DEFINITION

- A projection is a linear mapping of an R-dimensional signal to an S-dimensional one.

- Manifestations:
  - X-ray photographs
  - PET (Position Emission Tomography) images
PET images are usually around 128×128 pixels. Sinograms are around 256×192 pixels. The PET VI System has 72 detectors in the detector ring. Newer ones have a lot more.

- Nuclear cameras
- Line-spread functions
- SPECT (Single Photon Emitted Computed Tomography)

\[ p_\theta(\hat{u}) = \int x(\hat{u} \cos \theta + \hat{\nu} \sin \theta, -\hat{u} \sin \theta + \hat{\nu} \cos \theta) d\hat{\nu} \]

where \[
\begin{bmatrix}
u \\
u
\end{bmatrix} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
u \\
u
\end{bmatrix}
\]

- A projection is a series of line integrals across the object.
- By varying \( \theta \), a large number of projections can be taken.
- How can an object be recovered from a number of projections?

**THE PROJECTION SLICE THEOREM**

- The 2-D continuous – ‘Time’ Fourier transform can be defined as:

\[ X(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(u, v) e^{-j\mu u} e^{-j\nu v} du dv \]

\[ x(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\mu, \nu) e^{j\mu u} e^{j\nu v} d\mu dv \]

- If \( x(u, v) \) is rotated through an angle \( \theta_0 \), its Fourier transform rotated by the same angle.

Proof:

\[ X(\mu, \nu) \rightarrow \hat{X}(\omega, \theta) \]

in polar coordinates
\[ x(u, v) \rightarrow \hat{x}(r, \phi) \quad \text{in polar coordinates} \]

\[
\hat{X}(\omega, \theta) = \int_0^\pi \int_0^{\infty} \hat{x}(r, \theta) \exp[-j(\omega r \cos \theta \cos \phi + \omega r \sin \theta \sin \phi)] r dr d\phi
\]

\[ = \int_0^\pi \int_0^{\infty} \hat{x}(r, \theta) \exp[-j \omega r \cos(\phi - \theta)] r dr d\phi \]

\[ F\{\hat{x}(r, \phi + \theta_0)\} = \int_0^\pi \int_0^{\infty} \hat{x}(r, \phi + \theta_0) \exp[-j \omega r \cos(\phi - \theta)] r dr d\phi \]

Let \( \alpha = \phi + \theta_0 \rightarrow \phi = \alpha - \theta_0 \)

\[ F\{\hat{x}(r, \phi + \theta_0)\} = \int_0^\pi \int_0^{\infty} \hat{x}(r, \alpha) \exp[-j \omega r \cos(\alpha - (\theta + \theta_0))] r dr d\alpha \]

\[ = \hat{X}(\omega, \theta + \theta_0) \quad \text{End of Proof.} \]

- \( \hat{p}_o(\hat{u}) \leftrightarrow X(\mu, 0) \) because

\[ X(\mu, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(u, v) e^{-j\mu u} du dv \]

\[ = \int_{-\infty}^{\infty} e^{-j\mu u} \int_{-\infty}^{\infty} x(u, v) dv du \]

\[ = \int_{-\infty}^{\infty} e^{-j\mu u} p_o(u) du \]

- Combining the two previous results produces the Projection-Slice Theorem

\[ p_o(\hat{u}) \xleftarrow{1-D} \hat{X}(\omega, \theta) = X(\omega \cos \theta, \omega \sin \theta) \]
3-D PROJECTION PROBLEM

- 3-D reconstruction performed one plane at a time.

FOURIER-DOMAIN APPROACH

- Motivation
  - Projections provide transform samples on a polar raster
  - Interpolate to get samples on a rectangular raster (grid)

CONVOLUTION BACK-PROJECTION ALGORITHMS

- The interpolation process introduced errors. Back-projection takes a different approach.

\[
x(u, v) = \frac{1}{4 \pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\mu, \nu) \exp[j \mu u + j \nu v] \, d\mu \, d\nu
\]
Let’s again make a polar coordinate substitution, but only in the Fourier variables.

\[
x(u, v) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_\theta(\omega) \exp\left[ j \omega(u \cos \theta + v \sin \theta) \right] |\omega| d\omega d\theta
\]

\[
= \sum_i \Delta \theta_i \cdot \frac{1}{4\pi^2} \int_{-\infty}^{\infty} p_\theta(\omega) \exp\left[ j \omega(u \cos \theta_i + v \sin \theta_i) \right] |\omega| d\omega
\]

\[
= \sum_i \Delta \theta_i g_\theta(\hat{u})
\]

What is \( g_\theta(\hat{u}) \)?

- A 2-D sequence
- Uniform in \( \hat{v} \) direction
- As a function of \( \hat{u} \), it is a filtered projection.

\( \Rightarrow g_\theta(\hat{u}) \) is a back projected, filtered projection

\[
g_\theta(\hat{u}) = p_\theta(\hat{u}) * F^{-1}(|\omega|)
\]

**ALGORITHM**

1. Filter the projections.
2. Back-project the results in the appropriate directions.
3. Sum.
4. Correct the DC value:

\[
x(u, v) = \sum_i \Delta \theta_i g_\theta(\hat{u})
\]

where \( g_\theta(\hat{u}) = p_\theta(\hat{u}) * F^{-1}(|\omega|) \).