EE 313 Linear Signals & Systems (Fall 2017)

Solution Set for Homework #2 on Fourier Series

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1. **Prologue:** This problem reviews concepts introduced in the last homework, and shows how you can find the spectra for simple signals without calculating the full Fourier transform.

Solution:

a) $x(t) = 10 + 20\cos(2\pi(100)t + \pi/4) + 10\cos(2\pi(250)t)$

$= 10 + 20(e^{j(200\pi t+\pi/4)} + e^{-j(200\pi t+\pi/4)})/2 + 10(e^{j2\pi 250t} + e^{-j2\pi 250t})/2$

$= 10 + 10e^{j\pi/4}e^{j200\pi t} + 10e^{-j\pi/4}e^{-j200\pi t} + 5e^{j2\pi 250t} + 5e^{-j2\pi 250t}$

Now, the fundamental frequency of the signal is gcd(100,250) = 50Hz.

$\underline{f_0 = 50 \text{ Hz}}$.

$\underline{N = f_{max}/f_0 = 250 \text{ Hz} / 50 \text{ Hz} = 5}$

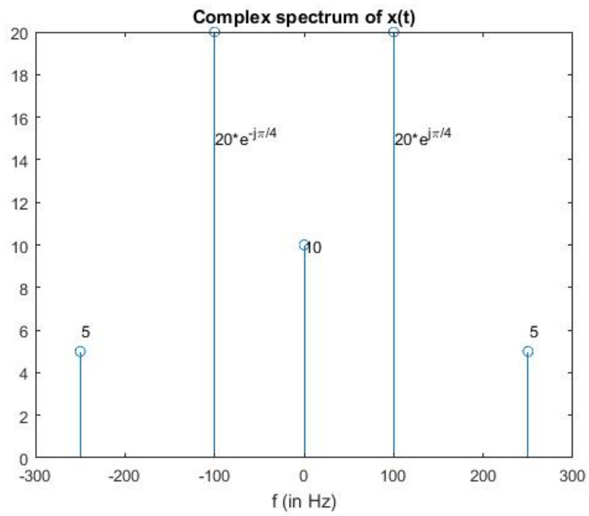
As we can see above, the non-zero spectral components occur at $k = 0$, $k = \pm N$ and $k = \pm 100\text{Hz}/f_0 = \pm 2$

From the equation above, we can read off the values of the a_k's (the values not specified below are all zero):

$\underline{a_0 = 10,\ a_2 = 10e^{j\pi/4},\ a_{-2} = 10e^{-j\pi/4},\ a_5 = 5,\ a_{-5} = 5}$

b) As shown above, the signal is periodic, with a fundamental frequency of f_0 = 50Hz. The fundamental period is thus $1/f_0$ = <u>0.02s</u>

c) Here's the plot of the spectrum



Epilogue: In part (a), the Fourier coefficient a_2 is the conjugate of a_{-2}, and likewise for a_5 and a_{-5}. Since each pair of Fourier coefficients for $k \neq 0$ has conjugate symmetry, and since a_0 is always real-valued, the signal $x(t)$ must be real-valued, and it is.

2. **Prologue:** This problem uses the Fourier analysis equation (Equation 3.26 in the book), and asks you to find the complex coefficients of the spectral lines for a square wave. A part of the question asks you to compute the coefficients of a time shifted version of a base signal, which uses some of the most important properties of the Fourier transform & series.

Solution:

a) $a_k = 1/T_0 \int_{-T_0/2}^{T_0/2} x(t) e^{-jkw_0 t} dt$, $\omega_0 = 2\pi/T_0$, $k \neq 0$

$= 1/T_0 \int_{-T_0/4}^{T_0/4} e^{-jkw_0 t} dt$

$= \frac{1}{T_0} \cdot \frac{-1}{jkw_0} \cdot e^{-jkw_0 t} \Big|_{-T_0/4}^{T_0/4}$

$= \frac{1}{T_0} \cdot \frac{-1}{jkw_0} \cdot (e^{-jkw_0 T_0/4} - e^{jkw_0 T_0/4})$

$= \frac{1}{T_0} \cdot \frac{2}{kw_0} \cdot \sin\left(\frac{kw_0 T_0}{4}\right)$

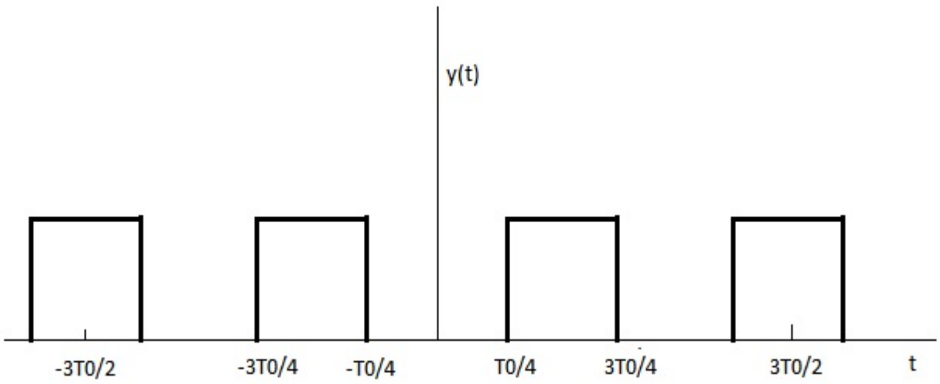
$= \frac{1}{k\pi} \cdot \sin\left(\frac{2\pi k}{4}\right) = \frac{1}{k\pi} \cdot \sin\left(\frac{\pi k}{2}\right)$ (Substituting the value of ω_0)

$$a_0 = 1/T_0 \int_{-T_0/2}^{T_0/2} x(t) dt$$

$$= 1/T_0 \int_{-T_0/4}^{T_0/4} dt = \frac{1}{T_0} \cdot \frac{T_0}{2} = \frac{1}{2}$$

$$So, a_0 = \frac{1}{2}, a_k = \frac{1}{k\pi} \cdot \sin\left(\frac{\pi k}{2}\right), k \neq 0$$

b) The amplitude of the rectangular pulses below is 2.



From P-3.14, we know that the Fourier coefficients scale as follows:

$x(t) \to a_k$ $\Rightarrow M\,x(t) \to M\,a_k$ (1)

$x(t) \to a_k$ $\Rightarrow x(t-t_0) \to a_k e^{-jk\omega_0 t_0}$ hello (2)

Applying (1) and (2) to the coefficients derived in the previous part, we get the coefficients b_k for the Fourier series of $y(t)$:

$$b_0 = 1, b_k = \frac{2}{k\pi} \cdot \sin\left(\frac{\pi k}{2}\right) \cdot e^{-jk\left(\frac{2\pi}{T_0}\right) \cdot T_0/2} = \frac{2}{k\pi} \cdot \sin\left(\frac{\pi k}{2}\right) \cdot e^{-jk\pi} = (-1)^k \cdot \frac{2}{k\pi} \cdot \sin\left(\frac{k\pi}{2}\right)$$

c) <u>Solution #1:</u> To plot $x(t)$, we can add up shifted rectangular pulses

In MATLAB, the `rectpuls(x)` command has value 1 for x in [-0.5, 0.5). Pulse width is 1.

For t in $[-T_0/4, 3T_0/4)$, our pulse is one for t in $[-T_0/4, T_0/4)$. Pulse width is $T_0/2$.

The MATLAB command would be `rectpuls(t / (T0/2));`

For t in $[3T_0/4, 7T_0/4)$, our pulse is shifted right by T_0: `rectpuls((t-T0) / (T0/2));`

We can add up each shifted pulse to define our signal over a finite interval of time.

```
f0 = 440;
T0 = 1/f0;
fs = 44100;
tmax = 3;
t = -T0/4 : 1/fs : tmax-T0/4;
timeoffsets = 0 : T0 : tmax;
x = zeros(1,length(t));
for t0 = timeoffsets
  x = x + rectpuls((t - t0) / (T0/2));
end
sound(x,fs)                       % Plays x(t)
x1 = cos(2*pi*f0*t);
pause(tmax+1);
sound(x1,fs);                     % Plays cosine at frequency f0
```

Solution #2: To plot $x(t)$, we could convert a sine wave to a rectangular pulse.
The sine wave oscillates from -1 to 1 inclusive.
The MATLAB function `sign(x)` returns 1 if $x > 0$, 0 if $x = 0$ and -1 if $x < 0$.
For the square wave, we'd like to have amplitude values in the interval [0, 1].
We can take the output of the sign function, add 1, and divide by 2 to get the values in [0, 1].

```
f0 = 440;
T0 = 1/f0;
fs = 44100;
tmax = 3;
t = -T0/4 : 1/fs : tmax-T0/4;
sineWave = sin(2*pi*f0*(t + T0/4));
sq_wave = sign(sineWave);
x2 = (1 + sq_wave)/2;
sound(x2,fs);
x1 = cos(2*pi*f0*t);
pause(tmax+1);
sound(x1,fs);
```

Compared to a cosine at a single frequency, $x(t)$ has more harmonics, leading to a 'richer' sound, whereas the cosine has just one frequency, which sounds 'thinner'.

Epilogue: Here we see that even a "simple" signal in the time domain (square wave), has an infinite number of harmonics in it. This shows an instance of a general rule of thumb, signals are usually easier to manipulate mathematically in the time or frequency domain.

3. **Prologue:** This problem introduces the chirp signal (Section 3-8), and the concept of instantaneous frequency. Chirp signals are widely used in audio, sonar, cellular, and other systems. More about that in the epilogue.

Solution:

$\psi(t) = \alpha t^2 + \beta t + \phi$

$\omega(t) = d\,\psi(t)/dt = 2\alpha t + \beta$

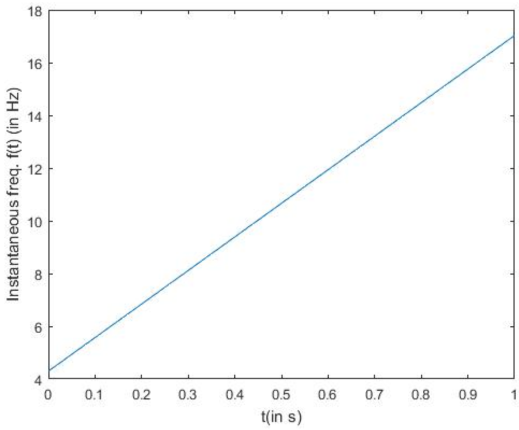
Starting frequency $\underline{\omega_1 = \omega(0) = \beta}$ rad/s.

Ending frequency $\underline{\omega_2 = \omega(T_2) = 2\alpha T_2 + \beta}$ rad/s.

b) $\psi(t) = 40t^2+27t+13$

$\omega(t) = d\,\psi(t)/dt$ = <u>80t + 27 rad/s.</u>

c) The time-frequency plot over $0s \le t \le 1s$ follows:



d) For $\psi(t) = \alpha t^2 + \beta t + \phi$ over $0 \le t \le T_2$, instantaneous frequencies go from β rad/s to $2\alpha T_2+\beta$ rad/s from part (a). So, for $\alpha = 800\pi$, $\beta = 540\pi$, $\phi = 260\pi$ and $T_2 = 3$s, the instantaneous frequencies go from 540π to 5340π rad/s, or from 270 to 2670 Hz. The maximum frequency f_{max} is 2670 Hz. The sampling rate $f_s > 2\,f_{max}$ and also needs to be a sampling rate supported by the audio playback system.

```
fs = 44100;
Ts = 1 / fs;
t = 0: Ts : 3;
y = real(exp(j.*(800.*pi.*t.^2+540.*pi.*t+260.*pi)));
sound(y,fs)
```

As expected, the signal shows linearly increasing frequency with time. Although one might be able to detect that the increase is linear, one should be able to hear an increase.

Epilogue: In audio systems, a chirp can be used to sweep through a range of audible frequencies to measure the sound quality. In active sonar systems, a chirp is transmitted over acoustic (sound) frequencies using an underwater speaker and the sonar receiver uses a microphone to listen for the return of the chirp signal to determine the location (angle, distance) of objects in the environment; in this context, the chirp is often called a 'ping'. In cellular systems, a transmitter sends a complex-valued chirp signal, a.k.a. a Zadoff-Chu sequence, and a receiver can used the received signal to estimate and compensate for the distortion in the channel. A chirp signal can also be used to estimate Doppler shift.

The concept of instantaneous frequency of a sinusoid varying with time is used extensively to carry a message signal in the frequency content of a sinusoid instead of in the amplitude. Examples include frequency modulation and phase modulation for analog continuous-time message signals, and frequency-shift keying and phase-shift keying for digital discrete-time messages. Phase-shift keying has recently gained a lot of attention in low-power Internet of Things sensors because of its incredible power effficient in transmitting bits over the air.

4. Prologue: This problem shows the effect of various mathematical operations on the frequency spectrum.

Solution:

a)

```
fs = 8000;
f0 = 440;
t = 0:1/fs:2;           % Generating vectors to play sound for 2s
x = cos(2*pi*f0*t);
y1 = x.*cos(2*pi*220*t);
y2 = x.^2;
y3 = x.^3;
sound(x,fs); pause(4);
sound(y1,fs); pause(4);
soundsc(y2,fs); pause(4);
sound(y3,fs);
```

$$y(t) = \cos(2\pi 440t)\cos(2\pi 220t)$$

$$y(t) = (e^{j2\pi 440t} + e^{-j2\pi 440t})(e^{j2\pi 220t} + e^{-j2\pi 220t})/4$$

$$= (e^{j2\pi 660t} + e^{-j2\pi 220t} + e^{j2\pi 220t} + e^{-j2\pi 660t})/4$$

$$= (\cos(2\pi 660t) + \cos(2\pi 220t))/2$$

So, y(t) contains the frequencies ±660Hz and ±220Hz.

As expected, we can hear a higher frequency sound (which is provided by the 660Hz component). The sound feels as if more than one frequency is present (less 'thin' than a pure cosine). Please note that 660 Hz is a harmonic of 220 Hz, and each individual may have a different perception of a tone and its harmonic.

b)

$$y(t) = cos^2(2\pi 440t)$$

$$y(t) = (\cos(2\pi 880t) + 1)/2$$

(Using the trig. identity $\cos(2\theta) = 2\cos^2(\theta) - 1$)

So, y(t) contains the frequencies ±880Hz.

Alternatively,

$$y(t) = cos^2(2\pi 440t)$$

$$y(t) = (e^{j2\pi 440t} + e^{-j2\pi 440t})^2/4$$

$$= \frac{e^{j2\pi 880t} + e^{-j2\pi 880t} + 2}{4} = \frac{1}{2} + \frac{\cos(2\pi 880t)}{2}$$

(Using the binomial expansion for $(a+b)^2 = a^2+b^2+2ab$)

We hear a higher frequency than what we got for part(a), which is provided by the 880Hz component. It sounds 'thin', which is because just one frequency is present in the waveform.

c)

$$y(t) = cos^3(2\pi 440t)$$

$$y(t) = (\cos(2\pi 1320t) + 3\cos(2\pi 440t))/4$$

(Using the trig. identity cos(3θ) = $4\cos^3$(θ) -3cos(θ))

Alternatively, one can use phasors to work the problem with needing a trig identity:

$$y(t) = (e^{j2\pi 440t} + e^{-j2\pi 440t})^3/8$$

$$= \frac{e^{j2\pi 1320t} + e^{-j2\pi 1320t} + 3e^{j2\pi 440t} + 3e^{-j2\pi 440t}}{8}$$

$$= \frac{3\cos(2\pi 440t)}{4} + \frac{\cos(2\pi 1320t)}{4}$$

(Using the binomial expansion for $(a+b)^3 = a^3+b^3+3a^2b+3ab^2$)

So, y(t) contains the frequencies ±1320Hz and ±440Hz.

This sounds as if it has a frequency between the signals in parts (a) & (b). This is probably because the component at 1320Hz has a relatively low power. This puts more of the signal power in the lower frequency (400Hz), making the sound seem low pitched.

Epilogue: The expansions above could also have been achieved by multiplying the terms by hand. The binomial expansion just provides a short-cut, and cuts down on algebraic mistakes. For reference, the binomial expansion formula is:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \quad \forall\, n \in W \ (\coloneqq set\ of\ whole\ numbers)$$

The spectral effects of multiplying signals in time will be revisited later, in much more detail, while studying the properties of continuous-time Fourier transforms.