Iterative Signal Restoration

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1 Introduction

1.1 The Problem

We measure $y(n)$ which is the result of linear distortion applied to the original signal $x(n)$ plus additive noise $e(n)$, as shown in Fig. 1. Given

1. $y(n) = D\{x(n)\}$,
2. linear operator D representing the distortion, and
3. prior knowledge about feasible $x(n)$,

the problem is to determine $x(n)$.

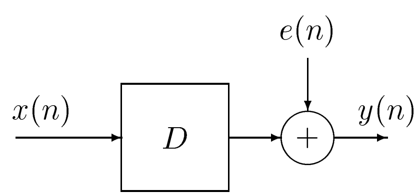

Figure 1: Block diagram for signal distortion in signal restoration.

1.2 Examples

- Constrained Deconvolution
- Shift-varying deconvolution
- Extrapolation of bandlimited signals
- Recovery of signals from Fourier transform magnitude or phase

2 Inverse Filter Solution

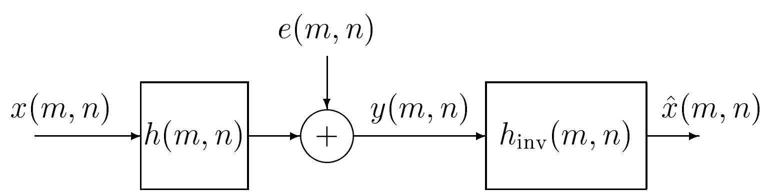

Figure 2: An inverse filtering solution for image restoration.

Fig. 2 shows an inverse filtering solution for image restoration where

$$H_{\text{inv}}(\omega_1, \omega_2) = \frac{1}{H(\omega_1, \omega_2)}$$

$$\hat{x}(m, n) = x(m, n) + e(m, n) * h_{\text{inv}}(m, n)$$

The resulting restoration is the original solution plus inverse filtered noise.

Disadvantages:

1. Noise is amplified at nulls of $H(\omega_1, \omega_2)$

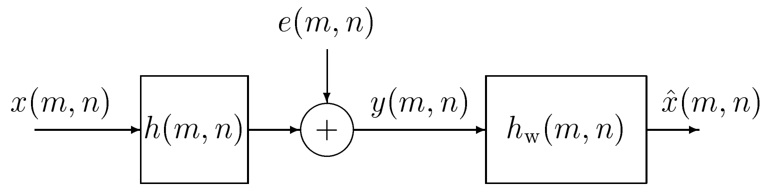

Figure 3: The Wiener filtering solution for image restoration.

2. Inverse filter may not exist

3. Inverse filter may be difficult to build

4. Properties of $x(m,n)$ are not used

3 Wiener Filter

The Wiener filter $h_{\mathrm{w}}(m,n)$ is the one that minimizes

$$E\{(x(m,n)-\hat{x}(m,n))^2\}$$

If the measurement noise is white, this gives

$$H_{\mathrm{w}}(\omega_1,\omega_2)=\frac{H^*(\omega_1,\omega_2)}{|H(\omega_1,\omega_2)|^2+\frac{1}{\mathrm{SNR}}}$$

The Wiener filtering approach is shown in Fig. 3.

Advantages

1. Begins to exploit signal

2. Controls output error

3. Straightforward to design

Disadvantages

1. Results often too blurred

2. Spatially invariant

4 Regularization

A more recent approach generalizes the Wiener solution. From the space of feasible solutions

$$\|y(m,n) - \hat{x}(m,n) * h(m,n)\|^2 \leq \epsilon^2$$

the regularized solution is the one which minimizes the quantity

$$\|f(m,n) * \hat{x}(m,n)\| \leq E^2$$

The solution to this problem is again a linear filter

$$H_{\text{reg}} = \frac{H^*(\omega_1,\omega_2)}{|H(\omega_1,\omega_2)|^2 + \left(\frac{\epsilon}{E}\right)^2 |F(\omega_1,\omega_2)|^2}$$

5 Van Citteri's Iteration (1931)

In the Fourier domain, the following identity must hold for all λ

$$X = X + \lambda\,(Y - HX)$$

Applying making successive substitutions,

$$\begin{aligned} X_0 &= \lambda Y \\ X_{k+1} &= X_k + \lambda(Y - HX_k) \\ & \lambda Y + (1 - \lambda H)X_k \\ & \lambda Y + GX_k \end{aligned}$$

Enumerating

$$\begin{aligned} X_0 &= \lambda Y \\ X_1 &= \lambda Y + \lambda GY \\ X_2 &= \lambda Y + \lambda GY + \lambda G^2 Y \\ \vdots &= \vdots \\ X_k &= \lambda \sum_{i=0}^{k} G^i Y \\ &= \frac{\lambda \left(1 - G^{k+1}\right) Y}{1 - G} \end{aligned}$$

If $\lim_{k \to \infty} G^{k+1} Y = 0$, then

$$X_\infty = \frac{\lambda Y}{1 - (1 - \lambda H)} = \frac{Y}{H}$$

which is the inverse filter solution. This iteration retains many of disadvantages of the inverse filter operation, but

- It can be terminated prior to convergence
- The inverse operator does not need to be built

For this iteration to converge, we need $|G(\omega_1, \omega_2)| < 1$, or equivalently

$$\begin{aligned}
|1 - \lambda H(\omega_1, \omega_2)| &< 1 \\
|1 - \lambda H(\omega_1, \omega_2)|^2 &< 1 \\
(1 - \Re e\{\lambda H(\omega_1, \omega_2)\})^2 + \Im m^2\{\lambda H(\omega_1, \omega_2)\} &< 1
\end{aligned}$$

which is a disk of radius 1 centered at $(1, 0)$ in the complex plane with axes $\Re e\{\lambda H(\omega_1, \omega_2)\}$ and $\Im m\{\lambda H(\omega_1, \omega_2)\}$.

For real-valued λ and $H(\omega_1, \omega_2)$,

$$\begin{array}{lcl}
-1 < & 1 - \lambda H(\omega_1, \omega_2) & < 1 \\
-2 < & -\lambda H(\omega_1, \omega_2) & < 0 \\
0 < & \lambda H(\omega_1, \omega_2) & < 2 \\
0 < & H(\omega_1, \omega_2) & < \dfrac{2}{\lambda}
\end{array}$$

The final condition is a common necessary condition for convergence in iterative linear systems, e.g. adaptive LMS FIR filters.

6 Constrained Restoration

Suppose $x(m, n) = Cx(m, n)$ for constraint matrix C. Examples are bandlimited signals, finite support, and positive amplitudes. The basic iteration can be modified:

$$x_i(n) = C\{x_{i-1}(n)\} + \lambda\,(y(n) - H\,C\{x_i(m, n)\})$$

$$x_o(m, n) = \lambda\,C\{y(m, n)\}$$

Iterative implementations of the Wiener and regularized solutions also exist.