EE 313 Linear Signals & Systems (Fall 2017)

Solution Set for Homework #3 on Sampling & Reconstruction

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1. **Prologue:** This question revisits the Fourier series coefficient computation with a somewhat more complex integration.

Solution: The given signal is:

$$x(t) = \begin{cases} \frac{4}{T_0}t & for\ \ 0 < t \le \frac{T_0}{4} \\ -\frac{4}{T_0}\ t & for -\frac{T_0}{4} \le t \le 0 \\ 0 \ \ otherwise \end{cases}$$

The Fourier series synthesis equation is $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_o kt}$ where $\omega_o = 2\pi f_0$.
Here, the fundamental frequency f_0 is the inverse of the fundamental period T_0.
The coefficient a_0 is the average value of $x(t)$ over the fundamental period:

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,dt = \frac{Area\ under\ the\ curve\ for\ one\ period}{T_0} = \frac{T_0/4}{T_0} = \frac{1}{4}$$

By the Fourier analysis formula, we get

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j\omega_o kt}\,dt = \frac{1}{T_0}\left[\int_0^{T_0/4} x(t)e^{-j\omega_o kt}\,dt + \int_{-T_0/4}^{0} x(t)e^{-j\omega_o kt}\,dt\right]$$

$$= \frac{1}{T_0}\left[\int_0^{T_0/4} x(t)e^{-j\omega_o kt}\,dt + \int_{-T_0/4}^{0} x(t)e^{-j\omega_o kt}\,dt\right]$$

$$= \frac{4}{{T_0}^2}\left[\int_0^{T_0/4} te^{-j\omega_o kt}\,dt + \int_{-T_0/4}^{0} -te^{-j\omega_o kt}\,dt\right] = I_1 + I_2$$

Now consider I_2.

$$I_2 = \frac{4}{{T_0}^2}\int_{-T_0/4}^{0} -te^{-j\omega_o kt}\,dt = \frac{4}{{T_0}^2}\int_{\frac{T_0}{4}}^{0} \tau e^{j\omega_o k\tau}\,(-d\tau)\ \ [Substituting\ \tau = -t]$$

$$= \frac{4}{{T_0}^2}\int_0^{\frac{T_0}{4}} \tau e^{j\omega_o k\tau}\,(d\tau)$$

Substituting back into (1), we have,

$$a_k = \frac{4}{{T_0}^2}\left[\int_0^{\frac{T_0}{4}} te^{-j\omega_o kt}\,dt + \int_0^{\frac{T_0}{4}} te^{j\omega_o kt}\,dt\right]$$

$$= \frac{4}{{T_0}^2}\int_0^{T_0/4} 2t\cos(\omega_o kt)\,dt = \frac{8}{{T_0}^2}\int_0^{T_0/4} t\cos(\omega_o kt)\,dt$$

$$= \frac{8}{{T_0}^2}\left[\frac{t\sin(\omega_o kt)}{\omega_o k}\Big|_0^{\frac{T_0}{4}} - \int_0^{\frac{T_0}{4}}\frac{\sin(\omega_o kt)}{\omega_o k}dt\right]\ \ (Integration\ by\ parts)$$

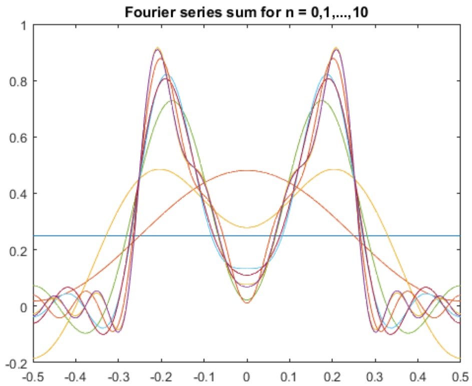
$$= \frac{8}{{T_0}^2}\left[\frac{t\sin(\omega_o kt)}{\omega_o k} + \frac{\cos(\omega_o kt)}{(\omega_o k)^2}\right]\Big|_0^{\frac{T_0}{4}}$$

$$= \frac{8}{{T_0}^2}\left[\frac{T_0\sin(\pi k/2)}{4\omega_o k} + \frac{\cos(\pi k/2)}{(\omega_o k)^2} - \frac{1}{(\omega_o k)^2}\right],k \neq 0\ \ (As\ \omega_o = \frac{2\pi}{T_0})$$

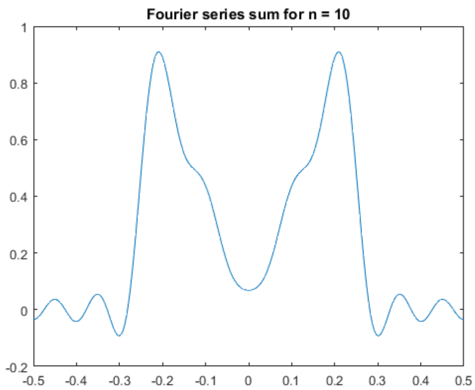
We can now use the Fourier synthesis formula to validate the Fourier series coefficients:

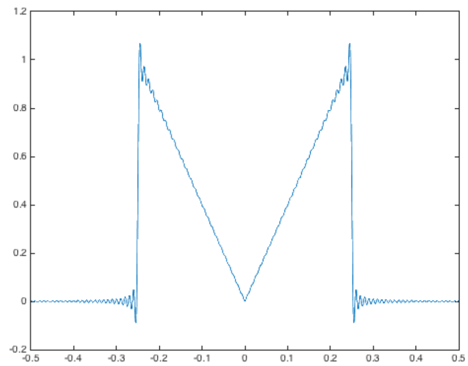
$$\hat{x}(t) = \sum_{k=-N}^{N} a_k e^{j\omega_o k t}$$

Below, we plot the synthesis for N = 0, 1, 2, …, 10. For N = 10, we have $2N+1=21$ terms.



The approximated function is shown using 21 Fourier synthesis terms (N=10) on the left and 201 Fourier synthesis terms (N=100) on the right:





On the right, Gibbs phenomena occur at the amplitude discontinuities at $-0.25T_0$ and $0.25T_0$. At these points, there is an 8.9% error at the value of 1 and 8.9% error in the value of 0.

MATLAB code:

```
% Fourier synthesis for square wave
% Prof. Brian L. Evans
% The University of Texas at Austin
% Fall 2017

% Pick a value for the period of x(t)
T0 = 1;
f0 = 1 / T0;
% Pick number of terms for Fourier synthesis
N = 100;
fmax = N * f0;
% Define a sampling rate for plotting
fs = 24 * fmax;
Ts = 1 / fs;
% Define samples in time for one period
t = -0.5*T0 : Ts : 0.5*T0;
% First Fourier synthesis term
a0 = 0.25;
x = a0 * ones(1, length(t));
figure;
plot(t, x);
```

```
hold on;
% Generate each pair of synthesis terms
for k = 1 : N
    % Define Fourier coefficients at k and -k
    akpos = 8*(sin(pi*k/2)/(4*2*pi*k)+ (cos(pi*k/2)-1)/((2*pi*k)^2));
    akneg = akpos;
    theta = j*2*pi*k*f0*t;
    x = x + akpos * exp(theta) + akneg * exp(-theta);
    % Plot Fourier synthesis for indices -k ... k
    plot(t, x);
end
hold off;
```

Epilogue: Think about why a_k and a_0 had to be calculated separately, even though there's just one common formula for calculating both of them. Can you think of a signal for which they would not have to be computed separately?

2. Prologue: This question goes over some of the basics of sampling, and helps you understand the difference between the concept of frequency in continuous & discrete time domains.

Solution: $x(t) = 10\cos(880\pi t + \varphi)$. Sampling time is T_s = 0.00001s, so $f_s = 10^4$ Hz.

a) $x(t)$ has a fundamental period of

$$T_0 = \frac{1}{f_0} = \frac{1}{440} s$$

We are taking samples every T_s seconds, so the nth sample occurs at time $t = n\,T_s$.

We solve for $nT_s = T_0$ to obtain n = 22.727 samples. Note: this is not an integer value.

b) $y(nT_s) = 10\cos(\omega_o nT_s + \varphi)$ and $x(nT_s) = 10\cos(880\pi nT_s + \varphi)$

Let $\omega_0 = 880\pi + \omega_1$ where $\omega_1 > 0$.

$y(nT_s) = 10\cos((880\pi + \omega_1)nT_s + \varphi) = 10\cos(880\pi nT_s + \omega_1 T_s n + \varphi)$

We would like $\omega_1 > 0$ so that $\omega_1 T_s = 2\pi$, which is $\omega_1 = 2\pi f_s = 20000\pi$.

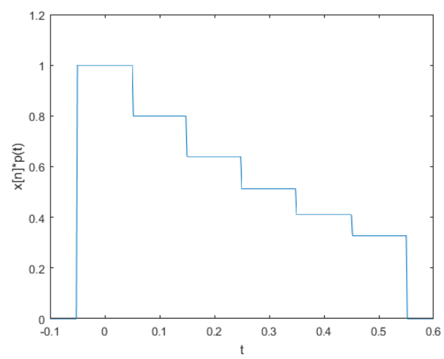
So, ω_0 = 20880π rad/s would satisfy the requirement, and so would 40880π, 60880π,… rad/s.

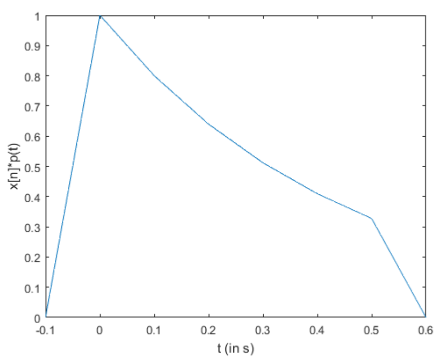
c) Using ω_0 = 20880π rad/s, we get $n = T_0/T_s = (2\pi)/(\omega_0 T_s)$ = 0.95785 samples.

Epilogue: Here, we see the Nyquist sampling theorem in action: The undersampled signal in parts (b) & (c) loses its continuous-time frequency information, whereas the oversampled signal in part (a) retains its continuous-time frequency information.

3. **Prologue**: This question starts concerns the reconstruction of a continuous-time signal from its discrete-time samples. It asks you to interpolate a short discrete-time signal using two different interpolation functions.

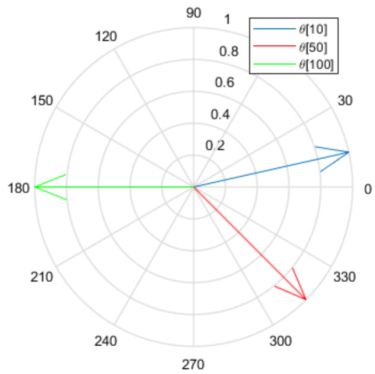
Solution: In the plots, $x[n]$ refers to the $y[n]$ specified in the question. Interpolation plots are given for part (a) sample-and-hold on the left and part (b) linear interpolation on the right:





Epilogue: Here we see that different interpolation functions give very different reconstructed signals. In lecture, we have also discussed sinc interpolation, which can perfectly reconstruct a bandlimited signal.
The summation that you evaluated in this question is a convolution sum. Convolution sums and integrals are extremely important in signal processing & mathematics, and will be revisited many times throughout this course.

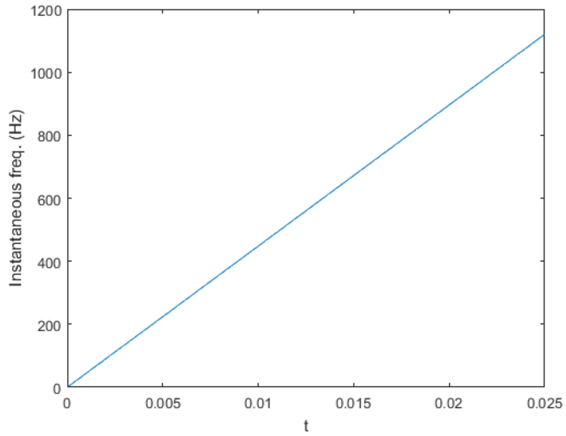
4. **Prologue**: This question goes over sampling (again), this time with chirp signals. Chirp signals are nice examples for studying the effects of different sampling frequencies, as you can get a very clear illustration of the frequency domain effects of sampling by looking at its spectrogram.
Solution: For this problem, the sampling rate f_s = 8000 Hz.
a) $\theta[n] = \pi\,(0.7 * 10^{-3})\, n^2$
$So, \theta[10] = 0.07\pi \; and \; \theta[50] = 1.75\pi \; and \; \theta[100] = 7\pi \;\equiv\; \pm\pi,$



b) Consider the signal $x(t) = \cos(2\pi f_0 t^2)$
Sampling this with a sampling period of T_s, we get $x[n] = \cos(2\pi f_0 (nT_s)^2) = \cos(2\pi f_0 T_s^2 n^2)$
Substituting the values in this problem, we get
$2\pi f_0 (1/(8\text{x}10^3))^2 = 0.7*10^{-3}\pi \;\; \Rightarrow f_0 = 22.4\text{x}10^3 \; s^{-1/2}$.
The units of $s^{-1/2}$ are needed to cancel with the units of T_s^2.

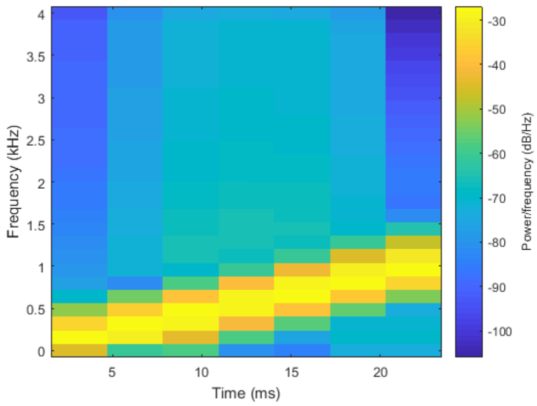
So, $x(t) = \cos(2\pi\,(22.4\text{x}10^3)\, t^2) = \cos(f(t))$
We know that instantaneous frequency of x(t) is $f_i(t) = df(t)/dt$
Substituting, we get $\underline{f_i(t) = (44.8*10^3)\, t \text{ Hz}}$

Now, by passing the discrete-time signal through a digital-to-analog (D/A) converter, we will recover the original signal if and only if the conditions of the Nyquist sampling theorem are satisfied. To check this, we need to calculate the maximum instantaneous frequency:
$T_{max} = n_{max} T_s = n_{max} / f_s = 200 / (8\text{x}10^3) = 0.025\text{s}$
The maximum instantaneous frequency is $\underline{f_{max} = (44.8\text{x}10^3)\, T_{max} = 1120 \text{ Hz}}$.
Since $f_s > 2 f_{max}$, Nyquist sampling theorem is satisfied holds, and we faithfully get back the continuous-time signal $x(t) = \cos(2\pi\,(22.4\text{x}10^3)\, t^2)$.

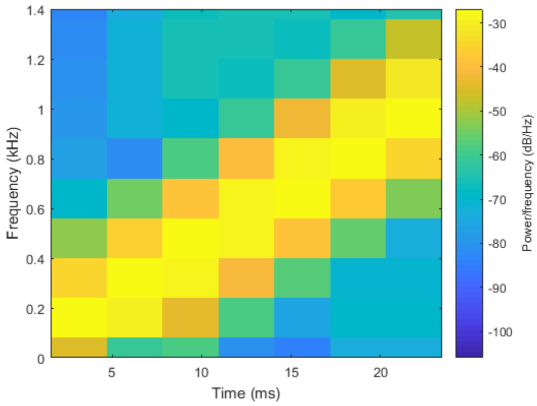


Code for plotting the spectrogram:

```
n = 1 : 200;
x = cos(pi*(0.7*10^(-3))*(n.^2));
fs = 8000;
blockSize = 50;
shift = blockSize/2;
spectrogram(x, blockSize, shift, blockSize, fs, 'yaxis');
```



Here's a magnified view of the above graph.



In this case, it is difficult to get a well-defined spectrogram because we only have 200 samples to work with and the chirp is increasing its frequency quickly over the 200 samples.

c) Consider the signal $x(t) = \cos(2\pi f_0 t)$

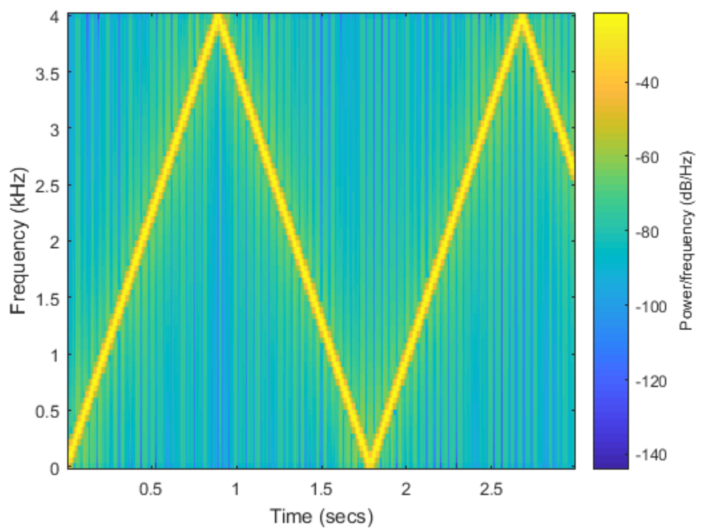
Sampling this with a sampling period of T_s, we get $x[n] = \cos(2\pi f_0 (n T_s)) = \cos(2\pi f_0 T_s n)$

Substituting the values in the question, we get,

$2 \pi f_0 (1/(8x10^3)) = 0.7\pi$ => $f_0 = 2.8x10^3$ Hz.
So, we hear an analog frequency of 2.8kHz. Note that this is irrespective of whether the original signal was sampled above the Nyquist rate or not.

d) Code taken directly from Prof. Evans' update to Tune-Up #4.

```
n = 1 : 24000;
x = cos(pi*(0.7*10^(-4))*(n.^2));
fs = 8000;
blockSize = 150;
shift = blockSize/2;
spectrogram(x, blockSize, shift, blockSize, fs, 'yaxis');
sound(x,fs)
```



To explain this plot, first calculate the maximum frequency without aliasing effects as we did in part (b) above. In this part, T_{max} = 24000 T_s = 24000 / f_s = 3 s and the maximum instantaneous frequency is $f_{max} = (44.8x10^2)\, T_{max}$ = 13.44 kHz which means $f_{max} / f_s = 1.68$. The sampling theorem is not followed for continuous-time frequencies at or above 4 kHz.

Due to sampling, continuous-time frequencies "turn around" at 0.5 f_s, decrease until they stop at f_s, increase until 1.5 f_s, "turn around" at 1.5 fs, etc. The ratio f_{max} / f_s shows the number of cycles in the plot, which is exactly what we can observe from the spectrogram.

Epilogue:
This problem shows that while continuous time frequency may increase indefinitely, once it is sampled (which is necessary for almost all applications), the effective frequency is restricted. After midterm #2, we will return to this issue using another type of Fourier analysis known as the continuous-time Fourier transform.