Reconstruction from Projections

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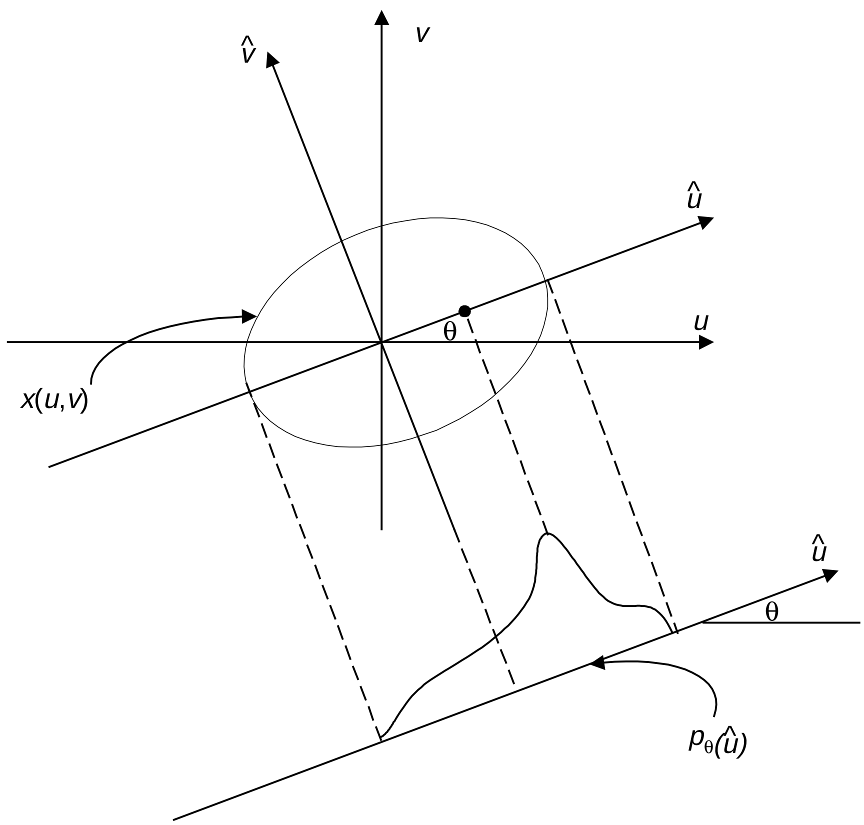
DEFINITION

- A projection is a linear mapping of an R-dimensional signal to an S-dimensional one.



- Manifestations:

- X-ray photographs

- PET (Position Emission Tomography) images

PET images are usually around 128×128 pixels. Sinograms are around 256×192 pixels. The PET VI System has 72 detectors in the detector ring. Newer ones have a lot more.

- Nuclear cameras
- Line-spread functions
- SPECT (Single Photon Emitted Computed Tomography)

• $p_{\theta}(\hat{u}) = \int x(\hat{u}\cos\theta + \hat{v}\sin\theta, -\hat{u}\sin\theta + \hat{v}\cos\theta)d\hat{v}$

where $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}$

- A projection is a series of line integrals across the object.
- By varying θ, a large number of projections can be taken.
- How can an object be recovered from a number of projections?

THE PROJECTION SLICE THEOREM

• The 2-D continuous – 'Time' Fourier transform can be defined as:

$$X(\mu,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(u,v)e^{-ju\mu}e^{-jv\nu}dudv$$

$$x(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\mu,\nu)e^{ju\mu}e^{jv\nu}d\mu d\nu$$

• If $x(u,v)$ is rotated through an angle θ_0, its Fourier transform rotated by the same angle.

Proof:

$X(\mu,\nu) \rightarrow \hat{X}(\omega,\theta)$ in polar coordinates

$x(u,v) \to \hat{x}(r,\phi)$ in polar coordinates

$$\hat{X}(\omega,\theta) = \int_0^{\pi} \int_{-\infty}^{\infty} \hat{x}(r,\theta) \exp[-j(\omega r \cos\theta \cos\phi + \omega r \sin\theta \sin\phi)] |r| dr d\phi$$

$$= \int_0^{\pi} \int_{-\infty}^{\infty} \hat{x}(r,\theta) \exp[-j\omega r \cos(\phi - \theta)] |r| dr d\phi$$

$$F\{\hat{x}(r,\phi+\theta_0)\} = \int_0^{\pi} \int_{-\infty}^{\infty} \hat{x}(r,\phi+\theta_0) \exp[-j\omega r \cos(\phi - \theta)] |r| dr d\phi$$

Let $\alpha = \phi + \theta_0 \to \phi = \alpha - \theta_0$

$$F\{\hat{x}(r,\phi+\theta_0)\} = \int_0^{\pi} \int_{-\infty}^{\infty} \hat{x}(r,\alpha) \exp[-j\omega r \cos(\alpha - (\theta + \theta_0))] |r| dr d\alpha$$

$$= \hat{X}(\omega,\theta+\theta_0)$$ End of Proof.

- $\hat{p}_0(\hat{u}) \longleftrightarrow X(\mu,0)$ because

$$X(\mu,0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(u,v) e^{-j\mu u} du dv$$

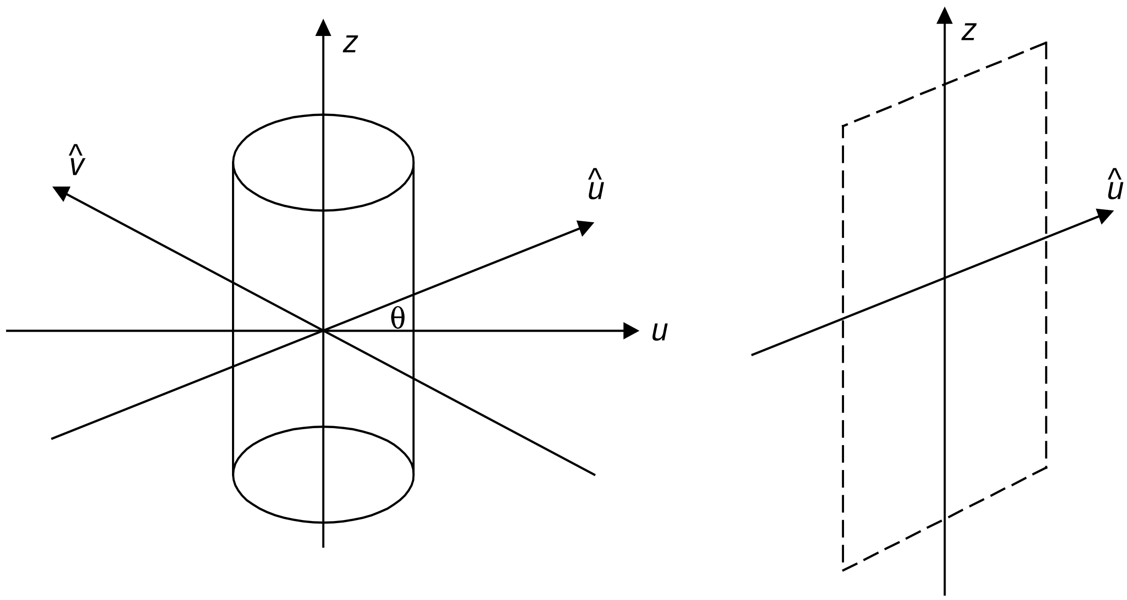
$$= \int_{-\infty}^{\infty} e^{-j\mu u} \int_{-\infty}^{\infty} x(u,v) dv du$$

$$= \int_{-\infty}^{\infty} e^{-j\mu u} p_0(u) du$$

- Combining the two previous results produces the Projection-Slice Theorem

$$p_\theta(\hat{u}) \xleftrightarrow{1-D} \hat{X}(\omega,\theta) = X(\omega\cos\theta, \omega\sin\theta)$$

3-D PROJECTION PROBLEM



- 3-D reconstruction performed one plane at a time.

FOURIER-DOMAIN APPROACH

- Motivation

- Projections provide transform samples on a polar raster

- Interpolate to get samples on a rectangular raster (grid)

CONVOLUTION BACK-PROJECTION ALGORITHMS

- The interpolation process introduced errors. Back-projection takes a different approach.

$$x(u,v) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\mu,\nu)\exp[j\mu u + j\nu v]\,d\mu\,d\nu$$

- Let's again make a polar coordinate substitution, but only in the Fourier variables.

$$x(u,v)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\pi}p_\theta(\omega)\exp[j\omega(u\cos\theta+v\sin\theta)]|\omega|d\omega d\theta$$

$$\approx\sum_i\Delta\theta_i\cdot\frac{1}{4\pi^2}\int_{-\infty}^{\infty}p_{\theta_i}(\omega)\exp[j\omega(u\cos\theta_i+v\sin\theta_i)]|\omega|d\omega$$

$$=\sum_i\Delta\theta_i g_{\theta_i}(\hat{u})$$

- What is $g_{\theta_i}(\hat{u})$?

- A 2-D sequence
- Uniform in $\hat{v}$ direction
- As a function of $\hat{u}$, it is a filtered projection.

$\Rightarrow g_{\theta_i}(\hat{u})$ is a back projected, filtered projection

$$g_{\theta_i}(\hat{u})=p_{\theta_i}(\hat{u})*F^{-1}\{|\omega|\}$$

ALGORITHM

1. Filter the projections.
2. Back-project the results in the appropriate directions.
3. Sum.
4. Correct the DC value:

$$x(u,v)=\sum_i\Delta\theta_i g_{\theta_i}(\hat{u})$$

where $g_{\theta_i}(\hat{u})=p_{\theta_i}(\hat{u})*F^{-1}\{|\omega|\}$.