EE 313 Linear Signals & Systems

Solution Set for HW#6 on IIR filters

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1. Prologue: This question asks you to calculate pole & zero locations for several systems, working from their basic definition. It also asks you to visualize the locations on the complex plane.

Solution: For each solution, one can use the zplane command in Matlab to generate the pole-zero plot for each difference equation or transfer function. The syntax is

```
zplane( feedforwardCoefficients, feedbackCoefficients )
```

a) $S_1: y[n] = 0.9y[n-1] + 0.5(x[n] + x[n-1])$

$$Y(z) = 0.9Y(z)z^{-1} + 0.5X(z)(1+z^{-1}) => H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-0.9z^{-1}}$$

So, $H(z)$ has a zero at *z = -1*, and a pole at *z = 0.9*. `zplane( [1 1], [1 -0.9] );`

This corresponds to the Pole-Zero Plot #2.

This is a lowpass filter. Frequency-domain plots appear at end of the solution.

Note: The region of convergence is |z| > 0.9.

b) $S_2: y[n] = -0.9y[n-1] + 9x[n] + 10x[n-1]$

$$Y(z) = -0.9Y(z)z^{-1} + X(z)(9+10z^{-1}) => H(z) = \frac{Y(z)}{X(z)} = \frac{9+10z^{-1}}{1+0.9z^{-1}}$$

So, $H(z)$ has a zero at *z = -10/9*, and a pole at *z = -0.9*. `zplane( [9 10], [1 0.9] );`

This corresponds to the Pole-Zero Plot #3.

This is an all-pass filter. See the course handout on all-pass filters at

http://users.ece.utexas.edu/~bevans/courses/signals/handouts/Appendix%20I%20All%20Pass%20Filters.pdf

Frequency-domain plots appear at end of the solution.

Note: The region of convergence is |z| > |-0.9| which is |z| > 0.9.

c) $S_3: H(z) = \frac{1}{2} \cdot \frac{1-z^{-1}}{1+0.9z^{-1}}$

So, *H(z)* has a zero at *z=1*, and a pole at *z = -0.9*. `zplane( [1 -1], [1 0.9] );`

This corresponds to Pole-Zero Plot #6.

This is a highpass filter. Frequency-domain plots appear at end of the solution.

Note: The region of convergence is |z| > |-0.9| which is |z| > 0.9.

d) $S_4: y[n] = \frac{1}{4}x[n] + x[n-1] + \frac{3}{2}x[n-2] + x[n-3] + \frac{1}{4}x[n-4]$

$$Y(z) = X(z)\left(\frac{1}{4} + z^{-1} + \frac{3}{2}z^{-2} + z^{-3} + \frac{1}{4}z^{-4}\right) => H(z) = \frac{1}{4} + z^{-1} + \frac{3}{2}z^{-2} + z^{-3} + \frac{1}{4}z^{-4}$$

Writing $H(z)$ in terms of z (instead of z^{-1}), we get

$$H(z) = \frac{0.25z^4 + z^3 + 1.5z^2 + z + 0.25}{z^4} = \frac{0.25(z^4 + 4z^3 + 6z^2 + 4z + 1)}{z^4} = \frac{0.25(z+1)^4}{z^4}.$$

So, $H(z)$ has 4 poles, all at *z=0,* and 4 zeroes, all at *z=-1.* `zplane( [0.25 1 1.5 1 0.25] );`

We see that none of the pole-zero plots shown meet these requirements. So, this <u>system does not correspond to any of the plots</u>.

This is lowpass filter. Frequency-domain plots appear at end of the solution.

Note: The region of convergence is z ≠ 0.

e) S_5: $H(z) = 1 - z^{-1} + z^{-2} - z^{-3} + z^{-4} = \frac{z^4 - z^3 + z^2 - z + 1}{z^4} := H_n(z)/H_d(z)$

So, $H(z)$ has 4 poles, all at *z=0. Note: The region of convergence is z ≠ 0.*

Since the numerator has only real coefficients and a degree of 4, the following must hold: (1) it has 4 roots, and (2) the roots are either real or occur in complex-conjugate pairs.

Let us consider the *n*th roots of unity $(1, \omega, \omega^2, \dots, \omega^{n-1})$. Specifically, let us take *n*=5.

We know that for any n, $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0.$ $\dots Property(1)$

Replacing ω with $-\omega$, we get, for n = 5,

$1 - (-\omega) + \omega^2 - (-\omega^3) + \omega^4 = 0$, which exactly matches the numerator polynomial. So, the roots of $H_n(z)$ are the 5th roots of -1.

Roots of $H_n(z)$: $= -\omega, -\omega^2, -\omega^3, -\omega^4 = -e^{\frac{j2\pi}{5}}, -e^{\frac{j4\pi}{5}}, -e^{\frac{j6\pi}{5}}, -e^{\frac{j8\pi}{5}} = e^{\frac{-j3\pi}{5}}, e^{\frac{-j\pi}{5}}, e^{\frac{j\pi}{5}}, e^{\frac{j3\pi}{5}}$

Using MATLAB to compute the zeroes,

```
roots( [1 -1 1 -1 1] )
```

we obtain

```
 -0.3090 + 0.9511j
 -0.3090 - 0.9511j
  0.8090 + 0.5878j
  0.8090 - 0.5878j
```

which matches with the answer above.

This corresponds to <u>Pole-Zero Plot #5</u>.

This is a highpass filter. Frequency-domain plots appear at end of the solution.

Note: The region of convergence is z ≠ 0.

f) S_6: $y[n] = x[n] + x[n-1] + x[n-2] + x[n-3]$

$\quad Y(z) = X(z)(1 + z^{-1} + z^{-2} + z^{-3})$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + z^{-1} + z^{-2} + z^{-3} = \frac{1 + z + z^2 + z^3}{z^3} := \frac{H_n(z)}{H_d(z)}$$

So, $H(z)$ has 3 poles, all at $z = 0$: `freqz( [1 1 1 1] );`

As in part (e), the following hold for the zeroes: (1) There are 3 zeroes, (2) They are either real or occur in complex conjugate pairs.

Using Property (1), we see that the zeroes of $H(z) = \omega, \omega^2, \omega^3$ *(the fourth roots of unity)*.

So, the zeroes are $e^{\frac{j\pi}{2}}, e^{\frac{2j\pi}{2}}, e^{\frac{3j\pi}{2}} = j, -1, -j$: `zplane( [1 1 1 1] );`

This corresponds to the <u>Pole-Zero Plot #4</u>.

This is a lowpass filter. Frequency-domain plots appear at end of the solution.

Note: The region of convergence is z ≠ 0.

g) $S_7 : y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4] + x[n-5]$

$$Y(z) = X(z)(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5})$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} = \frac{1 + z + z^2 + z^3 + z^4 + z^5}{z^5} := \frac{H_n(z)}{H_d(z)}$$

So, $H(z)$ has 5 poles, all at $z = 0$: `zplane( [1 1 1 1 1 1] );`

As in part (f), the following hold for the zeroes: (1) there are 5 zeroes, and (2) they are either real or occur in complex conjugate pairs.

Using Prop. (1), we see that the zeroes of $H(z) = \omega, \omega^2, \omega^3, \omega^4, \omega^5$ *(the sixth roots of unity)*.

So, the zeroes are $e^{\frac{j\pi}{3}}, e^{\frac{2j\pi}{3}}, e^{\frac{3j\pi}{3}}, e^{\frac{4j\pi}{3}}, e^{\frac{5j\pi}{3}} = e^{\frac{j\pi}{3}}, e^{\frac{2j\pi}{3}}, -1, e^{\frac{-j\pi}{3}}, e^{\frac{-2j\pi}{3}}$.

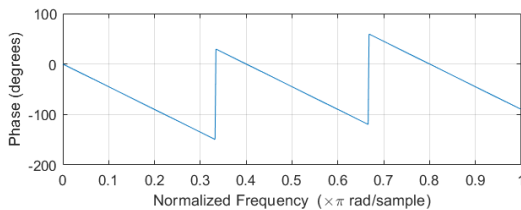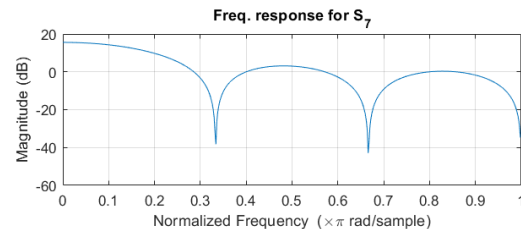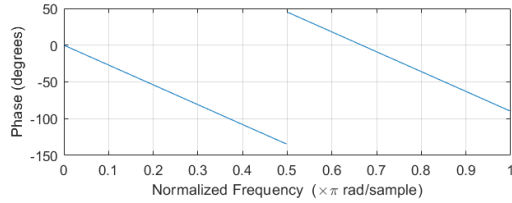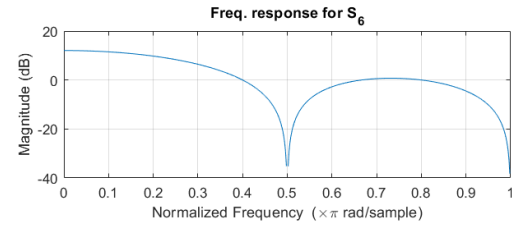This corresponds to the <u>Pole-Zero Plot #1</u>.

This is a lowpass filter. Frequency-domain plots appear at end of the solution.

Using MATLAB to compute the zeroes, we get,

Z=0.5+0.866j, 0.5-0.866j, -1, -0.5+0.866j, -0.5-0.866j, which matches with the answer above.

Frequency-Domain Plots. MATLAB code for plotting the frequency responses of the filters follows:

```
freqz([1,1],[1,-0.9]);
figure;
freqz([9,10],[1,0.9]);
figure;
freqz([1,-1],[1,0.9]);
figure;
freqz([0.25,1,1.5,1,0.25]);
figure;
freqz([1,-1,1,-1,1]);
figure;
freqz([1,1,1,1]);
figure;
freqz([1,1,1,1,1,1]);
```

Freq. response for S_1

Freq. response for S_2

Freq. response for S_3

Freq. response for S_4

Freq. response for S_5

Freq. response for S_6

Freq. response for S_7
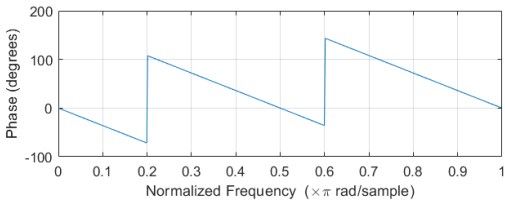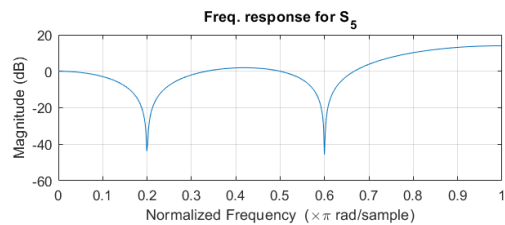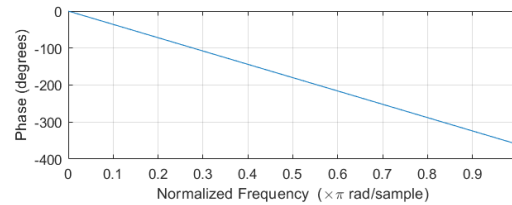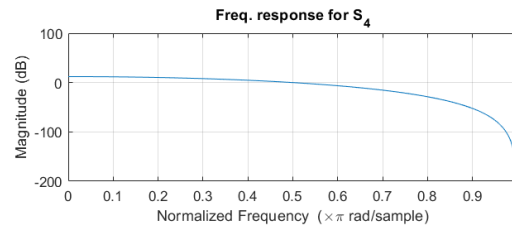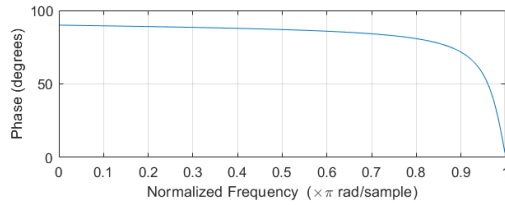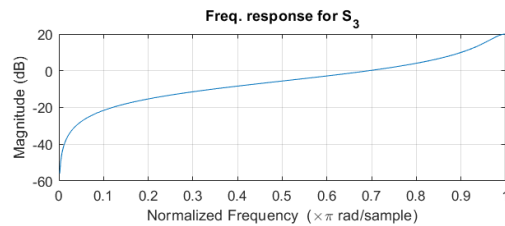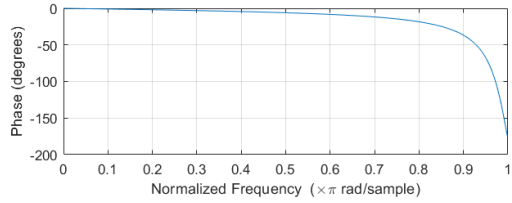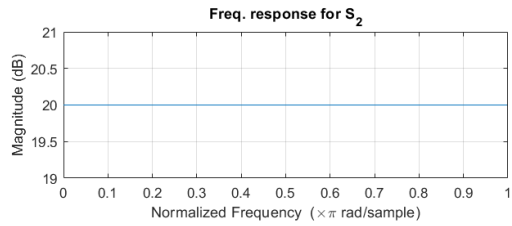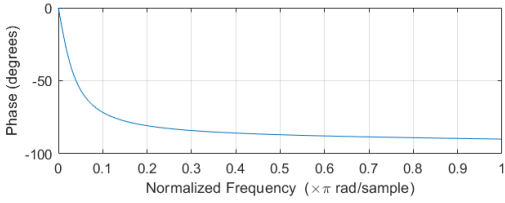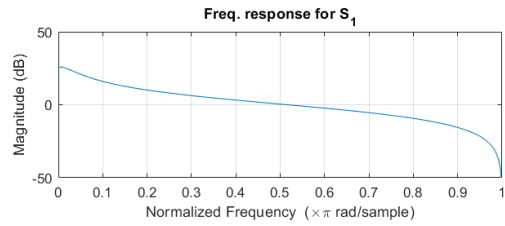
As we can see, the frequency selectivity of the filters is (in order): lowpass, allpass, highpass, lowpass, highpass, lowpass, lowpass.

Epilogue: As you can see above, some basic algebra helps a lot in finding the roots of the numerator & denominator polynomials. Another fact that you may have noticed is that the number of poles and zeroes is equal for all the systems in the question. This is something that holds for all LTI systems (counting zeroes and poles at ∞ as well).

2. **Prologue:** This question asks you to compute a simple IIR filter output by performing an analysis in the z-domain. It also deals with the steady state response of the system, which is explained in Section 8-8 of the *Signal Processing First* textbook.

Solution:

a) $y[n] = \frac{1}{2}y[n-1] + x[n]$ with the initial condition y[-1] = 0.

$$Y(Z) = \frac{1}{2}Y(z)z^{-1} + X(z) => Y(z)(1 - 0.5z^{-1}) = X(z)$$

$$X(z) = \sum_{n=-\infty}^{\infty} u[n]z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}.$$

$$Y(z) = \frac{1}{1-z^{-1}} \cdot \frac{1}{1-0.5z^{-1}} = \frac{A}{1-z^{-1}} + \frac{B}{1-0.5z^{-1}}$$

So, $A + B = 1, -0.5A - B = 0 => A = 2B => A = \frac{2}{3}, B = 1/3$.

$$Y(z) = \frac{2}{3} \cdot \frac{1}{1-z^{-1}} + \frac{1}{3} \cdot \frac{1}{1-0.5z^{-1}} => y[n] = \frac{2}{3}u[n] + \frac{1}{3} \cdot 0.5^n u[n].$$

(Using the z-transform for $a^n u[n]$ derived in homework problem 6.4 & lecture slide 11-5.).

b) $X(z) = \sum_{-\infty}^{\infty} e^{j\pi n/4}u[n]z^{-n} = \sum_{0}^{\infty} e^{j\pi n/4}z^{-n} = \frac{1}{1-\frac{e^{\frac{j\pi}{4}}}{z}}$

$$Y(z) = \frac{1}{1-e^{\frac{j\pi}{4}}z^{-1}} \cdot \frac{1}{1-0.5z^{-1}} = \frac{A}{1-e^{j\pi/4}z^{-1}} + \frac{B}{1-0.5z^{-1}} \ (say)$$

So, $A - 0.5Az^{-1} + B - e^{j\pi/4}z^{-1}B => A + B = 1, 0.5A + e^{j\pi/4}B = 0 => A = -2e^{j\pi/4}B$

$$B(1 - 2e^{j\pi/4}) = 1 => B = \frac{1}{1-2e^{j\pi/4}}, A = -\frac{2e^{\frac{j\pi}{4}}}{1-2e^{\frac{j\pi}{4}}}.$$

$$=> y[n] = \frac{1}{1-2e^{\frac{j\pi}{4}}}(-2e^{\frac{j\pi(n+1)}{4}}u[n] + 0.5^n u[n])$$

c) Part (c) from the book was not assigned, but its solution is provided for reference.

$$y[n] = \frac{1}{1-2e^{\frac{j\pi}{4}}}\left(-2e^{\frac{j\pi(n+1)}{4}}u[n] + 0.5^n u[n]\right) = \frac{1}{1-2e^{\frac{j\pi}{4}}}\left(-2e^{\frac{j\pi}{4}}e^{\frac{j\pi n}{4}}u[n]\right) + \frac{1}{1-2e^{\frac{j\pi}{4}}}(0.5^n u[n])$$

The second term dies out for large *n*, and is called the transient component. The first term remains constant in magnitude for all *n*, and hence represents the steady-state component.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 - 0.5z^{-1})} \Rightarrow H\left(e^{\frac{j\pi}{4}}\right) = -\frac{2e^{\frac{j\pi}{4}}}{1 - 2e^{\frac{j\pi}{4}}}$$

As we can see, the coefficient of $x[n]$ in the steady state component is exactly the same as the frequency response at $\omega = \pi/4$. So, both the magnitude and the phase response will be identical to that for the steady state coefficient.

d) Draw the block diagram for the input-output relationship for the system in the time domain:
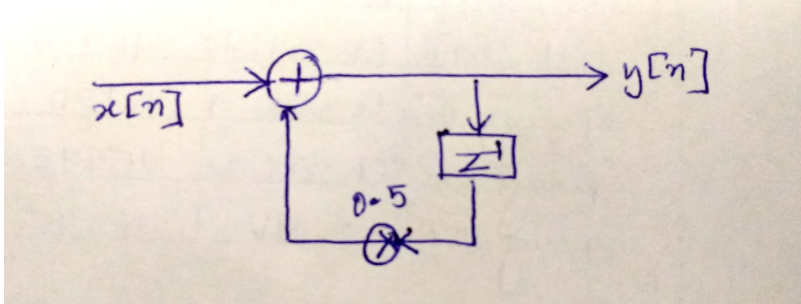
Epilogue: The system response to a sinusoidal signal starting at a point time (which holds for practical situations in which observations start at a point in time) contains transient and steady-state components. The transient component (in absolute value) becomes smaller as *n* increases.

3. **Prologue:** This question continues with concepts explored in the previous problems. In addition, it asks you to do inverse z-transforms by using properties introduced in the previous homework.
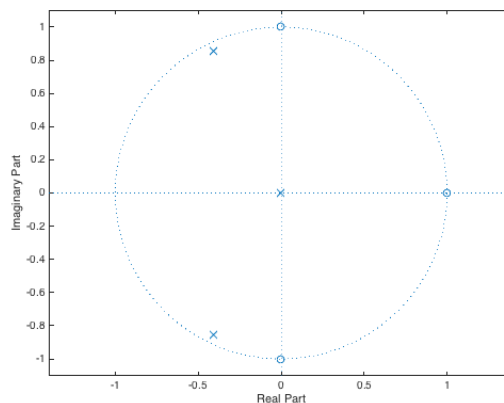
Solution:

a) $H(z) = \frac{Y(z)}{X(z)} = \frac{(1-z^{-1})(1-e^{\frac{j\pi}{2}}z^{-1})(1-e^{\frac{-j\pi}{2}}z^{-1})}{(1-0.9e^{\frac{j2\pi}{3}}z^{-1})(1-0.9e^{\frac{j2\pi}{3}}z^{-1})} = \frac{(1-z^{-1})(1-\left(e^{-\frac{j\pi}{2}}+e^{\frac{j\pi}{2}}\right)z^{-1}+z^{-2})}{(1-0.9\left(e^{\frac{j2\pi}{3}}+e^{\frac{-j2\pi}{3}}\right)z^{-1}+0.81z^{-2})}$

$$Y(z)(1 + 0.81z^{-2} + 0.9z^{-1}) = X(z)(1 - z^{-1})(1 + z^{-2}) = X(z)(1 - z^{-1} + z^{-2} - z^{-3})$$

$$y[n] + 0.9y[n - 1] + 0.81y[n - 2] = x[n] - x[n - 1] + x[n - 2] - x[n - 3]$$

b) For the pole-zero plot, we are using `zplane( [1 -1 1 -1], [1 0.81 0.9] )`

c) $x[n] = Ae^{j\phi}e^{j\omega_0 n} \Rightarrow X(e^{j\omega}) = Ae^{j\phi}\delta(\omega - \omega_0)$. Note that $\delta(\omega)$ is the Dirac delta functional, which is discussed in Section 9.2 of *Signal Processing First*.

Now, $y[n] = 0 \Rightarrow Y(e^{j\omega}) = 0 \ \forall \ \omega \in [-\pi, \pi] \Rightarrow H(e^{j\omega})X(e^{j\omega}) = 0$.

Since $X(e^{j\omega}) = 0 \ \forall \ \omega \neq \omega_0$, the only condition on $H(e^{j\omega})$ is that $H(e^{j\omega_0}) = 0$.

So, ω_0 is a zero of *H(z)* lying on the unit circle. From the plot above, we can read off the values of the zeros. So, $\omega_0 = 0, \pi/2, -\pi/2$.

d) We use the command `freqz( [1 -1 1 -1], [1 0.81 0.9] )`
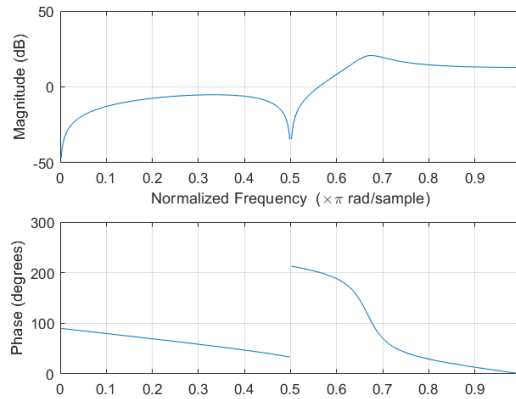
This is a high-pass filter. Note the discontinuity in the phase response at $\omega_0 = \pi/2$ which is a frequency component that is zeroed out according to the magnitude response at $\omega_0 = \pi/2$.

The code for generating the plots above would be:

```
num=[1,-1,1,-1];
den = [1,0.9,0.81];
zplane(num,den);
figure;
freqz(num,den);
```

Epilogue: What would happen in part (c) if some of the zeroes were outside the unit circle? How about if they were inside?

4. **Prologue:** This question asks you to find the convolution of two signals, this time by going to the z-domain. The time domain convolution is shown in Handout F on Canvas, which is also available at

http://users.ece.utexas.edu/~bevans/courses/signals/handouts/Appendix%20F%20Convolution%20Exp%20Sequences.pdf

Solution:

a) $x_1[n] = a^n u[n], x_2[n] = b^n u[n]$.

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]z^{-n} = \sum_{n=-\infty}^{\infty} u[n]a^n z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}} \ if \ |z| < |a|.$$

Similarly, $X_2(z) = \frac{1}{1-bz^{-1}} \ if \ |z| < |b|$. In general, constants *a* and *b* could be complex-valued.

Since convolution in the time domain corresponds to a multiplication in the *z*-domain, we get

$$Y(z) = \frac{1}{(1-az^{-1})(1-bz^{-1})} = \frac{A}{1-az^{-1}} + \frac{B}{1-bz^{-1}} \ for \ some \ A, B. \quad ... eqn.(1)$$

This approach is known as partial fractions decomposition.

We solve for A, B by putting the rightmost term over a common denominator and equating the numerators, we get $A + B - z^{-1}(bA + aB) = 1 \ \forall \ z$. This gives us two equations in two unknowns.

$\Rightarrow$ $A + B = 1, bA + aB = 0 \Rightarrow A = -\frac{aB}{b} \Rightarrow B\left(1 - \frac{a}{b}\right) = 1 \Rightarrow B = \frac{b}{b-a}, A = -\frac{a}{b-a}.$

$\Rightarrow$ $Y(z) = -\frac{a}{b-a} \cdot \frac{1}{1-az^{-1}} + \frac{b}{b-a} \cdot \frac{1}{1-bz^{-1}}$

$\Rightarrow$ $y[n] = -\frac{a}{b-a} \cdot a^n u[n] + \frac{b}{b-a} \cdot b^n u[n] = \frac{1}{b-a} \cdot (b^{n+1} - a^{n+1})u[n].$

b) Since $b = a$, $Y(z) = \frac{1}{(1-bz^{-1})^2}$.

Here are two approaches for using the inverse z-transform to find a closed-form expression for $y[n]$.

Approach #1: Use partial fractions decomposition. $Y(z)$ has two poles at $z = b$, and the partial fractions method for the case of repeated poles is not covered in the textbook. Here's the approach:

$$\frac{1}{(1 - bz^{-1})^2} = \frac{Az^{-1}}{(1 - bz^{-1})^2} + \frac{B}{1 - bz^{-1}} = \frac{Az^{-1} + B(1 - bz^{-1})}{(1 - bz^{-1})^2}$$

Put the two terms on the right-hand side over a common denominator and equate numerators:

$$Az^{-1} + B(1 - bz^{-1}) = 1$$

This gives two equations in the two unknowns: $A-b=0$ and $B=1$ which gives $A = b$ and $B = 1$.

$$Y(z) = \frac{bz^{-1}}{(1 - bz^{-1})^2} + \frac{1}{1 - bz^{-1}}$$

For the inverse z-transform for the leftmost term, we'll need to use a z-transform table, as mentioned in Approach #2 below.

$$y[n] = nb^n u[n] + b^n u[n] = (n + 1)b^n u[n]$$

Approach #2: Use z-transform tables and properties. Wikipedia has useful information on the z-transform at https://en.wikipedia.org/wiki/Z-transform. We seek to find the inverse transform of

$$Y(z) = \frac{1}{(1 - bz^{-1})^2}$$

The row 13 in the table of transform pairs will match the above form if we rewrite it as

$$Y(z) = \left(\frac{1}{b}z\right)\frac{bz^{-1}}{(1 - bz^{-1})^2} = \left(\frac{1}{b}z\right)X(z)$$

The inverse z-transform is given by row 13 as

$$x[n] = nb^n u[n]$$

In the z-transform, multiplying by constant ($1/b$) will become as scaling by the same amount in the time domain. Multiplying by z in the z-domain means advancing time by one sample.

$$y[n] = \frac{1}{b}x[n + 1] = \frac{1}{b}(n + 1)b^{n+1}u[n + 1]$$

Note that $(1/b)\, b^{n+1} = b^{-1}\, b^{n+1} = b^n$. We can write $u[n+1] = \delta[n+1] + u[n]$ and use $y[-1] = 0$:

$$y[n] = (n+1)\, b^n\, u[n]$$

Epilogue: This question goes over partial fraction decomposition, which is a commonly used technique when a difficult problem can be expressed as a weighted sum of simpler terms that are easier to solve. This technique is widely used in calculus, especially in integration.