AM-FM Image Models

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GOAL

- MODEL images as sums of AM-FM functions:

$$t(\mathbf{m}) = \sum_{i=1}^{K} a_i(\mathbf{m}) \exp[j\varphi_i(\mathbf{m})].$$

- ESTIMATE the dominant image modulations.
- COMPUTE multi-component AM-FM image representations.
- RECOVER the essential image structure from the computed representations.

OUTLINE

I. Introduction

II. Demodulation

III. Dominant Component Analysis

IV. Multi-Component Analysis

V. Conclusions

AM-FM IMAGE MODELING

- Image model:

$$\begin{aligned} t(\mathbf{m}) &= \sum_{i=1}^{K} a_i(\mathbf{m}) \exp[j\varphi_i(\mathbf{m})] \\ &= \sum_{i=1}^{K} t_i(\mathbf{m}) \end{aligned}$$

- Each $t_i(\mathbf{m})$ is an AM-FM COMPONENT.

- MODULATING FUNCTIONS of $t_i(\mathbf{m})$:
  - $a_i(\mathbf{m})$: amplitude modulation function.
  - $\nabla\varphi_i(\mathbf{m})$: frequency modulation function.

COMPARISON TO DFT

- DFT represents an $N \times N$ image as the sum of N^2 sinusoidal components, each having
  - constant amplitude.
  - constant frequency.
- Nonstationary structure represented by constructive and destructive INTERFERENCE between STATIONARY components.

- The modulating functions $a_i(\mathbf{m})$ and $\nabla\varphi_i(\mathbf{m})$ are permitted to vary smoothly across the image.
- EACH AM-FM component is capable of capturing SIGNIFICANT nonstationary structure.
- AM-FM model captures essential image structure using MUCH fewer than N^2 components.

1D BACKGROUND

- In 1977, Moorer
  - COMPUTED multi-component AM-FM representations for musical instrument signals.
  - CODED the representations and achieved compression ratios of 43:1.
  - RECONSTRUCTED the signals from the computed representations.
  - Assumed harmonically related components.
- 1D Teager-Kaiser operator introduced in 1990.
- Energy Separation Algorithm (ESA) developed by Maragos, Kaiser, & Quatieri (1991).
  - Used by MANY people for AM-FM speech modeling.

1D MULTI-COMPONENT MODELS

- Kumaresan, Sadasiv, Ramalingam, and Kaiser (1992-96):
  - Used Teager-Kaiser operator and matrix invariance properties.
  - Problems with $K > 4$ components.
  - Problems with $n > 1$ dimensions.
- Lu and Doerschuck (1996).
  - Used Kalman filters.
  - 1D formulation.

2D BACKGROUND

- Bovik, Clark, & Geisler characterized TEXTURE as a CARRIER of region information (1986).
  - MODELED textures as single-component AM-FM functions.
  - SEGMENTED texture using amplitude and phase of Gabor filter response.
- Bovik, et. al., estimated DOMINANT modulating functions via an iterative relaxation algorithm (1992).
  - SEGMENTED textures.
  - RECONSTRUCTED 3D surfaces.
  - Iteration was computationally EXPENSIVE.

2D AM-FM MODELS

- Knutsson, Westin, & Granlund (1994).
  - Used lognormal filters.
  - Estimated FM for a single component.
- Francos and Friedlander (1995-96).
  - Polynomial phase model.
  - Estimated polynomial order and coefficients.
  - Demonstrated on simple synthetics.

2D TEAGER-KAISER OPERATOR

- Introduced by Yu, Mitra, & Kaiser (1991).
- 2D ESA: Maragos, Bovik, & Quatieri (1992).
  - Computationally efficient.
  - Promised to replace iterative relaxation technique for estimating dominant modulations.
- Havlicek began work on the problem (1992).
- TROUBLE:
  - ESA cannot estimate SIGNED frequency.
  - SIGN of frequency is important in multi-D: embodies ORIENTATION.
- A NEW approach was needed.

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  1. reptile
  2. reptile baseband comp, comp 1
  3. reptile comp 2, comp 3
  4. reptile
  5. reptile comp 4, comp 5

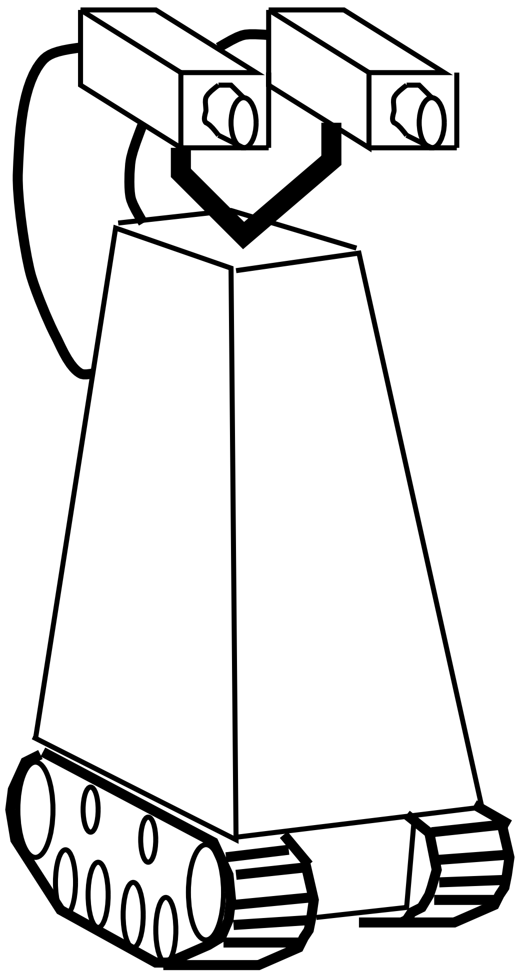
  ▶ look at how “fold” is in FM of comps 2, 3. Also in AM of baseband and comps 4, 5. Comp 1 is AM-FM harmonic of comp 3.

  6. All six reptile components.
  7. reptile + 6 comp recon
  8. reptile + 43 channelized comp recon

SIGNIFICANCE

- These results are AMAZING!!
- FIRST approach to
  - Estimate multi-D multi-component amplitude and frequency modulations.
  - Estimate multi-D multi-component FM with CORRECT signs.
  - Compute multi-component AM-FM representations for natural images.
  - Reconstruct from estimated modulations (in both 1D & 2D).

APPLICATIONS



- Texture-based scene segmentation. (Bovik, Clark, Geisler; Porat, Zeevi; Havlicek, Harding, Bovik)
- Phase-based computational stereopsis. (Chen, Bovik).
- 3-D shape from texture. (Super, Bovik, Klarquist).

APPLICATIONS. . .

- Spatio-spectral analysis.
- Texture modeling and synthesis.
- Future:
  - Image coding.
  - Video coding.
  - Multimedia and CD-ROM applications.

OUTLINE

I. Introduction

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REAL-VALUED IMAGES

- The AM-FM model for $t(\mathbf{m})$ is COMPLEX.
- To analyze a REAL-VALUED image $s(\mathbf{m})$, take

$$t(\mathbf{m}) = s(\mathbf{m}) + j\mathcal{H}[s(\mathbf{m})].$$

- $\mathcal{H}$ is the 2D discrete Hilbert transform.
- $t(\mathbf{m})$ is the complex ANALYTIC IMAGE associated with $s(\mathbf{m})$.
  - $\text{Re}\left[t(\mathbf{m})\right] = s(\mathbf{m})$.
- $\mathcal{F}[t(\mathbf{m})]$ is ZERO in quadrants II and III.
- Gives intuitive, physically meaningful interpretations for
  - instantaneous frequency of $s(\mathbf{m})$.
  - first moment of frequency in $s(\mathbf{m})$.

DEMODULATION

- Demodulation algorithm is NONLINEAR.
- Cross-component interference is a PROBLEM if $t(\mathbf{m})$ is MULTI-COMPONENT.
- Use a multiband bank of linear filters $g_p(\mathbf{m})$ to ISOLATE components on a POINTWISE basis.
  - The filters DO NOT need to isolate the components on a GLOBAL scale.
  - The response $y_p(\mathbf{m})$ of each filter $g_p(\mathbf{m})$ DOES need to be dominated by only ONE component at each image pixel.
- Suppose that component $t_i(\mathbf{m})$ dominates the response $y_p(\mathbf{m})$ of filter $g_p(\mathbf{m})$ at pixel $\mathbf{m}$.
- GOAL: use the response $y_p(\mathbf{m})$ to estimate the values of the modulating functions $a_i(\mathbf{m})$ and $\nabla\varphi_i(\mathbf{m})$ at pixel $\mathbf{m}$.

DEMODULATION...

- Demodulation algorithm depends on a *quasi-eigenfunction approximation* (QEA).
- Suppose G_p is a 2D LSI system.
- Exact response to $t_i(\mathbf{m})$:

$$y_p(\mathbf{m}) = \sum_{\mathbf{p}\in\mathbb{Z}^2} g_p(\mathbf{p}) t_i(\mathbf{m}-\mathbf{p}).$$

- QEA:

$$\widehat{y}_p(\mathbf{m}) = t_i(\mathbf{m}) G_p[\nabla\varphi_i(\mathbf{m})].$$

- QEA error generally small or negligible if
  - $g_p(\mathbf{m})$ spatially localized.
  - $a_i(\mathbf{m})$ and $\nabla\varphi_i(\mathbf{m})$ are smoothly varying, or LOCALLY COHERENT.

DEMODULATION ALGORITHM

- Recall:

$$\begin{aligned} y_p(\mathbf{m}) &= t(\mathbf{m}) * g_p(\mathbf{m}) \\ &\approx t_i(\mathbf{m}) * g_p(\mathbf{m}). \end{aligned}$$

- Horizontal component of $\nabla\widehat{\varphi}_i(m,n)$:

$$\arcsin\left[\frac{y_p(m+1,n) - y_p(m-1,n)}{2j\,y_p(m,n)}\right].$$

- Amplitude algorithm:

$$\widehat{a}_i(\mathbf{m}) = \left|\frac{y_p(\mathbf{m})}{G_p[\nabla\widehat{\varphi}_i(\mathbf{m})]}\right|.$$

FILTERBANK

- Physiology and psychophysics:
  - CHANNELS in visual cortex operate at 18 orientations and 4 magnitude frequencies.
  - Visual cortical cells function as 2D complex Gabor filters.
- For ANALYTIC images $t(\mathbf{m})$, only half of the orientations need be considered.
- 2D Gabor filters have OPTIMAL spatio-spectral localization.

FILTERBANK...

- Gabor filterbank:
  - ▶ Frequency response $G_p(\boldsymbol{\omega})$ is Gaussian.
  - ▶ Bandwidth = 1 Octave.
- Filterbank channels at 8 orientations and 5 magnitude frequencies.
- One Gaussian baseband channel to capture low-frequency structure.
- Two special HIGH-FREQUENCY channels.
- Frequency responses of adjacent filters intersect at half-peak.
- TOTAL number of channels = 43.

- Slides Here

  9. Filterbank in frequency domain.

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DOMINANT COMPONENT ANALYSIS

- At every pixel, estimate the modulating functions of the component that DOMINATES the local frequency spectrum at that pixel.

- The dominant component frequencies are the EMERGENT frequencies which characterize the local texture structure.

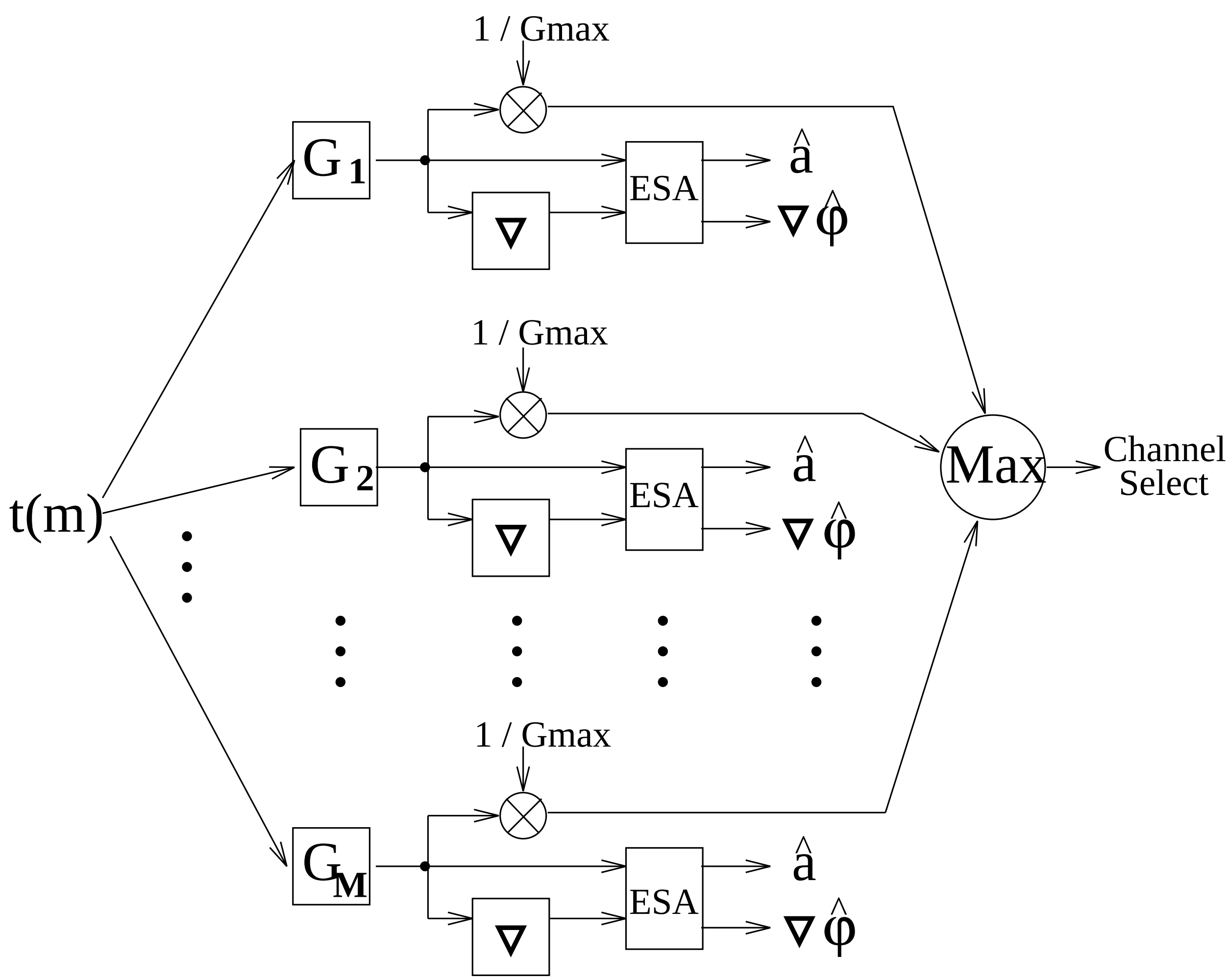
- At each pixel, the dominant component maximizes

$$\Psi_p(\mathbf{m}) = \frac{|y_p(\mathbf{m})|}{\max_{\boldsymbol{\omega}} |G_p(\boldsymbol{\omega})|}.$$

- Apply QEA:

$$\Psi_p(\mathbf{m}) \approx a_i(\mathbf{m}) \frac{\left|G_p[\nabla \varphi_i(\mathbf{m})]\right|}{\max_{\boldsymbol{\omega}} |G_p(\boldsymbol{\omega})|}.$$

BLOCK DIAGRAM



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10. Tree + dom comp recon
11. Raffia + dom comp recon
12. Burlap + dom recon

TEXTURE SEGMENTATION

- Apply LoG edge detector to dominant
  - Frequency magnitudes.
  - Frequency orientations.
  - Amplitudes.
- Segment along SIGNIFICANT zero crossings.

- Slides Here

13. MicaBurlap Mag Freq + Seg, $\sigma = 15$, $\tau = 1.5$

    PelletBean Mag Freq + Seg, $\sigma = 49$, No τ

14. WoodWood Arg Freq + Seg, $\sigma = 14$, $\tau = 11$

    PaperBurlap AM + Seg, $\sigma = 46.5$, No τ

APPLICATIONS

- Spatio-spectral analysis.
- Image segmentation.
- Phase-based stereo (Chen, Bovik).
- 3-D shape from texture (Super, Bovik, Klarquist).
- Biologically feasible model of mammalian visual function.

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COMPUTED REPRESENTATIONS

- Recall model: $t(\mathbf{m}) = \sum_{i=1}^{K} a_i(\mathbf{m}) \exp[j\varphi_i(\mathbf{m})]$.

- GOAL: compute an AM-FM REPRESENTATION for $t(\mathbf{m})$ by estimating the modulating functions of EACH component.

- NOTE: decomposition into components is NOT unique.

CHANNELIZED COMPONENTS

- Estimate modulating functions for ONE component from EACH filterbank channel.
- M-channel filterbank $\Rightarrow$ M-component representation.
- Consistent with biological vision models.
- Gives good reconstructions.

- Slides Here

15. reptile + recon (channelized comps)
16. raffia + recon (channelized comps)
17. tree + recon (channelized comps)
18. pebbles + recon (channelized comps)
19. Ocean City, NJ + recon (channelized comps)
20. Celebrity + recon (channelized comps)

CHANNELIZED COMPONENTS...

- NOT EFFICIENT:
  - Many images have FEWER than M components.
  - Often, many components are nearly zero over large regions.
  - Single elements of the image structure tend to be manifest in more than one channelized component.
- SHOULD be able to get BETTER results with FEWER components.

IMPROVED REPRESENTATIONS

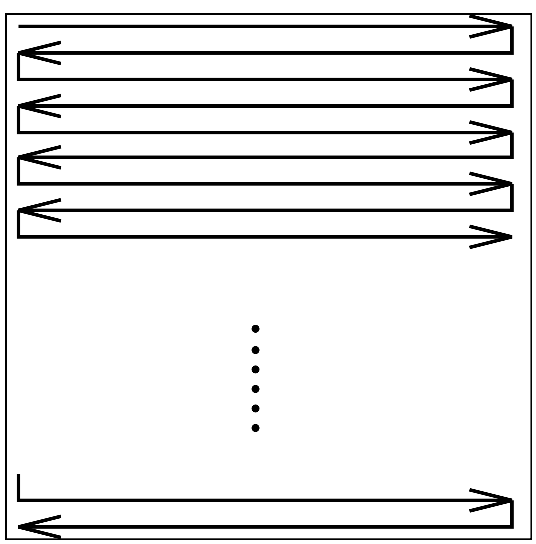
- Recall model: $t(\mathbf{m}) = \sum_{i=1}^{K} a_i(\mathbf{m}) \exp[j\varphi_i(\mathbf{m})]$.
- GOAL 1: compute an AM-FM REPRESENTATION for $t(\mathbf{m})$ by estimating the modulating functions of EACH component.
- GOAL 2: use fewer than M components.
- At each pixel, must determine
  - How many components are present.
  - Which filterbank channel to use for estimating the modulating functions of each component.

MULTIPLE COMPONENTS

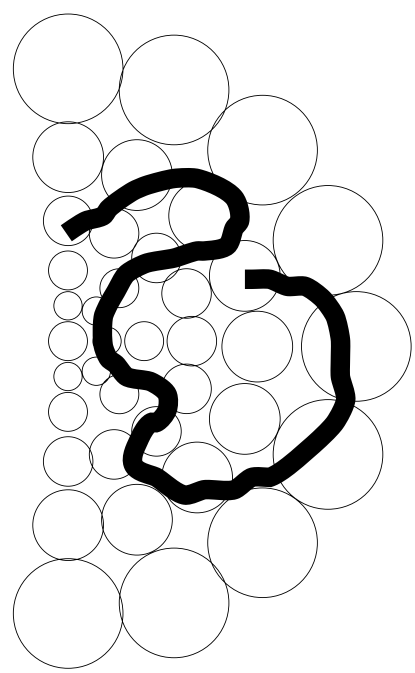
- Order the image pixels with a path function $\mathcal{P} : \mathbf{m} \longmapsto k$.

- Traverse pixels in order in image.

- Then $\nabla \varphi_i(k)$ maps out a path for EACH component:



Space

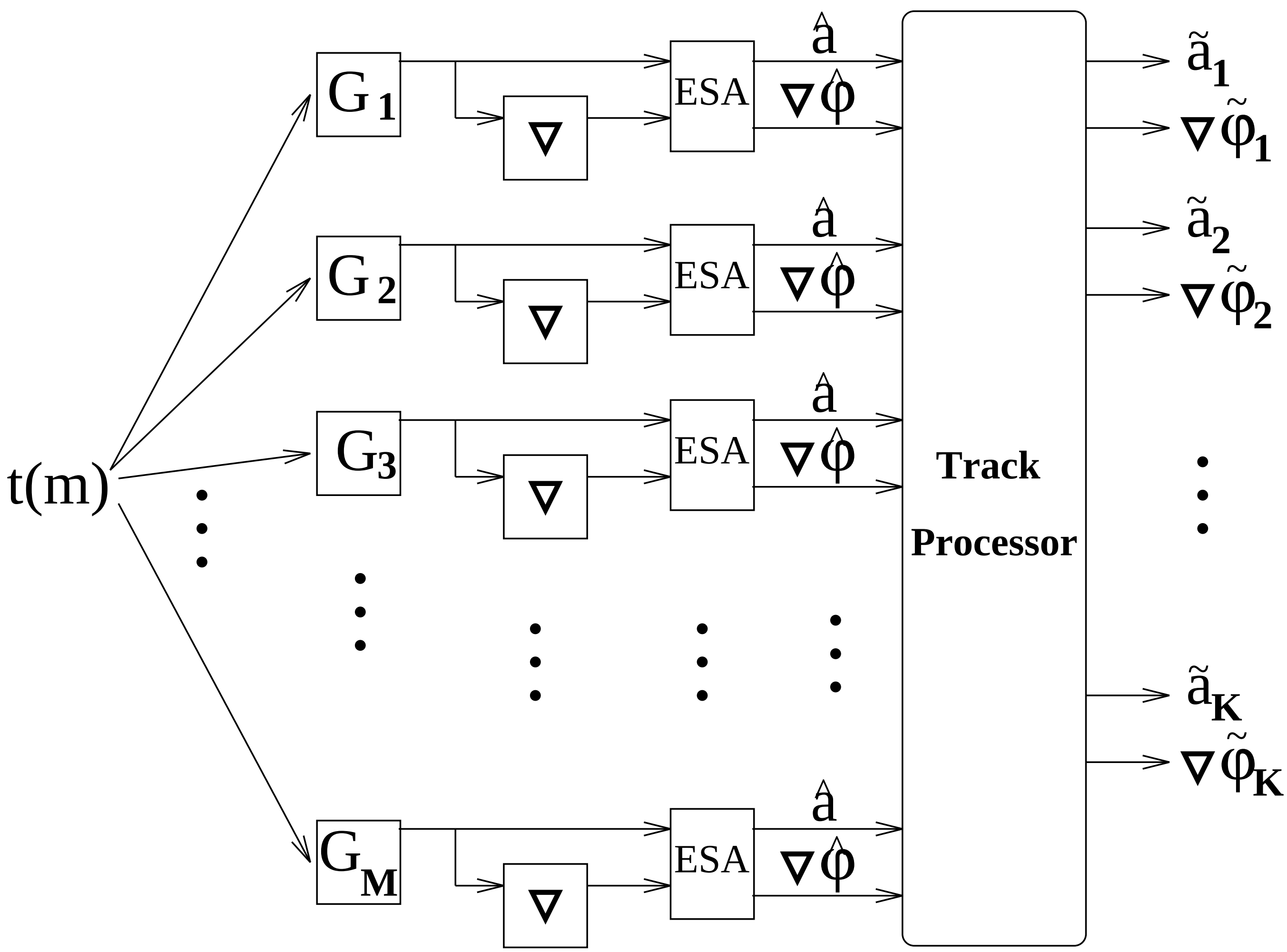


Frequency

TRACK PROCESSOR

- Expand modulating functions of $t_i(\mathbf{m})$ in Taylor series to obtain a canonical state-space model.
- Model higher-order derivatives of modulating functions AND errors in $\widehat{a}_i(\mathbf{m})$ and $\nabla\widehat{\varphi}_i(\mathbf{m})$ as STOCHASTIC PROCESSES.
- Use a Kalman filter to track the modulating functions of each component across the filterbank channel responses.
  - ▶ INPUTS the estimated modulating functions computed from the CHANNEL responses.
  - ▶ PREDICTS which channel should be used for estimating the modulating functions of each component.
  - ▶ OUTPUTS estimated modulating functions for each COMPONENT.

BLOCK DIAGRAM

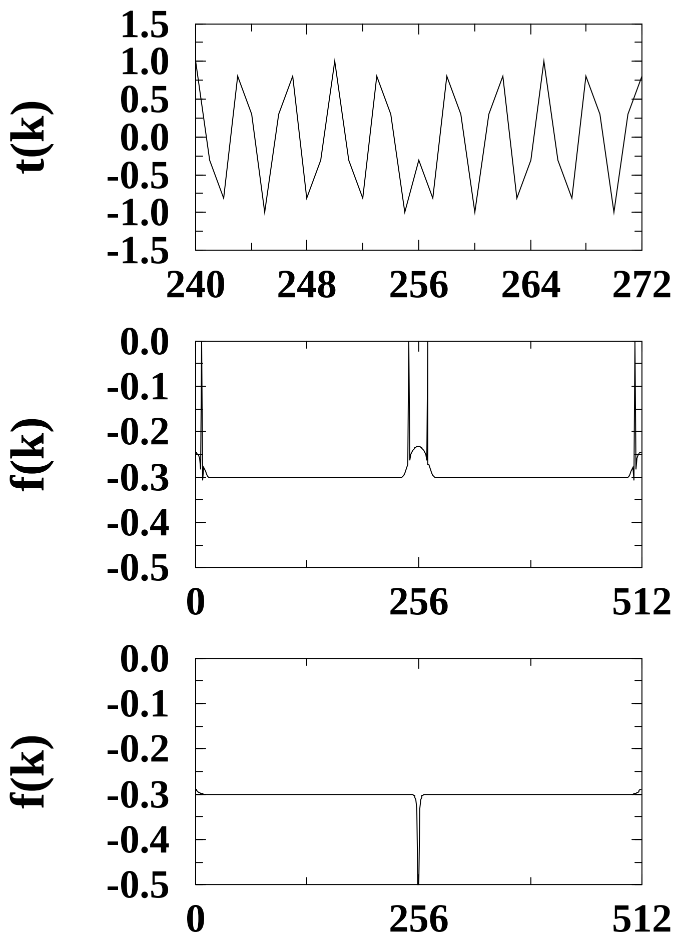


THE NEED FOR POSTFILTERING

- PROBLEM: image phase discontinuities cause
  - Abrupt, large-scale excursions in $\nabla\varphi_i(\mathbf{m})$.
  - Nontrivial QEA errors.
  - Absurd amplitude estimates.
- Frequency excursions can be UNBOUNDED.
- If the track processor follows a frequency excursion in $t_i(\mathbf{m})$, it
  - Rapidly updates from MANY channels.
  - Often updates from a channel that is dominated by another component.
  - The track of $t_i(\mathbf{m})$ is then lost.

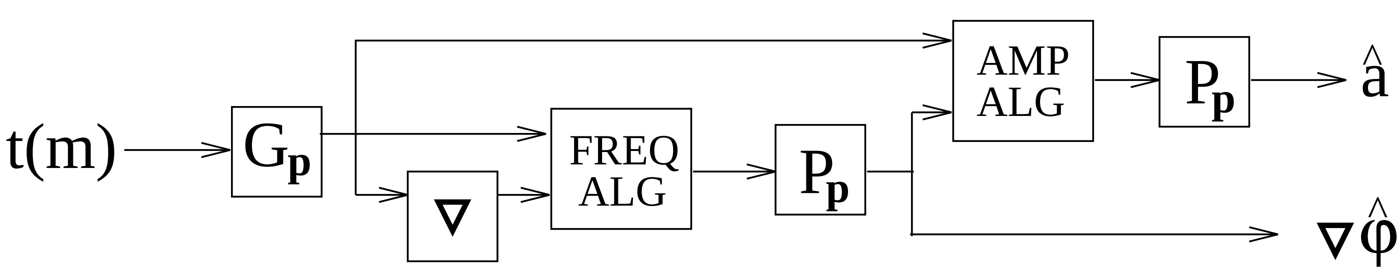
EXAMPLE PHASE DISCONTINUITY

- Input: 1D cosine, frequency -0.3 cycles/sample.
- Phase discontinuity of π radians at k=256.
- $f(k)$ is estimated frequency (1D).
- Octave filters with center frequencies of 0.217, 0.306 cycles/sample



POSTFILTERS

- Natural images are expected to contain phase discontinuities:
  - Occlusions.
  - Surface discontinuities, defects, deformations.
  - Shadows and specularities.
  - Noise.
- SOLUTION: Post-process the estimated modulating functions delivered by $g_p(\mathbf{m})$ with a low-pass Gaussian smoothing filter $P_p(\boldsymbol{\omega})$.
- Postfiltered channel model:



- Slides Here

21. reptile + recon (baseband + 5 tracked comps)
22. straw + recon (baseband + 8 tracked comps)
23. recons of 2 tracked straw comps
24. recons of 2 tracked straw comps
25. pellets + recon (baseband + 12 tracked comps)
26. recons of 2 tracked pellets comps
27. recons of 2 tracked pellets comps
28. burlap + recon (baseband + 8 tracked comps)
29. recons of 2 tracked burlap comps
30. raffia + recon (baseband + 8 tracked comps)
31. ice + recon (baseband + 12 tracked comps)

APPLICATIONS

- Nonstationary spatio-spectral analysis.
- Image and video coding.
- Current track processor has difficulty with
  - Regionally supported structure
  - images containing multiple textured objects.
- Investigation of IMPROVED tracking approaches is ongoing.

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CONTRIBUTION

- Developed a NEW theory of multidimensional AM-FM signal modeling.
  - Applicable in ARBITRARY dimensions.
  - Continuous AND discrete theories.
- FIRST to:
  - estimate multi-D multi-component AM-FM.
  - estimate multi-D FM with CORRECT signs.
  - comprehensively treat multi-D analytic signal.
  - treat multi-D QEA theory.
- Developed two computational paradigms in 2D. FIRST to:
  - compute multi-component AM-FM representations for natural images.
  - reconstruct from estimated modulations (in both 1D & 2D).

CONCLUSIONS

- Estimated dominant image modulations.
- Computed AM-FM representations.
- Reconstructed from the representations.
- USEFUL in PRACTICAL engineering systems:
  - ▶ Texture segmentation.
  - ▶ Phase based computational stereopsis.
  - ▶ 3D shape from texture.
  - ▶ Spatio-spectral analysis.
- Future work:
  - ▶ Improved tracking approaches.
  - ▶ Color images.
  - ▶ Video and color video.
  - ▶ Image and video coding.

MODEL AMBIGUITIES

- Suppose $s(\mathbf{m})$ is a real-valued image.
- Real model: $s(\mathbf{m}) = a(\mathbf{m})\cos[\varphi(\mathbf{m})]$.
  - Take

$$\begin{aligned} a_1(\mathbf{m}) &= s_{\max} = \sup_{\mathbf{m}} |s(\mathbf{m})| \\ \varphi_1(\mathbf{m}) &= \arccos\left[s(\mathbf{m})/s_{\max}\right] \\ a_2(\mathbf{m}) &= |s(\mathbf{m})| \\ \varphi_2(\mathbf{m}) &= \arccos\left[\operatorname{sgn} s(\mathbf{m})\right] \end{aligned}$$

  - Then,

$$\begin{aligned} s(\mathbf{m}) &= a_1(\mathbf{m})\cos[\varphi_1(\mathbf{m})] \\ &= a_2(\mathbf{m})\cos[\varphi_2(\mathbf{m})]. \end{aligned}$$

- The decomposition into components also is NOT unique.

QEA ERROR

- The ERROR in the QEA is

$$\varepsilon(\mathbf{m}) = \left| y_p(\mathbf{m}) - \widehat{y}_p(\mathbf{m}) \right|.$$

- Write a vector component-wise using angle brackets: $\mathbf{m} = [m_1 \; m_2 \; \ldots \; m_n]^T = \langle m_k \rangle_k$.

- Let $a_{\max} = \sup_{\mathbf{m} \in \mathbb{Z}^n} a_i(\mathbf{m})$.

- Then

$$\varepsilon(\mathbf{m}) \leq \Big(||g_p||_{\ell^1} - |g_p(\mathbf{0})| \Big) \mathfrak{L}^1(a_i) + a_{\max} \Big\langle \boldsymbol{\Delta}_k(g_p) \Big\rangle_k^T \Big\langle \mathfrak{S}_k^1(\varphi_i) \Big\rangle_k.$$

- Provides a uniform global bound on the error.

- The functionals $\mathfrak{L}^1(a_i)$ and $\mathfrak{S}_k^1(\varphi_i)$ quantify the GLOBAL SMOOTHNESS of $t_i(\mathbf{m})$.

- The functionals $||g_p||_{\ell^1}$ and $\boldsymbol{\Delta}_k(g_p)$ quantify the SPREAD, or LOCALIZATION, of $g_p(\mathbf{m})$.

ERROR BOUND FUNCTIONALS

- Let P^m denote the set of vector-valued paths $\boldsymbol{\sigma}(s)$ through $\mathbb{R}^n$ such that EACH component of $\boldsymbol{\sigma}(s)$ is a polynomial in s of degree m or less.

- Then the functional $\mathfrak{L}^1(a_i)$ may be expressed as

$$\mathfrak{L}^1(a_i) = \sup_{\boldsymbol{\sigma} \in P^1} \left| \int_{\boldsymbol{\sigma}} \nabla a_i(\mathbf{x}) \cdot d\mathbf{x} \right| .$$

ERROR BOUND FUNCTIONALS...

- Let $\varphi_{i,x_k}(\mathbf{x}) = \frac{\partial}{\partial x_k}\varphi_i(\mathbf{x})$.
- Then

$$\mathfrak{S}_k^1(\varphi_i) = \sup_{\boldsymbol{\sigma}\in P^1} \int_{\boldsymbol{\sigma}} |\nabla\varphi_{i,x_k}(\mathbf{x})|\, ds.$$

- In the n-dimensional discrete ℓ^q-space, $||g_p||_{\ell^q}$ is the usual norm of g_p with respect to the counting measure:

$$||g_p||_{\ell^q} = \left\{ \sum_{\mathbf{m}\in\mathbb{Z}^n} |g_p(\mathbf{m})|^q \right\}^{\frac{1}{q}}.$$

- When $q = 1$, $||g_p||_{\ell^q}$ is the absolute sum of $g_p(\mathbf{m})$.

ERROR BOUND FUNCTIONALS...

- Generalized energy moment functional $\boldsymbol{\Delta}_k(g_p)$:

$$\boldsymbol{\Delta}_k(g_p) = \sum_{\mathbf{m}\in\mathbb{Z}^n} \left|\mathbf{m}\mathbf{e}_k^T\mathbf{m}\right| |g_p(\mathbf{m})|,$$

  - $\mathbf{e}_k$: unit vector in x_k direction.
  - Note: $\left|\mathbf{m}\mathbf{e}_k^T\mathbf{m}\right| = |m_k||\mathbf{m}|$.

UNFILTERED DEMODULATION

- Amplitude: $a_i(\mathbf{m}) = |t_i(\mathbf{m})|$.
- Let $\mathbf{e}_1$ be a horizontal unit vector and $\mathbf{e}_2$ be a vertical unit vector.
- Let $k = 1$ or $k = 2$.
- Then, $\mathbf{e}_k^T \nabla \varphi_i(\mathbf{m})$ is one component of $\nabla \varphi_i(\mathbf{m})$.
- Let $h_k(\mathbf{m}) = \delta(\mathbf{m} + n_1 \mathbf{e}_k) + q\delta(\mathbf{m} + n_2 \mathbf{e}_k)$.
- Let $n_1 = 1$ and $n_2 = q = -1$. Apply QEA:

$$\mathbf{e}_k^T \nabla \widehat{\varphi}_i(\mathbf{m}) = \arcsin\left[\frac{t_i(\mathbf{m} + \mathbf{e}_k) - t_i(\mathbf{m} - \mathbf{e}_k)}{2jt_i(\mathbf{m})}\right]$$

- Let $n_1 = q = 1$ and $n_2 = -1$. Apply QEA:

$$\mathbf{e}_k^T \nabla \widehat{\varphi}_i(\mathbf{m}) = \arccos\left[\frac{t_i(\mathbf{m} + \mathbf{e}_k) + t_i(\mathbf{m} - \mathbf{e}_k)}{2t_i(\mathbf{m})}\right]$$

- Together, these algorithms place the estimated frequencies to within 2π radians.

STATE-SPACE MODEL

- Let ρ denote continuous arc length along $\mathcal{P}$.
- Restrict the derivatives of the modulating functions of $t_i(\rho)$ to the 1D lattice:

$$a_i'(k) = \frac{\partial}{\partial k} a_i(k) = \frac{\partial}{\partial \rho} a_i(\rho) \bigg|_{\rho=k}$$

- Expand the modulating functions in first-order Taylor series, *e.g.*

$$a_i(k+1) = a_i(k) + a_i'(k) + \int_k^{k+1} (k+1-\rho) a_i''(\rho) d\rho.$$

- Expand the derivatives of the modulating functions in zeroth-order Taylor series, *e.g.*

$$a_i'(k+1) = a_i'(k) + \int_k^{k+1} a_i''(\rho) d\rho.$$

STATE-SPACE MODEL...

- Consider $a_i(\mathbf{x})$ and $\varphi_i(\mathbf{x})$ to be homogeneous, m.s. differentiable random fields; let $\varphi_i(\mathbf{x})$ be *quadrant symmetric*.

- Model the six Taylor series integrals with six noise processes — $u_a(k)$, $u_{\varphi_x}(k)$, $u_{\varphi_y}(k)$; $\nu_a(k)$, $\nu_{\varphi_x}(k)$, $\nu_{\varphi_y}(k)$ — called *Modulation Accelerations* or *MA's* (local averages of modulating function second derivatives).

- Model errors in filtered demodulation algorithm with uncorrelated *Measurement Noises (MN's)* $n_a(k)$, $n_{\varphi_x}(k)$, and $n_{\varphi_y}(k)$.

STATE-SPACE MODEL...

- Rearrange the six Taylor series and write together to obtain the statistical state-space component model

$$\begin{bmatrix} a_i(k+1) \\ a_i'(k+1) \\ \varphi_i^x(k+1) \\ \varphi_i^{x\prime}(k+1) \\ \varphi_i^y(k+1) \\ \varphi_i^{y\prime}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_i(k) \\ a_i'(k) \\ \varphi_i^x(k) \\ \varphi_i^{x\prime}(k) \\ \varphi_i^y(k) \\ \varphi_i^{y\prime}(k) \end{bmatrix} + \begin{bmatrix} u_a(k) \\ \nu_a(k) \\ u_{\varphi_x}(k) \\ \nu_{\varphi_x}(k) \\ u_{\varphi_y}(k) \\ \nu_{\varphi_y}(k) \end{bmatrix}$$

$$\mathbf{Y}(k) = \begin{bmatrix} a_i(k) \\ \varphi_i^x(k) \\ \varphi_i^y(k) \end{bmatrix}$$

- Observation equation:

$$\begin{bmatrix} \widehat{a}(k) \\ \widehat{\varphi^x}(k) \\ \widehat{\varphi^y}(k) \end{bmatrix} = \begin{bmatrix} a_i(k) \\ \varphi_i^x(k) \\ \varphi_i^y(k) \end{bmatrix} + \begin{bmatrix} n_a(k) \\ n_{\varphi_x}(k) \\ n_{\varphi_y}(k) \end{bmatrix}$$

PATH FUNCTION

- The TRACK PROCESSOR processes image pixels in the order specified by $\mathcal{P}$: