EE 313 Linear Signals & Systems

Solution Set for Homework #5 on
Frequency Response of FIR Filters & Z-transforms

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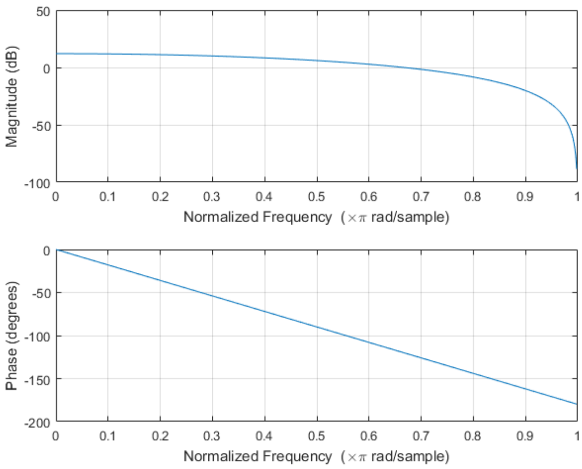
1. **Prologue:** This problem revisits lowpass filters that you have had seen on several occasions, including homework assignments, tune-ups, and lecture. This problem provides more detail into the discrete-time Fourier transform, and asks you to calculate it analytically.
Solution: a) $h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$
The discrete-time Fourier transform (frequency response) is

$$H(e^{j\hat{\omega}}) = \sum_{n=0}^{M} h[n]e^{-j\hat{\omega}n} = \sum_{n=0}^{M} (\delta[n] + 2\delta[n-1] + \delta[n-2])e^{-j\hat{\omega}n}$$
$$= \sum_{n=0}^{M} \delta[n]e^{-j\hat{\omega}n} + 2\sum_{n=0}^{M} \delta[n-1]e^{-j\hat{\omega}n} + \sum_{n=0}^{M} \delta[n-2]e^{-j\hat{\omega}n}$$
$$= 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$

b) We factor the signal's discrete-time Fourier transform into magnitude-phase form in the same way that lecture slide 9-5 does:
$H(e^{j\hat{\omega}}) = 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} = e^{-j\hat{\omega}}(e^{j\hat{\omega}} + 2 + e^{-j\hat{\omega}}) = e^{-j\hat{\omega}}(2 + 2\cos(\hat{\omega}))$
So, $|H(e^{j\hat{\omega}})| = 2 + 2\cos(\hat{\omega})$ and $\angle\left(H(e^{j\hat{\omega}})\right) = -\hat{\omega}$.
Although not asked, the phase is linear with slope of -1. Hence, the group delay through the FIR filter is 1 sample.
To plot: use `freqz([1,2,1])`



c)
$x[n] = 10 + 4\cos\left(0.5\pi n + \frac{\pi}{4}\right)$
We evaluate the effect of the LTI FIR filter on each frequency component of *x*[*n*].
The signal *x*[*n*] has frequency components at $\hat{\omega} = 0$ and $\hat{\omega} = \pm 0.5\pi$.
The frequency response at $\hat{\omega} = 0$ has value $H(e^{j0}) = 1 + 2 + 1 = 4$ and at $\hat{\omega} = 0.5\pi$ is
$H\left(e^{j\frac{\pi}{2}}\right) = 1 + 2e^{-j\frac{\pi}{2}} + 1e^{-j\pi} = 1 + 2e^{-j\frac{\pi}{2}} - 1 = 2e^{-j\frac{\pi}{2}}$
$y[n] = (4)10 + \left(2e^{-j\frac{\pi}{2}}\right) 4\cos\left(0.5\pi n + \frac{\pi}{4}\right) = 40 + 8\cos\left(0.5\pi n - \frac{\pi}{4}\right)$

d) $x[n] = \delta[n]$

$y[n] = x[n] + 2x[n-1] + x[n-2] = \delta[n] + 2\delta[n-1] + \delta[n-2]$

So, $y[n] = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ 1, & n = 2 \\ 0, & else \end{cases}$

e) $x[n] = u[n]$

$y[n] = x[n] + 2x[n-1] + x[n-2] = u[n] + 2u[n-1] + u[n-2]$

So, $y[n] = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ 3, & n = 1 \\ 4, & n > 1 \end{cases}$

Epilogue: Some of the above parts were easier to work in the frequency domain and others were easier to work in the time domain.

2. **Prologue:** You'll be deriving the impulse response for various LTI filters given their frequency responses. This is one way to compute the inverse Fourier transform.

Solution:

a)

$$H(e^{j\hat{\omega}}) = \sum_{n=0}^{M} h[n]e^{-j\hat{\omega}n} = 1 + 2e^{-j3\hat{\omega}}$$

This is satisfied by $h[n] = \begin{cases} 1, & n = 0 \\ 2, & n = 3 \\ 0, & else \end{cases}$. So, $h[n] = \delta[n] + 2\delta[n-3]$.

b)

$$H(e^{j\hat{\omega}}) = \sum_{n=0}^{M} h[n]e^{-j\hat{\omega}n} = 2e^{-j3\hat{\omega}}\cos(w) = e^{-j3\hat{\omega}}\left(e^{j\hat{\omega}} + e^{-j\hat{\omega}}\right) = e^{-j2\hat{\omega}} + e^{-j4\hat{\omega}}$$

This is satisfied by $h[n] = \begin{cases} 1, & n = 2 \\ 1, & n = 4 \\ 0, & else \end{cases}$. So, $h[n] = \delta[n-2] + \delta[n-4]$.

c)

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k} = e^{-j4.5\hat{\omega}}\frac{\sin(5\hat{\omega})}{\sin\left(\frac{\hat{\omega}}{2}\right)} = \frac{e^{-j4.5\hat{\omega}}\left(e^{j5\hat{\omega}} - e^{-j5\hat{\omega}}\right)}{e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}}$$

$$e^{j\frac{1}{2}\hat{\omega}} - e^{-j9.5\hat{\omega}} = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k}\left(e^{j\frac{1}{2}\hat{\omega}} - e^{-j\frac{1}{2}\hat{\omega}}\right)$$

Expanding the right side of the equation,

$$\sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k}\left(e^{j\frac{1}{2}\hat{\omega}} - e^{-j\frac{1}{2}\hat{\omega}}\right)$$
$$= h[0]e^{j\frac{1}{2}\hat{\omega}} - h[0]e^{-j\frac{1}{2}\hat{\omega}} + h[1]e^{-j\frac{1}{2}\hat{\omega}} - h[1]e^{-j\frac{3}{2}\hat{\omega}} + \cdots + h[9]e^{-j8.5\hat{\omega}} - h[9]e^{-j9.5\hat{\omega}} + \cdots$$

To make the $e^{j\hat{\omega}/2}$ terms equal on the left and right sides, $h[0] = 1$. Also, $h[1] - h[0] = 0$, so $h[1] = 1$. Likewise, $h[n] = 1$ for $n = 2, 3, \ldots, 8$. To make the $-e^{-j9.5\hat{\omega}}$ term equal on the left and right sides, $h[9] = 1$. All other values for $h[n]$ are zero.

Combining the above, we obtain $h[n] = u[n] - u[n-10]$. This filter is a 10-point running sum filter. It is also an averaging filter whose impulse response has been scaled by 10.

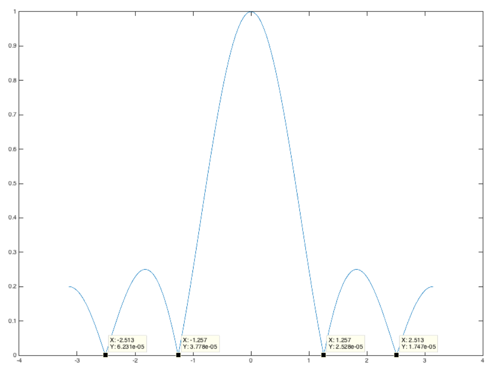
Epilogue: Part (c) can also have been solved by matching the form of the frequency response to that of a 10-point running sum filter.

3. **Prologue:** This question takes you back to the concepts that you learned before the midterm #1 exam, and adds some complexity by processing the signal after discretizing it.
Solution:
a) $h[n] = \delta[n] \;=>\; H(e^{j\hat{\omega}}) = 1$. All discrete-time frequencies pass through the LTI system unchanged. If the signal is not aliased by the Ideal C-to-D converter, then it can be faithfully reconstructed by the Ideal D-to-C converter. The Nyquist theorem says that $f_s > 2f_0$.
Alternately, $f_0 < \frac{1}{2} f_s$ which means frequencies up to $\frac{1}{2} f_s$ can be faithfully reconstructed.
$w_0 = 2\pi(500) = 2\pi f_0 => f_0 = 500\text{Hz}$. So, $f_s > 1000$ Hz.

b) $h[n] = \delta[n-10] \;=>\; H(e^{j\hat{\omega}}) = e^{-j10\hat{\omega}}$.
$x(t) = 10 + 20\cos\left(\omega_0 t + \frac{\pi}{3}\right) => x[n] = 10 + 20\cos\,(\omega_0 n T_s + \frac{\pi}{3})$
$y[n] = x[n-10] = 10 + 20\cos\left(\omega_0 (n-10) T_s + \frac{\pi}{3}\right) = 10 + 20\cos\left(\omega_0 n T_s + \frac{\pi}{3} - 10\omega_0 T_s\right)$
$So, y(t) = 10 + 20\cos(\omega_0 (t - 10T_s) + \frac{\pi}{3})$
$Comparing, we\ get, 10T_s = 0.001 => T_s = 0.0001s => f_s = 10\text{kHz}.$
To get this signal, we additionally want the signal to pass unaliased through the C/D & D/C converters. Hence, we want our signal to be sampled above the Nyquist rate.
$f_s > 2f_0 => f_0 < 5\text{kHz} => \omega_0 < 10000\pi$ rad/s.

c) Frequency response of the LTI system is given as $H(e^{j\hat{\omega}}) = \frac{\sin(\frac{5}{2}\hat{\omega})}{5\sin(\frac{1}{2}\hat{\omega})} e^{-j2\hat{\omega}}$

From (6.27), this is the frequency response of a 5-point running-average filter (p. 145).
We are given $x(t) = 10 + 20\cos\left(\omega_0 t + \frac{\pi}{3}\right)$ and would like *y*(*t*) = *A* for all values of *t*.
The Ideal C-to-D block samples *x*(*t*) to give $x[n] = 10 + 20\cos\,(\hat{\omega}_0 n + \frac{\pi}{3})$ where $\hat{\omega}_0 = \frac{\omega_0}{f_s}$.
We seek to filter out the cosine term $20\cos\,(\hat{\omega}_0 n + \frac{\pi}{3})$ completely. This will happen at values of $\hat{\omega}_0$ where magnitude response of $H(e^{j\hat{\omega}})$ goes to zero.
Here is the plot of the magnitude response of $H(e^{j\hat{\omega}})$:




```
w = -pi : 0.0001 : pi;
H = diric(w, 5) .* exp(-j*2*w);
plot(w, abs(H));
```

On page 147, under item (d) in the upper left-hand corner, the zero values of the magnitude response will occur at integer multiples of 2π/5 except at zero frequency. In the figure window in MATLAB, I used the data cursor tool to estimate the discrete-time frequency values at which the magnitude response goes to zero over $-\pi < \hat{\omega} \leq \pi$: -2.513, -1.257, 1.257 and 2.513. These correspond to -4π/5, -2π/5, 2π/5, and 4π/5.

Since $\hat{\omega}_0 = \frac{\omega_0}{f_s}$, we have $\omega_0 = \hat{\omega}_0 f_s$.

<u>To give *y*(*t*) = 10:</u>

- Two positive values are possible: $\omega_0 = \left(\frac{2}{5}\pi\right)(2000\text{Hz}) = 800\pi$ and $\omega_0 = 1600\pi$.
- Two negative values are also possible: $\omega_0 = -800\pi$ and $\omega_0 = -1600\pi$.
- If we were to include the aliases, then any of the four frequencies shifted by an integer multiple of 4000π would also work.

<u>To give *y*(*t*) = 20:</u>

- $\omega_0 = 0$
- ω_0 can be any integer multiple of 2 π f_s, i.e. any integer multiple of 4000 π.

Epilogue: The system introduced in this question can be used to describe a wide variety of user I/O devices, simply by changing $H(e^{j\hat{\omega}})$. Can you think of some commonly used devices that can be (roughly) modelled using this block diagram?

4. **Prologue:** This question introduces the *z*-transform, which is a generalization of the discrete-time Fourier transform to the entire complex plane. It starts you off on simple properties of the transform, which will be very useful in the future.
Solution:
Time delay property: $x[n] \rightarrow X(z) => x[n-n_0] \rightarrow X(z)z^{-n_0}$.
Superposition property: $x_i[n] \rightarrow X_i(z) => \sum a_i x_i[n] \rightarrow \sum a_i X_i(z)$.

a) $x_1[n] = \delta[n] => X_1(z) = \sum \delta[n] z^{-n} = 1$.

b) $x_2[n] = \delta[n-1] => X_2(z) = X_1(z)z^{-1} = z^{-1}$.

c) $x_3[n] = \delta[n-7] => X_3(z) = X_1(z)z^{-7} = z^{-7}$.

d) $x_4[n] = 2\delta[n] - 3\delta[n-1] + 4\delta[n-3]$
$X_4(z) = 2X_1(z) - 3X_1(z)z^{-1} + 4X_1(z)z^{-3} = 2 - 3z^{-1} + 4z^{-3}$.

Epilogue: In taking discrete-time Fourier transforms, z-transforms, and other related transforms, it is common to use transform pairs and properties instead of the definition.