Solution Set for Homework #1 on Sinusoidal Signals

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1. ***Prologue***: This problem helps you to identify the points of interest in a sinusoidal signal, and calculate the parameters of the waveform based on your observations. It relies on the definitions given in Sec. 2-3 of the *Signal Processing First* textbook.

   Solution:

   To estimate the frequency, we would first estimate the period and the invert the period. We could estimate a period in the plot by measuring the amount of time it takes to go from a peak to the next peak, or from one valley to the next. The plot has peaks at times t_0 = -20 ms, t_1 = 5 ms, t_2 = 30 ms. One period has elapsed from t_0 to t_1 and two periods have elapsed from t_0 to t_2. The period is about 25 ms long.
   $\omega_0 = 2\pi/T = 80\pi = 251.327$ rad/s.
   To calculate A, consider the peak-to-peak amplitude (X_{pp}) of the waveform which is the difference between the maximum value and the minimum value:
   $X_{pp} = 20 - (-20) = 40$
   So, $A = X_{pp}/2 = 20$.
   We are left with one unknown, ϕ. We can pick any point in time to give us one equation in one unknown: $x(t) = A\cos(\omega_0 t + \phi)$
   - *Approach #1:* If we pick $t = 0$, then $x(0) = 20\cos(\phi) \approx 5$ and
   $\phi \approx \cos^{-1}(5/20) = 1.3181$ (1)
   Recall that $\cos(-\phi) = \cos(\phi)$, so ϕ could be +1.318 rad or -1.318 rad. Also, any addition of ϕ and a multiple of 2π would also be a valid answer.
   To decide between the values, consider the slope at $t = 0$, which is positive. (2)
   $x(t) = A\cos(\omega_0 t + \phi)$ => $x'(t) = -A\,\omega_0 \sin(\omega_0 t + \phi)$ => $x'(0) = -A\,\omega_0 \sin(\phi)$
   So, condition (2) implies that $x'(0) > 0$ => $\sin(\phi) < 0$.
   Out of the two values in (1), only $\phi = -1.318$ rad satisfies this condition.
   - *Approach #2.* Pick t = 5 ms where there is a peak. Then,
   $x(5\text{ms}) = 20\cos(2\pi(5\text{ms}/25\text{ms}) + \phi) = 20$ => $\cos(2\pi/5 + \phi) = 1$ =>
   $2\pi/5 + \phi = 0$ => $\phi = -2\pi/5 = -1.2566$ rad

   Epilogue: There is nothing special about the time points considered for the solution. The points were chosen due to convenience, and the problem could have been done by taking other points as well. Additionally, the value of the phase ϕ is not unique (See prob. 4 on this solution set for an illustration of this).

2. ***Prologue***: This problem highlights a property of complex numbers on the unit circle, which has many implications. Euler's formula is given in Sec. 2-5.1 of *Signal Processing First*.

   Solution: Proof:

   $(\cos\theta + j\sin\theta)^n = (e^{j\theta})^n$ [Applying Euler's formula]
   $(e^{j\theta})^n = e^{jn\theta}$
   $e^{jn\theta} = \cos(n\theta) + j\sin(n\theta)$ [Applying Euler's formula] Q.E.D.

For $3/5 + j\,4/5$, $\theta = \cos^{-1}(3/5) = 0.9273$ rad

$(3/5 + j\,4/5)^{100} = \cos(100*0.9273) + j\sin(100*0.9273)$ = $\underline{0.0530 - j\,0.9986}$

Checking answer in MATLAB: `(3/5 + j*4/5)^100` gives `0.0525 - 0.9986j`

Epilogue: This formula makes it easy to deal with powers of complex numbers. A plot of the complex roots of unity helps visualize some of the implications of this formula.

3. ***Prologue***: This question shows a standard way of plotting sinusoids in MATLAB. It also requires you to identify which parts of the code are mathematically interesting.

Solution: The lines which define the sinusoid are:

zz = 300*exp(j*(2*pi*F0*(tt-0.75))); (1)

xx = real(zz);

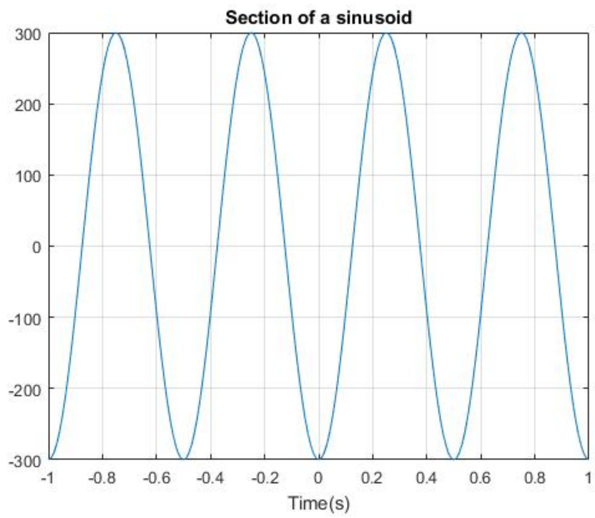
Using Euler's formula on (1), we get,

xx = 300*cos(2*pi*F0*tt – 2*pi*F0*0.75)

In continuous time, with F0 = 2 Hz, this becomes

$x(t) = 300\cos(2\pi(2)t - 2\pi(2)\,0.75)$ = $\underline{200\cos(12.566\,t - 9.425)}$

Using the MATLAB code in the problem, here's the resulting plot of the sinusoidal signal:



Epilogue: This code shows a particular method of generating a sinusoid in MATLAB. A sinusoid can also be generated by plotting the cosine (or sine) for a given array of time points. The Euler representation of sinusoids was part of homework problem 1.2.

4. ***Prologue***: This problem gives an insight into the phase of a sinusoid. It also explicitly makes it clear that the value of the phase is not unique.

Solution: The phase $\phi = -2\pi f_0 t_1 = -2\pi t_1/T_0$ (unique up to a multiple of 2π).

a) $\Phi = -2\pi(-2)/8 = \pi/2$.

Hence, the statement is <u>true</u>.

b) $\Phi = -2\pi(3)/8 = -3\pi/4$.

Hence, $\phi = -3\pi/4$ or $-3\pi/4 + 2\pi = 5\pi/4$.

The statement is <u>false</u>.

c) Φ = -2π(7)/8 = -7π/4.
Hence, ϕ = -7π/4 or -7π/4 + 2π = π/4.
The statement is true.

Epilogue: Phase values are often considered to be in [-π, π] or [0, 2π]. However, this is just a matter of common practice, and any other interval of length 2π would do just as well.

5. ***Prologue***: This problem changes the signal representation from the time domain to the frequency domain. It also deals with periodicity while adding sinusoidal signals, which is explained in Sec. 3-3 of the *Signal Processing First* textbook.

Solution:

a) x(t) = 10*cos(800*pi*t + pi/4) + 7*cos(1200*pi*t – pi/3) – 3*cos(1600*pi*t)

$$= 10.\frac{e^{j(800*pi*t+\frac{pi}{4})}+e^{-j(800*pi*t+\frac{pi}{4})}}{2} + 7.\frac{e^{j(1200*pi*t-pi/3)}+e^{-j(1200*pi*t-pi/3)}}{2} - 3.\frac{e^{j(1600*pi*t)}+e^{-j(1600*pi*t)}}{2}$$

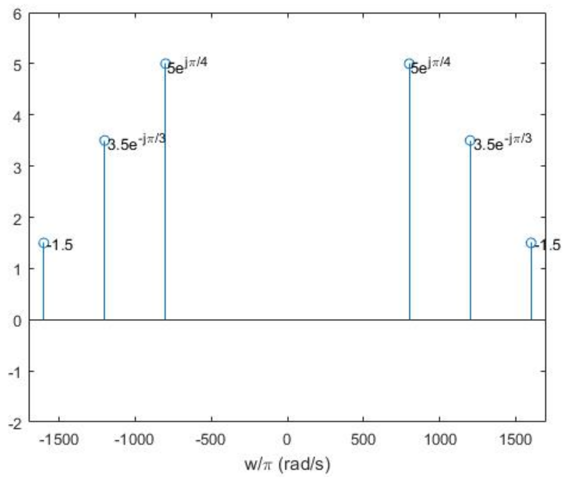


Fig 1: Spectrum plotted with complex amplitudes.

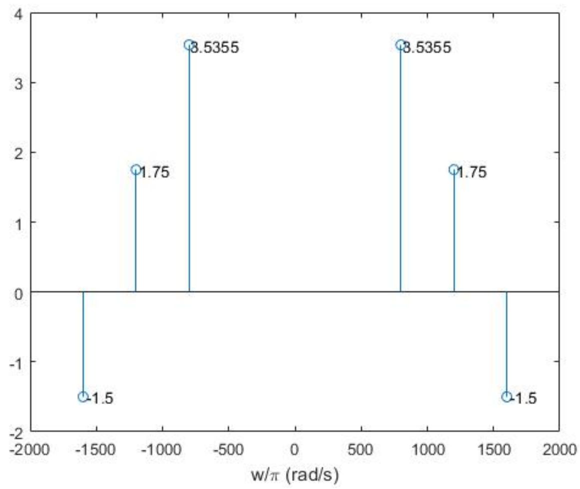


Fig 2: Spectrum plotted with real part of complex amplitudes.

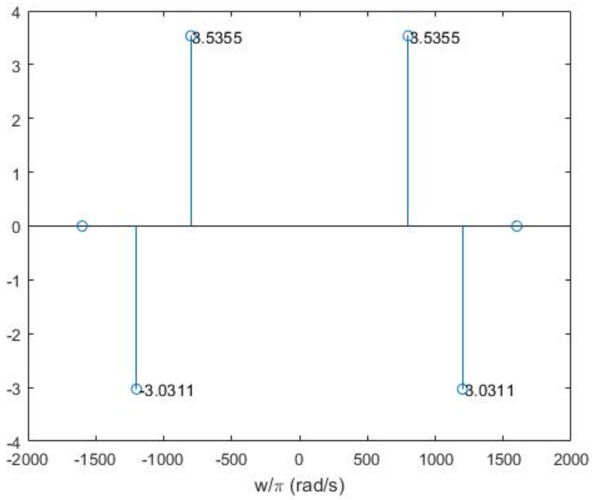


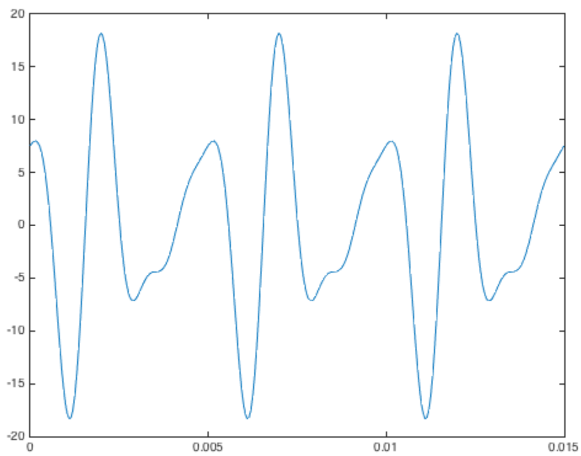
Fig 3: Spectrum plotted with imaginary parts of complex amplitudes.

b) Yes, $x(t)$ is periodic. The fundamental frequency ω_0 in rad/s of $x(t)$ is gcd(800π, 1200π, 1600π) = 400π which means $T_0 = 2\pi / \omega_0 = 2\pi/(400\pi) = 1/200$ s = 0.005 s = 5 ms

c) Adding 5 cos(1000πt + π/2) adds another frequency component to the signal. So, the a line for $2.5e^{j*pi/2}$ at ω = 1000 π and a line for $2.5e^{j*pi/2}$ at ω = -1000 π are added to the spectrum. $y(t)$ is periodic. The fundamental freq. ω_0 of $y(t)$ is gcd (400π, 1000π) = 200π which means $T_0 = 2\pi/\omega_y = 2\pi/(200\pi) = 0.01$s = 10 ms

d) We set the sampling rate $f_s = 24 f_{max}$ for plotting where f_{max} = 800 Hz from part (b).

```
f0 = 200;             % fundamental frequency in part(b) in Hz
T0 = 1/f0;            % fundamental period in part (b) in seconds
fmax = 800;           % maximum frequency in Hz in x(t)
fs = 24*fmax;         % set sampling rate fs = 24 fmax for plotting
Ts = 1/fs;            % sampling time is the inverse of sampling rate
t = 0 : Ts : 3*T0;    % plot three periods of x(t)
x = 10*cos(800*pi*t+pi/4)+7*cos(1200*pi*t-pi/3)-3*cos(1600*pi*t);
plot(t, x);
```



Epilogue: The fact that sinusoids, and by extension, complex exponentials, are single frequency signals is what underlies the Fourier series and the Fourier transform. Here, we also see that the frequency domain symmetry around ω = 0 rad/s for a real-valued time-domain signal. This observation will be revisited later when studying Fourier transforms.