Spring 2014 EE 445S Real-Time Digital Signal Processing Laboratory Prof. Evans

Homework #0 Solutions on Review of Signals and Systems Material

Problem 0.1. Continuous-Time Sinusoidal Generation.

In practice, we cannot generate a two-sided sinusoid $\cos(2\pi f_c t)$, nor can we wait until the end of time to observe a one-sided sinusoid $\cos(2\pi f_c t)\, u(t)$.

In the lab, we can turn on a signal generator for a short time and observe the output in the time domain on an oscilloscope or in the frequency domain using a spectrum analyzer.

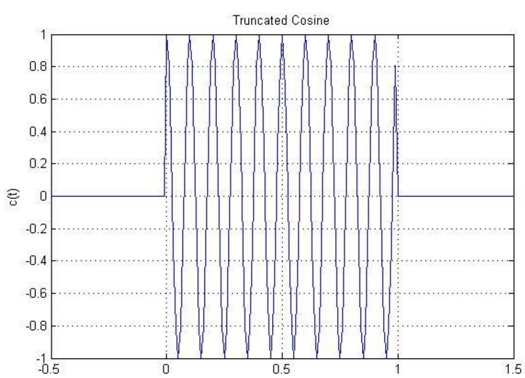
Consider a finite-duration cosine that is on from 0 sec to 1 sec given by the equation

$$c(t) = \cos(2\pi f_c t)\, \mathrm{rect}(t - ½)$$

where f_c is the carrier frequency (in Hz).

Part (a). Using MATLAB, LabVIEW Mathscript or GNU Octave, plot $c(t)$ for $-0.5 < t < 1.5$ for f_c = 10 Hz. Turn in your code and plot. You may find the rectpuls command useful. Give a formula for the Fourier transform of $c(t)$ for a general value of f_c.

Solution: Use MATLAB to plot the signal $c(t) = \cos(2\pi f_0 t)\, \mathrm{rect}(t - ½)$ for $-0.5 < t < 1.5$. We'll need to pick a sampling rate f_s so that $f_s > 2 f_{max}$ to obey the sampling theorem. The value of f_{max} could be computed using the answer in part (d) but we know from the modulation property of the Fourier transform that f_{max} is f_0 plus the bandwidth of $\mathrm{rect}(t - ½)$. Let's overample: f_s = 100 Hz.



MATLAB code

```
Ts = 0.01;
t= -0.5 : Ts : 1.5;
f0 = 10;
x = cos(2*pi*f0*t);
h = rectpuls(t-0.5);
c = x .* h;
figure
plot(t,c)
grid
title('Truncated Cosine')
xlabel('t')
ylabel('c(t)')
```

Part (b). Sketch by hand the magnitude of the Fourier transform of $c(t)$ for a general value of f_c. Using MATLAB, LabVIEW Mathscript or GNU Octave, plot the magnitude of the Fourier transform of $c(t)$ for f_c = 8 Hz. Turn in your code and plot.

Solution: Taking the Fourier transform of $c(t)$ we obtain

$$C(\omega)=\mathscr{F}\left\{\cos(\omega_0 t)\,\mathrm{rect}\left(t-\frac{1}{2}\right)\right\}=\frac{1}{2\pi}\mathscr{F}\{\cos(\omega_0 t)\}*\mathscr{F}\left\{\mathrm{rect}\left(t-\frac{1}{2}\right)\right\}$$

by using the Fourier transform property that multiplication in the time domain is convolution in the frequency domain. We then lookup the Fourier transforms of cosine and rectangular pulse:

$$C(\omega)=\frac{1}{2\pi}\left(\pi\,\delta(\omega-\omega_0)+\pi\,\delta(\omega+\omega_0)\right)*\left(e^{-j\omega/2}\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)\right)$$

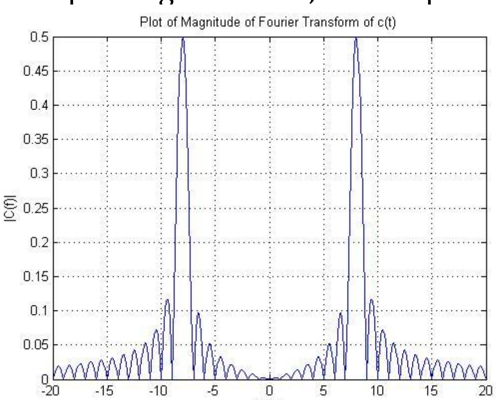
where sinc(x) = sin(π x) / (π x). Then,

$$\begin{aligned}
C(\omega) &= \frac{1}{2}\delta(\omega-\omega_0)*e^{-j\omega/2}\mathrm{sinc}(\frac{\omega}{2\pi})+\frac{1}{2}\delta(\omega+\omega_0)*e^{-j\omega/2}\mathrm{sinc}(\frac{\omega}{2\pi}) \\
&= \frac{1}{2}e^{-j\frac{\omega-\omega_0}{2}}\mathrm{sinc}(\frac{\omega-\omega_0}{2\pi})+\frac{1}{2}e^{-j\frac{\omega+\omega_0}{2}}\mathrm{sinc}(\frac{\omega+\omega_0}{2\pi}) \quad \text{(sifting property)} \\
&= \frac{1}{2}e^{-j(\frac{\omega-\omega 0}{2})}\mathrm{sinc}(\frac{\omega-\omega_0}{2\pi})+\frac{1}{2}e^{-j(\frac{\omega+\omega_0}{2})}\mathrm{sinc}(\frac{\omega+\omega_0}{2\pi})
\end{aligned}$$

When plotting by hand, the first term on the right hand side of the equation is centered at $f_0=\omega_0/2\pi$, and the second is centered at $-f_0=-\omega_0/2\pi$. We can then take the absolute value:

$$\begin{aligned}
|C(\omega)| &= \frac{1}{2}|e^{-j(\frac{\omega-\omega_0}{2})}\mathrm{sinc}(\frac{\omega-\omega_0}{2\pi})+e^{-j(\frac{\omega+\omega_0}{2})}\mathrm{sinc}(\frac{\omega+\omega_0}{2\pi})| \\
&\le \frac{1}{2}|e^{-j(\frac{\omega-\omega_0}{2})}\mathrm{sinc}(\frac{\omega-\omega_0}{2\pi})|+\frac{1}{2}|e^{-j(\frac{\omega+\omega_0}{2})}\mathrm{sinc}(\frac{\omega+\omega_0}{2\pi})| \\
&= \frac{1}{2}|\mathrm{sinc}(\frac{\omega-\omega_0}{2\pi})|+\frac{1}{2}|\mathrm{sinc}(\frac{\omega+\omega_0}{2\pi})|
\end{aligned}$$

When plotting in Matlab, we can plot the full formula for the Fourier transform magnitude.



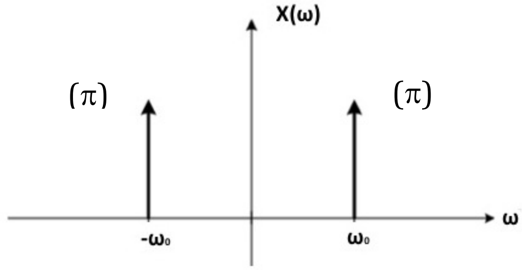
MATLAB code

```
f0=8;
f=[-20:0.1:20];
C = 0.5*exp(-j*pi*(f-f0)).*sinc(f-f0) +
0.5*exp(-j*pi*(f+f0)).*sinc(f+f0);
plot(f,abs(C))
grid
title('Plot of Magnitude of Fourier
Transform of c(t)')
xlabel('f (Hz)')
ylabel('|C(f)|')
```

A spectrum analyzer would display the above magnitude spectrum plus noise.

Part (c). Describe the differences between the magnitude of the Fourier transforms of $c(t)$ and a two-sided cosine of the same frequency. What is the bandwidth of each signal?

Solution: Bandwidth is the non-zero extent in positive frequencies of the signal's spectrum. Fourier transform of the two-sided cosine $x(t) = \cos(2\,\pi\, f_0\, t)$ is $X(\omega) = \pi\,(\delta(\omega+\omega_0) + \delta(\omega-\omega_0))$:



Each Dirac delta has width of zero and area of π.   Bandwidth is zero.

From the plot in part (c), the bandwidth of $C(\omega)$ is not zero. This is because of the multiplication of the two-sided cosine by rect(t - ½) in the time domain.

Each of the following is a valid method to estimate the bandwidth of bandpass signal $C(\omega)$:

- Set amplitude threshold of magnitude spectrum at an arbitrary point below which amplitudes are treated as if zero. If we use 0.15 as the threshold for the plot in part (c), then we can eyeball the estimated bandwidth of $C(\omega)$ to be 1.5 Hz.
- Measure width of the mainlobe in the magnitude spectrum between the two zero crossings on either side of the carrier frequency. For the magnitude spectrum in part (c), we can eyeball the estimated bandwidth to be 2 Hz.
- Estimate the power bandwidth to capture 90% of the area under the power spectrum. With a center frequency of 8 Hz, a bandwidth of 20 Hz would only occupy 55% of the total area. This approach does not work all that well in this case. More discussion next.

First, we define a function called myfun in file myfun.m to compute $|C(f)|$:

```
function mag_C = myfun(f)
f0=8;
mag_C = abs(0.5*exp(-j*pi*(f-f0)).*sinc(f-f0) + ...
0.5*exp(-j*pi*(f+f0)).*sinc(f+f0));
```

Next, we can approximate the total area under the magnitude spectrum by using 1000 Hz in place of ∞ and using the numerical integration function quad:

```
area = quad(@myfun,0,1000);
```

Finally, frequencies 0 to 20 Hz only contain 55% of the area:

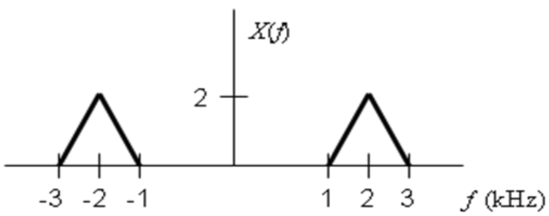
```
area1 = quad(@myfun,0,20);
```

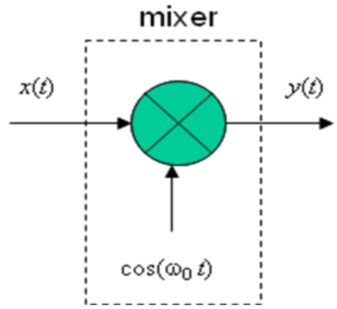
We are better off eyeballing the bandwidth from the plot of the magnitude spectrum.

Problem 0.2. Downconversion.

A signal $x(t)$ is input to a mixer to produce the output $y(t)$ where $y(t) = x(t)\cos(\omega_0\, t)$ and

$\omega_0 = 2\,\pi\,f_0$ and f_0 = 2 kHz. A block diagram of the mixer is shown below on the left. The Fourier transform of $x(t)$ is shown below on the left.





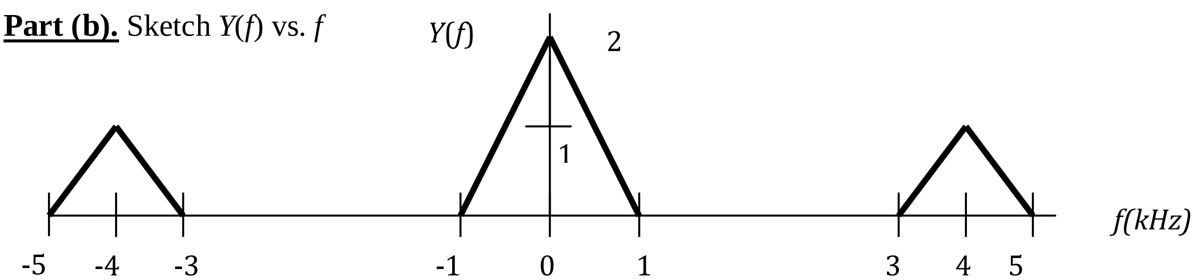
Part (a). Using Fourier transform properties, derive an expression for $Y(f)$ in terms of $X(f)$.

Solution: Fourier transform of $y(t) = x(t)\cos(\omega_0\, t)$ is $Y(\omega) = \dfrac{1}{2\pi}\,\Im\,[\cos(\omega_0\, t)] * X(\omega)$

Since $\cos(\omega_0\, t) \overset{\Im}{\leftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$, $Y(\omega) = \dfrac{1}{2}(X(\omega - \omega_0) + X(\omega - \omega_0))$

In a similar way, $Y(f) = \dfrac{1}{2}(\delta(f - f_0) + \delta(f + f_0)) * X(f) = \dfrac{1}{2}(X(f - f_0) + X(f + f_0))$

Part (b). Sketch $Y(f)$ vs. f



Part (c). What operation would you apply to the signal $y(t)$ in part (b) to obtain a baseband signal? The process of extracting the baseband signal from a bandpass signal is known as *downconversion*.

Solution: Baseband signal may be obtained by applying a lowpass filter to $y(t)$ that passes frequencies in [-1,1] kHz and attenuates frequencies at and above 3 kHz in absolute value.

Problem 0.3. Sampling in Continuous Time.

Sampling the amplitude of an analog, continuous-time signal $f(t)$ every T_s seconds can be modeled in continuous time as $y(t) = f(t)\, p(t)$ where $p(t)$ is the impulse train defined by

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

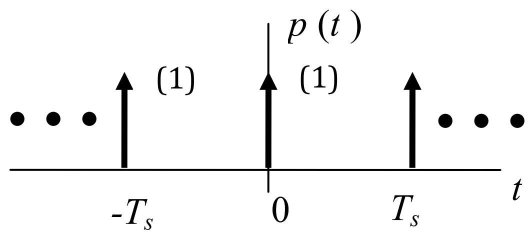
T_s is known as the sampling duration. The Fourier series expansion of the impulse train is

$$p(t) = \frac{1}{T_s}\left(1 + 2\cos(\omega_s\ t) + 2\cos(2\ \omega_s\ t) + \ldots\right)$$

where $\omega_s = 2\ \pi\ /\ T_s$ is the sampling rate in units of radians per second.

Part (a). Plot the impulse train.

Solution: Plot of the impulse train $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$



The notation (A) means that the area under the Dirac delta is A. This is important because the value of the Dirac delta at the origin is undefined.

Part (b). Note that in part (a), $p(t)$ is periodic. What is the period?

Solution: A signal $x(t)$ is periodic with period T if $x(t) = x(t - T)$ for all t. The smallest value of T for which $x(t)$ is periodic is called the fundamental period. The impulse train $p(t)$ is periodic with fundamental period T_s. The fundamental period is used in the Fourier series expansion of the

$$p(t) = \frac{1}{T_s}\left(1 + 2\cos(\omega_s\ t) + 2\cos(2\ \omega_s\ t) + \ldots\right)$$

impulse train where the sampling rate in rad/s is $\omega_s = 2\ \pi\ /\ T_s$:

Part (c). Using the Fourier series representation of $p(t)$ given above, please give a formula for $P(\omega)$, which is the Fourier transform of $p(t)$. Express your answer for $P(\omega)$ as an impulse train in the Fourier domain.

Solution: We apply the Fourier transform to $p(t)$ given immediately above to compute

$$P(\omega) = \frac{2\pi}{T_s}\left(\delta(\omega) + \delta(\omega - \omega_s) + \delta(\omega + \omega_s) + \delta(\omega - 2\omega_s) + \delta(\omega + 2\omega_s) + \ldots\right)$$

by using the transform pairs $\cos(\omega_s t) \overset{\mathfrak{F}}{\leftrightarrow} \pi[\delta(\omega - \omega_s) + \delta(\omega + \omega_s)]$ and $1 \overset{\mathfrak{F}}{\leftrightarrow} 2\pi\delta(\omega)$

Thus $P(\omega) = \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$

Part (d). What is the spacing of adjacent impulses in the impulse train in $P(\omega)$ with respect to frequency ω in rad/s?

Solution: The spacing of the impulse train $P(\omega)$ in ω is ω_s, which is the distance between adjacent impulses in the Fourier domain.

Problem 0.4. Discrete-Time Sinusoidal Generation.

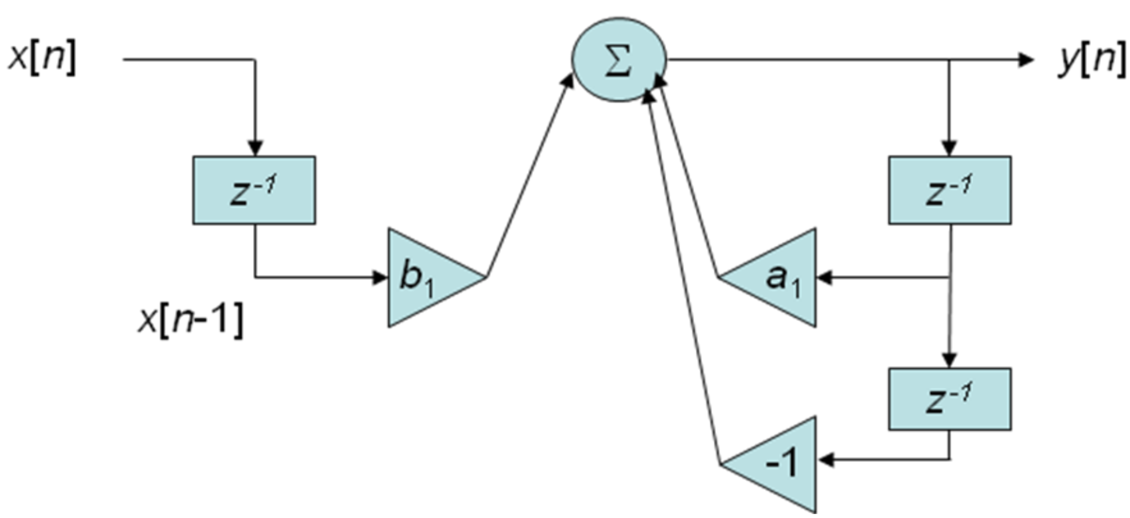
Consider a causal discrete-time linear time-invariant system with input $x[n]$ and output $y[n]$ being governed by the following difference equation:

$$y[n] = (2 \cos \omega_0)\, y[n-1] - y[n-2] + (\sin \omega_0)\, x[n-1]$$

The impulse response of the above system is a ***causal sinusoid*** with discrete-time frequency ω_0 in units of rad/sample. The value of ω_0 would normally be in the interval $[-\pi, \pi)$. You will be implementing the above difference equation in C in lab #2 on a programmable digital signal processor for real-time sinusoidal generation.

Part (a). Draw the block diagram for this system using add (or summation), multiplication (or gain), and delay blocks. Please label delay blocks with the text z^{-M} to denote a delay of M samples. Use arrowheads to indicate direction of the flow of signals.

Solution: Block diagram of the given filter with input $x[n]$ and output $y[n]$ and with coefficients $b_1 = \sin \omega_0$ and $a_1 = 2 \cos \omega_0$:



Part (b). Please state all initial conditions. Please give values for the initial conditions to satisfy the stated system properties.

Solution: Initial conditions can be found by calculating the first few output values:

$$y[0] = (2\cos\omega_0)\, y[-1] - y[-2] + (\sin\omega_0)\, x[-1]$$

$$y[1] = (2\cos\omega_0)\, y[0] - y[-1] + (\sin\omega_0)\, x[0]$$

Hence, the initial conditions are given by $y[-1]$, $x[-1]$ and $y[-2]$. These values correspond to the initial values in the delay-by-one-sample blocks, which are denoted by z^{-1}.

The impulse response is given as $h[n] = \sin(\omega_0 n)\, u[n]$. That is, for $x[n] = \delta[n]$, the output is $y[n] = h[n]$, which gives $y[-1] = h[-1] = 0$ and $y[-2] = h[-2] = 0$ and $x[-1] = \delta[-1] = 0$,

For the system to satisfy the linearity property, the initial conditions must be zero. Same goes for the system property of time-invariance. We will show this on slides 3-11 and 3-12.

Part (c). Find the equation for the transfer function in the z-domain, including the region of convergence.

Solution: $y[n] = (2\cos\omega_0)\, y[n-1] - y[n-2] + (\sin\omega_0)\, x[n-1]$

Taking z-transform of both sides, we get

$$Y(z) = (2\cos\omega_0)\, z^{-1}\, Y(z) - z^{-2}\, Y(z) + (\sin\omega_0)\, z^{-1} X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{(\sin\omega_0)\, z^{-1}}{1 - 2(\cos\omega_0)\, z^{-1} + z^{-2}} \Rightarrow H(z) = \frac{(\sin\omega_0)\, z^{-1}}{1 - 2(\cos\omega_0)\, z^{-1} + z^{-2}}$$

To find the region of convergence, we need to find the poles of this transfer function:

$$1 - (2\cos\omega_0)\, z^{-1} + z^{-2} = 0$$

By multiplying each side by z^2 (assuming that $z \neq 0$):

$$z^2 - (2\cos\omega_0)\, z + 1 = 0$$

Roots are located at ½ (-b ± sqrt(Δ)). Here,

$$\Delta = 4\cos^2\omega_0 - 4 = -4\sin^2\omega_0$$

Since $\Delta < 0$, there are two complex roots:

$$x_1 = \frac{2\cos\omega_0 + 2j\sin\omega_0}{2} = \cos\omega_0 + j\sin\omega_0 ; x_2 = \frac{2\cos\omega_0 - 2j\sin\omega_0}{2} = \cos\omega_0 - j\sin\omega_0$$

Both poles have a magnitude of 1. As a result, they lie on the unit circle in the z-plane.

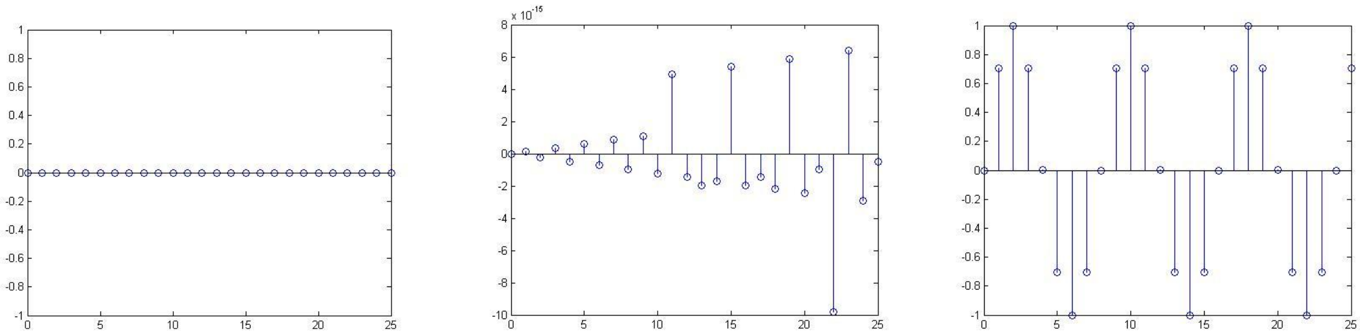
Since the system is causal, the region of convergence will be outside of the circle of radius equal to the magnitude of the pole with the greatest magnitude, i.e. $|z| > 1$.

Part (d). Compute the inverse z-transform of the transfer function in part (c) to find the impulse response of the system.

Solution: By using inverse z-transform tables, the above $H(z)$ with the given region of convergence will have an impulse response of $h[n] = \sin(\omega_0\ n)\ u[n]$.

Part (e). Using MATLAB, LabVIEW Mathscript or GNU Octave, plot the impulse response obtained in part (d) for ω_0 equal to 0, π, and a value in the interval $(0, \pi)$ of your choosing. Turn in your code and plots.

Solution: ω_0 is 0 rad/sample (left), π rad/sample (center), and π/4 rad/sample (right):



Sample MATLAB code for $\omega_0 = \pi/4$:

```
w0 = pi/4;
n = [0:25];
u = stepfun(n,0);
x = sin(w0*n);
h = x.*u;
figure, stem(n,h);
```

In the middle plot, which is for ω_0 **=** π rad/sample, all amplitude values should have been zero. Non-zero values are due to numerical error in computing $\sin(\omega_0\ n)$ for $0 \le n \le 25$ in floating-point arithmetic. The values are on the order of 10^{-15}, which is very close to zero.