EE 313 Linear Signals & Systems

Solution Set for HW#7 on Continuous Time Signals & Systems

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Here are several useful properties of the Dirac delta functional.

a) <u>*Dirac delta*</u>: $\int_{-\infty}^{\infty} \delta(x)dx = 1$

b) <u>*Sifting property:*</u> $\int_a^b f(x)\delta(x-x_0)dx = \begin{cases} f(x_0), x_0 \in [a,b] \\ 0 \quad\quad , otherwise \end{cases}$

c) <u>*Even symmetry:*</u> $\delta(x) = \delta(-x)$

d) <u>*Relationship to the unit step function.*</u> $\frac{d}{dx}u(x) = \delta(x)$.

Here are several comments about bounded-input bounded-output (BIBO) stability:

e) <u>*BIBO Stability:*</u> *If input x(t) is bounded in amplitude, i.e.* $|x(t)| \leq B$ *for a finite value B, then output y(t) is always bounded in amplitude, i.e.*$|y(t)| \leq B_1$ *for a finite value* B_1*. This definition does not require the system to be LTI.*

f) <u>*BIBO stability for LTI systems:*</u> *For a continuous-time LTI system with an impulse response h(t), BIBO stability reduces to* $\int_{-\infty}^{\infty}|h(t)|dt < \infty$*. A derivation is given in problem 3 below.*

g) <u>*BIBO stability for FIR filters:*</u> *From f), it immediately follows that FIR filters are always BIBO stable (if |h(t)| < ∞ for all t). This is also reflected in the fact that all the poles of an FIR filter are at z=0 (inside the unit circle), which implies stability.*

h) <u>*Convolution:*</u> *Let c(t) = x(t)*y(t) =>* $c(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)y(t)d\tau$

1. Solution:

a) $y(t) = e^{x(t+2)}$

i) This system fails the all-zero input test. That is, when *x*(*t*) = 0 for all *t*, *y*(*t*) = 1 instead of 0. Linearity does not hold.

Alternate answer: Let $x_1(t) = a\,x(t) => y_1(t) = e^{x_1(t+2)} = e^{a\,x(t+2)} = (y(t))^a$
So, $y_1(t) \neq ay(t)$. So, homogeneity doesn't hold & the system is not linear.

ii) $x_{shifted}(t) = x(t-t_0) => y_{shifted}(t) = e^{x_{shifted}(t+2)} = e^{x(t-t_0+2)} = y(t-t_0)$
So, $y_{shifted}(t) = y(t-t_0)$. So, the system is time-invariant. Note that the system is pointwise, and all pointwise systems are time-invariant.

iii) When $|x(t)| \leq B \quad \forall\, t$, it means that $-B \leq x(t) \leq B \quad \forall\, t$ which in turn means $e^{-B} \leq x(t) \leq e^{B} \quad \forall\, t$. So, a bounded input generates a bounded output and hence the system is stable.

iv) *y*(*t*) is a function of a future value of *x*(*t*) viz. *x*(*t*+2). So, this system is not causal.

b) $y(t) = \cos\,(w_c t + x(t))$

i) This system fails the all-zero input test. That is, when *x*(*t*) = 0 for all *t*, then $y(t) = \cos\,(w_c t)$ which is not zero for all *t*.

Alternate solution: Let $x_1(t) = a\,x(t) => y_1(t) = \cos(w_c t + x_1(t)) = \cos(w_c t + a\,x(t))$. So, $y_1(t) \neq a\,y(t)$. So, homogeneity doesn't hold & system is not linear.

ii) $x_{shifted}(t) = x(t-t_0) => y_{shifted}(t) = \cos\left(w_c t + x_{shifted}(t)\right) = \cos\,(w_c t + x(t-t_0))$ and $y(t-t_0) = \cos(w_c(t-t_0) + x(t-t_0)) \neq y_s(t)$. The system is time-varying.

iii) Regardless of the value of $x(t), -1 \leq y(t) \leq 1 \ \ \forall\, t$ since *y*(*t*) is a cosine.
So, a bounded input would generate a bounded output and hence the system is BIBO stable.

iv) *y*(*t*) is a function of only the current value of *x*(*t*). So, this system is causal.

c) $y(t) = [A + x(t)]\cos(w_c t)$

i) This system fails the all-zero input test. That is, when *x*(*t*) = 0 for all *t*, we have $y(t) = A\cos(w_c t)$ instead of 0. Linearity does not hold.
Alternate answer: Let $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$
$ax_1(t) + bx_2(t) \to y(t) => y(t) = [A + ax_1(t) + bx_2(t)]\cos(w_c(t))$
$y(t) = [aA + ax_1(t)]\cos(w_c t) + [bA + bx_2(t)]\cos(w_c t) + A(1 - a - b)$
$\therefore y(t) = ay_1(t) + by_2(t) + A(1 - a - b) \neq ay_1(t) + by_2(t) \ \ \forall\, A \neq 0.$
So, the system is not linear.

ii) Let $x_{shifted}(t) = x(t - t_0)$
$\therefore y_{shifted}(t) = [A + x_{shifted}(t)]\cos(w_c t) = [A + x(t - t_0)]\cos(w_c t)$
$y(t - t_0) = [A + x(t - t_0)]\cos(w_c(t - t_0)) \neq y_s(t).$
So, the system is time-varying.

iii) If $|x(t)| \leq B \ \ \forall\, t$, then $|y(t)| = |[A + x(t)]\cos(w_c t)| = |A + x(t)|\,|\cos(w_c t)|$
So, a bounded input generates a bounded output and hence the s/m is stable.

iv) Clearly, y(t) is a function of the current value of x(t). So, this s/m is causal.

d) System computes the even part of the input signal: $y(t) = \frac{x(t)+x(-t)}{2}$

i) System passes the all-zero input test; that is, when *x*(*t*) = 0, *y*(*t*) = 0. So, we have to prove that the property either holds or does not hold.
Let $x_1(t) \to y_1(t) \,\&\, x_2(t) \to y_2(t).$
$ax_1(t) + bx_2(t) \to \frac{(ax_1(t)+bx_2(t))+(ax_1(-t)+bx_2(-t))}{2} := y(t)$
So, $y(t) = \frac{a(x_1(t)+x_1(-t))}{2} + \frac{b(x_2(t)+x_2(-t))}{2} = ay_1(t) + by_2(t)$
So, the system satisfies the linearity property and is hence, linear.

ii) When *x*(*t*) = sin(2π*t*), we have *y*(*t*) = 0. Let τ = ¼, $x_{shifted}(t)$ = *x*(*t*-τ) = sin(2π (*t*– ¼)) = sin(2π*t*– π/2) = cos(2π*t*) and $y_{shifted}(t)$ = cos(2π*t*) but this does not equal *y*(*t*-τ).
Alternate solution: Let $x_{shifted}(t) = x(t - t_0) =>,$
$y_{shifted}(t) = \frac{x_s(t)+x_s(-t)}{2} = \frac{x(t-t_0)+x(-t-t_0)}{2}$
$$y(t - t_0) = \frac{x(t - t_0) + x(-(t - t_0))}{2} = \frac{x(t - t_0) + x(-t + t_0)}{2}.$$
$So, y_{shifted}(t) \neq y(t - t_0)$. The system is not time-invariant.

iii) If $|x(t)| \leq B\ \forall t$, then $|y(t)| = \left|\frac{x(t)+x(-t)}{2}\right| = \frac{1}{2}|x(t) + x(-t)| \leq \frac{1}{2}|x(t)| + \frac{1}{2}|x(-t)| \leq \frac{1}{2}B + \frac{1}{2}B$. Hence, $|y(t)| \leq B$. Yes, system is BIBO stable.

iv) Let *t* = -1. Then $y(-1) = \frac{x(-1)+x(1)}{2}$, which clearly depends on a future value of *x*(*t*) viz. *x*(1). System is not causal.

2. Solution:

a) $\int_0^{10} e^{-(t-4)} u(t - 4)\delta(t - 5)\, dt$
$= e^{-(5-4)} u(5 - 4)\ \ (Using\ sifting\ property\ of\ \delta(t))$
$= e^{-1}u(1) = e^{-1}$

b) $\int_{-\infty}^{t-5} \delta(\tau - 1)d\tau = \begin{cases} 1, & -\infty < 1 \le t-5 \\ 0, & else \end{cases} = \begin{cases} 1, & t \ge 6 \\ 0, & else \end{cases}$

(Using $\int_a^b \delta(t)dt = \begin{cases} 1, & 0 \in [a,b], b > a \\ 0, & otherwise \end{cases}$)

c) $\frac{d}{dt}\left(e^{-(t-4)}u(t-4)\right) = -e^{-(t-4)}u(t-4) + e^{-(t-4)}\delta(t-4)$

$$= e^{-(t-4)}(\delta(t-4) - u(t-4))$$

(Using $\frac{d}{dt}(u(t)) = \delta(t)$)

d) $\delta(t-1) * \delta(t-2) * \delta(t) = (\delta(t-1) * \delta(t-2)) * \delta(t)$

$$= \left(\int_{-\infty}^{\infty} \delta(\tau-1)\delta(t-\tau-2)d\tau\right) * \delta(t)$$

$$= \delta(t-3) * \delta(t) = \left(\int_{-\infty}^{\infty} \delta(\tau)\delta(t-\tau-3)d\tau\right) = \delta(t-3)$$

3. **Solution:**

A continuous-time LTI system is bounded-input bounded-output (BIBO) stable if its impulse response *h*(*t*) satisfies $\int |h(t)|dt < \infty$

For a derivation of this condition, let $x(t)$ be any bounded input. Then $|x(t)| \le B$ for some positive *B*. Then,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right| \le \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|d\tau \le \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|d\tau$$

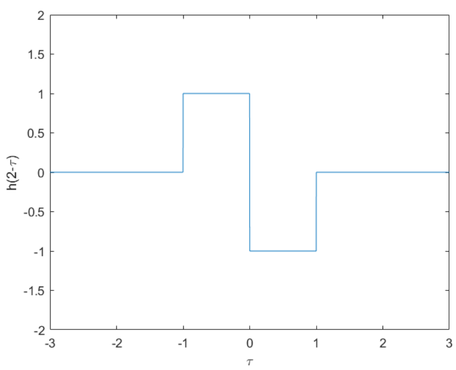
$$|y(t)| \le B\int_{-\infty}^{\infty} |h(\tau)|d\tau$$

So, $\int_{-\infty}^{\infty}|h(\tau)|d\tau < \infty \Rightarrow |y(t)| < \infty \ \forall\, t\ (Condition\ for\ BIBO\ stability)$.)

a) $\int_{-\infty}^{\infty}|h(\tau)|d\tau = 1 + 1 = 2$

Since $\int_{-\infty}^{\infty}|h(\tau)|d\tau < \infty$, the system is BIBO stable.

b)



c) $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau \Rightarrow y(2) = \int u(\tau)h(2-\tau)d\tau$

$So, y(2) = \int_0^{\infty} h(2-\tau)d\tau$. Hence, *y*(2) can be calculated by summing the area under the graph in part (b) over the interval $[0,\infty)$. So, $y(2) = -1$.

d) $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$

$$= \int_0^{\infty} h(t-\tau)d\tau = \int_t^{-\infty} h(\tau')(-d\tau') \quad (Substituting\ \tau' = t-\tau)$$

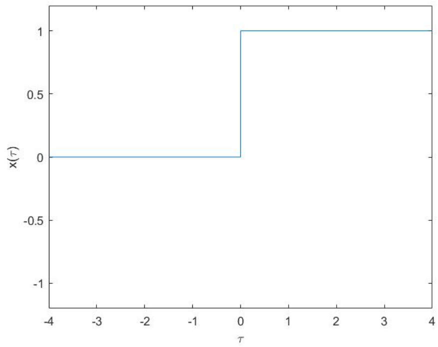
$$= \int_{-\infty}^{t} h(\tau')d\tau' = \begin{cases} 0 & ,t < 1 \\ -(t-1) & ,1 \le t < 2 \\ (t-2)-1 & ,2 \le t < 3 \\ 0 & ,3 \le t \end{cases} = \begin{cases} 0 & ,t \in (-\infty, 1) \cup [3, \infty) \\ 1-t & ,t \in [1,2) \\ t-3 & ,t \in [2,3) \end{cases}$$

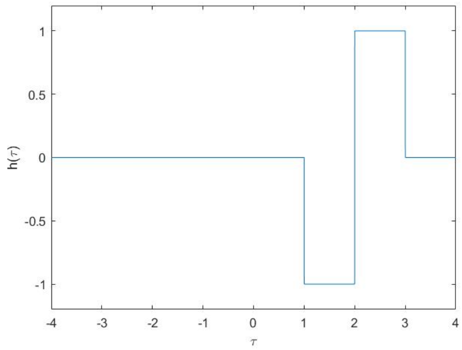
So, $T_1 = 1$ & $T_2 = 3$.

Graphically, this integration is worked out as follows:

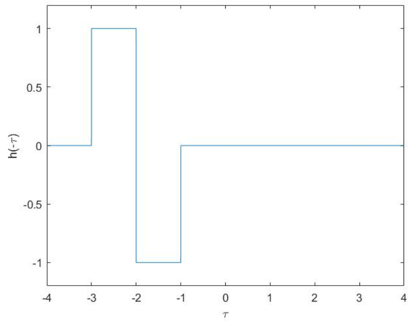
Convolution Integral formula: $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

Step 1: Take $x(\tau)$ and $h(\tau)$.



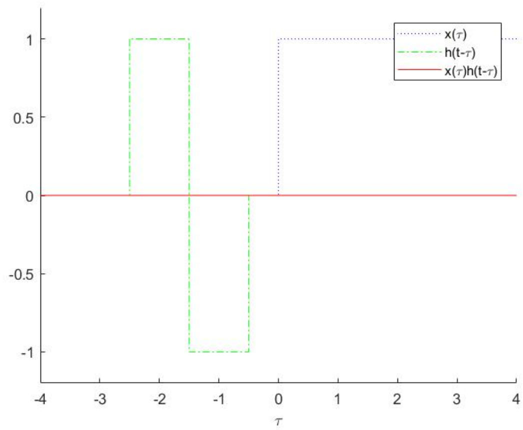


Step 2: Flip $h(\tau)$ around the y-axis to generate $h(-\tau)$.

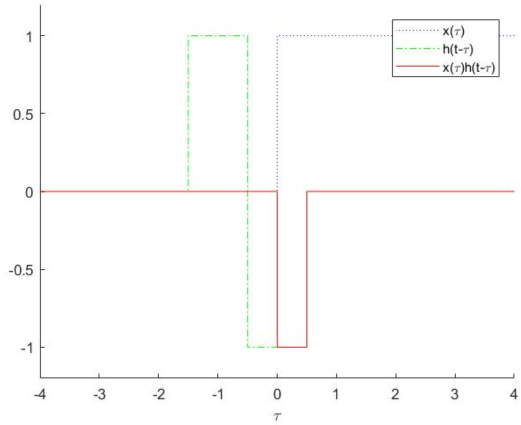


Step 3: Shift $h(-\tau)$ right by t to generate $h(t-\tau)$. Multiply it by $x(\tau)$ to generate $x(\tau)h(t-\tau)$ (the function being integrated in the convolution).

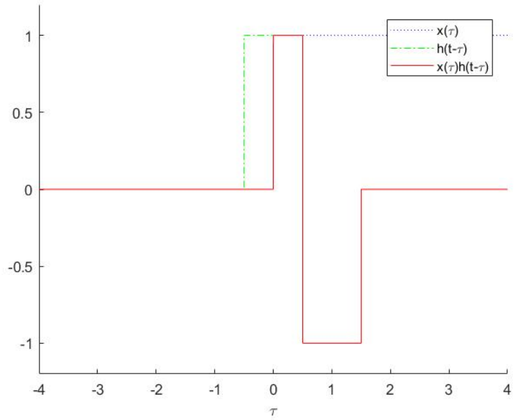
Plots for each of the four intervals of interest are shown below.



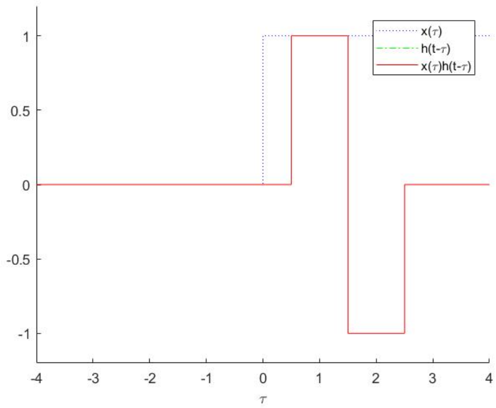
t < 1, no overlap



1<= t < 2 , overlap of length t-1.



2<= t < 3, overlap of length t-1.



3 <= t, overlap of length t-1.

Step 4: Find the area under the plot for $x(\tau)h(t-\tau)$. This step performs the integration from $-\infty$ to ∞.

Looking at the plot,

For t<1, there is no overlap, and the net area is 0.

For 1<= t < 2, the net area is –(t-1) = 1-t.

For 2<= t < 3, the net area is –1*1+1(t-1-1) = -1+t-2 = t-3.

For 3<= t, the net area is –1*1+1*1 = 0.

This matches the answer derived above.

4. **Solution:**

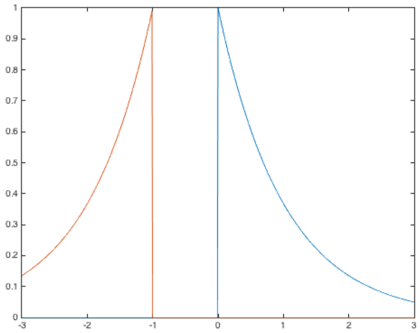
$y(t) = e^{-at}u(t) * e^{-at}u(t)$

$$= \int_{-\infty}^{\infty} e^{-a\tau} u(\tau) e^{-a(t-\tau)} u(t-\tau) d\tau$$
$$= \int_{-\infty}^{\infty} e^{-at} u(\tau) u(t-\tau) d\tau$$

The term $u(\tau)$ is 1 when $\tau \geq 0$ and 0 otherwise. So, the lower limit in the integral becomes 0. The term $u(t-\tau)$ is 1 when $t-\tau \geq 0$ and 0 otherwise. With $t \geq \tau$ and $\tau \geq 0$, we have $t \geq \tau \geq 0$. So, the upper limit in the integral becomes t. For $t > 0$,

$$y(t) = \int_0^t e^{-at} d\tau = e^{-at} \int_0^t d\tau = t e^{-at}$$

When $t < 0$,



and there is no overlap between $e^{-a\tau}u(\tau)$ and $e^{-a(t-\tau)}u(t-\tau)$.

The complete answer is

$$y(t) = t e^{-at} u(t)$$

Comparing this with Exercise 9.4 on page 265 of the *Signal Processing First* textbook, we see that $y(t) = \lim_{b \to a} y_{ex}(t)$, which is the expected behavior.

Also, compare this answer to the discrete-time version of the problem in the Handout F on Convolution of Two Exponential Sequences in discrete time at

http://users.ece.utexas.edu/~bevans/courses/signals/handouts/Appendix%20F%20Convolution%20Exp%20Sequences.pdf