Ekstrom*) A general recursive digital filter is stable if and only if its 2-D complex-cepstrum exists and has the same minimum angle support as the original sequence.

Stabilization of unstable recursive digital filters

- In the 1-D case this is very simple
- $|3z_1^{-1} - 1| = |z_1^{-1} - 3|$, a reflection of the root did not change the magnitude spectrum.
- Factor denominator polynomial and reflect the roots inside the unit circle.
- Fundamental curse of multidimensional digital signal processing: no polynomial factorization algorithm
- Proposed methods
  - Double Planar Least Squares Inversion [*Shanks, Treitel and Reddy*]
  - Discrete Hilbert Transform [*Read, Treitel, Reddy*]

Double Planar Least Squares Inversion

- PLSI of a coefficient array C is an array P such that

  1. $C * P \approx U$, U is the unit pulse array (of all ones)
  2. $C * P = G$ such that $U - G$ is minimized in least squares sense.

- *Shank's conjecture*: Given an arbitrary real, finite array C, any PLSI of C is minimum phase, and the applying PLSI twice to C yields minimum phase with the same magnitude spectrum as C.

- Proof of "modified" Shank's conjecture [*Reddy*]

Stabilization and Stability Testing Unified: The Multidimensional DHT

- Continuous Hilbert Transform theory involves theory of singular integrals and m-D extensions are very complicated [*Besikovitch, Calderon and Zygmund*]

- DHT is the relation between the real and imaginary parts of the Fourier Transform of a causal sequence.

$$\Im m\Big[X(\vec{\mathbf{f}})\Big] = -jDFT\left(t(\vec{\mathbf{i}})IDFT\left\{\Re e\Big[X(\vec{\mathbf{f}})\Big]\right\})\right)$$

- If we assume that the complex cepstrum is causal,

$$\Phi(\vec{\mathbf{f}}) = -jDFT\left(t(\vec{\mathbf{i}})IDFT\left\{\log|X(\vec{\mathbf{f}})|\right\})\right)$$

- Expression for $t(\vec{\mathbf{i}})$ very complicated [*Damera-Venkata, Venkataraman, Hrishikesh and Reddy*] and reduces to $\text{sgn}(i)$ in the 1-D case.

Stabilization via DHT

- To stabilize $b(\vec{i})$

1. Find $\Phi(\vec{\mathbf{f}})$, the minimum phase response
2. Evaluate $B_H(\vec{\mathbf{f}}) = |B(\vec{\mathbf{f}})|e^{j\Phi(\vec{\mathbf{f}})}$
3. Take multidimensional inverse FFT
4. Truncate $b_H(\vec{i})$ coefficients to same support as $b(\vec{i})$
5. Use a large size FFT for higher coefficient accuracy

Stabilization via DHT: Example

Example: Consider $B(z_1, z_2, z_3)$ given by:

$$\begin{aligned} B(z_1, z_2, z_3) &= (z_1 - 0.5)(z_2 + 2)(z_3 - 0.75) \\ &= z_1 z_2 z_3 + 2z_1 z_3 - 0.5z_2 z_3 - 0.75z_1 z_2 \\ &\quad -1.5z_1 + 0.375z_2 - z_3 + 0.75 \end{aligned}$$

$\downarrow_{DHT}$

$$\begin{aligned} B_{NT}(z_1, z_2, z_3) &= 0.375z_1 z_2 z_3 + 0.75z_1 z_3 - 0.75z_2 z_3 \\ &\quad -0.5z_1 z_2 - z_1 + z_2 - 1.5z_3 + 2 \\ &= (0.5z_1 - 1)(z_2 + 2)(0.75z_3 - 1) \end{aligned}$$

Useful Theorems: [*Damera-Venkata, Venkataraman, Hrishikesh and Reddy*]

- Multidimensional minimum phase if it exists is unique
- *If the given m-D polynomial $B(\vec{\mathbf{z}})$ is factorizable, then the transformed polynomial $B_{NT}(\vec{\mathbf{z}})$ is also factorizable, and the factors of the transformed polynomial are transformed versions of the factors of the given m-D polynomial.*
- *The m-D polynomial $B_{NT}(\vec{\mathbf{z}})$ of any causal m-D polynomial $B(\vec{\mathbf{z}})$, not having zeros on the unit hypercircle is stable.*
- *Minimum phase polynomials are fixed points of the multidimensional DHT*

Stability Testing using the DHT

- It is required to ascertain whether array **B** is stable or not.
  1. Apply the DHT to obtain array **A**.
  2. Compare arrays **B** and **A**.
     - If $\mathbf{B} \equiv \mathbf{A}$,then **B** is a stable array.
     - If $\mathbf{B} \not\equiv \mathbf{A}$, then **B** is unstable.

Stability Testing Example

$$
\begin{aligned}
B(z_1, z_2, z_3) &= 0.95z_1z_2z_3 - 0.7z_1z_2 \\
&\quad -0.5z_2z_3 + 2z_3z_1 - 1.5z_1 \\
&\quad +0.375z_2 - z_3 + 0.75
\end{aligned}
$$

$\downarrow_{DHT}$

$$
\begin{aligned}
A(z_1, z_2, z_3) &= 0.3563z_1z_2z3 \\
&\quad -0.4727z_1z_2 - 0.7172z_2z_3 \\
&\quad +0.7545z_3z_1 - 1.0059z_1 \\
&\quad +0.9527z_2 - 1.4971z_3 + 2.0001
\end{aligned}
$$

$\downarrow_{DHT}$

$$
\begin{aligned}
A'(z_1, z_2, z_3) &= 0.3563z_1z_2z3 \\
&\quad -0.4727z_1z_2 - 0.7172z_2z_3 \\
&\quad +0.7545z_3z_1 - 1.0059z_1 \\
&\quad +.9527z_2 - 1.4971z_3 + 2.0001
\end{aligned}
$$

$B(z_1, z_2, z_3)$ is unstable, while $A(z_1, z_2, z_3)$ is stable.