For a one sided discrete event system where without loss of generality $T \subseteq [0, \infty)$ define *cantor metric*

$$d(s_1, s_2) = \frac{1}{2^t}$$

where t is the smallest tag in T such that $s_1 \mid^t \neq s_2 \mid^t$, or if $s_1 = s_2$ then $d(s_1, s_2) = 0$.

An *open neighbourhood* in this metric space is the set of all the signals which have the same prefix. More formally, for $a \in S$, $r \in \Re$, $0 < r \leq 1$

$$\begin{aligned} N(a, r) &= \{\underline{s} : d(a, \underline{s}) < r\} \\ &= \{\underline{s} : \underline{s}|^{\tau} = a|^{\tau}\} \end{aligned}$$

where $\tau = \log_2\left(\frac{1}{r}\right)$.

Problem: Let S_d be the set of one sided discrete event signals and the cantor metric is defined. Show that S_d is Huasdorff, i.e., given $s, s' \in S_d$, $s \neq s'$ show that there exist open sets U_1 and U_2 such that $s \in U_1$, $s' \in U_2$ and $U_1 \cap U_2 = \phi$

We can also describe causality in the cantor metric space

- **Causality** $d(F(s), F(s')) \leq d(s, s')$
- **Strict Causality** $d(F(s), F(s')) < d(s, s')$
- **Delta Causality** $\exists k < 1, k = 2^{-\Delta}$ such that $d(F(s), F(s')) \leq k d(s, s')$ (F is a *contraction mapping*)

We now investigate whether a causal system placed in feedback configuration deterministic. We will show that if the system is strictly causal then the feedback configuration is guaranteed to have at most one behaviour, i.e., it can have either a unique behaviour or no behaviour at all. We will also show that if the system is delta causal then the feedback configuration has a unique behaviour and we can systematically find it.

We will first show that the cantor metric is indeed a metric. We need to show that it satisfies the four conditions that any metric must satisfy. For $s, s', s'' \in S_d$,

1. $d(s, s') = d(s', s)$ by definition
2. $d(s, s') \geq 0$ by definition
3. $d(s, s') = 0$ iff $s = s'$ by definition
4. Triangle inequality $d(s, s') + d(s', s'') \geq d(s, s'')$

We will prove a stronger condition than the last condition by showing that $\max(d(s,s'), d(s',s'') \geq d(s,s''))$, i.e., the cantor metric is an *ultra metric*.

Proof Without loss of generality let $d(s,s') \geq d(s',s'')$. $\exists\, \tau_1 \tau_2$ such that $s|^{\tau_1} = s'|^{\tau_1}$ and $s'|^{\tau_2} = s''|^{\tau_2}$. Since $d(s,s') \geq d(s',s'')$, $\tau_2 \geq \tau_1$. Hence $\exists\, \tau_3 \geq \tau_1$ such that $s|^{\tau_3} = s''|^{\tau_3}$. Hence $d(s,s') \geq d(s,s'')$.

Composing Functional Processes

- **Parallel Composition** The parallel composition of functional (causal, strictly causal or delta causal) processes is functional (causal, strictly causal or delta causal).
- **Cascade Composition** The cascade composition of functional (causal, strictly causal or delta causal) processes is functional (causal, strictly causal or delta causal).
- **Source Composition** The parallel composition of functional (causal, strictly causal or delta causal) processes and source processes is functional (causal, strictly causal or delta causal) if all the source processes are determinate (determinate, strictly causal, delta causal).
- **Feedback Composition** We will modify the general feedback configuration slightly. We replace the input with equivalent determinate source process. We also make all the output only signals as inputs. For $f : S^N \to S^N$, define the *semantics* to be a fixed point of f, i.e., s such that $f(s) = s$. If no or one signal satisfies this semantic, then the system is determinate otherwise it is indeterminate.
  - If f is strictly causal, then it has at most one fixed point. Hence the feedback composition is determinate.
  - *(Banach fixed point theorem)* If the metric space is complete and f is delta causal, then it has exactly one fixed point and that fixed point can be found by starting with any signal tuple s_0 and finding the limit of $s_1 = f(s_0), s_2 = f(s_1) \ldots$
  - If the metric space is compact (for instance if V is finite and time is discrete), then f only needs to be strictly causal to apply the Banach fixed point theorem.

Lemma Let (S_d, d) be some metric space and let the function $f : S^d \to S^d$ be strictly causal, i.e., $d(f(s), f(s') < d(s,s') \forall s, s' \in S_d$. Then if $f(s_1) = s_1$ and $f(s_2) = s_2$ then $s_1 = s_2$.

Proof Assume $s_1 \neq s_2$ then $d(f(s), f(s') < d(s,s') \forall s, s' \in S_d$ implies $d(s,s') < d(s,s')$ which is a contradiction. Hence $s_1 = s_2$.

Why do we have to restrict ourself to strict causality? Consider the following system with a delta causal process f. Define an input output process pair such that output is the same as input. Now this composite process is no longer delta causal but it is causal nevertheless. Even though f is delta causal but the overall system is causal and in the given feedback configuration, it is indeterminate.

Now consider the following example. The process has two inputs s_1 and s_2 and an output s_3. $T = [0, \infty), V = \{1, 2, 3 \ldots\}$
foreach $e = (\tau, v) \in s_1$, let $e' = (\tau + 1, v + 1) \in s_3$
foreach $e = (\tau, v) \in s_2$, let $e' = (\tau + 1/v^2, v + 1) \in s_3$
if collision, choose s_1.
Now consider this process in a feedback configuration with the output s_3 fed back to s_2 and the input at $s_1 = \{(i, 1), i = 0, 1, 2, \ldots\}$. Clearly the system is a discrete event system and obeys strict causality but in the feedback configuration, the behaviour of the system is not discrete event. Hence strict causality need not preclude absence of behaviour.